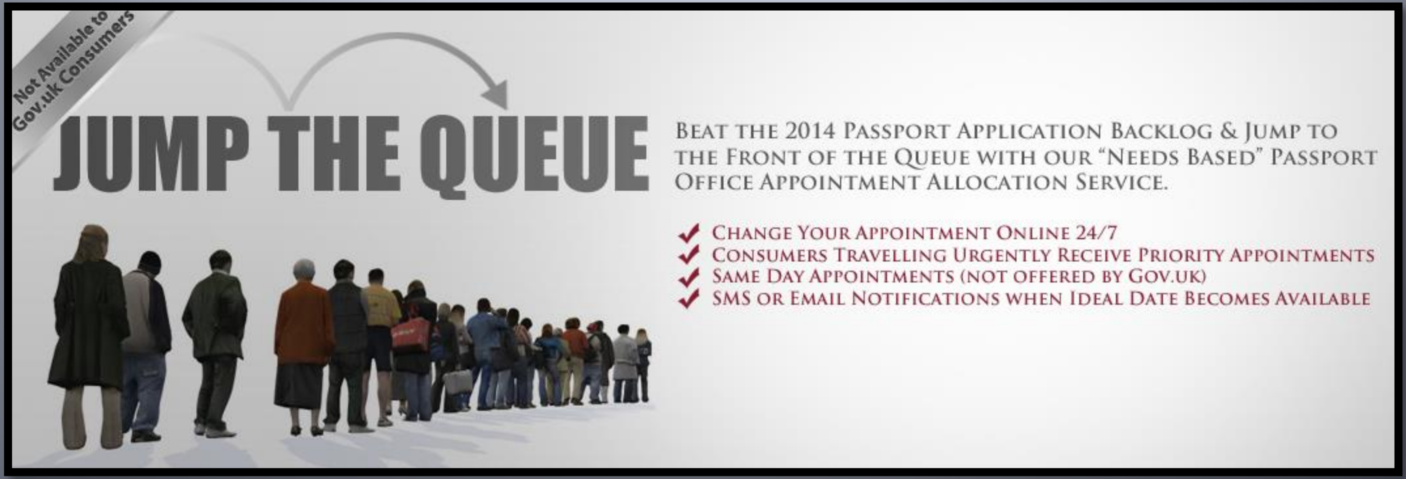
Priority Queues and Heaps

Objectives

- Define the ADT priority queue
- Define the partially ordered property
- Define a heap
- Implement a heap using an array
- Implement the heap sort algorithm

Priority Queue ADT



ADT Priority Queue

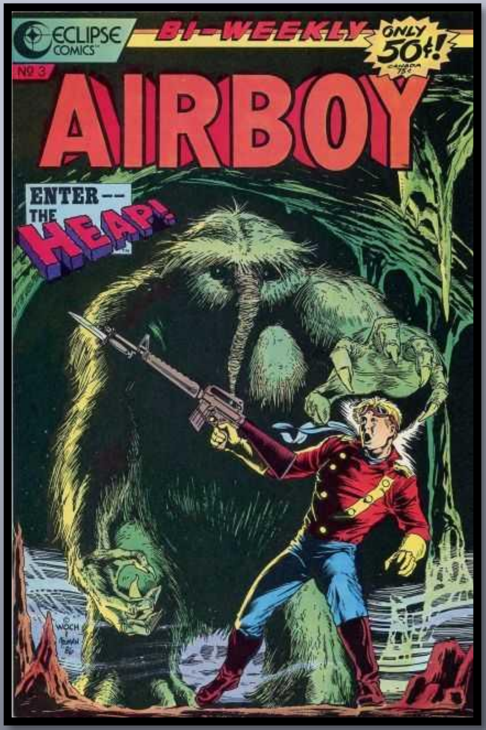
- Items in a priority queue have a *priority*
  - Not necessarily numerical
  - Could be lowest first or highest first
- The highest priority item is removed first
- Priority queue operations
  - Insert
  - Remove in priority queue order
  - Both operations should be performed in at most $O(\log n)$ time

Implementing a Priority Queue

- Items have to be removed in priority order
  - This can only be done efficiently if the items are ordered in some way
- One option would be to use a balanced binary search tree
  - Binary search trees are fully ordered and insertion and removal can be implemented in $O(\log n)$ time
    - Some operations (e.g. removal) are complex
    - Although operations are $O(\log n)$ they require quite a lot of structural overhead
- There is a much simpler binary tree solution

Heap ADT

A complete, partially ordered, binary tree



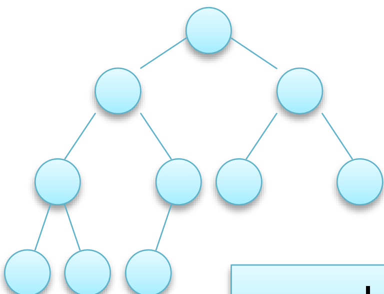
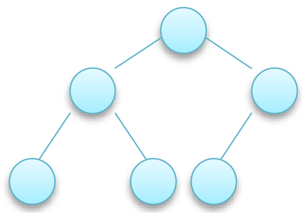
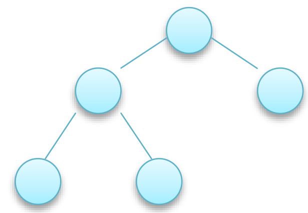
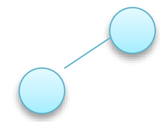
Tree Summary

