Lecture will start @ 2pm EST ☺

TECHNICAL ASSESSMENT

If you have not taken the technical assessment yet, please take it now on Canvas.

Do not look up any answers!

INTRO TO QUANTUM COMPUTING

LECTURE #4

MORE VECTORS & INTRO TO MATRICES

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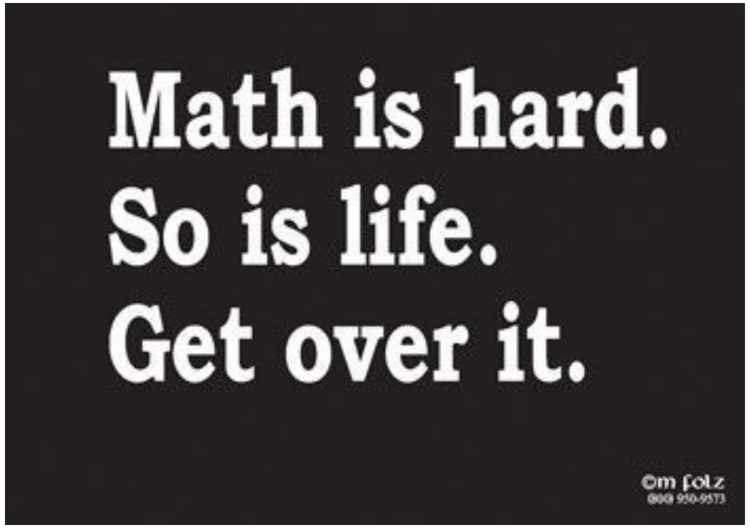
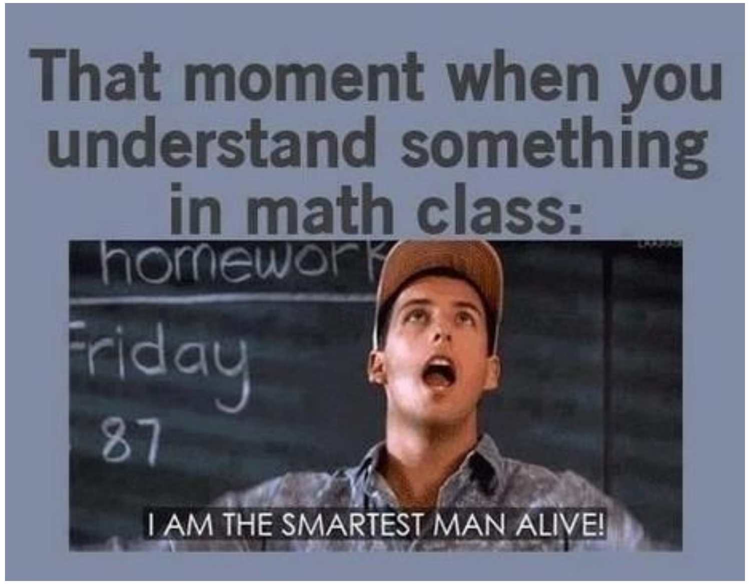
11/8/2020

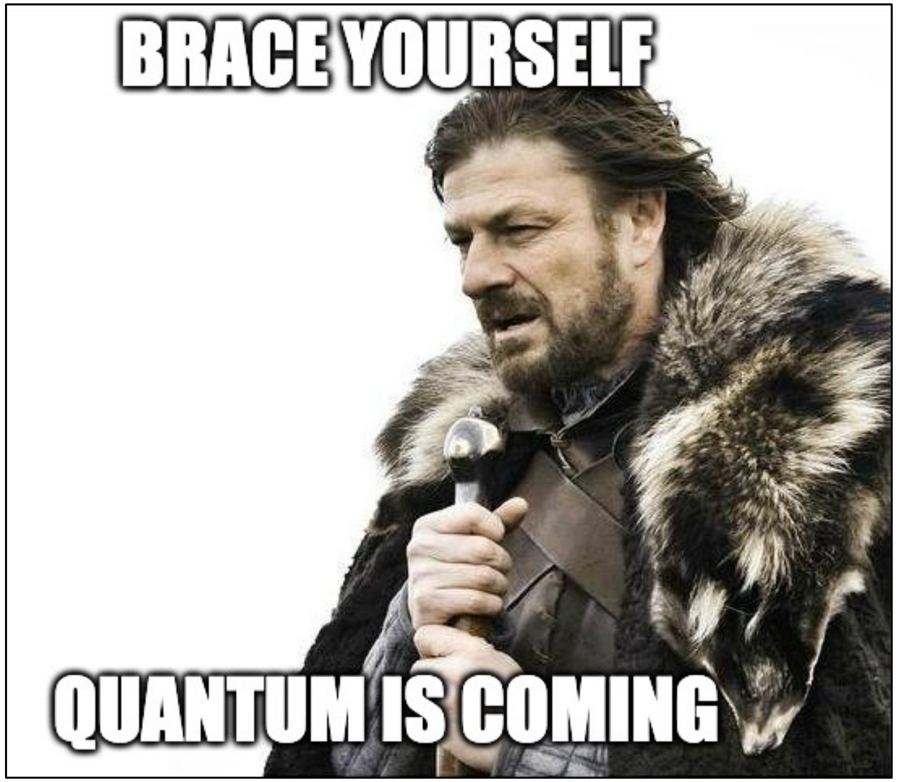
MORE MATH MOTIVATION?

