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Problem-2

$x \in \mathbb{R}^n \quad b \in \mathbb{R}^n \quad A \in \mathbb{R}^{n \times n}$

$f: \mathbb{R}^n \to \mathbb{R} \quad f(x) = b^T x + x^T A x$

a) Gradient:

$$\frac{\partial f}{\partial x} = \frac{\partial (b^T x)}{\partial x} + \frac{\partial (x^T A x)}{\partial x}$$

$$b^T x = b_1 x_1 + b_2 x_2 + \cdots + b_n x_n$$

$$\frac{\partial (b^T x)}{\partial x} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = b$$

$$\frac{\partial (x^T A x)}{\partial x} = \frac{\partial (x^T A^T x)}{\partial x}$$

[Since $x^T A x$ is scalar;
$(x^T A x)^T = x^T A x$
$(x^T A^T x) = x^T A x$]

$$= \frac{\partial x^T}{\partial x}(A^T x) + \frac{\partial x^T}{\partial x}(Ax)$$

$$= A^T x + Ax$$

$$= (A^T + A)x$$

Gradient $\Rightarrow \boxed{\nabla f = b + (A + A^T)x}$

If $A \to$ symmetric: $\nabla f = b + 2Ax$

Hessian :

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(b + (A + A^T) x \right)$$

$$= 0 + (A + A^T)$$

$$\boxed{\nabla^2 f(x) = A + A^T} \Rightarrow \text{Hessian .}$$

If $A \rightarrow$ symmetric : $\nabla^2 f = 2A$

b) Taylor Series:

First Order: $f(x_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0}^{T} \cdot (x - x_0)$

$\Rightarrow b^T(x_0) + x_0^T A x_0 + \left[b + (A + A^T) x_0\right]^T (x - x_0)$

$\Rightarrow x_0 = 0$

$\Rightarrow b^T x$

$\boxed{1^{st} \text{ order} \Rightarrow b^T x}$ $\Rightarrow$

Second Order: $F(x_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0} (x - x_0) + \frac{1}{2} \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0} (x - x_0)^2$

$\Rightarrow$ Firstorder $+ \frac{1}{2} \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0} (x - x_0)^2$

$\Rightarrow b^T x + \frac{1}{2} (A + A^T)(x - 0)^2$

$\boxed{2^{nd} \text{ order} \Rightarrow b^T x + \frac{1}{2} x^T (A + A^T) x}$

$\Rightarrow$ This is an exact approximation because the function is quadratic.

c) $A \in \mathbb{R}^{n \times n}$

$\therefore$ Necessary and sufficient conditions for A to be positive definite matrix:

↳ A needs to symmetric matrix

↳ $x^T A x > 0 \quad \forall x \neq 0, x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}$

d) Necessary and sufficient condition for A to have full rank:

↳ Determinant of $A \neq 0 \Rightarrow |A| \neq 0$

e) $y \in \mathbb{R}^n$ and $y \neq 0$ s.t $A^T y = 0$.

$A^T y = 0 \Rightarrow y \in N(A^T)$ y belongs to null space of A^T

From principal of orthogonality

$$C(A) \perp N(A^T)$$

For $Ax = b$ to have a solution $b \in C(A)$

$\therefore$ $\boxed{\text{Condition on } b\text{: } \; b^T y = 0}$

Problem-3

N : Number of different groceries. $i = [1, N]$

M : Number of nutrients contained. $j = [1, M]$

a_{ij} : quantity of nutrition j in food i

b_j : necessary quantity of nutrition j in a month

C_i : Price of food i

x_i : Quantity of food type i

⇒ Solution:

→ Objective: Minimize $\sum_{i=1}^{N} x_i C_i$

Constraints: $\boxed{\sum_{i=1}^{N} x_i a_{ij} - b_j \geq 0}$ $\forall j = 1, 2 \ldots M$

$\boxed{x_i \geq 0}$ $\forall i = 1, 2 \ldots, N$

[∵ Quantity of food cannot be negative]