ADSP Books

1> Digital Signal Processing by Proakis (for sampling, DSP etc)
2> Adaptive Signal Processing by Simon Haykin
3> DSP by A Nagoor Kani (Module 01, 02(some part))

Refer: gaussianwaves.com (for module 02)

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Advanced Digital Signal Processing

Module 01 : Multirate Signal Processing

Multirate

↙ ↘

Upsampling (L)   Downsampling (M)

↓

Insert zeroes in time domain

For (L-1) inserted zeroes

$$x(n) \longrightarrow x\left(\frac{n}{L}\right)$$

Example

$L = 3$

$$\therefore y(n) = x\left(\frac{n}{3}\right)$$

$n = 0$

$y(0) = x(0)$ ✓

$y(1) = x(1/3) \longrightarrow 0$

$y(2) = x(2/3) \longrightarrow 0$

$y(3) = x(3/3) = x(1)$

Similiarly $y(4) = y(5) = 0$ & $y(6) = x(2)$ ... and so on

Hence

$$\boxed{\begin{aligned} y(n) &= x\left(\frac{n}{L}\right) \quad \forall \ \frac{n}{L} \in Z \\ &= 0 \quad \text{elsewhere} \end{aligned}} \quad \text{— (1)}$$

Take the Z-transform

$$Y(z) = \sum x\left(\frac{n}{L}\right) z^{-n}$$

where $\frac{n}{L} = k$

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$$Y(z) = \sum x(k) z^{-Lk}$$

$$= \sum x(k) (z^{L})^{-k}$$

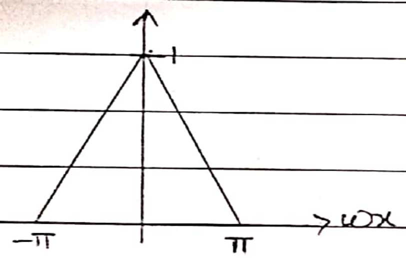
$$= X(z^{L})$$

$$Y(e^{j\omega}) = X(e^{j\omega L})$$

Note :	
$\omega = 2\pi f = 2\pi \frac{F}{F_s}$	$F_y = LF_x \longrightarrow$ upsampling
$\omega_x = 2\pi \frac{F}{F_x}$	$\therefore \omega_y = \frac{\omega_x}{L}$
$\omega_y = 2\pi \frac{F}{F_y}$	

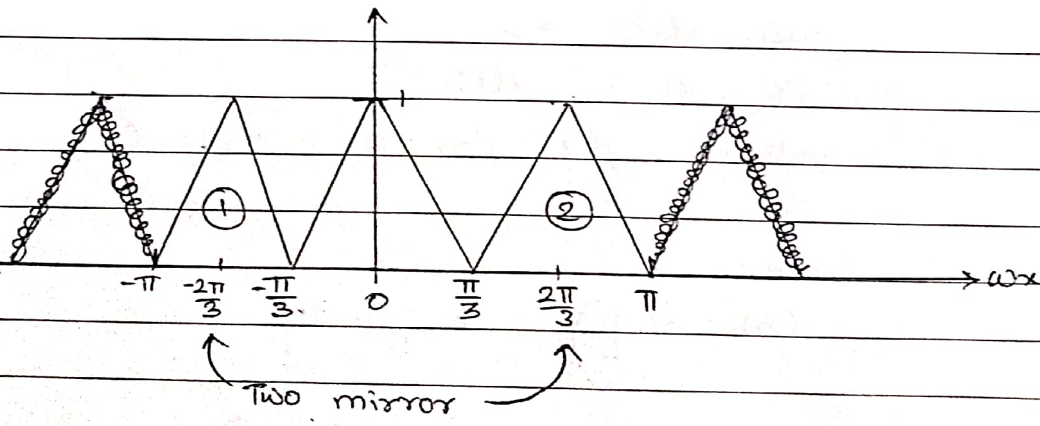
Q]

For $L=3$



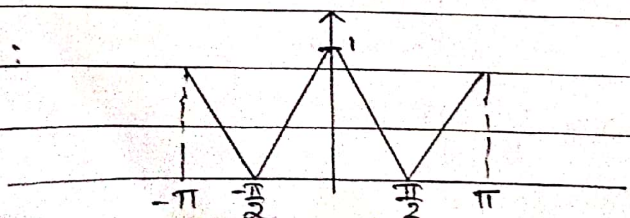
Draw the upsampled signal

Ans:-



Process: $\boxed{x(n)} \rightarrow \boxed{\text{Upsampling}} \rightarrow \boxed{\text{Filter } H(z)} \rightarrow X(z)$

For $L=2$:



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Downsampling

Here $F_y = \frac{F_x}{M}$

$\therefore \boxed{\omega_y = M\omega_x}$

- So, in the case of downsampling, we are taking every $(M-1)^{th}$ sample

$$\boxed{y(n) = x(nM)}$$

Example : For $n = 0, 1$, $M = 3$.

$n=0$ $\quad y(0) = x(0)$

$n=1$ $\quad y(1) = x(3)$

$n=2$ $\quad y(2) = x(6)$

$\therefore y(n) = \{x(0), x(3), x(6), x(9), \ldots\ldots\}$

Sampling freq $= \frac{f_s}{M}$

$f_{max} = \frac{f_s}{2}$

In our case, $f_s = f_s/M$

$\therefore f_{max} = \frac{f_s}{2M}$

$\therefore \omega_{max} = \frac{2\pi}{f_s} f_s/2M = \boxed{\frac{\pi}{M}}$

- If in the input, ω is limited to π/M, only then is the Nyquist criteria satisfied.

$f_s \geq 2 f_{max}$

$\therefore f_{max} \leq \frac{f_s}{2}$

/ /

- In downsampling, since we take samples after leaving (M-1) samples, the sampling frequency changes from fs to fs/M

$$f_{max} \le \frac{f_s}{2} \le \frac{f_s}{2M}$$

- Now let us see the effect of f_{max} on ω

$$\omega_{max} = 2\pi \frac{F}{f_s} = 2\pi \frac{f_s/2M}{f_s} = \frac{\pi}{M}$$

Therefore, when we are doing down-sampling

$$0 \le \omega_x \le \pi/M$$

$$0 \le \omega_y \le \pi$$

Process:

$\boxed{y(n)} \longrightarrow \boxed{\text{Filter } H(z)} \longrightarrow \boxed{\text{Downsampling}} \longrightarrow Y(z)$

- Derivation of Downsampling **(v.imp)

- The formula for discrete inverse fourier transform (IFT)

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{\frac{j2\pi Kn}{N}} \quad \text{--- (1)}$$

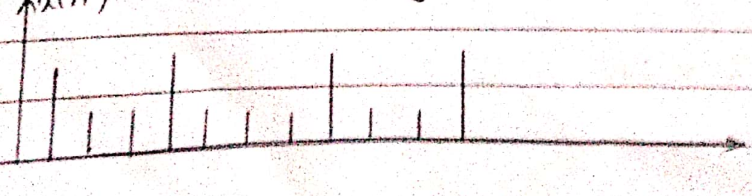
- Also we know that

$$y(n) = x(nM) \quad \text{--- (2)}$$

Consider M = 2

$$\therefore y(n) = \hat{x}(2n) \quad \text{--- (3)}$$

Consider the following graphs



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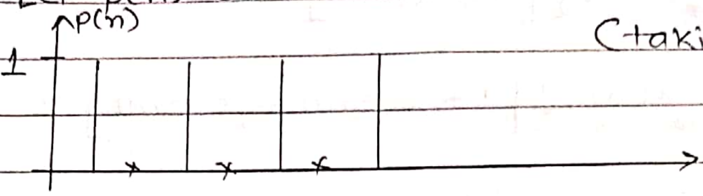
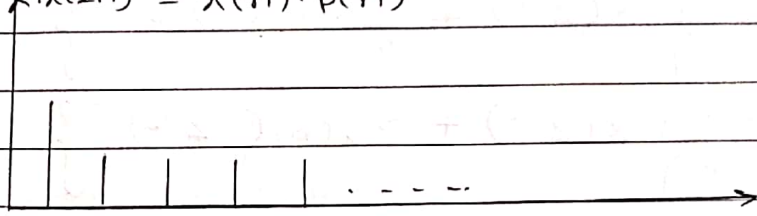
Let $p(n)$ be

(taking every 2^{nd} element)