Glass half empty…



Glass half full…





(Next lecture!!!)

WHY ALL THE MATH?

Quantum mechanics is a beautiful generalization of the laws of probability: a generalization based on the 2-norm rather than the 1-norm, and on complex numbers rather than nonnegative real numbers. It can be studied completely separately from its applications to physics (and indeed, doing so provides a good starting point for learning the physical applications later). This generalized probability theory leads naturally to a new model of computation – the quantum computing model – that challenges ideas about computation once considered a priori, and that theoretical computer scientists might have been driven to invent for their own purposes, even if there were no relation to physics. In short, while quantum mechanics was invented a century ago to solve technical problems in physics, today it can be fruitfully explained from an extremely different perspective: as part of the history of ideas, in math, logic, computation, and philosophy, about the limits of the knowable.

- Professor Scott Aaronson (UT Austin)

(excerpt from *Quantum Computing Since Democritus*)

Image Source: ETH Zurich

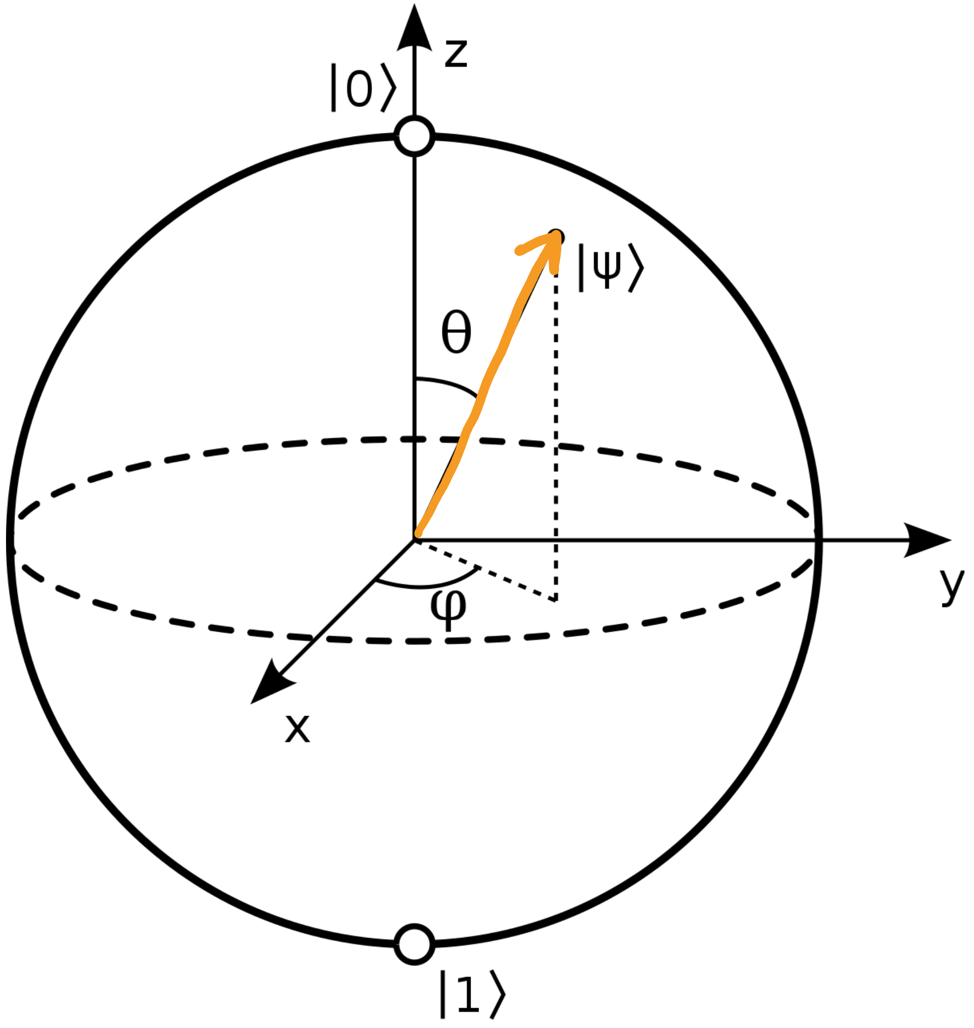
TODAY'S LECTURE

1. More Vectors
    a) Vector Shape
    b) Transpose
    c) Inner Product
    d) Normalization
    e) Conjugate Transpose
    f) Complex Inner Product
    g) Linear Combinations

2. Intro to Matrices
    a) Matrix Notation and Shape
    b) Matrix Operations
        a) Matrix Addition
        b) Matrix-Scalar Multiplication
        c) Matrix-Vector Multiplication
        d) Matrix-Matrix Multiplication
        e) Matrix Transpose/Complex Conjugate
    c) Solving Linear Systems of Equations
        a) The Identity Matrix
        b) Matrix Inversion

MORE VECTORS

WHAT DO VECTORS MEAN FOR QUANTUM COMPUTING?