- A *tree* is a connected graph made up of nodes and edges
  - With exactly one less edge than the number of nodes
- A tree has a root
  - The first node in the tree
- A tree has leaves
  - Nodes that have no children
- A binary tree is a tree with at most two children per node

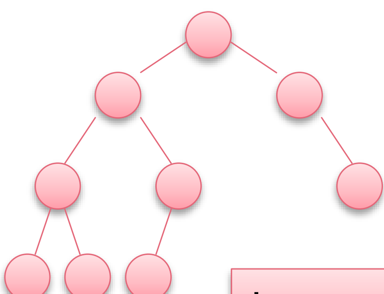
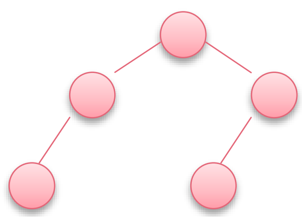
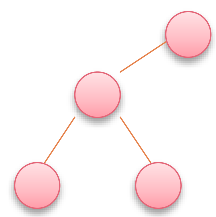
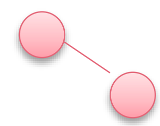
Heaps

- A *heap* is binary tree with two properties
- Heaps are *complete*
  - All levels, except the bottom, must be completely filled in
  - The leaves on the bottom level are as far to the left as possible
- Heaps are *partially ordered*
  - For a *max* heap – the value of a node is at least as large as its children's values
  - For a *min* heap – the value of a node is no greater than its children's values