Hence $x(n) \cdot p(n)$ is

$\hat{x}(2n) = x(n) \cdot p(n)$



Now take the z transform

$$Y(z) = \sum_{m=-\infty}^{\infty} y(m) z^{-m} \qquad —(4)$$

$$= \sum_{m=-\infty}^{\infty} \hat{x}(2m) z^{-m}$$

Let $2m = n$

$$\therefore Y(z) = \sum_{n=-\infty}^{\infty} \hat{x}(n) z^{-n/2}$$

$$Y(z) = \sum_{-\infty}^{\infty} x(n) \cdot p(n) z^{-n/2} \qquad —(5)$$

Now

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{\frac{j 2\pi n K}{N}} \qquad —(6)$$

$$p(n) = \frac{1}{2} \sum_{K=0}^{1} (1) e^{\frac{j 2\pi n K}{2}}$$

$$\therefore p(n) = \frac{1}{2} \sum_{K=0}^{1} e^{\frac{j 2\pi n K}{2}} \qquad —(7)$$

Substitute eq^n (7) in (5)

$$\therefore Y(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot \frac{1}{2} \sum_{K=0}^{1} e^{\frac{j 2\pi n K}{2}} \cdot z^{-n/2} \qquad (\ast\ast\ast)$$

general derivation done later

Expand the summation of $p(n)$ to get

$$Y(z) = \sum_{n=-\infty}^{\infty} x(n)\cdot\frac{1}{2}\left[e^{0} + e^{jn\pi}\right]\cdot z^{-n/2}$$

$$Y(z) = \sum_{n=-\infty}^{\infty} x(n)\cdot\frac{1}{2}\left[1 + \cos n\pi + j\sin n\pi\right]\cdot z^{-n/2}$$

$$Y(z) = \frac{1}{2}\left\{\sum_{n=-\infty}^{\infty} x(n) z^{-n/2} + \sum_{n=-\infty}^{\infty} x(n) e^{jn\pi} z^{-n/2}\right\}$$

$$Y(z) = \frac{1}{2}\left\{X(z^{1/2}) + \sum_{m} x(n)(-1)^{n} z^{-n/2}\right\}$$

$$= \frac{1}{2}\left\{X(z^{1/2}) + \sum x(n)(-z^{1/2})^{-n}\right\}$$

$$= \frac{1}{2}\left\{X(z^{1/2}) + X(-z^{1/2})\right\}$$

Hence

$$Y(e^{jw}) = \frac{1}{2}\left\{X(e^{jw/2}) + X(-e^{jw/2})\right\} \quad \text{— (8)}$$

The above expression is for $M=2$. Consider the general case $M = M$

$$\therefore \boxed{Y(e^{jw}) = \frac{1}{M}\left\{X(e^{jw/M}) + X(-e^{jw/M})\right\}} \;\star\star \quad \text{— (9)}$$

★ Polyphase Decomposition

- The expression for a FIR filter is

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$\therefore H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + \dots$$

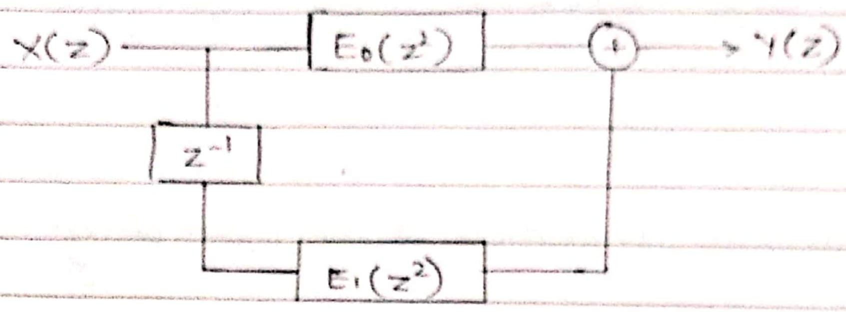
Now leaving alternate elements

$$H(z) = \left(h(0) + h(2)z^{-2} + h(4)z^{-4} + \dots\right) + \left(h(1)z^{-1} + h(3)z^{-3} + h(5)z^{-5} + \dots\right)$$

$$H(z) = \left(h(0) + h(2)z^{-2} + h(4)z^{-4}\right) + z^{-1}\left(h(1) + h(3)z^{-2} + h(5)z^{-4}\right)$$

$H(z) = E_0(z^2) + z^{-1}E_1(z^2)$

This can be represented as



Example : For L=4

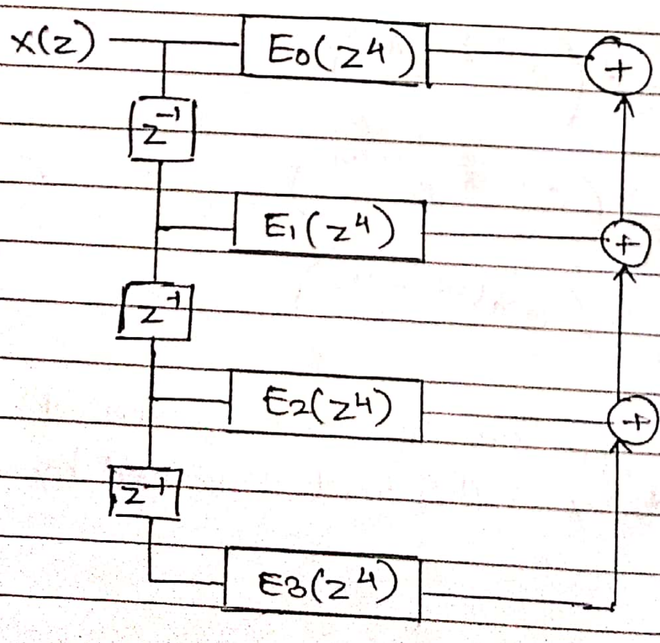
Q] $H(z) = 0.3 + 0.6z^{-1} + 0.7z^{-2} + 0.18z^{-3} + 0.85z^{-4} + 0.25z^{-5}$
$+ 0.28z^{-6} + 0.42z^{-7} + 0.89z^{-8}$

Ans: For L=4, take every 4th term i.e skip 3

$H(z) = (0.3 + 0.85z^{-4} + 0.89z^{-8}) + z^{-1}(0.6 + 0.25z^{-4})$
$+ z^{-2}(0.7 + 0.28z^{-4}) + z^{-3}(0.18 + 0.42z^{-4})$

$\therefore H(z) = E_0(z^4) + z^{-1}E_1(z^4) + z^{-2}E_2(z^4) + z^{-3}E_3(z^4)$

This can be represented as



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General Form :-

$$H(z) = E_0(z^{l}) + z^{-1}E_1(z^{l}) + z^{-2}E_2(z^{l}) + \dots z^{-(l-1)}E_{l-1}(z^{l})$$

$$\boxed{H(z) = \sum_{m=0}^{l-1} z^{-m} E_m(z^{l})} \ \star\star$$

(★★★) - General derivation of downsampling

We know that for $M=2$

$$p(n) = \frac{1}{2}\sum_{k=0}^{1} e^{+j\frac{2\pi nk}{2}} \quad ; \quad Y(z) = \sum_{n=-\infty}^{\infty} x(n)\, p(n)\, z^{-nk}$$

$- (A)$

For a general M

$$Y(z) = \sum_{n} x(n) \cdot \frac{1}{M}\sum_{k=0}^{M-1} e^{\frac{j2\pi nk}{M}} \cdot z^{-n/M}$$

$$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1}\left[\sum_{n=-\infty}^{\infty} x(n)\, e^{\frac{j2\pi nk}{M}} \cdot z^{-n/M}\right] \quad -(B)$$

$$X(z) = \sum_{m=-\infty}^{\infty} x(n)\, z^{-n}$$

$$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1}\left[\sum_{n=-\infty}^{\infty} x(n)\left\{ e^{\frac{-j2\pi k}{M}} \cdot z^{1/m}\right\}^{-n}\right]$$

$$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{\frac{-j2\pi k}{M}} z^{1/M}\right) \quad -(C)$$

$$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{-j\frac{2\pi k}{M}} e^{j\omega/M}\right)$$

$$= \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{\frac{j}{M}(\omega - 2\pi k)}\right) \quad -(D)$$

- In downsampling, the spectrum is scaled by a factor of $1/M$, shifted by $2\pi k$ and stretched by a factor of M.

