Graphs 1

Graph Terminology and Traversals

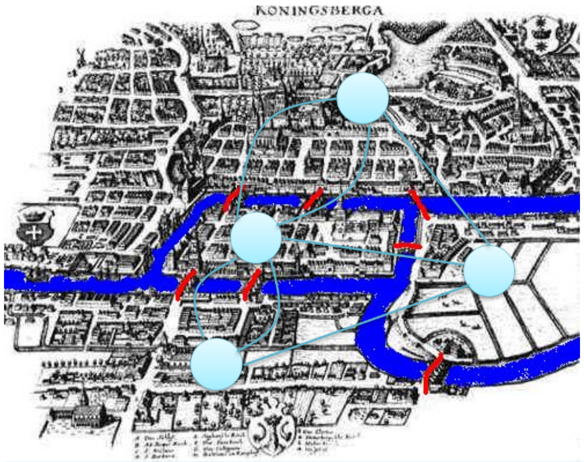
Objectives

- Understand graph terminology
- Implement graphs using
  - Adjacency lists and
  - Adjacency matrices
- Perform graph searches
  - Depth first search
  - Breadth first search
- Perform shortest-path algorithms
  - Dijkstra's algorithm
  - A* algorithm

Graph Terminology

Graph Theory and Euler

- Graph theory is often considered to have been born with Leonhard Euler
  - In 1736 he solved the *Konigsberg bridge problem*
- Konigsberg was a city in Eastern Prussia
  - Renamed Kalinigrad when East Prussia was divided between Poland and Russia in 1945
  - Konigsberg had seven bridges in its centre
    - The inhabitants of Konigsberg liked to see if it was possible to walk across each bridge just once and then return to where they started
  - Euler proved it was impossible to do this
    - As part of this proof, he represented the problem as a graph



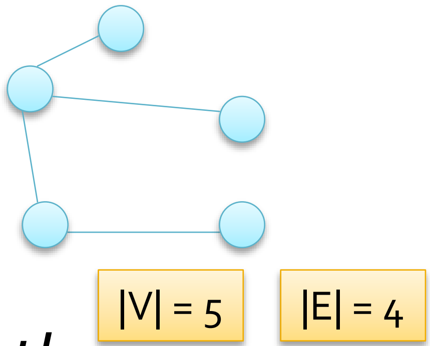
- The edges represent bridges
- The Konigsberg graph is a *multigraph*
- Multigraphs allow multiple edges between the same two vertices

Graph Uses

- Graphs are used as representations of many different types of problems
  - Network configuration
  - Airline flight booking
  - Pathfinding algorithms
  - Database dependencies
  - Task scheduling
  - Critical path analysis
  - …

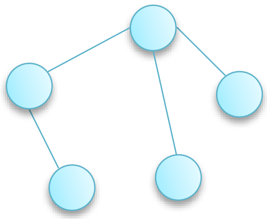
Graph Terminology

- A graph consists of two sets
  - A set *V* of *vertices* (or nodes) and
  - A set *E* of *edges* that connect vertices
  - |*V*| is the size of *V*, |*E*| the size of *E*
- Two vertices may be connected by a *path*
  - A sequence of edges that begins at one vertex and ends at the other
    - A *simple path* does not pass through the same vertex twice
    - A *cycle* is a path that starts and ends at the same vertex
    - The graph shown here is acyclic



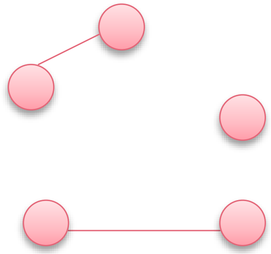
Connected and Unconnected Graphs

- A *connected* graph is one where every pair of distinct vertices has a *path* between them
  - An unconnected graph does not
- A *complete* graph is one where every pair of vertices has an *edge* between them
- A graph cannot have multiple edges between the same pair of vertices
- A graph cannot have *self edges*, an edge from and to the same vertex