Complete Binary Trees

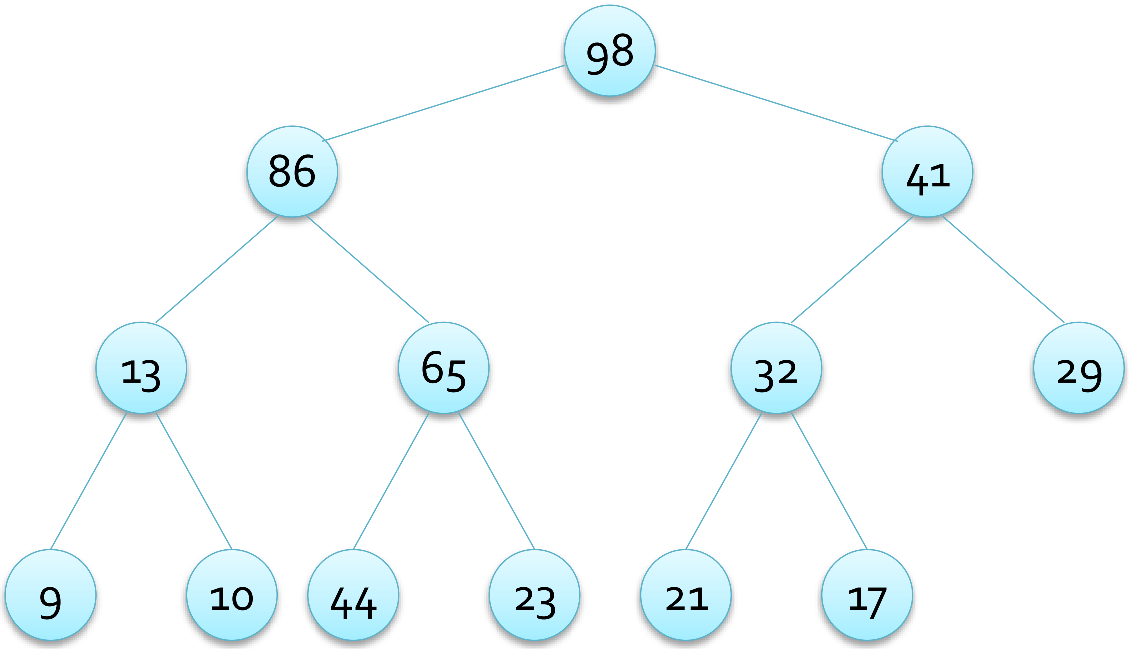





complete binary trees

incomplete binary trees

Partially Ordered Tree – max heap



Heaps are *not* fully ordered, an in-order traversal would result in

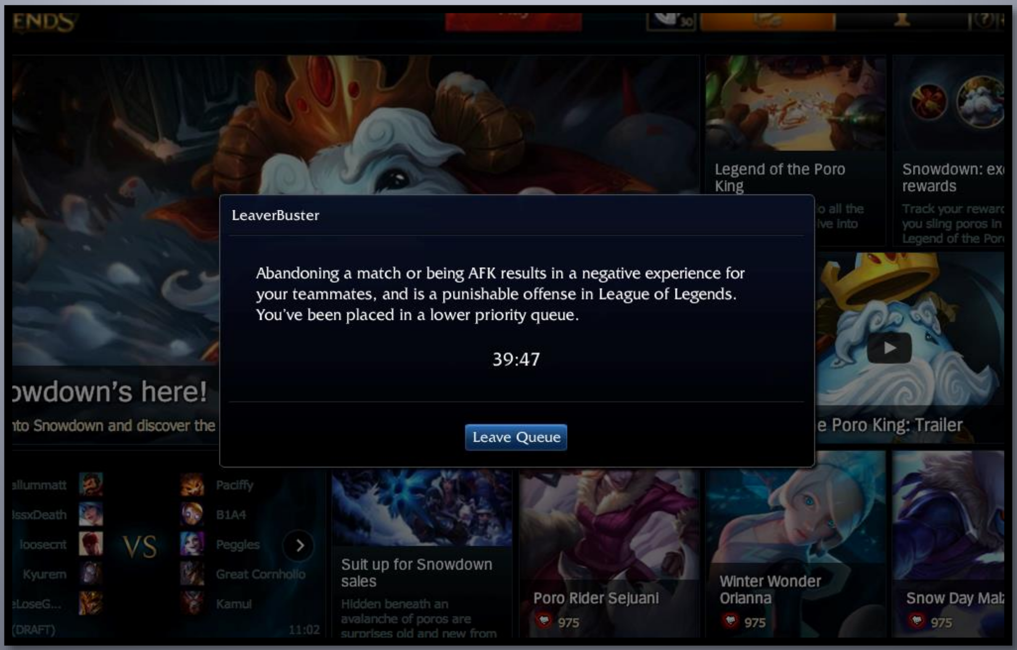
9, 13, 10, 86, 44, 65, 23, 98, 21, 32, 17, 41, 29

Priority Queues and Heaps

- A heap can be used to implement a priority queue
- The item at the top of the heap must always be the highest priority value
  - Because of the partial ordering property
- Implement priority queue operations:
  - Insertions – insert an item into a heap
  - Removal – remove and return the heap's root
  - For both operations preserve the heap property

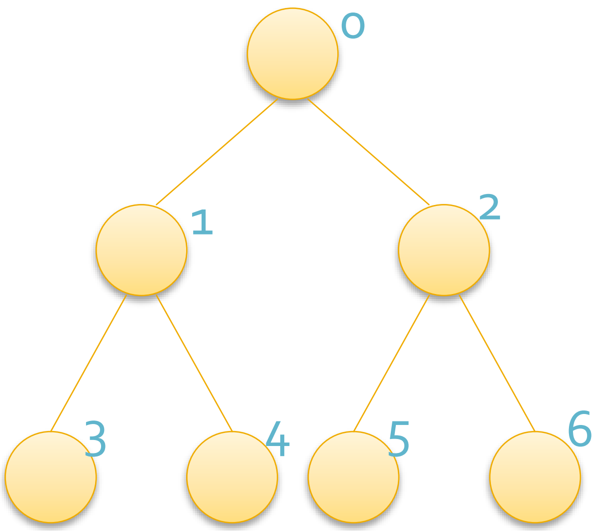
Priority Queue Implementation

Using an Array to Implement a Heap



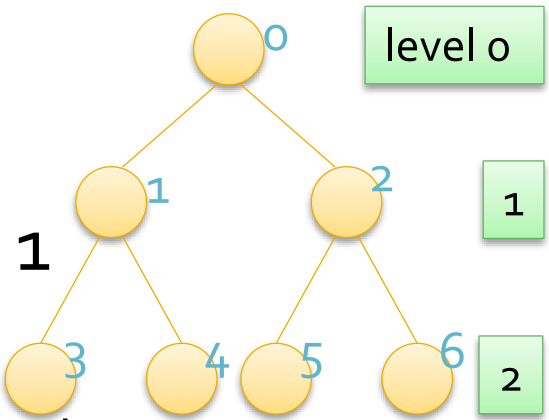
Heap Implementation

- Heaps can be implemented using *arrays*
- There is a natural method of indexing tree nodes
  - Index nodes from top to bottom and left to right
  - Because heaps are *complete* binary trees there can be no gaps in the array



Referencing Nodes

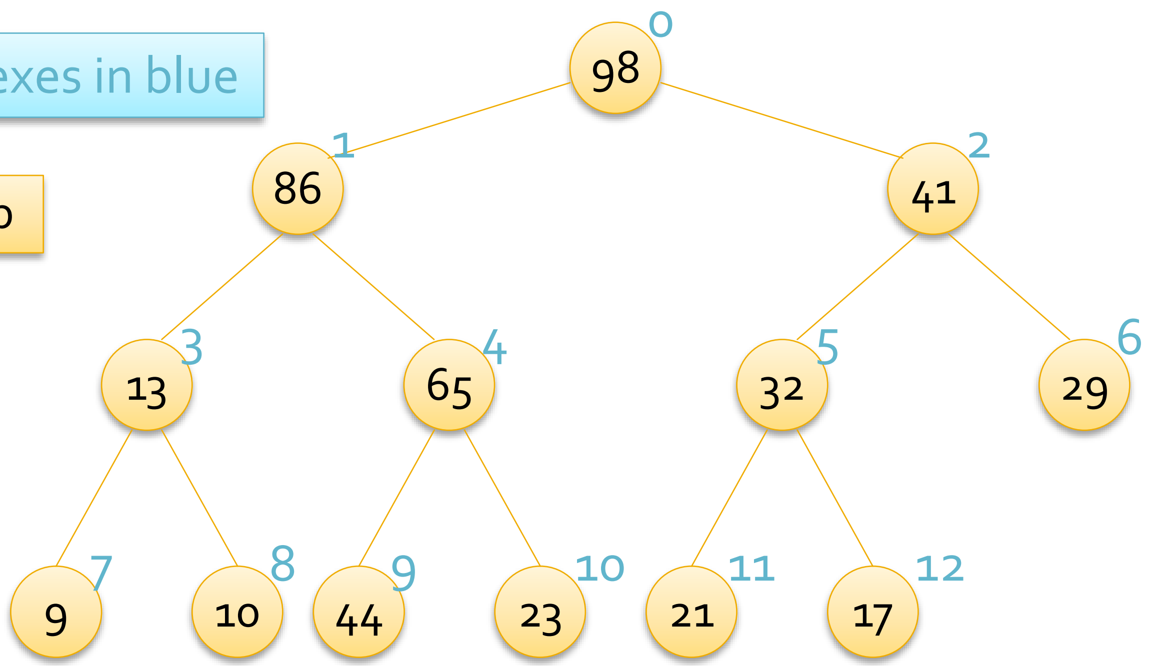
- It will be necessary to find the index of the parents of a node
  - Or the children of a node
- The array is indexed from 0 to n - 1
  - Each level's nodes are indexed from
    - $2^{level} - 1$ to $2^{level+1} - 2$ (where the root is level 0)
  - The children of a node i, are the array elements indexed at $2i + 1$ and $2i + 2$
  - The parent of a node i, is the array element indexed at $(i - 1) / 2$