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Now for $M=2$

$$Y(z) = \frac{1}{2}\sum_{k=0}^{1} X\left(e^{\frac{1}{2}(\omega-2\pi k)}\right)$$

$$= \frac{1}{2}\left\{X(e^{j\omega/2}) + X(-e^{+j\omega/2})\right\}$$

- **Note**: For practical purposes, we ignore the shifts & take $Y(z) = \frac{1}{M}X(e^{j\omega/M})$

Hence: note that

$$x(n) \longrightarrow \boxed{\downarrow M} \longrightarrow x(nM)$$

$X(z)$ $\qquad \frac{1}{M}X(z^{1/M}) = \frac{1}{M}X(e^{j\omega/M})$

$\omega \rightarrow$ stretched by a factor of M

$$x(n) \longrightarrow \boxed{\uparrow L} \longrightarrow x(n/L)$$

$X(z)$ $\qquad X(z^L) = X(e^{j\omega L})$

$\omega \rightarrow$ compressed by L

✪ Identities for Upsampling & Downsampling

I] Identities for Upsampling

$$x(n) - \boxed{H(z)} - \boxed{\uparrow L} - y(n) \equiv x(n) - \boxed{\uparrow L} - \boxed{H(z^L)} - y(n)$$

Consider the LHS

$$x(n) - \boxed{H(z)} \xrightarrow{V(n),\ V(z)} \boxed{\uparrow L} - y(n)$$

We can write

$$V(z) = H(z)\cdot X(z) \qquad -① $$

$\left(\because \frac{V(z)}{X(z)} = H(z) \rightarrow \text{transfer func}^n\right)$

Now $v(n) \longrightarrow \boxed{\uparrow L} \longrightarrow y(n)$

$\quad\;\; V(z) \qquad\qquad\qquad\quad Y(z)$

From the above diagram

$$Y(z) = V(z^L) \qquad — (2)$$

From (1)

$V(z) = H(z) \cdot X(z)$

$V(z^L) = H(z^L) \cdot X(z^L)$

Hence

$$\boxed{LHS = Y(z) = H(z^L) \cdot X(z^L)} ** \qquad — (LHS)$$

Consider the RHS

$$x(n) \longrightarrow \boxed{\uparrow L} \xrightarrow{v(n)} \boxed{H(z^L)} \longrightarrow y(n)$$

$X(z) \qquad\qquad V(z) \qquad\qquad\qquad Y(z)$

Now for $x(n) \longrightarrow \boxed{\uparrow L} \longrightarrow v(n)$

$\qquad\qquad X(z) \qquad\qquad\qquad V(z)$

Here $V(z) = X(z^L)$ — (3)

Now

$$\boxed{\uparrow L} \longrightarrow \boxed{H(z^L)} \longrightarrow y(n)$$

$v(n) \qquad\qquad\qquad\qquad Y(z)$

$V(z)$

Here, $Y(z) = V(z) \cdot H(z^L)$ — (4)

from (3),

$$\boxed{RHS = Y(z) = X(z^L) \cdot H(z^L)} \qquad — (RHS)$$

Hence $\underline{LHS = RHS}$

P.T.O

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II] Identity for Downsampling

$x(n) \rightarrow \boxed{\downarrow M} \rightarrow \boxed{H(z)} \rightarrow y(n) \equiv x(n) \rightarrow \boxed{H(z^M)} \rightarrow \boxed{\downarrow M} \rightarrow y(n)$

Consider the LHS

$x(n) \rightarrow \boxed{\downarrow M} \xrightarrow[V(z)]{v(n)} \boxed{H(z)} \rightarrow y(n)$

We can write

$$V(z) = \frac{1}{M} X(z^{1/M}) \qquad - (1)$$

$$Y(z) = V(z) \cdot H(z) \qquad - (2)$$

$$\boxed{Y(z) = \frac{1}{M} X(z^{1/M}) \cdot H(z)} \qquad - \text{LHS}$$

Consider the RHS

$x(n) \rightarrow \boxed{H(z^M)} \xrightarrow[V(z)]{v(n)} \boxed{\downarrow M} \rightarrow Y(n)$

We can write

$$V(z) = X(z) \cdot H(z^M) \qquad - (1)$$

and

$$Y(z) = \frac{1}{M} \left\{ X(z^{1/M}) H(z^{M/M}) \right\} \qquad - (2)$$

$$\boxed{Y(z) = \frac{1}{M} X(z^{1/M}) H(z)} \qquad - \text{RHS}$$

Hence LHS = RHS

{ *** Assignment 01: Applications of Multirate Signal Processing.
Submission → 18/01/2020 }

Q] Prove the following identities:-

1) $x(n) \rightarrow \boxed{z^{-1}} \xrightarrow[V(z)]{v(n)} \boxed{\downarrow D} \rightarrow y(n) \equiv x(n) \rightarrow \boxed{\downarrow D} \xrightarrow[V(z)]{v(n)} \boxed{z^{-1/D}} \rightarrow y(n)$

Ans:- LHS:

$$V(z) = X(z) \cdot z^{-1}$$

$$Y(z) = \frac{1}{D} X(z^{1/D}) z^{-1/D} \quad — ①$$

RHS :

$$V(z) = \frac{1}{D} X(^{1/D})$$

$$Y(z) = \frac{1}{D} X(z^{1/D}) \cdot z^{-1/D} \quad — ②$$

Hence LHS = RHS

2) $x(n) \longrightarrow \boxed{\uparrow I_1} \xrightarrow{V(z)} \boxed{\uparrow I_2} \longrightarrow y(n) \equiv x(n) \longrightarrow \boxed{\uparrow I} \longrightarrow y(n)$

$I = I_1 I_2$

Ans:- LHS :

$$V(z) = X(z^{I_1})$$

$$Y(z) = X\left((z^{I_1})^{I_2}\right) = X(z^{I_1 I_2}) \quad — ①$$

RHS :

$$Y(z) = X(z^{I}) = X(z^{I_1 I_2}) \quad — ②$$

Hence LHS = RHS

★ Polyphase Decomposition Continued

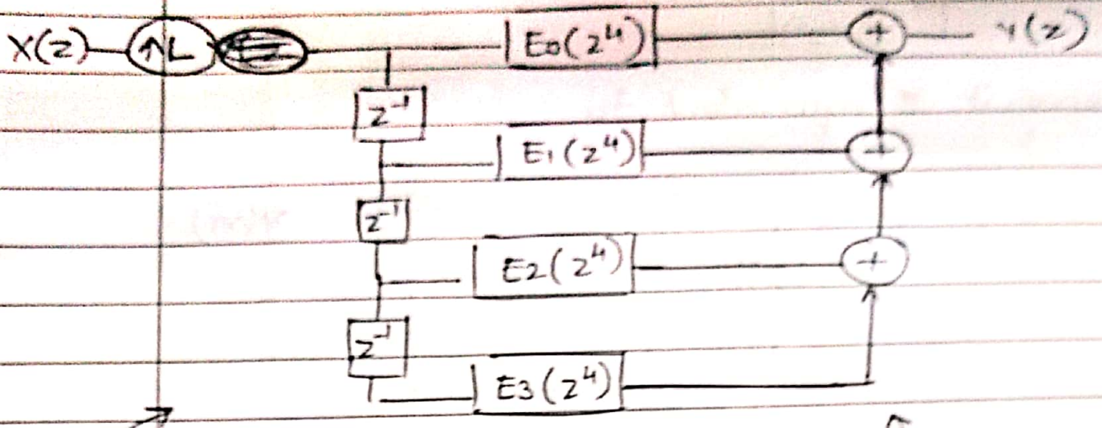
Ex: Given transfer function of an FIR filter

$$H(z) = 0.2 + 0.7z^{-1} + 0.8z^{-2} + 0.15z^{-3} + 0.6z^{-4} + 0.32z^{-5} + 0.5z^{-6} + 0.6z^{-7} + 0.9z^{-8}$$

Ans For L = 4 (Decomposition into 4 sections)

$$H(z) = (0.2 + 0.6z^{-4} + 0.9z^{-8}) + z^{-1}(0.7 + 0.32z^{-4}) + z^{-2}(0.8 + 0.5z^{-4}) + z^{-3}(0.15 + 0.6z^{-7})$$

$$\therefore H(z) = E_0(z^4) + z^{-1}E_1(z^4) + z^{-2}E_2(z^4) + z^{-3}E_3(z^4)$$

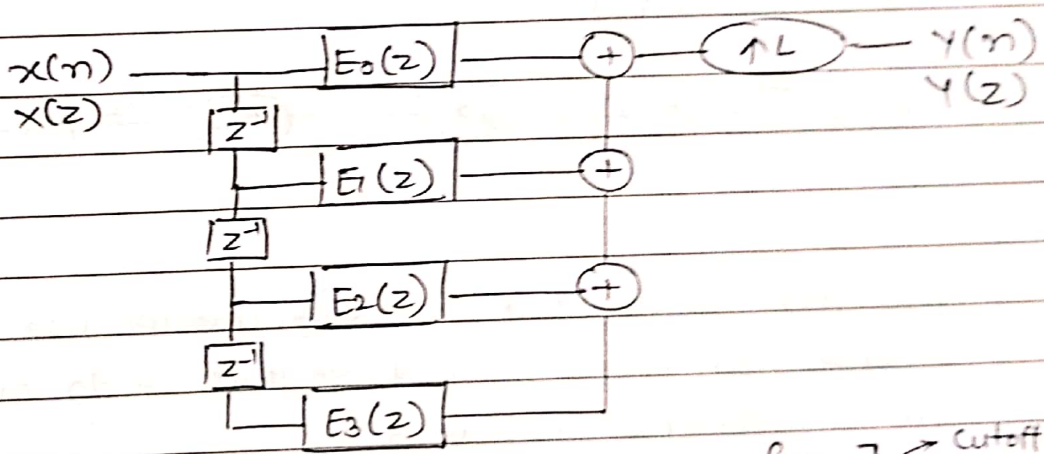


Polyphase of an Interpolator

The identity can be written as

$$x(n) \to \boxed{\uparrow L} \to \boxed{H(z^L)} \equiv x(n) \to \boxed{H(z)} \to \boxed{\uparrow L}$$