Qubits are two-level quantum systems that lie in the Bloch sphere and their states can be represented as vectors:

$$\vec{\Psi} = \begin{pmatrix} \cos(\theta/2) \\ e^{i\varphi} \sin(\theta/2) \end{pmatrix}$$

VECTOR REVIEW

Vector Representation:

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Vector Addition:

$$\vec{a} + \vec{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

Vector-Scalar Multiplication:

$$c * \vec{v} = \begin{pmatrix} c * v_1 \\ c * v_2 \\ \vdots \\ c * v_n \end{pmatrix}$$

Vector Magnitude:

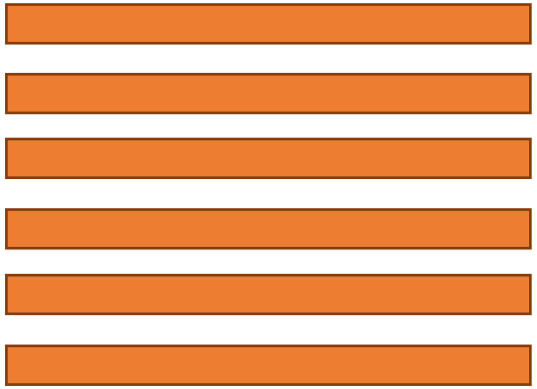
$$||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

SHAPE

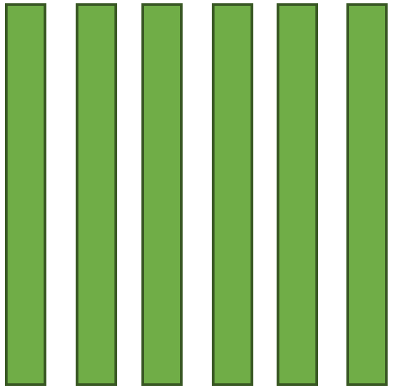
Today we are going to be talking a lot about ***shape***!

Turns out it is a very important idea for vector and matrix manipulation…

ROWS



COLUMNS



VECTOR SHAPE

What is the *shape* of vector?

VECTOR SHAPE: $(\# \text{ rows} \times \# \text{ cols})$

	COLUMN VECTOR	**ROW VECTOR**
	$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$	$\begin{pmatrix} r_1 & r_2 & \cdots & r_m \end{pmatrix}$
SHAPE:	$(n \times 1)$	$(1 \times m)$

QUANTUM PRACTICE TIME!

Give the shape of the following vectors.

(1) $\vec{a} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$

(2) $\vec{b} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

(3) $\vec{c} = \begin{pmatrix} 13 \end{pmatrix}$

(4) $\vec{d} = \begin{pmatrix} 5 \\ 3 \\ 2 \\ 1 \end{pmatrix}$

(5) $\vec{e} = \begin{pmatrix} 9 & 8 & 7 & 6 & 5 \end{pmatrix}$

QUANTUM PRACTICE SOLUTIONS

Give the shape of the following vectors.

(1) $\vec{a} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ (2×1)

(2) $\vec{b} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ (1×3)

(3) $\vec{c} = (13)$ (1×1)

(4) $\vec{d} = \begin{pmatrix} 5 \\ 3 \\ 2 \\ 1 \end{pmatrix}$ (4×1)

(5) $\vec{e} = \begin{pmatrix} 9 & 8 & 7 & 6 & 5 \end{pmatrix}$ (1×5)

VECTOR TRANSPOSE

The ***transpose*** is an operation which flips the shape of a vector.

(Rows become columns and columns become rows.)

$(n \times 1)$ $(1 \times n)$

If $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$, then its transpose is $\vec{v}^T = (v_1 \quad v_2 \quad \cdots \quad v_n)$

If $\vec{w} = (w_1 \quad w_2 \quad \cdots \quad w_n)$, then its transpose is $\vec{w}^T = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$

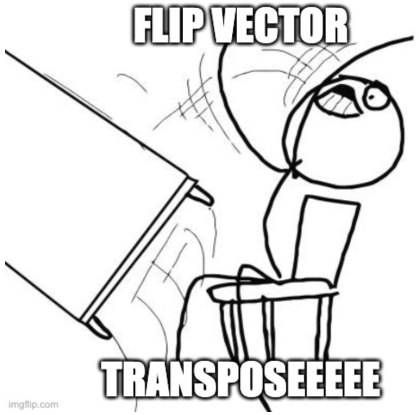
$(1 \times n)$ $(n \times 1)$

Note: The transpose does not change anything about the vector geometrically. It just changes the shape.

VECTOR TRANSPOSE

TRANSPOSE

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{pmatrix} \quad (k \times 1)$$



$$\begin{pmatrix} v_1 & v_2 & \cdots & v_k \end{pmatrix} \quad (1 \times k)$$

TRANSPOSE

$$\begin{pmatrix} v_1 & v_2 & \cdots & v_k \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{pmatrix}$$

TRANSPOSE

QUANTUM PRACTICE TIME!

Write out the transpose of the 3 following vectors.

(1) $\vec{a} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ $\qquad \vec{a}^T = ?$

(2) $\vec{b} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ $\qquad \vec{b}^T = ?$

(3) $\vec{c} = (13)$ $\qquad \vec{c}^T = ?$

QUANTUM PRACTICE SOLUTIONS

Write out the transpose of the 3 following vectors.