Heap Array Example

indexes in blue

Heap



index	0	1	2	3	4	5	6	7	8	9	10	11	12
value	98	86	41	13	65	32	29	9	10	44	23	21	17

The heap is represented by an array

Heap Insertion

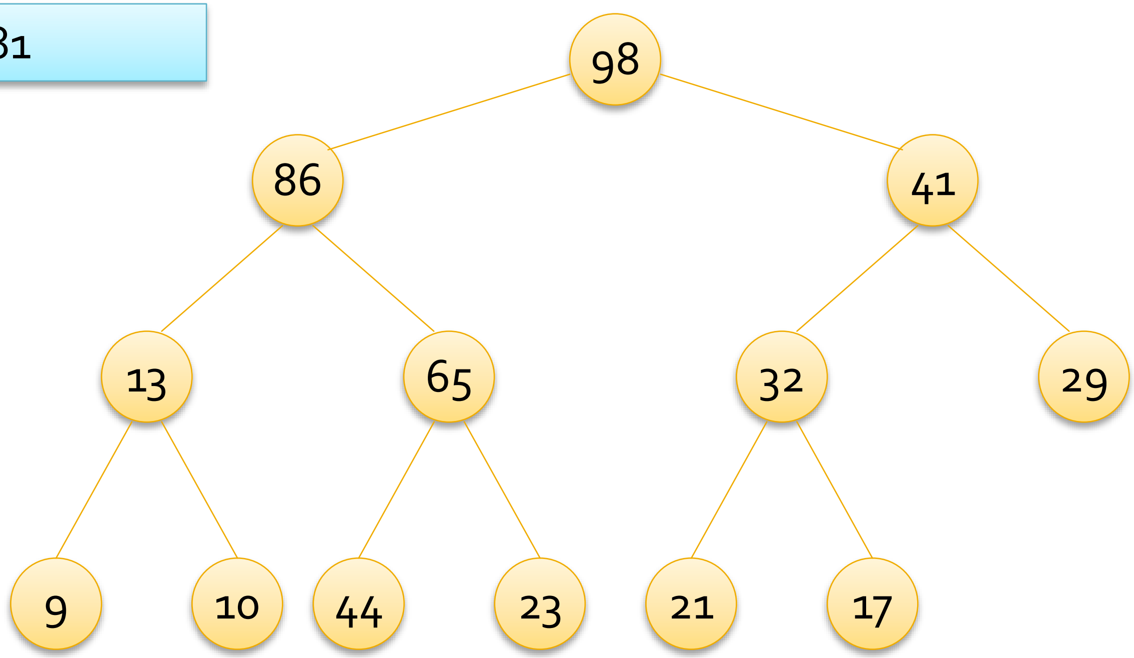
- On insertion the heap properties have to be maintained, remember that
  - A heap is a complete binary tree and
  - A partially ordered binary tree
- There are two general strategies that could be used to maintain the heap properties
  - Make sure that the tree is complete and then fix the ordering or
  - Make sure the ordering is correct first
  - Which is better?

Heap Insertion Sketch

- The insertion algorithm first ensures that the tree is complete
  - Make the new item the first available (left-most) leaf on the bottom level
  - i.e. the first free element in the underlying array
- Fix the partial ordering
  - Compare the new value to its parent
  - Swap them if the new value is greater than the parent
  - Repeat until this is not the case
    - Referred to as *bubbling up*, or *trickling up*