Hence the above diagram is equivalent to



[Note: H(z) is called an anti-imaging filter] → Cutoff at $\left(\frac{\pi}{L}\right)$

Ex Polyphase of a decimator

We use the identity for downsampling

$$\boxed{H(z^M)} \to \boxed{\downarrow M} \equiv \boxed{\downarrow M} \to \boxed{H(z)}$$

Hence the polyphase of a decimator is

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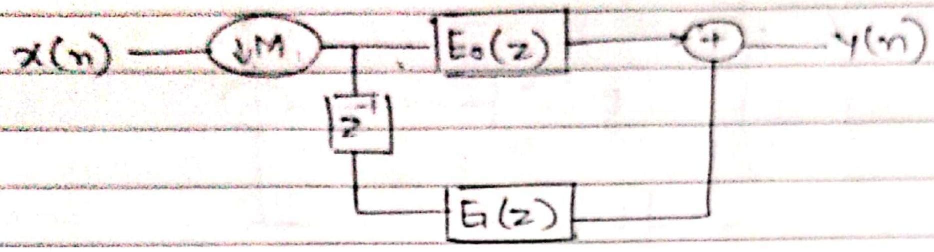
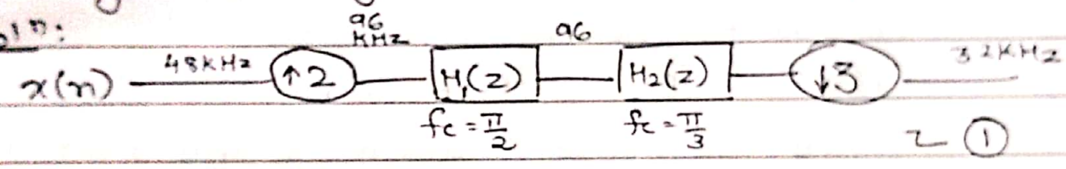
$H(z) = E_0(z^2) + z^{-1}E_1(z^2)$

Hence, it is equivalent to

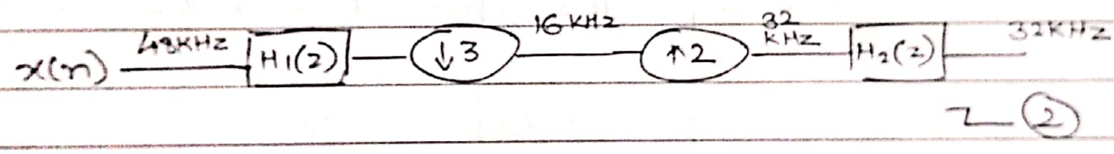


Ex: Digital audio signal at 48kHz to be sampled at 32kHz

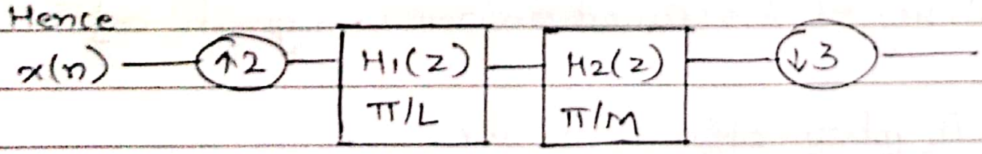
Sol^n:

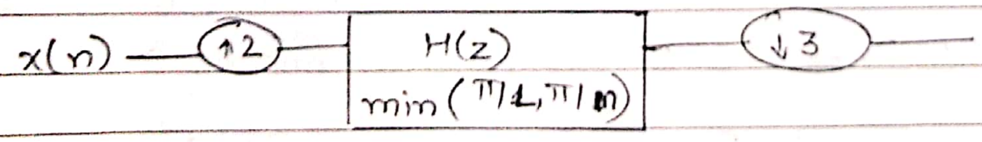


Alternate method



- Now from methods ① and ②, we will use the Interpolator first method, so that we do not miss out on any spectral components.
- Hence





For the fractional polyphase

$$\boxed{L/M = 2/3}$$

→ Note: Upsampling & Downsampling are linear AND Time Variant

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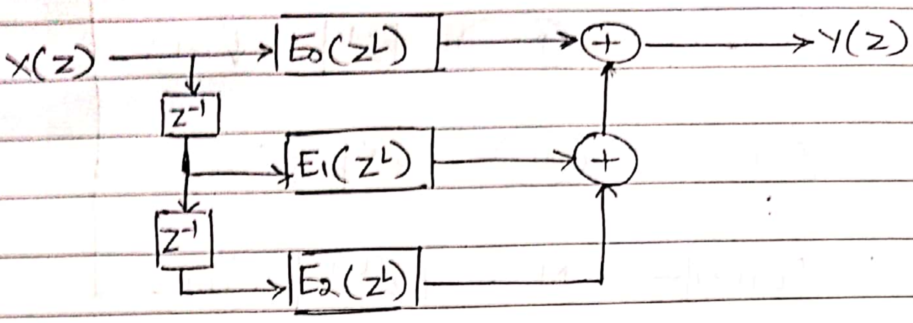
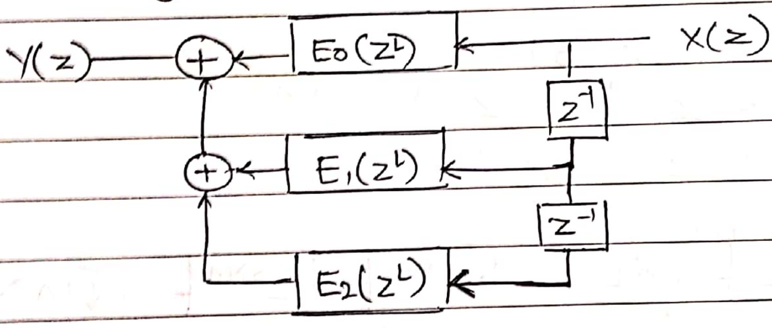
Polyphase Structure II

Consider the polyphase structure I

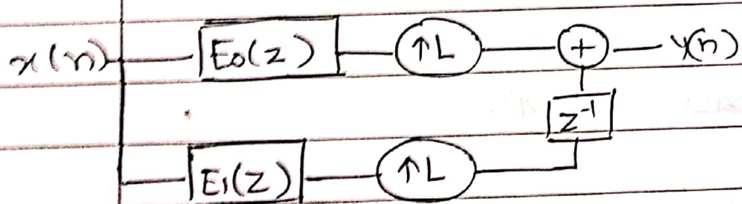


Transposing this to structure II

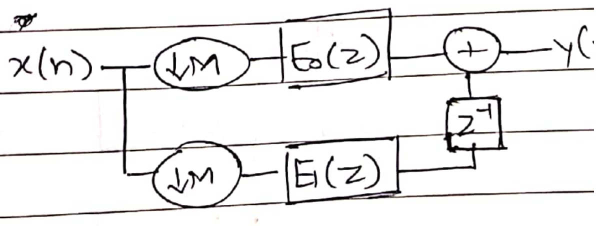


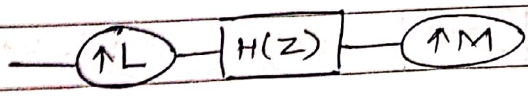
Rational Sampling Rate Convertor

Upsampler

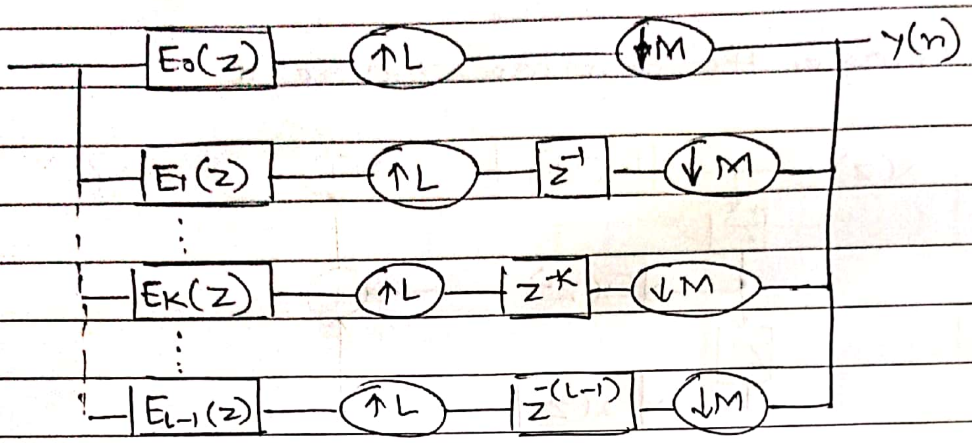


Downsampler





The system will be shown as



- L and M are relatively prime

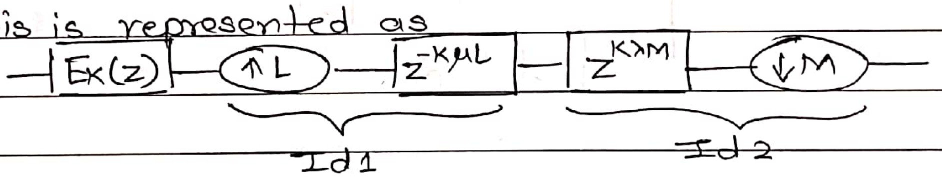
$\mu L - \lambda M = 1$

(Eg : $L = 2$, $M = 3$)