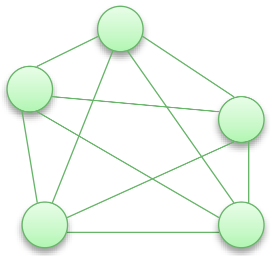


connected graph

and a tree



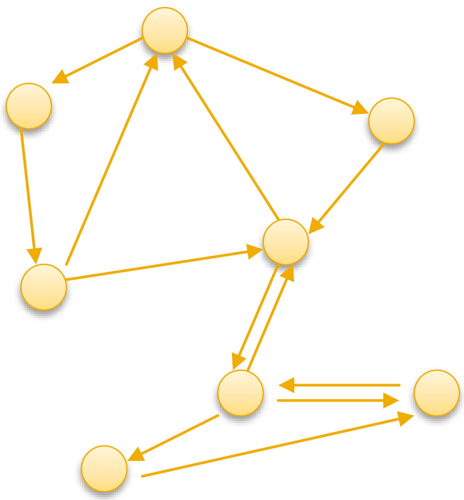
unconnected graph



complete graph

Directed Graphs

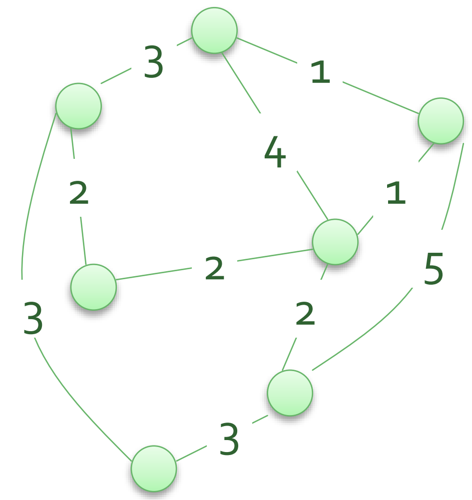
- In a *directed graph* (or digraph) each edge has a direction and is called a directed edge
- A directed edge can only be traveled in one direction
- A pair of vertices in a digraph may have two edges between them, one in each direction



directed graph

Weighted Graphs

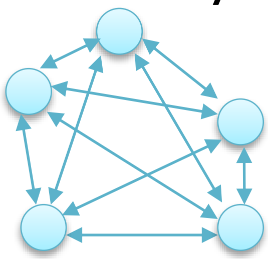
- In a *weighted graph* each edge is assigned a weight
  - Edges are labeled with their weights
- Each edge's weight represents the cost to travel along that edge
  - The cost could be distance, time, money or some other measure
  - The cost depends on the underlying problem

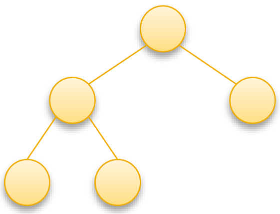


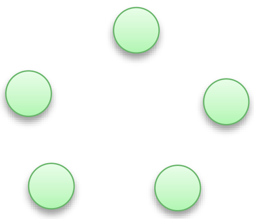
weighted graph

Numbers of Vertices and Edges

- If a graph has v vertices, how many edges does it have?
  - If every vertex is connected to every other vertex, and the graph is directed
    - $v^2 - v$
  - If the graph is a tree
    - $v - 1$
  - Minimum number of edges
    - 0







Basic Graph Operations

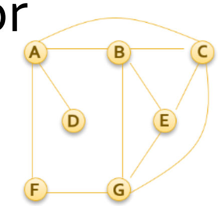
- Create an empty graph
- Test to see if a graph is empty
- Determine the number of vertices in a graph
- Determine the number of edges in a graph
- Determine if an edge exists between two vertices
  - and in a weighted graph determine its weight
- Insert a vertex
  - each vertex is assumed to have a distinct search key
- Insert an edge
- Remove a vertex, and its associated edges
- Remove an edge
- Return a vertex with a given key

Graph Implementation

- There are two common implementations of graphs
  - Both implementations require a list of all vertices in the set of vertices, V
  - The implementations differ in how edges are recorded
- Adjacency matrices
  - Provide fast lookup of individual edges
  - But waste space for sparse graphs
- Adjacency lists
  - Are more space efficient for sparse graphs
  - Can efficiently find all the neighbours of a vertex

