Assignment 1

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Fundamentals and Review

Exercise 1: Likelihood Estimation

Problem 1

The geometric distribution has pdf:
$P(x|\theta) = \theta(1-\theta)^{x-1}$, for $x = 1, 2, 3, \ldots$
and likelihood function:
$L(\theta|x) = (1-\theta)^{x_1-1}\theta(1-\theta)^{x_2-1}\theta(1-\theta)^{x_3-1}\theta \ldots (1-\theta)^{x_n-1}\theta = \theta^n(1-\theta)^{\sum_1^n x_i - n}$
Taking the log of the likelihood function produces:
$log(L(\theta|x)) = nlog(\theta) + ((\sum_1^n x_i) - n)log(1-\theta)$
Differentiating and setting the result equal to zero produces:
$\frac{d(log(L(\theta|x)))}{d\theta} = \frac{n}{\theta} - \frac{(\sum_1^n x_i)-n}{1-\theta} = 0$
$\frac{n}{\theta} = \frac{(\sum_1^n x_i)-n}{1-\theta}$
$n(1-\theta) = \theta((\sum_1^n x_i) - n))$
$n - n\theta = \theta(\sum_1^n x_i) - n\theta$
$n = \theta(\sum_1^n x_i)$
$\hat{\theta} = \frac{n}{\sum_1^n x_i}$
This is equivaent to the number of successes divided by the total number of trials.

Problem 2

The uniform distribution on $[a, b]$ has pdf: $P(x|a, b) = \frac{1}{b-a}$ and likelihood function:

$$L(a, b|x) = \begin{cases} (\frac{1}{b-a})^n & \text{if all } x_i \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

We don't need to take the log and differentiate because the likelihood is at its maximum when all the samples are contained the the interval and the quantity $b - a$ is minimized, giving us the estimates:
$\hat{a} = min(x_i)$
$\hat{b} = max(x_i)$

Exercise 2: Loss Functions

Problem 1

If the data is described by the linear model $y = \vec{x}\vec{w} + \vec{\epsilon}$ where $\vec{\epsilon} \sim N(0, \sigma^2)$ then the squared error (L2) loss can be described as:
$\sum_{i=1}^N (y_i - \hat{y_i})^2 = \sum_{i=1}^N (y_i - \hat{x_i}w)^2$

The log likelihood of the model is as follows:
$log(L(y|x, w)) = \sum_{i=1}^N log(N(y_i|x_iw, \sigma^2))$
$\sum_{i=1}^N log(\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{(y_i - x_iw)^2}{2\sigma^2}}) = -\frac{N}{2}log(2\pi\sigma^2) - \sum_{i=1}^N \frac{(y_i - x_iw)^2}{2\sigma^2}$
Since σ^2 is known, we only need to maximize the second term above:

$-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(y_i - x_i w)^2$ again since σ^2 is known, we specifically would maximize the following:
$-\sum_{i=1}^{N}(y_i - x_i w)^2$
This is equivalent to the minimization of the squared error loss above.

Problem 2

The mean absolute error can be described as:
$\frac{1}{n}\sum_{i=1}^{n}|y_i - \hat{y}_i|$

The log likelihood of a $Y \sim LaPlace(\theta)$ is:
$log(L(\theta|y)) = -nlog(2\theta) - \frac{1}{\theta}\sum_{i=1}^{n}|y_i - \mu|$
If we take the derivative and set it equal to 0 we find:
$\frac{d(log(L(\theta|y)))}{d\theta} = -\frac{n}{2\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n}|y_i - \mu| = 0$
The estimate for θ becomes:
$\hat{\theta} = \frac{2}{n}\sum_{i=1}^{n}|y_i - \mu|$ Since the sum of absolute differences is the same as the sum of differences from the mean, this is equivalent to the mean absolute error above.

Exercise 3: Decision Rules

Problem 1

For squared error loss the risk function would be:
$R(\theta, \delta) = Var_\theta \delta(X) + (E_\theta \delta(X) + (Bias_\theta \delta(X))^2$ If the decision rule is unbiased then only the variance would need to be minimized to minimize the risk. If X has mean μ and variance $\sigma^2 < \infty$ and we know that the decision rule needs to be unbiased then we know that it must satisfy $E_\theta X = \theta$ for all θ :
$E_\theta(X - \theta)^2 = Var_\theta X$ Since we are only restricted to considering unbiased decision rules, we know that minimizing the risk is equivalent to minimizing the variance. In this case $\bar{X}$, or μ , satisfies the equation above, where the mean squared error equals the variance. No other value for θ further minimizes the variance so we know that the mean is the optimal decision rule.