(1) $\vec{a} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ $\qquad \vec{a}^T = \begin{pmatrix} 7 & 2 \end{pmatrix}$

(2) $\vec{b} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ $\qquad \vec{b}^T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

(3) $\vec{c} = (13)$ $\qquad \vec{c}^T = (13)$

THE INNER PRODUCT

What is the ***inner product*** of two real vectors?

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}\,\vec{w}^T = \sum_{i=1}^{n} v_i w_i$$

$\rightarrow v_1w_1 + v_2w_2 + v_3w_3$

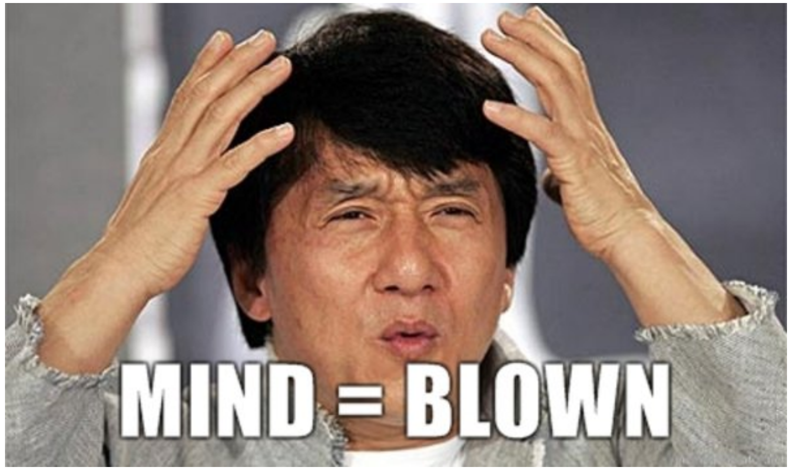
where
$\vec{v}, \vec{w} \in \mathbb{R}^n$
are row vectors

$(\cdot \; \cdot \; \cdot \; \cdot \; \cdot)$

Note: In literature, you might also hear the inner product referred to as the ***dot product*** (denoted $\vec{v} \cdot \vec{w}$) or ***scalar product*** (since it maps vectors to a scalar!).

Turns out it's a lot more than just that, though. It is a:

1. **vector x vector** to **scalar** mapping
2. tool for calculating vector **magnitude**
3. Tool for vector **normalization**
4. tool for **geometrically comparing** vectors
5. tool for determining vector **orthogonality**



1. THE INNER PRODUCT – VECTOR TO SCALAR MAPPING

Let's start off by seeing why this is true…

$$\langle \vec{v}, \vec{w} \rangle = \underbrace{\vec{v}\,\vec{w}^T}_{\text{row vectors}} = \sum_{i=1}^{n} v_i w_i$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^T \vec{w}$$

$$= \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

$$= \sum_{i=1}^{3} v_i w_i$$

1. THE INNER PRODUCT – VECTOR TO SCALAR MAPPING

Let's work through a quick example…

$$\vec{a} = (1 \quad 2 \quad 3) \qquad \vec{b} = (4 \quad 5 \quad 6)$$

$$\langle \vec{a}, \vec{b} \rangle = \vec{a}\,\vec{b}^{T} = (1 \quad 2 \quad 3)\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 1*4 + 2*5 + 3*6$$

$$= 4 + 10 + 18$$

$$= 14 + 18 = \underline{\underline{32}}$$

Now, what if we take the inner product of a vector with itself?

$$\langle \vec{v}, \vec{v} \rangle = \vec{v}\vec{v}^T = \sum_{i=1}^{n} v_i v_i = \sum_{i=1}^{n} {v_i}^2 = \|\vec{v}\|^2$$

where
$\vec{v} \in \mathbb{R}^n$

$\|\vec{v}\| = \sqrt{\sum v_i^2}$

The inner product of a vector with itself gives us the magnitude squared of the vector!

Note: the magnitude of the vector is also called its ***norm***.

QUANTUM PRACTICE TIME!

Calculate the norm/magnitude of the following vectors, using an inner product:

(1) $$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(2) $$\vec{w} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

QUANTUM PRACTICE SOLUTIONS

Calculate the norm/magnitude of the following vectors, using an inner product:

(1) $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{\vec{v}^T \vec{v}} = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$(1 \ 2 \ 3) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(2) $\vec{w} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

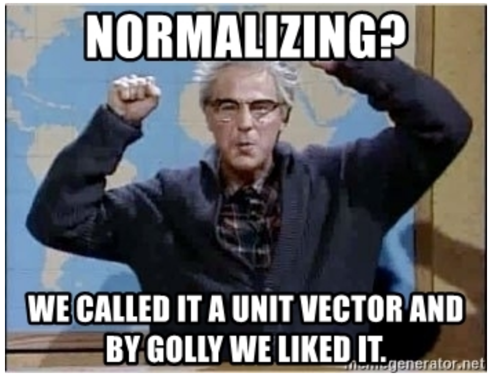
$$\|\vec{w}\| = \sqrt{\langle \vec{w}, \vec{w} \rangle} = \sqrt{\vec{w}\vec{w}^T} = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$$

3. THE INNER PRODUCT –VECTOR NORMALIZATION

A vector is ***normalized*** if it has a magnitude of 1.

Note: a normalized vector is often called a ***unit vector***.