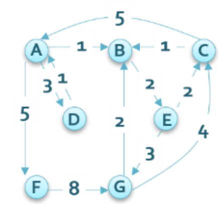
Adjacency Matrix

- The edges are recorded in an $|V| * |V|$ matrix
- In an unweighted graph entries are
  - 1 when there is an edge between vertices or
  - 0 when there is no edge between vertices
- In a weighted graph entries are either
  - The edge weight if there is an edge between vertices
  - Infinity when there is no edge between vertices
- Adjacency matrix performance
  - Looking up an edge requires O(1) time
  - Finding all neighbours of a vertex requires $O(|V|)$ time
  - The matrix requires $|V|^2$ space



line of symmetry

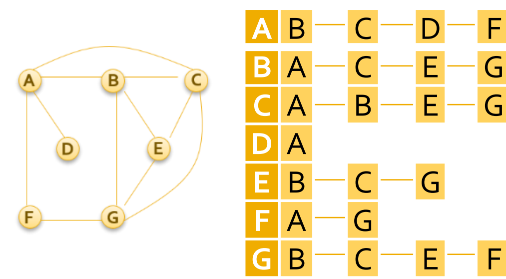
	A	B	C	D	E	F	G
A	0	1	1	1	0	1	0
B	1	0	1	0	1	0	1
C	1	1	0	0	1	0	1
D	1	0	0	0	0	0	0
E	0	1	1	0	0	0	1
F	1	0	0	0	0	0	1
G	0	1	1	0	1	1	0



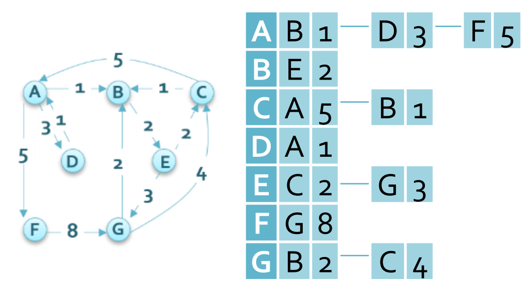
	A	B	C	D	E	F	G
A	∞	1	∞	3	∞	5	∞
B	∞	∞	∞	∞	2	∞	∞
C	5	1	∞	∞	∞	∞	∞
D	1	∞	∞	∞	∞	∞	∞
E	∞	∞	2	∞	∞	∞	3
F	∞	∞	∞	∞	∞	∞	8
G	∞	2	4	∞	∞	∞	∞

Adjacency Lists

- The edges are recorded in an array size $|V|$ of linked lists
- In an unweighted graph a list at index *i* records keys of vertices adjacent to vertex *i*



- In a weighted graph a list at index *i* contains pairs
  - Which record vertex keys (of vertices adjacent to *i*)
  - And their associated edge weights



- Adjacency List Performance
  - Looking up an edge requires time proportional to the average number of edges
  - Finding all vertices adjacent to a given vertex also takes time proportional to the average number of edges
  - The list requires $O(|E|)$ space

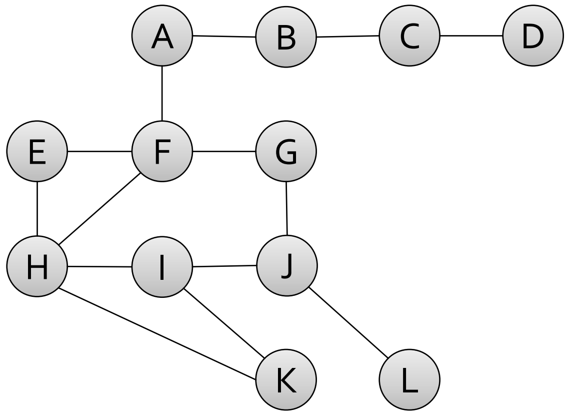
Traversals

Graph Traversals

- A graph traversal algorithm visits all of the vertices that can be reached from the start node
  - If the graph is not connected some of the vertices will not be visited
  - Therefore, a graph traversal algorithm can be used to see if a graph is connected
- Vertices should be marked as *visited*
  - Otherwise, a traversal will go into an infinite loop if the graph contains a cycle