Heap Insertion Example

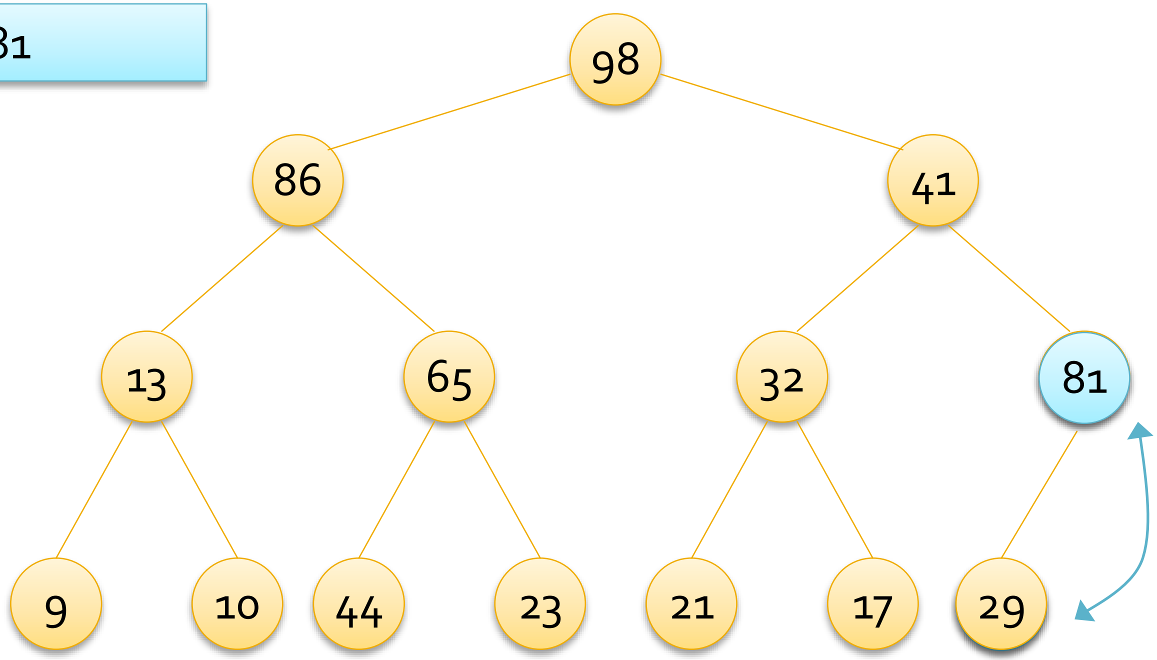
Insert 81



index	0	1	2	3	4	5	6	7	8	9	10	11	12	13
value	98	86	41	13	65	32	29	9	10	44	23	21	17	

Heap Insertion Example

Insert 81

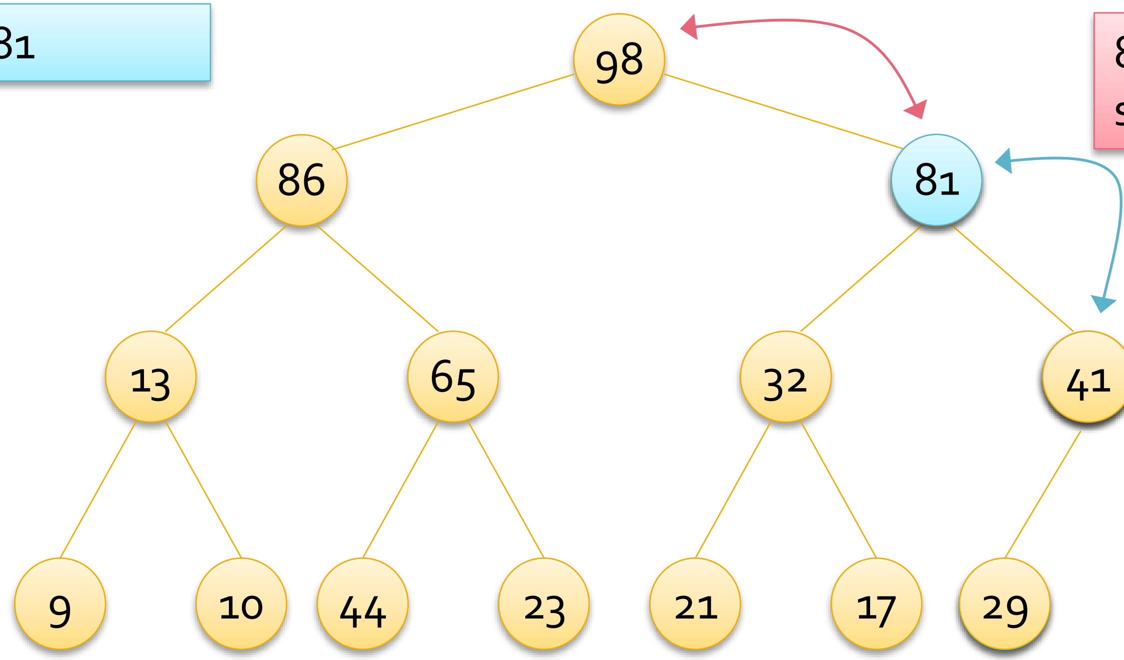


(13-1)/2 = 6

index	0	1	2	3	4	5	6	7	8	9	10	11	12	13
value	98	86	41	13	65	32	81	9	10	44	23	21	17	29