$z^{-K} = z^{-K(1)} = z^{-K(\mu L - \lambda M)}$

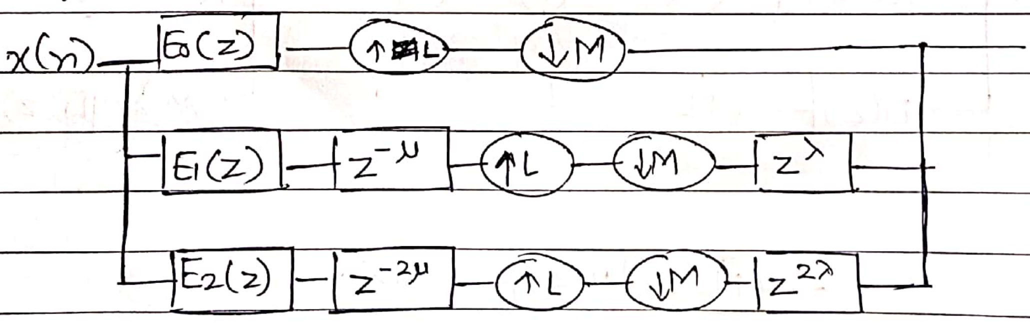
$z^{-K} = z^{-K\mu L} \cdot z^{K\lambda M}$

This is represented as

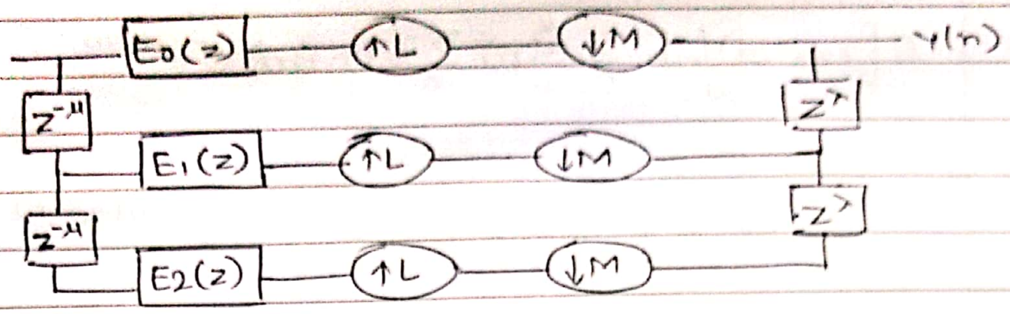


Applying the identity

$\Downarrow$



Final Structure



Module 02

Power Spectral Estimation

- The formula for Fourier transform is given as

$$X(e^{j\omega}) = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$\therefore\ X(2\pi f) = X(f) = \sum x(n) e^{-j2\pi n f}$$

- According to Parseval's theorem

$$\boxed{\sum |x(n)|^2 = \sum |X(f)|^2}$$

where $|X(f)|^2 \rightarrow$ Energy spectrum

The graph of $|X(f)|$ vs f is called the "Periodogram"

- Autocorrelation is used for random data signals. It is given, for signal $x(n)$, as

$$\boxed{r_{xx}(m) = \sum_{n=-\infty}^{\infty} x^*(n) \cdot x(n+m)}$$

shift the funcn by m

Now, taking the Fourier transform

$$F[r_{xx}(m)] = \sum_{m=-\infty}^{\infty} r_{xx}(m)\, e^{-j2\pi f m}$$

$$= \sum_m \sum_n x^*(n) \cdot x(n+m) \cdot e^{-j2\pi fm}$$

$$F[r_{xx}(m)] = \sum_n x^*(n) \sum_m x(n+m) e^{-j2\pi fm}$$

Let $n+m = k$

$$\therefore F[r_{xx}(m)] = \sum_n x^*(n) \sum_k x(k) e^{-j2\pi f(k-n)}$$

$$= \sum_n x^*(n) \left\{ \sum_k x(k) e^{-j2\pi fk} \right\} e^{j2\pi fn}$$

$$= \sum_n x^*(n) \cdot X(f)\, e^{j2\pi fn}$$

$$= X(f) \sum x^*(n) e^{j2\pi fn}$$

$$F[r_{xx}(m)] = X(f) \left\{ \sum x(n) e^{-j2\pi fn} \right\}^*$$

$$= X(f) \cdot X^*(f)$$

$$\boxed{F[r_{xx}(m)] = |X(f)|^2} \; **$$

- Hence to find the power spectral density, we calculate the Fourier transform of the autocorrelation funᶜⁿ.

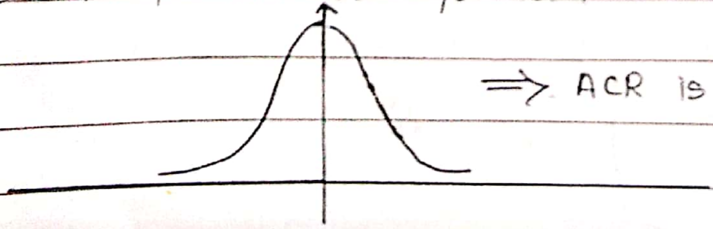
$$r_{xx}(m) = \sum_{n=-\infty}^{n=\infty} x^*(n)\, x(n+m)$$

$$= \lim_{N \to \infty} \frac{1}{(2N+1)} \sum_{n=-N}^{N} x^*(n) \cdot x(n+m)$$

Hence simplifying this

$$\boxed{r_{xx}(m) = \frac{1}{N} \sum_{n=0}^{N-1} x^*(n) \cdot x(n+m)} \; **$$

- Arguement of the above formula



$\Rightarrow$ ACR is an even funcⁿ

$$\text{Periodogram} = F[r_{xx}(m)]$$

$$= F\left[\frac{1}{N}\sum_{n=0}^{N-1} x^*(n)\,x(n+m)\right]$$

Power of periodogram $P(f) = \frac{1}{N} F\left\{ \sum x^*(n)\, x(n+m) \right\}$

$$= \frac{|X(f)|^2}{N}$$

(*) Methods to obtain Energy Spectrum or Periodogram

Methods
- Non-Parametric
  - Bartlett
  - Welch
  - Blackman Tuckey
- Parametric
  - AR
  - ARMA
  - MA
  - . . . .

A] Bartlett

→ Divide $x(n)$ into K samples $(0$ to $K-1)$ / K segments
Each part $x_i(n)$ will have $n = 0$ to .... $(M-1)$

→ Calculate the periodogram for each $(0 \ldots\ldots M)$ samples by ACR formula

→ Average out all the L samples

$x_i(n) = x(n+iM)$; $i = 0$ to $L-1$ (or 0 to $K-1$)

for each i $\{n = 0$ to ... $(M-1)\}$

$\boxed{N = ML \text{ or } N = MK}$

- Bartlett uses averaging, Welch uses modified averaging & Blackman Tuckey uses smoothening of a periodogram.

We have $\boxed{N = MK}$ for Bartlett

$$P_{xx}^{(i)}(f) = \frac{1}{M}\left[\sum_{n=0}^{M-1} x(n)\, e^{-j2\pi fn}\right]$$

We get values $P_{xx}^{(0)}(f)$, $P_{xx}^{(1)}(f)$, $P_{xx}^{(2)}(f)$ and so on.....
upto $P_{xx}^{(K-1)}(f)$

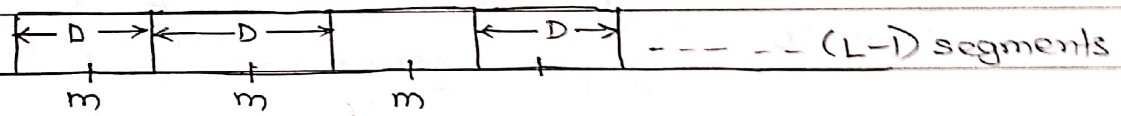
Hence the Bartlett func^n^ is given as **

$$\boxed{P_x^{(B)}(f) = \frac{1}{K}\sum_{i=0}^{K-1} P_{xx}^{(i)}(f)}$$

→ Periodogram
(Power in the freq. domain)

B] Welch Method

- This method uses overlapping windows. We will only consider the case of 50% overlap.