How can we use the inner product to ***normalize*** a vector?



$$\frac{\vec{v}}{\sqrt{\langle \vec{v}, \vec{v} \rangle}} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\|\vec{v}\| = \sqrt{\vec{v}^T \vec{v}} = \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$$

$$\vec{w} = \frac{\vec{v}}{\|\vec{v}\|} = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

$$\|\vec{w}\| = \sqrt{\vec{w}^T \vec{w}} = \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2} = \sqrt{\frac{1}{5} + \frac{4}{5}} = \sqrt{\frac{5}{5}} = \sqrt{1} = 1$$

3. THE INNER PRODUCT –VECTOR NORMALIZATION

$$|x|$$

$$||\vec{v}||$$

$$\frac{\vec{v}}{||\vec{v}||}$$

3. THE INNER PRODUCT –VECTOR NORMALIZATION

Let's check if the following vector is normalized…

$$\vec{w} = \begin{pmatrix} \sqrt{\frac{2}{5}} & \sqrt{\frac{3}{5}} \end{pmatrix}$$

QUANTUM PRACTICE TIME!

State whether the following vectors is normalized. If not, what is the normalized vector?

(1) $\vec{v} = \begin{pmatrix} 2 & 3 \end{pmatrix}$

QUANTUM PRACTICE TIME!

State whether the following vector is normalized. If not, what is the normalized vector?

(1) $\vec{v} = \begin{pmatrix} 2 & 3 \end{pmatrix}$

$$\|\vec{v}\| = \sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$$

Since $\sqrt{13} \neq 1$, the vector $\vec{v}$ is *not* normalized. The normalized vector is:

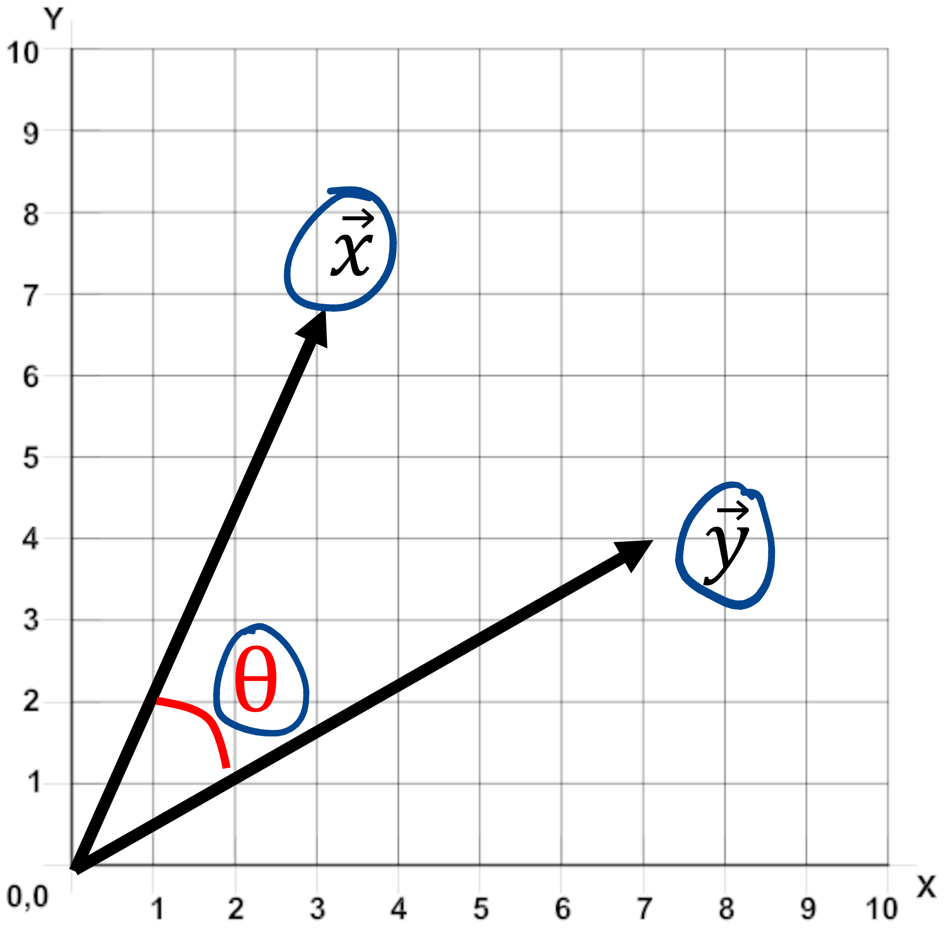
$$\vec{w} = \frac{\vec{v}}{\|\vec{v}\|} = \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \qquad \|\vec{w}\| = 1$$

What does the inner product mean geometrically, though?

$$\theta = \cos^{-1}\left(\frac{\langle \vec{x}, \vec{y} \rangle}{\underbrace{\|\vec{x}\|}_{\langle \vec{x}, \vec{x} \rangle}\underbrace{\|\vec{y}\|}_{\langle \vec{y}, \vec{y} \rangle}}\right)$$

$$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\|\|\vec{y}\| \cos(\theta)$$

where $\theta = \angle(\vec{x}, \vec{y})$ is the angle between $\vec{x}$ and $\vec{y}$



The inner product tells us the angle between two vectors!