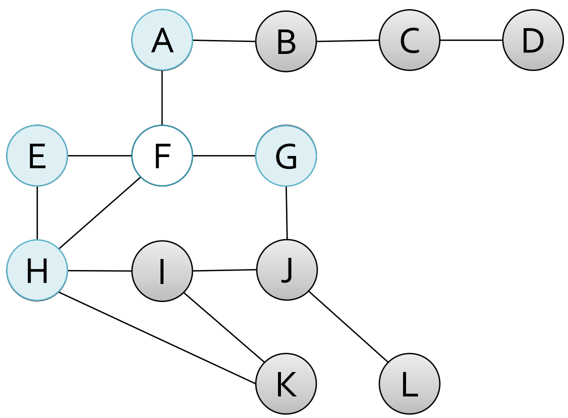
Breadth First Search 1

- Visit a vertex, *v*
  - Visit all adjacent vertices
  - Before considering next
- Uses a *queue* to store vertices
  - Queue are FIFO
- BFS:
  - visit and insert start
  - while (*q* not empty)
    - peek at front vertex, *v*
    - if *v* has unvisited neighbour visit it and insert it in *q*
    - else remove *v* from *q*
  - end while



Breadth First Search 2

- Visit a vertex, *v*
  - Visit all adjacent vertices
  - Before considering next
- Uses a *queue* to store vertices
  - Queue are FIFO
- BFS:
  - visit and insert start
  - while (*q* not empty)
    - peek at front vertex, *v*
    - if *v* has unvisited neighbour visit it and insert it in *q*
    - else remove *v* from *q*
  - end while

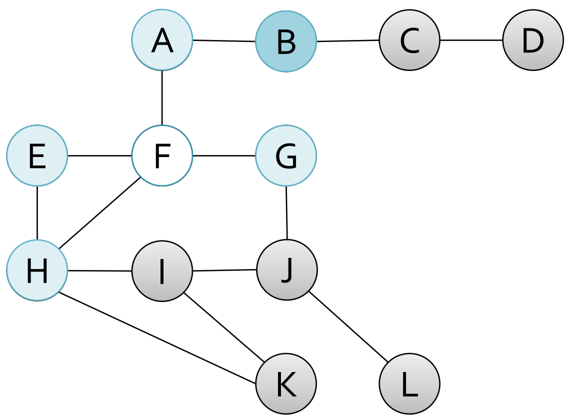


queue: F A E G H

visited: F A E G H

Breadth First Search 3

- Visit a vertex, *v*
  - Visit all adjacent vertices
  - Before considering next
- Uses a *queue* to store vertices
  - Queue are FIFO
- BFS:
  - visit and insert start
  - while (*q* not empty)
    - peek at front vertex, *v*
    - if *v* has unvisited neighbour visit it and insert it in *q*
    - else remove *v* from *q*
  - end while

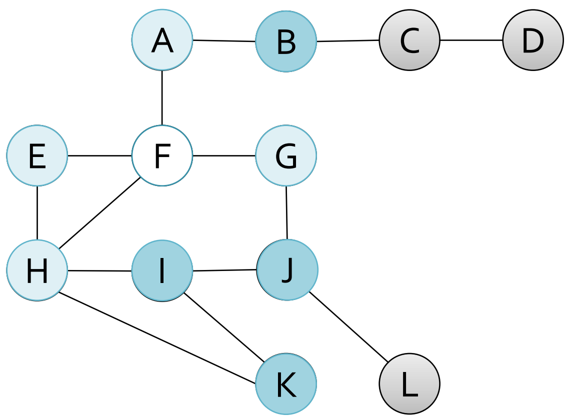


queue: A E G H B

visited: F A E G H B

Breadth First Search 4

- Visit a vertex, *v*
  - Visit all adjacent vertices
  - Before considering next
- Uses a *queue* to store vertices
  - Queue are FIFO
- BFS:
  - visit and insert start
  - while (*q* not empty)
    - peek at front vertex, *v*
    - if *v* has unvisited neighbour visit it and insert it in *q*
    - else remove *v* from *q*
  - end while

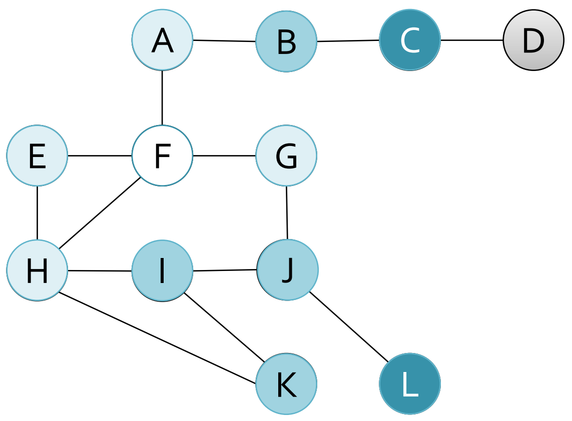


queue: G H B J I K

visited: F A E G H B J I K

Breadth First Search 5

- Visit a vertex, *v*
  - Visit all adjacent vertices
  - Before considering next
- Uses a *queue* to store vertices
  - Queue are FIFO
- BFS:
  - visit and insert start
  - while (*q* not empty)
    - peek at front vertex, *v*
    - if *v* has unvisited neighbour visit it and insert it in *q*
    - else remove *v* from *q*
  - end while