Problem 2

The risk function can be defined as:
$R(\theta, \delta) = E_\theta L(\theta, \delta(X))$
In absolute (L1) loss, the loss function is:
$L(\theta, \delta(X)) = |\theta - \delta(X)|$
We know that the variance is finite, so we can assume that the distribution of X goes to zero eventually. We will call these bounds (A, B). Now can say: $E|\theta - \delta(X)| = \int_A^{\delta(x)}(\delta(x) - \theta)dF + \int_{\delta(x)}^{B}(\theta - \delta(x))dF$
Using Leibniz's Rule we have:
$\frac{\partial E}{\partial \delta(x)} = \int_A^{\delta(x)}(1)dF + (\delta(x) - \delta(x))(1) - (\delta(x) - A)(0) + \int_{\delta(x)}^{B}(-1))dF + (B - \delta(x))(0) - (\delta(x) - \delta(x))(1)$
$= \int_A^{\delta(x)} dF - \int_{\delta(x)}^{B} dF$
If we set this derivative equal to zero we find that:
$\int_A^{\delta(x)} dF = \int_{\delta(x)}^{B} dF$
Therefore, $\delta(x)$ must be a value that allows the $P(X < \delta(x)) = P(X > \delta(x))$. This occurs at the median of the distribution.

Exercise 4: Convexity

Problem 1

The cross entropy loss is defined as $L(y,p) = -(ylog(p) + (1-y)log(1-p))$ where $p = 1/(1+exp(-\beta x))$
$L = -(ylog(\frac{1}{1+exp(-\beta x)}) + (1-y)log(1 - \frac{1}{1+exp(-\beta x)}))$
$= -(ylog(1) - ylog(1+exp(-\beta x)) + (1-y)log(\frac{exp(-\beta x)}{1+exp(-\beta x)}))$
$= -(-ylog(1+exp(-\beta x)) + log(\frac{exp(-\beta x)}{1+exp(-\beta x)}) - ylog(\frac{exp(-\beta x)}{1+exp(-\beta x)}))$
$= -(-ylog(1+exp(-\beta x)) + log(exp(-\beta x)) - log(1+exp(-\beta x)) - ylog(exp(-\beta x)) + ylog(1+exp(-\beta x)))$
$= -(log(exp(-\beta x)) - log(1+exp(-\beta x)) - ylog(exp(-\beta x))) = -(-\beta x - log(1+exp(-\beta x)) - y(-\beta x))$
$= -(-\beta x - log(1+exp(-\beta x)) + y\beta x)$
$= \beta x + log(1+exp(-\beta x)) - y\beta x$
$\frac{\partial L}{\partial \beta} = \frac{-xexp(-\beta x)}{1+exp(-\beta x)} + x - xy$
$\frac{\partial^2 L}{\partial \beta^2} = \frac{x^2 exp(\beta x)}{(exp(\beta x)+1)^2}$
If x is fixed, this second derivative with respect to β is ≥ 0 for all β , meaning L is convex with respect to β .

Problem 2

The mean squared error loss is defined as $L(y,p) = (y-p)^2$
$= (y - \frac{1}{1+exp(-\beta x)})$
$\frac{\partial L}{\partial \beta} = \frac{-2xexp(-x\beta)(y - \frac{1}{1+exp(-x\beta)})}{(1+exp(-x\beta))^2}$
$= \frac{2xexp(x\beta)((y-1)exp(x\beta)+y)}{(exp(x\beta)+1)^3}$
$\frac{\partial^2 L}{\partial \beta^2} = -\frac{2x((exp(x\beta)+1)^3(x(y-1)exp(2x\beta)+xexp(x\beta)((y-1)exp(x\beta)+y)) - 3x(exp(x\beta)+1)^2((y-1)exp(x\beta)+y)exp(2x\beta))}{(exp(x\beta)+1)^6}$
$= \frac{2x^2 exp(x\beta)((y-1)exp(2x\beta)+2exp(x\beta)-y)}{(exp(x\beta)+1)^4}$
If x is fixed, this second derivative with respect to β is not always ≥ 0 for all β , meaning L is NOT convex with respect to β
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Exercise 5: Decision Boundaries