5. THE INNER PRODUCT –VECTOR ORTHOGONALITY

Let's consider some possibilities when $\vec{x}$ and $\vec{y}$ are unit vectors ($\|\vec{x}\| = \|\vec{y}\| = 1$)...

$\boxed{\langle \vec{x}, \vec{y} \rangle = \cos(\theta)}$

PARALLEL

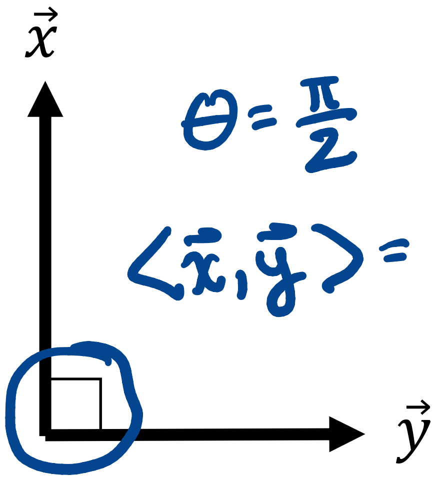
$\theta = 0$

$\langle \vec{x}, \vec{y} \rangle = \cos(\theta)$
$= \cos(0)$
$= 1$

$\theta = 0°$

$\langle \vec{x}, \vec{y} \rangle = 1$

ORTHOGONAL
(PERPENDICULAR)



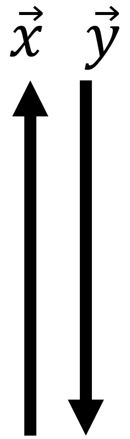
$\theta = \frac{\pi}{2}$

$\langle \vec{x}, \vec{y} \rangle = \cos\left(\frac{\pi}{2}\right) = 0$

$\theta = 90°$

$\langle \vec{x}, \vec{y} \rangle = 0$

ANTI-PARALLEL



$\theta = \pi$

$\langle \vec{x}, \vec{y} \rangle = \cos(\pi)$
$= -1$

$\theta = 180°$

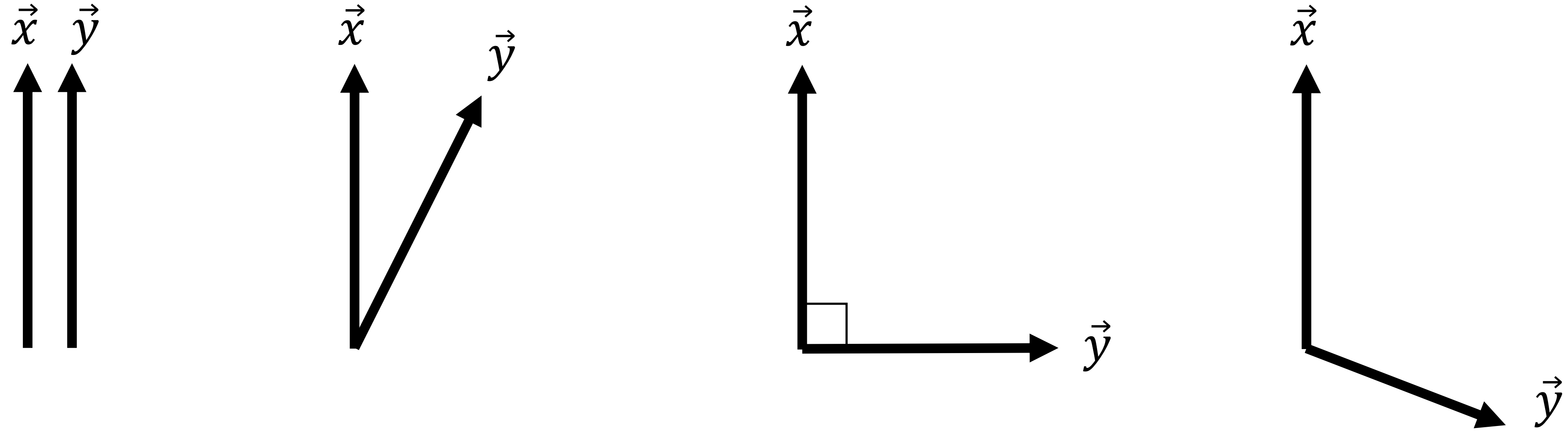
$\langle \vec{x}, \vec{y} \rangle = -1$

The inner product is a *similarity measure* between vectors!

5. THE INNER PRODUCT –VECTOR ORTHOGONALITY

Let's consider some possibilities when $\vec{x}$ and $\vec{y}$ are unit vectors ($\|\vec{x}\| = \|\vec{y}\| = 1$)… $\langle \vec{x}, \vec{y} \rangle = \cos(\theta)$

PARALLEL **ORTHOGONAL (PERPENDICULAR)** **ANTI-PARALLEL**



PARALLEL		ORTHOGONAL		ANTI-PARALLEL
$\theta = 0°$	$0° < \theta < 90°$	$\theta = 90°$	$90° < \theta < 180°$	$\theta = 180°$
$\langle \vec{x}, \vec{y} \rangle = 1$	$0 < \langle \vec{x}, \vec{y} \rangle < 1$	$\boxed{\langle \vec{x}, \vec{y} \rangle = 0}$	$-1 < \langle \vec{x}, \vec{y} \rangle < 0$	$\langle \vec{x}, \vec{y} \rangle = -1$

The inner product is a *similarity measure* between vectors!

