

ME598/494 Homework 2

1. (20 points) Show that the stationary point (zero gradient) of the function

$$f(x_1, x_2) = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$$

is a saddle (with indefinite Hessian).

Find the directions of downslopes away from the saddle. To do this, use Taylor's expansion at the saddle point to show that

$$f(x_1, x_2) = f(1, 1) + (a\partial x_1 - b\partial x_2)(c\partial x_1 - d\partial x_2),$$

with some constants a, b, c, d and $\partial x_i = x_i - 1$ for $i = 1, 2$. Then the directions of downslopes are such $(\partial x_1, \partial x_2)$ that

$$f(x_1, x_2) - f(1, 1) = (a\partial x_1 - b\partial x_2)(c\partial x_1 - d\partial x_2) < 0.$$

2. (a) (10 points) Find the point in the plane $x_1 + 2x_2 + 3x_3 = 1$ in \mathbb{R}^3 that is nearest to the point $(-1, 0, 1)^T$. Is this a convex problem? Hint: Convert the problem into an unconstrained problem using $x_1 + 2x_2 + 3x_3 = 1$.
 (b) (40 points) Implement the gradient descent and Newton's algorithm for solving the problem. Attach your codes in the report, along with a short summary of your findings. The summary should include: (1) The initial points tested; (2) corresponding solutions; (3) A log-linear convergence plot.
3. (5 points) Prove that a hyperplane is a convex set. Hint: A hyperplane in \mathbb{R}^n can be expressed as: $\mathbf{a}^T \mathbf{x} = c$ for $\mathbf{x} \in \mathbb{R}^n$, where \mathbf{a} is the normal direction of the hyperplane and c is some constant.
4. (15 points) Consider the following illumination problem:

$$\min_{\mathbf{p}} \max_k \{h(\mathbf{a}_k^T \mathbf{p}, I_t)\}$$

$$\text{subject to: } 0 \leq p_i \leq p_{\max},$$

where $\mathbf{p} := [p_1, \dots, p_n]^T$ are the power output of the n lamps, \mathbf{a}_k for $k = 1, \dots, m$ are fixed parameters for the m mirrors, I_t the target intensity level. $h(I, I_t)$ is defined as follows:

$$h(I, I_t) = \begin{cases} I_t/I & \text{if } I \leq I_t \\ I/I_t & \text{if } I_t \leq I \end{cases}$$

- (a) (5 points) Show that the problem is convex.
 - (b) (5 points) If we require the overall power output of any of the 10 lamps to be less than p^* , will the problem have a unique solution?
 - (c) (5 points) If we require no more than 10 lamps to be switched on ($p > 0$), will the problem have a unique solution?
5. (10 points) Let $c(x)$ be the cost of producing x amount of product A and assume that $c(x)$ is differentiable everywhere. Let y be the price set for the product. Assuming that the product is sold out. The total profit is defined as

$$c^*(y) = \max_x \{xy - c(x)\}.$$

Show that $c^*(y)$ is a convex function with respect to y .