Problem 1

$f_\theta(x) = \frac{1}{1+exp(-(B_0+B_1x_1))}$ If we take a look at the case when n=1 with $\theta_0 = 0$ and $\theta_1 = 1$, then:
$f_\theta(x) = \frac{1}{1+exp(-x_1)}$
In this case the decision threshold for $f_\theta(0) = \frac{1}{2}$
In the case when $\theta_0 = 0$ and $\theta_1 = -2$, then:
$f_\theta(x) = \frac{1}{1+exp(-2x_1)}$
In this case the decision threshold is still $f_\theta(0) = \frac{1}{2}$
In the case when $\theta_0 = 0.5$ and $\theta_1 = 1$, then:
$f_\theta(x) = \frac{1}{1+exp(-0.5-x_1)}$
In this case the decision threshold for $f_\theta(0) = \frac{1}{1+exp(-0.5)}$

Problem 2

logit = log odds
$log(\frac{x}{1-x})$
$= log(\frac{1/(1+exp(-Bx))}{1-(1/(1+exp(-Bx)))})$
$= log(\frac{1/(1+exp(-Bx))}{exp(-Bx)/(1+exp(-Bx))})$

$= log(\frac{1+exp(-Bx)}{(1+exp(-Bx))exp(-Bx)})$
$= log(\frac{1}{exp(-Bx)})$
$= log(1) - log(exp(-Bx))$
$= 0 - (-Bx)$
$= Bx$

Parametric Learning

Exercise 6: Sufficient Statistic

The pdf of X_i for $i \in (1, n)$ is:
$f_n(x|\mu) = \prod_{i=1}^{n} \frac{1}{(2\pi)^{1/2}\sigma} exp(\frac{-(x_i-\mu)^2}{2\sigma^2})$
to use the factorization theorem (whereby $T(X)$ is a sufficient statistic for a given θ if the join pdf can be factorized as $F_n(x|\theta) = u(x)v[T(x), \theta]$), this can be rewritten as the product of a function that does not depend on μ and one that depends only on $\sum_{i=1}^{n} x_i$:
$f_n(x|\mu) = \frac{1}{(2\pi)^{1/2}\sigma^n} exp(\frac{\sum_{i=1}^{n} x_i^2}{2\sigma^2}) exp(\frac{\mu}{\sigma} \sum_{i=1}^{n} x_i - \frac{n\mu^2}{2\sigma^2})$
Based on the theorem we can say that $T = \sum_{i=1}^{n} X_i = n\bar{x}$, so that $\bar{X}$ or function of $\bar{X}$ is a sufficient statistic for μ .

Exercise 7: Ancillarity

We will use $Z_1, \ldots, Z_n$ iid observations from $F(x)$ such that $X_1 = Z_1 + \theta, \ldots, X_n = Z_n + \theta$ and $min_i(X_i) = min_i(Z_i + \theta), max_i(X_i) = max_i(Z_i + \theta)$. The cdf of $R = max_i(X_i) - min_i(X_i)$ can be described as:
$F_R(r|\theta) = P_\theta(R \leq r)$
$F_R(r|\theta) = P_\theta(max_i X_i - min_i X_i \leq r)$
$F_R(r|\theta) = P_\theta(max_i(X_i + \theta) - min_i(Z_i + \theta) \leq r)$
$F_R(r|\theta) = P_\theta(max_i Z_i - min_i Z_i \leq r)$
Since the distribution of Z_i doesn't depend on θ , neither does the probability above which means $R = max_i(X_i) - min_i(X_i) = max_i(Z_i) - min_i(Z_i)$ is an ancillary statistic.

Parametric Learning

Exercise 8: Completeness

In this case $f(x|\mu) = (\frac{1}{\mu\sqrt{2\pi}})^n exp(\frac{-(x_i-\mu)^2}{2\mu^2})$
$f(x|\mu) = (\frac{1}{2\pi})^n exp(\frac{-n}{2})(\frac{1}{\mu})^n exp(\frac{-x_i^2}{2\mu^2}) exp(\frac{nx_i}{\mu})$
By the factorization theorem (whereby $T(X)$ is a sufficient statistic for a given θ if the join pdf can be factorized as $F_n(x|\theta) = u(x)v[T(x), \theta]$), $T(X) = (\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i^2)$ is a sufficient statistic.
We know $E(\sum_{i=1}^{n} x_i) = E(n\bar{x}) = n\mu$ and $E(x_i^2) = \sigma^2 + \mu^2 = 2\mu^2$ so that if $g(T) = n\sum_{i=1}^{n} x_i^2 - 2(\sum_{i=1}^{n} x_i)^2$
$E_\mu[g(T)] = nE[x_1^2 + \ldots + x_n^2] - 2E[\sum_{i=1}^{n} x_i]E[\sum_{i=1}^{n} x_i]$
$E_\mu[g(T)] = nE[x_1^2] + \ldots + nE[x_n^2] - 2(n\mu)^2$
$E_\mu[g(T)] = 2n^2\mu^2 - 2n^2\mu^2 = 0$
Now $P_\mu(n\sum_{i=1}^{n} x_i^2 - 2(\sum_{i=1}^{n} x_i)^2 = 0)$ so:
$P_\mu(n\sum_{i=1}^{n} x_i^2 = 2(\sum_{i=1}^{n} x_i)^2) \neq 1$ for all μ which means it is not a complete statistic.