QUANTUM PRACTICE TIME!

What is the inner product of and angle between the following vectors?

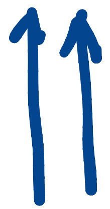
$$\vec{v} = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

QUANTUM PRACTICE SOLUTION

What is the inner product of and angle between the following vectors?

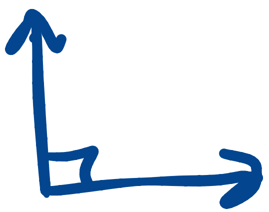
$$\vec{v} = \begin{pmatrix} \underset{x}{1} & \underset{y}{0} \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} \underset{x}{0} & \underset{y}{1} \end{pmatrix}$$

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}\,\vec{w}^T = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \cancel{1(0)}^{\,0} + \cancel{0(1)}^{\,0} = 0$$



$\theta = 0$

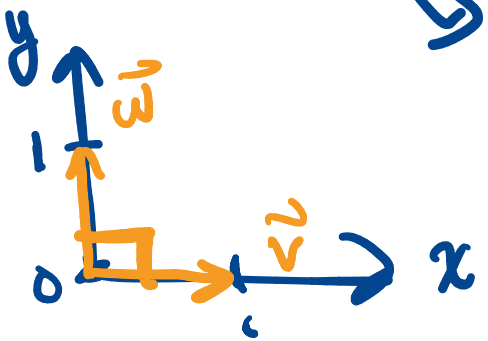
$\langle\ \rangle = 1$



$\rightarrow \theta = \pi/2$

$\langle\ \rangle = 0$

$\theta = \frac{\pi}{2}$



$$\langle \vec{v}, \vec{w} \rangle = \cos(\theta)$$

$$\theta = \cos^{-1}(\langle \vec{v}, \vec{w} \rangle)$$

$$= \cos^{-1}(0) = \pi/2$$

CONJUGATE TRANSPOSE

Question: What if our vectors contain complex numbers?

$$\boxed{\vec{v}^{\dagger} = (\vec{v}^{T})^{*} = (\vec{v}^{*})^{T}}$$

$$\vec{w} = \begin{pmatrix} 3e^{-i\theta} \\ 4+2i \end{pmatrix}$$

$$\bar{z} = z^{*}$$

$$\bar{\vec{v}} = \vec{v}^{*}$$

$$\vec{w}^{\dagger} = \begin{pmatrix} 3e^{+i\theta} & 4-2i \end{pmatrix}$$

Note: The complex conjugate is sometimes also called the Hermitian conjugate, denoted $\vec{v}^{H}$.

QUANTUM PRACTICE TIME!

Find the conjugate transpose of the following vectors.

(1)

$$\vec{a} = \begin{pmatrix} 5+3i \\ 2-i \\ 3e^{i\theta} \\ 4e^{-i\mu} \end{pmatrix}$$

$$\vec{a}^{\dagger} =$$

(2)

$$\vec{b} = \begin{pmatrix} 1 & 2i & -3i \end{pmatrix}$$

$$\vec{b}^{\dagger} =$$

QUANTUM PRACTICE SOLUTIONS

Find the complex conjugate of the following vectors.

(1)

$$\vec{a} = \begin{pmatrix} 5+3i \\ 2-i \\ 3e^{i\theta} \\ 4e^{-i\mu} \end{pmatrix}$$

$$\vec{a}^{\dagger} = (5-3i \quad 2+i \quad 3e^{-i\theta} \quad 4e^{+i\mu})$$

(2)

$$\vec{b} = (1 \quad 2i \quad -3i)$$

$$\vec{b}^{\dagger} = \begin{pmatrix} 1 \\ -2i \\ +3i \end{pmatrix}$$

INNER PRODUCT MAGNITUDE FOR COMPLEX NUMBERS?

What happens if we try to use the standard inner product to calculate the magnitude of a complex vector?

$$\vec{w} = \begin{pmatrix} 5e^{i\theta/3} \\ 2+i \end{pmatrix}$$

$$\|\vec{w}\| = \sqrt{\langle \vec{w}, \vec{w} \rangle} = \sqrt{(5e^{i\theta/3})^2 + (2+i)^2} = \sqrt{25e^{i2\theta/3} + \underline{4} + 2i + 2i + \underbrace{i^2}_{\underline{-1}}}$$

STANDARD

$$= \underbrace{\sqrt{25e^{i2\theta/3} + 3 + 4i}}$$

THE COMPLEX INNER PRODUCT

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^{\dagger}\vec{w} = v_1^* w_1 + \cdots + v_n^* w_n = \sum_{i=1}^{n} v_i^* w_i$$

where
$\vec{v}, \vec{w} \in \mathbb{C}^n$
are column vectors

$$\vec{w} = \begin{pmatrix} 5e^{i\theta/3} \\ 2+i \end{pmatrix}$$

$$\langle \vec{w}, \vec{w} \rangle = \vec{w}^{\dagger}\vec{w} = \begin{pmatrix} 5e^{-i\theta/3} & 2-i \end{pmatrix} \begin{pmatrix} 5e^{i\theta/3} \\ 2+i \end{pmatrix}$$

$$= \left(5e^{-i\theta/3}\right)\left(5e^{+i\theta/3}\right) + (2-i)(2+i)$$

$$= 5 * 5 + 4 \cancel{-2i} \cancel{+2i} - \underbrace{i^2}_{-1}$$

$$= 25 + 4 + 1 = 25 + 5 = 30$$

$$\|\vec{w}\| = \sqrt{\langle \vec{w}, \vec{w} \rangle}$$

$$= \sqrt{30} \longleftarrow \text{REAL!}$$

LINEAR COMBINATIONS

Physicists after enough math classes..