Heap Insertion Example

Insert 81



81 is less than 98 so finished

(6-1)/2 = 2

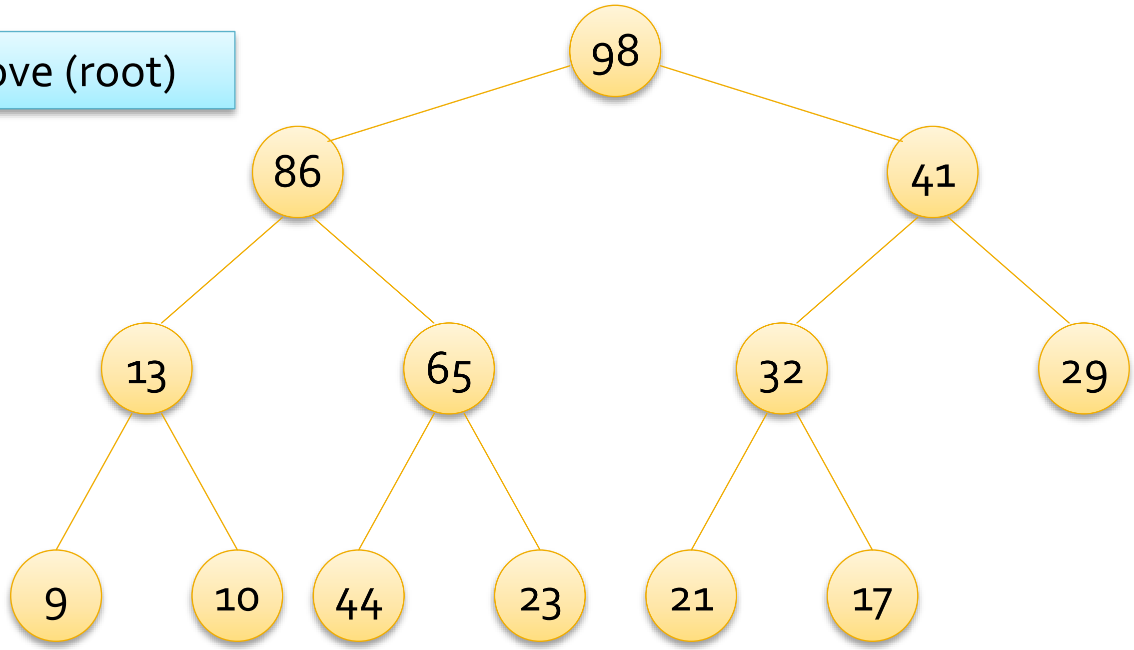
index	0	1	2	3	4	5	6	7	8	9	10	11	12	13
value	98	86	81	13	65	32	41	9	10	44	23	21	17	29

Heap Removal

- Make a temporary copy of the root's data
- Similarly to the insertion algorithm, first ensure that the heap remains complete
  - Replace the root node with the right-most leaf
  - i.e. the highest (occupied) index in the array
- Swap the new root with its largest valued child until the partially ordered property holds
  - i.e. *bubble down*
- Return the root's data

Heap Array Example

Remove (root)

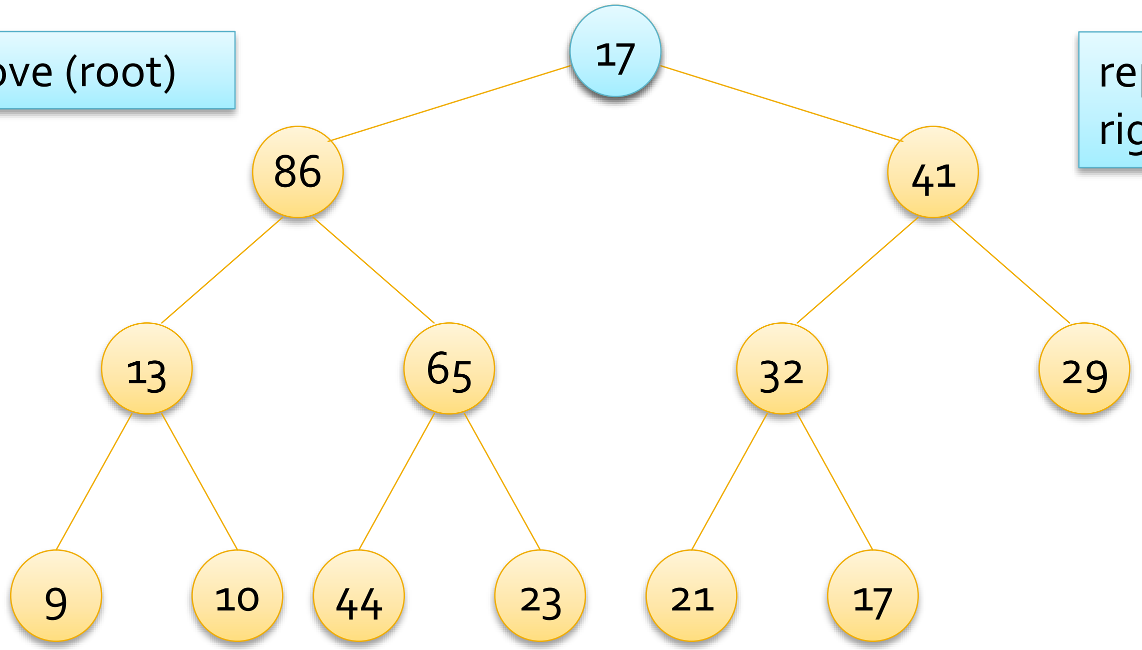


index	0	1	2	3	4	5	6	7	8	9	10	11	12
value	98	86	41	13	65	32	29	9	10	44	23	21	17

Heap Array Example

Remove (root)