Here $\boxed{x_i(n) = x(n + iD)}$

- Steps in Welch method: ① Overlap ② Averaging
- We can write the Welch periodogram func^n^ as

$$\boxed{P_x^{(W)}(f) = \frac{1}{L}\sum_{i=0}^{L-1} P_{xx}^{(i)}(f)}$$

where

$$\boxed{P_{xx}^{(i)} = \frac{1}{D}\left[\sum_{n=0}^{D-1} x(n)\, e^{-j2\pi fn}\right]}$$

- **Note:**
In Bartlett, we have K segments of length M each
In Welch, we have L segments of length (M/2) each
$\therefore \boxed{K = 2L}$

- Each FFT → $\frac{M}{2}\log_2 M$ → computations

→ Side lobe peaks are indicative of the spectral leakage
→ Main lobe width is -"- -"- -"- frequency resolution

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In Bartlett	In Welch
Computations $= \frac{N}{m}\left(\cancel{\log}\ \frac{m}{2}\log_2 M\right)$	Computations $= \frac{N}{m/2}\left(\frac{m}{2}\log_2 m\right)$
$= \frac{N}{2}\log_2 M$	$= N\log_2 M$

	Window Type	Main lobe width	Side lobe peaks → spectral leakage	
①	Rectangular	$2\pi/N$	-13	
②	Hamming	$4\pi/N$	-41	
③	Hanning	$4\pi/N$	-31	
④	Blackman	$6\pi/N$	-57	→ Poor freq. resolution
⑤	Bartlett	$4\pi/N$	-25	

Mean value of the ACR function

$$E[ACR] = E[r_{xx}(m)] = E\left\{\frac{1}{N}\sum_{n=0}^{N-1} x^*(m)\cdot x(n+m)\right\}$$

$$= \frac{1}{N}\sum_{n=0}^{N-M-1} E\left\{x^*(n)\cdot x(n+m)\right\}$$

$$= \frac{1}{N}\sum_{n=0}^{N-M-1} r_{xx}(m)$$

$$= \frac{r_{xx}(m)(N-m)}{N}$$

$$\therefore E[r_{xx}(m)] = r_{xx}(m)\left[1-\frac{m}{N}\right]$$

Hence $\boxed{E[ACR] = ACR\cdot w(n)}$

↳ Window function

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Perchance

$$E[Per] = E\left\{ \sum_{m=0}^{N-1} r_{xx}(m) e^{-j2\pi fm} \right\}$$

$$= \sum_{m=0}^{N-1} E\left[r_{xx}(m) \cdot e^{-j2\pi fm} \right]$$

$$= \sum_{m=0}^{N-1} r_{xx}(m) \cdot \omega(m) \cdot e^{-j2\pi fm}$$

$$\boxed{E[Per] = P(f) * \omega(f)}$$ {multiplication in time domain is convolution in freq domain}

Hence expected value of power = true power

Note:

$r_{xx}(m) = E[x^*(n) \cdot x(n+m)]$

$r_{xx}(0) = E[x^*(n) \cdot x(n)]$

$= E|x(n)|^2$

For mean value $= 0$

$\boxed{r_{xx}(0) = \text{variance}[x(n)]}$ **

- LTI System

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$y(t) = x(t) * h(t)$

$Y(t) = \int_{-\infty}^{\infty} h(\alpha) \cdot x(t-\alpha) \cdot d\alpha$

$R_{xy} = E[x(t) \cdot y(t)]$

$= E\left\{ x(t) \cdot \int_{-\infty}^{\infty} h(\alpha) \cdot x(t-\alpha) d\alpha \right\}$

$R_{xy} = E\left[\int_{-\infty}^{\infty} h(\alpha) x(t-\alpha) x(t+\tau) d\alpha \right]$

$= \int_{-\infty}^{\infty} h(\alpha) R_{xx}(t-\alpha-t-\tau) d\alpha$

$$= \int_{-\infty}^{\infty} h(\alpha) R_{xx}(-\alpha-\tau)\, d\alpha$$

$$= \int_{-\infty}^{+\infty} h(\alpha) R_{xx}(t+\tau)\, d\alpha$$

$$\boxed{R_{xy} = h(-\tau) * R_{xx}(\tau)}$$

Now, similiarly R_{YY} can be given as

$$R_{YY} = E[Y(t) \cdot Y(t)]$$

$$= \int_{-\infty}^{\infty} h(\alpha) \cdot R_{xy}(-\alpha-\tau)\, d\alpha$$

$$= \int_{-\infty}^{\infty} h(\alpha) R_{xy}(t+\tau)\, d\alpha$$

$$\boxed{R_{YY} = h(-\tau) * R_{xy}(\tau)}$$

Note: Remember that

$$\boxed{R_{xx}(m) = E[x(n) \cdot x(n+m)]} \rightarrow \text{ACR}$$

where $m \rightarrow$ lags/delay

and ACR $\rightarrow$ area betn the sequence/random process with itself and its delayed versions

(Why? $\rightarrow$ ① See that the PDF remains the same
② Mean should be constant)

The above two reasons imply that

$$\boxed{E[x(t) \cdot x(t+\tau)] = R_{xx}(\tau)} \;\star\star$$

$\hookrightarrow$ ACR depends ONLY on the 'delay' and not on the time constants

The actual equation is

$$\boxed{E[X(t_1) \cdot X(t_2)] = R_{xx}(t_1 - t_2)} \;\star\star\star$$

where $t_2 = t_1 + \tau$

Hence

$$\boxed{E[X(t_1) \cdot X(t_1+\tau)] = R_{xx}(\tau)} \;\star\star\star$$

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(*) White Noise [$w(n)$]

1. It contains all the frequencies (noise freq)
2. Noise is random
3. WN contains/is high frequencies
4. ★ It can be defined as the sequence of uncorrelated random variables
5. ★ The mean of white noise is always taken as zero
   i.e $E[w(n)] = \text{mean} = 0$
6. ★ The variance of white noise $V[w(n)]$ is constant i.e
   $V[w(n)] = \sigma_w^2 = \text{constant}$.

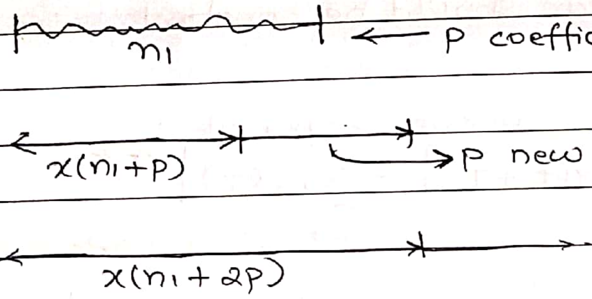
Hence $\boxed{R_{ww}(0) = \sigma_w^2}$

[*] Periodogram : Parametric Methods

$w(n) \longrightarrow \boxed{A_p\ ?} \longrightarrow x(n)$

$A_p \rightarrow$ filter (p → order of the filter)

I] Auto-Regressive (AR) Method



Auto — Regressive
↙ on its own ↘ fitting a set of points

- With random coefficient n_1 ~~bein~~ at the beginning & the coefficients are being updated everytime a new prediction is made

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$$w(n) \longrightarrow \boxed{A_p} \longrightarrow x(n)$$

$$W(z) \qquad A_p(z) \qquad X(z)$$

where

$$A_p(z) = \frac{X(z)}{W(z)} \qquad (p \rightarrow \text{order of the filter})$$

$$\boxed{\sum_{K=0}^{p} a_K \cdot x(n-K) = w(n)} \longrightarrow \text{AR eq}^n \qquad \text{— (1)}$$

Predicting $(p+1)$ coefficients from $w(n)$

→ Note that: $a_0, a_1, a_2 \ldots a_p$ are the $(p+1)$ filter coefficients

Solve the above equation so that we get the final values of $a_0, a_1 \ldots a_p$.