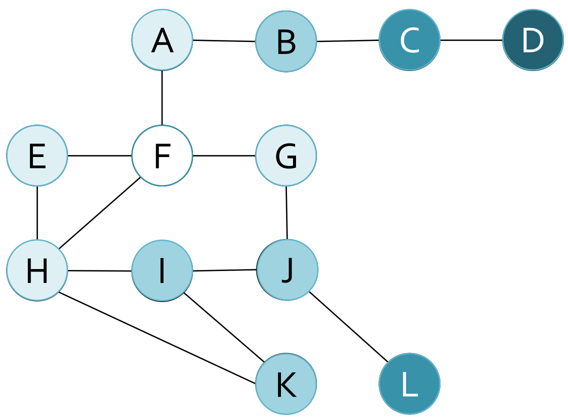


queue: B J I K C L

visited: F A E G H B J I K C L

Breadth First Search 6

- Visit a vertex, *v*
  - Visit all adjacent vertices
  - Before considering next
- Uses a *queue* to store vertices
  - Queue are FIFO
- BFS:
  - visit and insert start
  - while (*q* not empty)
    - peek at front vertex, *v*
    - if *v* has unvisited neighbour visit it and insert it in *q*
    - else remove *v* from *q*
  - end while

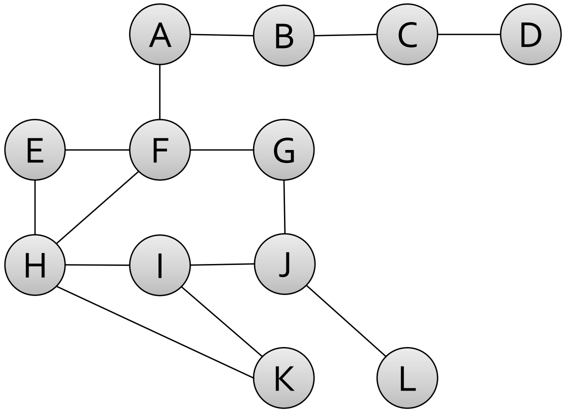


queue: C L D

visited: F A E G H B J I K C L D

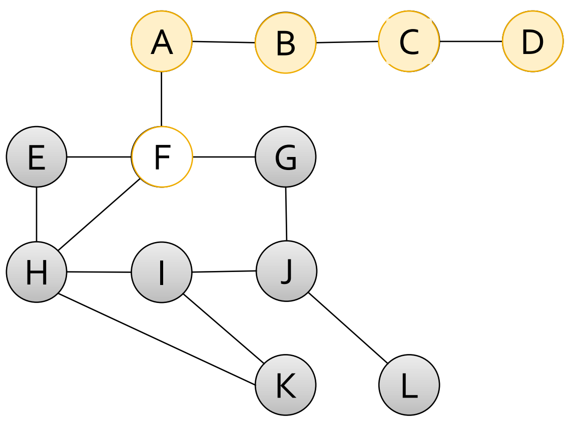
Depth First Search 1

- Visit a vertex, *v*
  - Follow a path from *v* to its end
    - Before following another path
- Uses a *stack* to store vertices
  - Stacks are LIFO
- DFS:
  - visit and insert start
  - while (*st* not empty)
    - peek at front vertex, *v*
    - if *v* has unvisited neighbour visit it and insert it in *st*
    - else remove v from *st*
  - end while



Depth First Search 2

- Visit a vertex, *v*
  - Follow a path from *v* to its end
    - Before following another path
- Uses a *stack* to store vertices
  - Stacks are LIFO
- DFS :
  - visit and insert start
  - while (*st* not empty)
    - peek at front vertex, *v*
    - if *v* has unvisited neighbour visit it and insert it in *st*
    - else remove v from *st*
  - end while