replace root with right-most leaf

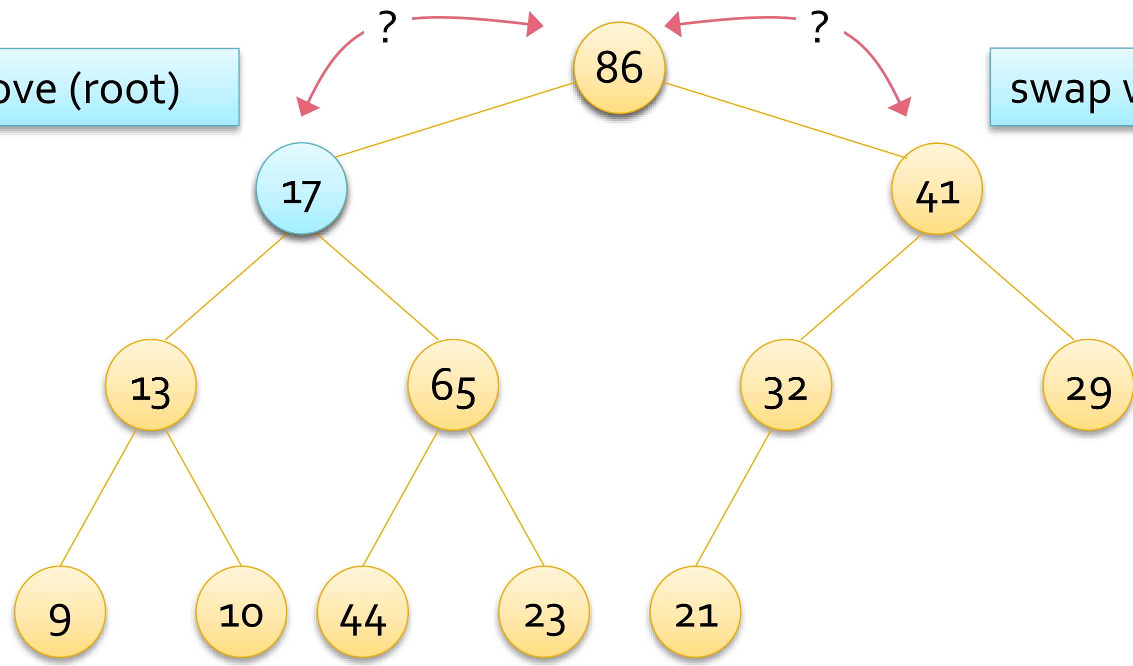


index	0	1	2	3	4	5	6	7	8	9	10	11	12
value	17	86	41	13	65	32	29	9	10	44	23	21	

Heap Array Example

Remove (root)