Exercise 9: Regular Exponential Family

The exponential family of probability distributions have the following general form:
$p(x|\eta) = h(x)exp(\eta^T T(x) - A(\eta))$ The probability mass function of a Poisson random variable is:
$p(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$
which can also be written as:
$p(x|\lambda) = \frac{1}{x!}exp(xlog\lambda - \lambda)$
Using this form it is clear that the Poisson is part of the regular exponential family:
$\eta = log\lambda$
$T(x) = x$
$A(\eta) = \lambda = e^\eta$
$h(x) = \frac{1}{x!}$

Exercise 10: Regular Exponential Family

If X is distributed according to a canonical exponential family the pdf can be expressed as:
$p(x|n) = h(x)exp(\eta T(x) - B(\eta))$
The moment generating function (MGF) of $T(X)$ is then:
$M_T(s) = E[exp(sT(x))|\eta] = exp(B(s+\eta) - B(\eta))$ for s near 0.
It follows from the moment functions that:
$E[T(x)|\eta] = \frac{\partial B(\eta)}{\partial \eta}$
$Var[T(x)|\eta] = \frac{\partial^2 B(\eta)}{\partial \eta^2}$
It then follows that the Covariance with respect to η could be expressed as:
$Cov_\eta(T_i(X), T_j(X)) = \frac{\partial^2 B(\eta)}{\partial \eta_i \partial \eta_j}$ for $i, j \in \{1, 2, \dots, n\}$

Exercise 11: Delta Method

$g(X_n) = g(\theta) + g'(\theta)(X_n - \theta)$
In this case:
$g(p) = p(1-p)$ and $g'(p) = 1 - 2p$
$g(\bar{x}) = g(p) + g'(p)(\bar{x} - p)$
rearrange and multiply by $\sqrt{n}$:
$\sqrt{n}[g(\bar{x}) - g(p)] = g'(p)\sqrt{n}[\bar{x} - p]$
By the definition of a delta function:
$\sqrt{n}[\bar{x} - p] \to^D N(0, p(1-p))$
by Slutsky Theorem:
$\sqrt{n}[g(\bar{x}) - g(p)] \to^D N(0, p(1-p)[g'(p)]^2)$
$\sqrt{n}[\bar{x}(1-\bar{x}) - p(1-p)] \to^D N(0, p(1-p)[g'(p)]^2)$
$\sqrt{n}[\bar{x}(1-\bar{x}) - p(1-p)] \to^D N(0, p(1-p)[1-2p]^2)$
$\hat{\tau} \sim N(p(1-p), \frac{p(1-p)(1-2p)^2}{n})$

Information Theory

Exercise 12: Joint Entropy

Problem 1

$H(X,Y) = -\sum_{x \in X}\sum_{y \in Y} P(x,y)log_2(P(x,y))$
$= -(2(\frac{1}{4}log_2(\frac{1}{4})) + 2(\frac{1}{6}log_2(\frac{1}{6})) + 2(\frac{1}{12}log_2(\frac{1}{12})))$
≈ 2.46

Problem 2

Marginal Distribution:

$$P(X) = 1/3, \text{if } x \in \{0,1,2\}$$

$H(X) = -\sum_{x \in X} p(x) log_2(p(x)) = -3(\frac{1}{3} log_2(\frac{1}{3})) \approx 1.58$

Conditional Entropy:

$H(Y|X) = -\sum_{x \in X, y \in Y} p(x,y) log(\frac{p(x,y)}{p(x)})$
$= -(2(\frac{1}{4} log_2(\frac{3}{4})) + 2(\frac{1}{6} log_2(\frac{3}{6})) + 2(\frac{1}{12} log_2(\frac{3}{12}))$
≈ 0.87

Problem 3

By the Chain Rule: $H(Y|X) = H(X,Y) - H(X)$
Using results from above:
$0.87 \approx 2.46 - 1.58$
So, the Chain rule is satisfied using the results from above.