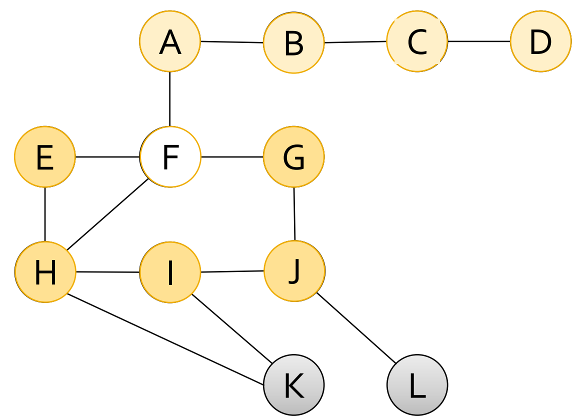
visited

F
A
B
C
D

D
C
B
A
F
stack

Depth First Search 3

- Visit a vertex, *v*
  - Follow a path from *v* to its end
    - Before following another path
- Uses a *stack* to store vertices
  - Stacks are LIFO
- DFS :
  - visit and insert start
  - while (*st* not empty)
    - peek at front vertex, *v*
    - if *v* has unvisited neighbour visit it and insert it in *st*
    - else remove v from *st*
  - end while



visited

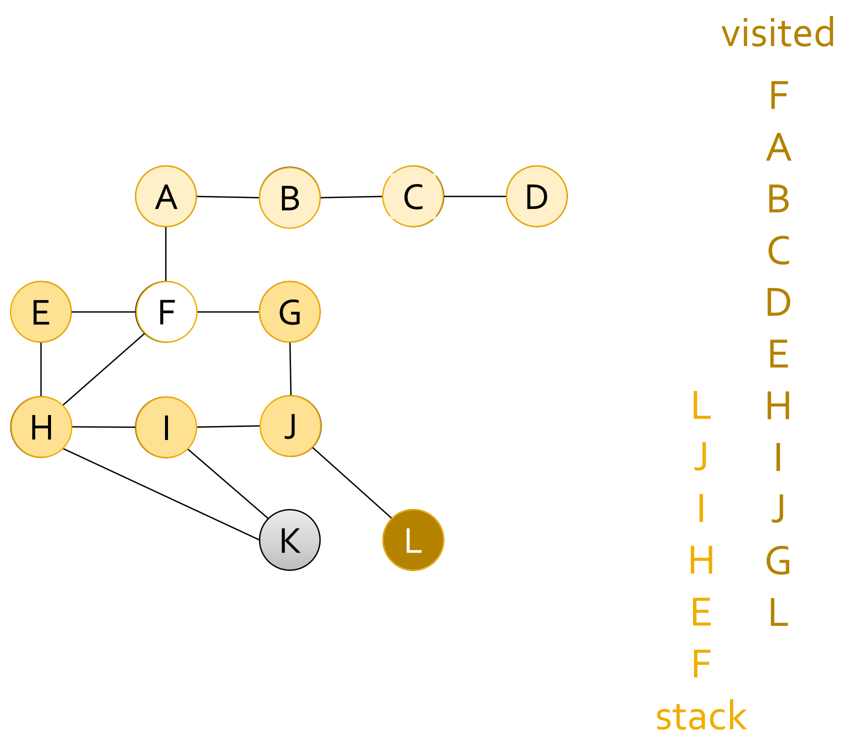
F
A
B
C
D
E
H
I
J
G

G
J
I
H
E
F

stack

Depth First Search 4

- Visit a vertex, *v*
  - Follow a path from *v* to its end
    - Before following another path
- Uses a *stack* to store vertices
  - Stacks are LIFO
- DFS :
  - visit and insert start
  - while (*st* not empty)
    - peek at front vertex, *v*
    - if *v* has unvisited neighbour visit it and insert it in *st*
    - else remove v from *st*
  - end while



Depth First Search 5

- Visit a vertex, *v*
    - Follow a path from *v* to its end
        - Before following another path
- Uses a *stack* to store vertices
    - Stacks are LIFO
- DFS :
    - visit and insert start
    - while (*st* not empty)
        - peek at front vertex, *v*
        - if *v* has unvisited neighbour visit it and insert it in *st*
        - else remove v from *st*
    - end while