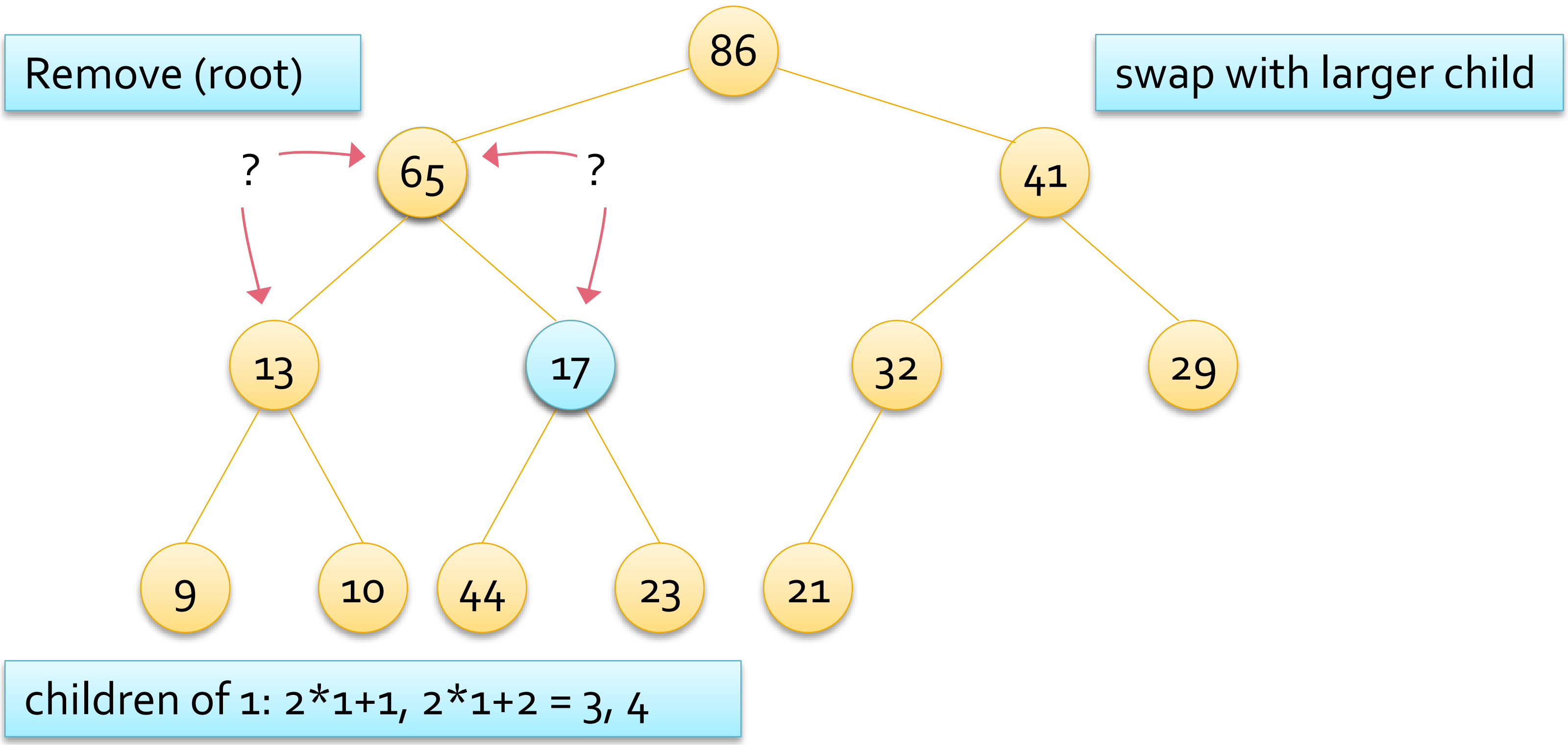
swap with larger child



children of root: 2*0+1, 2*0+2 = 1, 2

index	0	1	2	3	4	5	6	7	8	9	10	11	12
value	86	17	41	13	65	32	29	9	10	44	23	21	

Heap Array Example

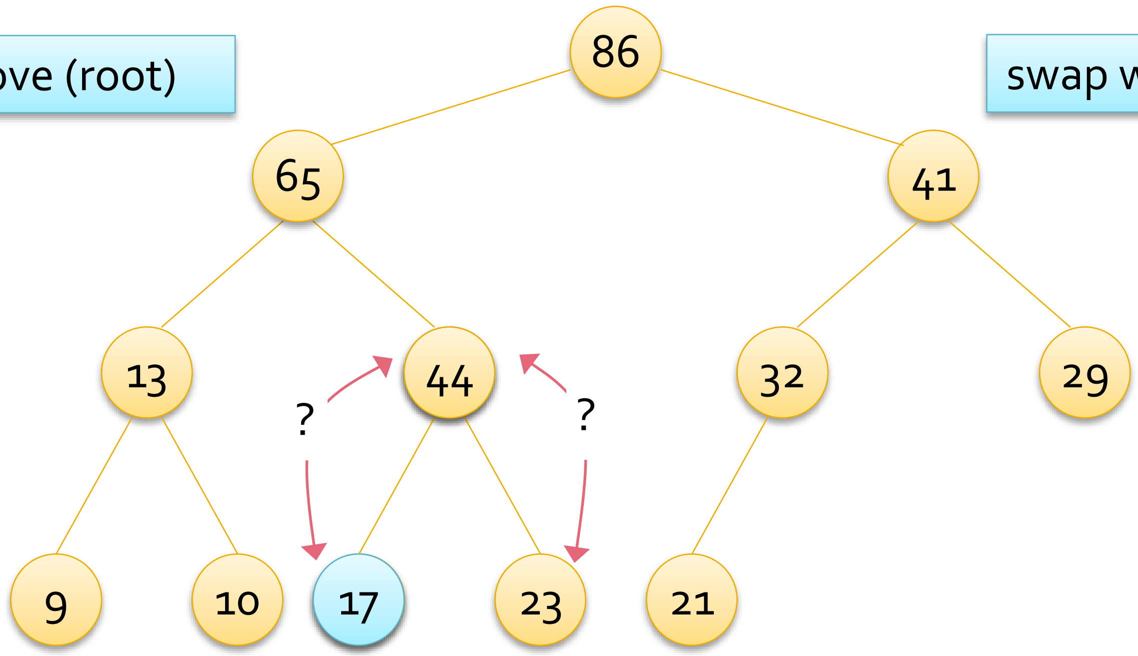


index	0	1	2	3	4	5	6	7	8	9	10	11	12
value	86	65	41	13	17	32	29	9	10	44	23	21	

Heap Array Example

Remove (root)

swap with larger child



children of 4: 2*4+1, 2*4+2 = 9, 10

index	0	1	2	3	4	5	6	7	8	9	10	11	12
value	86	65	41	13	44	32	29	9	10	17	23	21	

Bubble Up and Bubble Down

- Helper functions are usually written for preserving the heap property
  - *bubbleUp* ensures that the heap property is preserved from the start node up to the root
  - *bubbleDown* ensures that the heap property is preserved from the start node down to the leaves
- These functions may be implemented recursively or iteratively

BubbleUp Algorithm

```
void bubbleUp(int i){
  int parent = (i - 1) / 2;
  if (i > 0 && arr[i] > arr[parent]){
    int temp = arr[i];
    arr[i] = arr[parent];
    arr[parent] = temp;
    bubbleUp(parent);
  }
  // no else - implicit base case
}
```

Insertion Algorithm

```
void insert(int x){
  arr[size] = x;
  bubbleUp(size);
  size++;
}
```

Heap Efficiency

- Both insertion and removal into a heap involve at most *height* swaps
  - For insertion at most *height* comparisons
    - To bubble up the array
  - For removal at most *height* * 2 comparisons
    - To bubble down the array (have to compare two children)
- Height of a complete binary tree is $\lfloor \log_2(n) \rfloor$
  - Both insertion and removal are therefore $O(\log n)$