→ Note: $\boxed{a_0 = 1}$ "always"

Hence the eq^n becomes

$$\cancel{a_0}^{1} x(n) + \sum_{K=1}^{p} a_K \cdot x(n-K) = w(n)$$

$$\Rightarrow \boxed{x(n) + \sum_{K=1}^{p} a_K \cdot x(n-K) = w(n)} \qquad \text{— (2)}$$

This can be solved as shown

Step 1: We know $\sum_{K=0}^{p} a_K \cdot x(n-K) = w(n)$

Multiply on both sides by $x(n-l)$

$$\therefore \sum_{K=0}^{p} a_K \cdot x(n-K) \cdot x(n-l) = w(n) \cdot x(n-l)$$

$$\sum_{K=0}^{p} a_K \cdot E[x(n-K) \cdot x(n-l)] = E[w(n) \cdot x(n-l)]$$

$$\sum_{K=0}^{p} a_K \cdot r_{xx}(l-K) = r_{wx}(l) \qquad \text{— (3)}$$

To find a_K, we need to find ~~$r_{wx}(l)$~~ $r_{wx}(l)$

P.T.O

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Step 2: To get $r_{wx}(l)$

$$\sum_{K=0}^{p} a_K \cdot x(n-K) = w(n)$$

Replace n by $(n-l)$

$$\therefore \sum_{K=0}^{p} a_K \cdot x(n-K-l) = w(n-l)$$

$$x(n-l) + \sum_{K=1}^{p} a_K \cdot x(n-l-K) = w(n-l)$$

$$x(n-l) = w(n-l) - \sum_{K=1}^{p} a_K \cdot x(n-l-K) \qquad \text{— (4)}$$

To find $r_{wx}(l)$ in Step 1, need $x(n-l)$ because

$$r_{wx}(l) = E[w(n)\cdot x(n-l)]$$

$$r_{wx}(l) = E\left\{ w(n) \cdot \left[w(n-l) - \sum_{K=1}^{p} a_K \cdot x(n-l-K) \right] \right\}$$

$$= E\left\{ w(n)\cdot w(n-l) - w(n) \sum_{K=1}^{p} a_K\, x(n-l-K) \right\}$$

$$= E\left\{ w(n)\cdot w(n-l) \right\} - \sum_{K=1}^{p} a_K \underbrace{E\left\{ w(n)\cdot x(n-l-K) \right\}}_{=0}$$

Hence:

$$\boxed{r_{wx}(l) = r_{ww}(l)} \qquad \text{— (5)}$$

For $l=0$

$$\boxed{r_{wx}(l=0) = r_{ww}(0) = \sigma_w^2 = \text{variance}}$$

Hence

$$\left.\begin{aligned} r_{wx}(l) &= \sigma_w^2 && \text{for } l=0 \\ &= 0 && \text{for } l>0 \end{aligned}\right\} \qquad \text{— (6)}$$

In matrix form, we will first refer to

$$\sum_{K=0}^{p} a_K \cdot r_{xx}(l-K) = 0 \qquad (\text{for } l>0) \qquad \text{— (7)}$$

$$\therefore a_0 r_{xx}(l) + \sum_{K=1}^{p} a_K \cdot r_{xx}(l-K) = 0$$

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Hence: $\boxed{\sum_{k=1}^{p} a_k \cdot r_{xx}(l-k) = -r_{xx}(l)} \quad \forall\ l>0$ —(8)

For $l=1$

$$\sum_{k=1}^{p} a_k \cdot r_{xx}(1-k) = -r_{xx}(1)$$

$$a_1 r_{xx}(0) + a_2 r_{xx}(-1) + a_3 r_{xx}(-2) + \dots a_p r_{xx}(1-p) = r_{xx}(-1$$

For $l=2$

$$\sum_{k=1}^{p} a_k \cdot r_{xx}(2-k) = -r_{xx}(2)$$

$$a_1 \cdot r_{xx}(1) + a_2 \cdot r_{xx}(0) + \dots \dots a_p r_{xx}(2-p) = r_{xx}(-2)$$

Hence, we can form a matrix as shown below

$$\begin{array}{l} l=1 \\ l=2 \\ \vdots \\ \vdots \\ \vdots \end{array} \begin{bmatrix} r_{xx}(0) & r_{xx}(-1) & \dots & r_{xx}(1-p) \\ r_{xx}(1) & r_{xx}(0) & & r_{xx}(2-p) \\ r_{xx}(2) & & \ddots & \vdots \\ \vdots & & & \vdots \\ r_{xx}(p-1) & \dots & & r_{xx}(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} -r_{xx}(1) \\ -r_{xx}(2) \\ -r_{xx}(3) \\ \vdots \\ -r_{xx}(p-1) \end{bmatrix}$$

"Toeplitz matrix"

Note:

$$S_{xx}(\omega) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow S_{yy}(\omega)$$

i.e

$$\boxed{S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)}$$

We know that

$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$

$S_{yy}(\omega) \rightarrow$ PSD of o/P

When an input $x(n)$ is applied

$S_{xx}(\omega) \rightarrow$ PSD of i/P signal

$$S_{xx} \text{ — } \boxed{H(\omega)} \text{ — } S_{yy}$$

- If white noise is applied at the i/P, we should know $S_{xx}(\omega)$.
For white noise,
P.S.D $S_{xx}(\omega) = \sigma_{\omega}^2$ = variance (constant)

- The concept of white noise is introduced in speech processing applications i.e if the glotal excitation is taken as white noise, then the speech signal can be recovered from the white noise provided that we know the coefficients of the filter.

Method 2 : Least Squares Method

- We take $w(n) = x(n) - \sum_{k=1}^{p} (-a_k) x(n-k)$

OR

$$w(n) = x(n) - \sum_{k=1}^{p} a_k . x(n-k) \quad \{\text{where } a_k \text{ is } -ve\}$$

- Note that $w(n)$ is treated as the error between the actual & predicted value.

$$[w(n)]^2 = \left[x(n) - \sum_{k=1}^{p} a_k \cdot x(n-k)\right]^2$$

$$\therefore E[w(n)]^2 = E\left[x(n) - \sum_{k=1}^{p} a_k \cdot x(n-k)\right]^2$$

$$= E\left\{[x(n)]^2 - 2x(n)\sum_{k=1}^{p} a_k \cdot x(n-k) + \sum_{k=1}^{p} a_p(k) \cdot x(n-k) \times \sum_{l=1}^{p} a_p(l) \cdot x(n-l)\right\}$$

$$= E\left\{ [x(n)]^2 - 2\sum_{k=1}^{p} a_p(k)\cdot x(n)x(n-k) + \sum_{k=1}^{p} a_p(k)x(n-k)\cdot \sum_{l=1}^{p} a_p(l)\,x(n-l) \right\}$$

$$= E[x(n)]^2 - 2\sum_{k=1}^{p} a_p(k)E[x(n)\cdot x(n-k)] + \sum_{k=1}^{p}\sum_{l=1}^{p} a_p(k)\cdot a_p(l)\cdot E[x(n-k)\cdot x(n-l)]$$

Hence

$$E[w(n)]^2 = r_{xx}(0) - 2\sum_{k=1}^{p} a_p(k) r_{xx}(k) + \sum\sum a_p(k)\cdot a_p(l)\, r_{xx}(l-k)$$

$$\therefore \frac{d}{da}\left[E(w(n))^2\right] = \underset{\text{I}}{0} - \underbrace{2\frac{\partial}{\partial a}\left[\sum_{k=1}^{p} a_p(k)\cdot r_{xx}(k)\right]}_{\text{II}} + \underbrace{\sum_{k=1}^{p}\sum_{l=1}^{p} \frac{\partial a_p(k)\cdot a_p(l)\cdot r_{xx}(l-k)}{\partial a}}_{\text{III}}$$

$$a_p(1) r_{xx}(1) + a_p(2)\cdot r_{xx}(2) + \cdots\cdots\cdots a_p(p)\cdot r_{xx}(p)$$

$$= [r_{xx}(1) + r_{xx}(2) + \cdots\cdots\cdots r_{xx}(p)] \begin{bmatrix} a_p(1) \\ a_p(2) \\ \vdots \\ a_p(p) \end{bmatrix}$$

$$= r_{xx}^T \cdot a$$

$$\therefore \text{II} = -2\frac{\partial}{\partial a}(r_{xx}^T \cdot a) = \boxed{-2r_{xx}}$$

vector $y \longrightarrow$ vector a $[a_1 \quad a_2 \quad a_3]$

$$\frac{\partial y}{\partial a} = \begin{bmatrix} \partial y/\partial a_1 \\ \partial y/\partial a_2 \\ \partial y/\partial a_3 \\ \vdots \end{bmatrix}$$