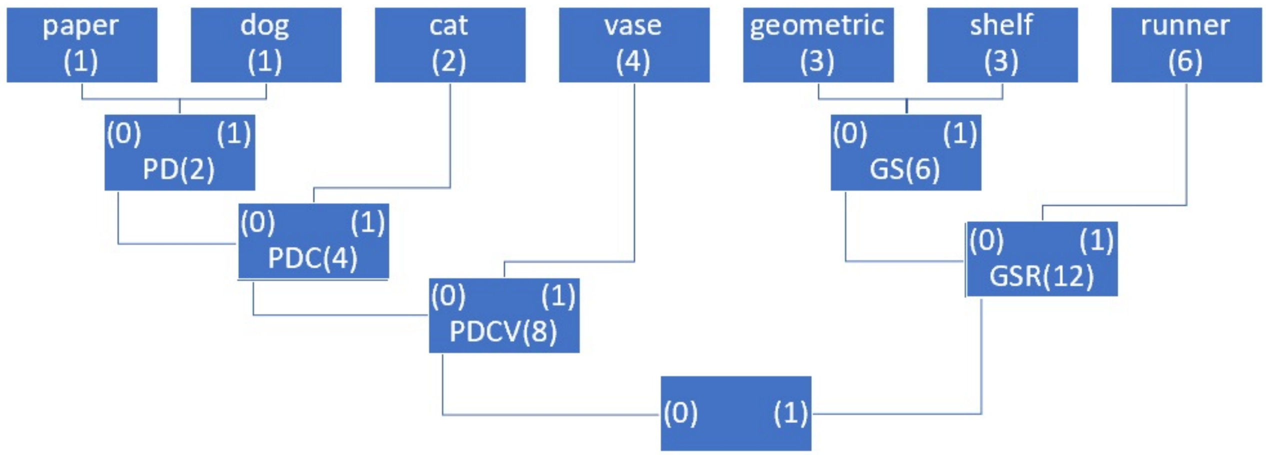
Exercise 13: Differential Entropy

$H(X) = -\int_{-\infty}^{\infty} N(x|\mu,\Sigma) ln(N(x|\mu,\Sigma))dx = -E[ln(N(x|\mu,\Sigma))]$
$= -E[ln(\frac{1}{\sqrt{(2\pi)^D|\Sigma|}} e^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu))}]$
where D is the dimension of x.
$= \frac{D}{2} ln(2\pi) + \frac{1}{2} ln(|\Sigma|) + \frac{1}{2} E[(x-\mu)^T \Sigma^{-1}(x-\mu)]$
Using the trick $trace(x^T\Sigma^{-1}x) = trace(\Sigma^-1xx^T)$, we can simplify this to:
$= \frac{D}{2} ln(2\pi) + \frac{1}{2} ln(|\Sigma|) + \frac{1}{2} E[trace((x-\mu)^T \Sigma^{-1}(x-\mu))]$
$= \frac{D}{2} ln(2\pi) + \frac{1}{2} ln(|\Sigma|) + \frac{1}{2} E[trace(\Sigma^{-1}(x-\mu)(x-\mu)^T)]$
$= \frac{D}{2} ln(2\pi) + \frac{1}{2} ln(|\Sigma|) + \frac{1}{2} trace(E[\Sigma^{-1}(x-\mu)(x-\mu)^T])$
$= \frac{D}{2} ln(2\pi) + \frac{1}{2} ln(|\Sigma|) + \frac{1}{2} trace(\Sigma^{-1} E[(x-\mu)(x-\mu)^T])$
$= \frac{D}{2} ln(2\pi) + \frac{1}{2} ln(|\Sigma|) + \frac{1}{2} trace(\Sigma^{-1}\Sigma)$
$= \frac{D}{2} ln(2\pi) + \frac{1}{2} ln(|\Sigma|) + \frac{1}{2} trace(I) = \frac{D}{2} ln(2\pi) + \frac{1}{2} ln(|\Sigma|) + \frac{D}{2} = \frac{1}{2}(Dln(2\pi) + ln(|\Sigma|) + D)$

Interview Questions

Huffman coding and probability trees

Exercise 1



Average bits: 52/20=2.6

NAME	HUFFMAN ENCODING	BITS	FREQUENCY	TOTAL BITS
Paper	0000	4	1	4
Dog	0001	4	1	4
Cat	001	3	2	6
Vase	01	2	4	8
Geometric	100	3	3	9
Shelf	101	3	3	9
Runner	11	2	6	12

Grand Total: 52