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$$\text{III} = \frac{\partial}{\partial a} \sum_k \sum_l a_p(k) \cdot a_p(l) \cdot r_{xx}(l-k)$$

$$= \frac{\partial}{\partial a} \cdot \sum_k a_p(k) \cdot \sum_l a_p(l) \cdot r_{xx}(l-k)$$

$$a_p(1) \times \sum_l a_p(l) \cdot r_{xx}(l-1)$$

$$a_p(2) \times \sum_l a_p(l) \cdot r_{xx}(l-2)$$

$$\vdots$$

$$a_p(p) \times \sum_l a_p(l) \cdot r_{xx}(l-p)$$

$$\Rightarrow a_p(1)\left[a_p(1) \cdot r_{xx}(0) + a_p(2) \cdot r_{xx}(1) + \cdots\cdots a_p(p) \cdot r_{xx}(p-1)\right]$$

$$a_p(2)\left[a_p(1) \cdot r_{xx}(-1) + a_p(2) \cdot r_{xx}(0) + \cdots\cdots a_p(p) \cdot r_{xx}(p-2)\right]$$

$$a_p(3)\left[\right.$$

$$\vdots$$

$$a_p(p)\left[a_p(1) \cdot r_{xx}(1-p) + a_p(2) \cdot r_{xx}(2-p) \cdots\cdots a_p(p) \cdot r_{xx}(0)\right]$$

This gives

$$\begin{bmatrix} a_p(1) \\ a_p(2) \\ \vdots \\ a_p(p) \end{bmatrix}_{p\times 1} \begin{bmatrix} r_{xx}(0) & r_{xx}(1) & \cdots & r_{xx}(p-1) \\ r_{xx}(-1) & r_{xx}(0) & & \vdots \\ \vdots & & \ddots & \vdots \\ r_{xx}(1-p) & & & r_{xx}(0) \end{bmatrix} \begin{bmatrix} a_p(1) \\ a_p(2) \\ \vdots \\ a_p(p) \end{bmatrix}$$

This can be modified as shown

$$\begin{bmatrix} a_p(1) & a_p(2) & \cdots & a_p(p) \end{bmatrix} \begin{bmatrix} r_{xx}(0) & r_{xx}(1) & \cdots & r_{xx}(p-1) \\ r_{xx}(-1) & r_{xx}(0) & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ r_{xx}(1-p) & & & r_{xx}(0) \end{bmatrix} \begin{bmatrix} a_p(1) \\ a_p(2) \\ \vdots \\ a_p(p) \end{bmatrix}$$

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Hence we write it as

$$\boxed{\text{III} = a^T \cdot R \cdot a} \; **$$

Now III $= \frac{\partial}{\partial \bar{a}} \left[a^T R a \right]$

$$= R \frac{\partial}{\partial \bar{a}} \left[\bar{a} \cdot a \right]$$

where $a^T \cdot a = \left[a_p^2(1) + a_p^2(2) + a_p^2(3) + \cdots a_p^2(p) \right]$

Note that the vector differentiation is w.r.t all the components

$$\therefore \frac{\partial}{\partial \bar{a}} (a^T a) = \begin{bmatrix} \partial/\partial a_p(1) \{ a_p^2(1) \} \\ \partial/\partial a_p(2) \{ a_p^2(2) \} \\ \vdots \\ \partial/\partial a_p(p) \{ a_p^2(p) \} \end{bmatrix} = \begin{bmatrix} 2a_p(1) \\ 2a_p(2) \\ \vdots \\ 2a_p(p) \end{bmatrix}$$

$$= 2a$$

$$\therefore \text{III} = R \frac{\partial}{\partial \bar{a}} (a^T a) = \boxed{2Ra} \; ***$$

Hence

$$\frac{\partial [w(n)]^2}{\partial a} = 0 - 2r_{xx} + 2Ra$$

Minimization of noise $\Rightarrow$

$$\frac{\partial [w(n)]^2}{\partial a} = 0$$

$$\therefore R \cdot a = r_{xx}$$

$$\boxed{a = -R^{-1} \cdot r_{xx}} \; ***$$

matrix vector

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- Note: This method can be problematic because
① Inverse of the matrix might not exist
② Ill conditioned matrix

Ⓐ Interpretation of the AR process in the Z domain

We have

$$w(n) = x(n) - \sum_{K=1}^{p} a_p(K) \cdot x(n-K)$$

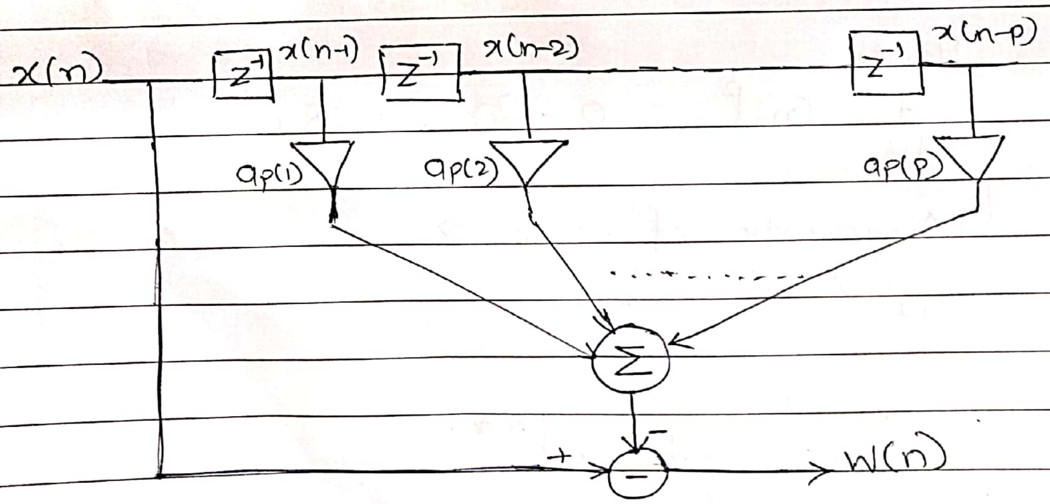
Taking the Z transform

$$W(z) = X(z) - \sum_{K=1}^{p} a_p(K) \cdot X(z) \cdot z^{-K}$$

$$\Rightarrow W(z) = X(z)\left\{ 1 - \sum_{K=1}^{p} a_p(K) \cdot z^{-K} \right\}$$

Hence $$\frac{X(z)}{W(z)} = \frac{1}{\left[1 - \sum_{K=1}^{p} a_p(K) z^{-K}\right]} \quad \text{— Ⓐ}$$

where $\frac{X(z)}{W(z)} = H(z) \longrightarrow \boxed{\text{all pole filter}}$



- In the case of AR model $H(e^{j\omega})$ is an all-pole filter because from the equation Ⓐ, we can see that $H(z) = \dfrac{1}{\left\{1 - \sum_{k=1}^{p} a_p z^{-k}\right\}}$

- An all-pole filter is used to find the peaks in the filter. Therefore, the AR model is used in the periodogram so that once we determine the peaks, we can get the amplitude at peaks and hence the power.

- To calculate the parameters of the AR model, we need to know $[1 \;\; a_p(1) \;\; a_p(2) \;\ldots\ldots\; a_p(p)]$ where p is the order of the AR process.

Method 3: Levinson-Durbin Method to calculate the parameters of the AR model

$$\underset{p=1}{\begin{bmatrix} 1 \\ a_1(1) \end{bmatrix}} \qquad \underset{p=2}{\begin{bmatrix} 1 \\ a_2(1) \\ a_2(2) \end{bmatrix}} \qquad \underset{p=3}{\begin{bmatrix} 1 \\ a_3(1) \\ a_3(2) \\ a_3(3) \end{bmatrix}}$$

$$\begin{bmatrix} 1 \\ K_1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ a_2(1) \\ K_2 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ a_3(1) \\ a_3(2) \\ K_3 \end{bmatrix}$$

- Algorithm

$$E^{(0)} = R(0)$$

$$K_i = R(i) - \sum_{j=1}^{i-1} a_j(i-1)\, R(i-j)$$

$$\text{for } 1 \leq i \leq p$$

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$a_i(i) = K_i$

$a_j(i) = a_j(i-1) - K_i \cdot a_{i-j}(i-1)$
where $1 \le j \le i-1$

$E^{(i)} = (1-K_i^2)E^{(i-1)}$
where $E \rightarrow$ error

- Solving the levinson - Durbin process

Step 1: For $p=1$
$E^{(0)} = R(0)$
$i=1$
$K_1 = R(1) - 0 = R(1)$
$a_1(1) = K_1$
$E^{(1)} = (1-K_1^2)E^{(0)}$
$\therefore E^{(1)} = (1-K_1^2)R(0)$

Step 2 : For $p=2$
$E^{(1)} = (1-K_1^2) \cdot R(0)$
$i=1$
$K_1 = R(1) - 0 = R(1)$
$a_1(1) = K_1$

$i=2$
$K_2 = R(2) - \sum_{j=1}^{1} a_j(i-1) R(i-j)$

$K_2 = R(2) - a_1(1) \cdot R(1)$

where $1 \le i \le 2$; $1 \le j \le 1$
$a_2(2) = K_2$

$a_2(1) = a_1(1) - K_2 \cdot a_1(1)$

$E^{(2)} = (1-K_2^2)E^{(1)}$

$E^{(2)} = (1-K_2^2)(1-K_1^2)R(0)$