A ***linear combination*** of a set of terms is simply the addition of those terms multiplied by scalar coefficients.

For example, a linear combination of fruits…

$= 6 *$  $+10 *$  $+6 *$  $+ \cdots$

$$\underbrace{6s + 10g + 6b}$$

LINEAR COMBINATIONS

In the case of vectors, a ***linear combination*** is simply a weighted sum of vectors.

$$\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \sum_{i=1}^{n} a_i\vec{v}_i$$

$$\hat{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

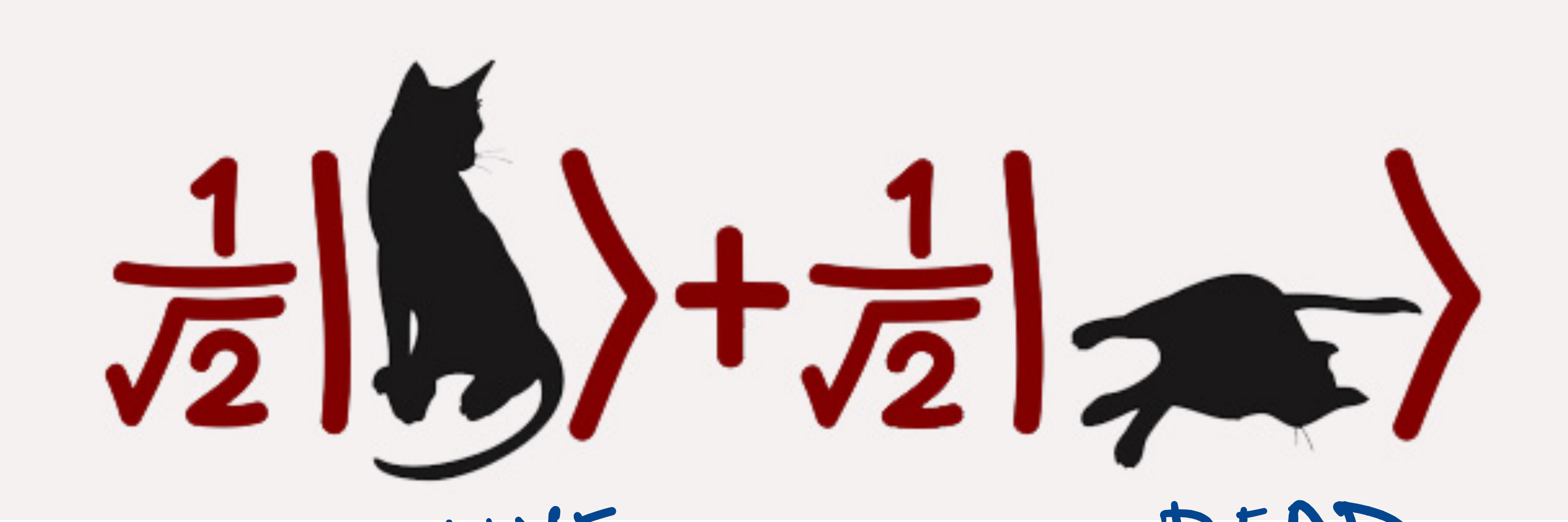
$$\hat{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 5 * \hat{x} + 4 * \hat{y}$$

$$= 5\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

LINEAR COMBINATIONS

In the case of quantum states, a ***superposition*** is simply a linear combination of quantum states!!



ALIVE DEAD

MORE VECTORS OVERVIEW

VECTOR SHAPE

$$(\ \#\ \text{rows} \times \#\ \text{cols}\ )$$

CONJUGATE TRANSPOSE

~~**COMPLEX CONJUGATION**~~

$$\vec{v}^{\dagger} = (\vec{v}^{T})^{*} = (\vec{v}^{*})^{T}$$

INNER PRODUCT

$$\langle \vec{v}, \vec{w} \rangle = \underbrace{\vec{v}^{\dagger}\vec{w}} = v_1^* w_1 + \cdots + v_n^* w_n = \underbrace{\sum_{i=1}^{n} v_i^* w_i} = \underbrace{\|\vec{v}\|\|\vec{w}\| \cos(\theta)}$$

↑↑ 1

0

↓↑ -1

NORMALIZATION UNIT VECTORS

$$\frac{\vec{v}}{\sqrt{\langle \vec{v}, \vec{v} \rangle}} = \frac{\vec{v}}{\|\vec{v}\|}$$

LINEAR COMBINATION ← ADDITION + MULTIPLICATION

$$\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \sum_{i=1}^{n} a_i \vec{v}_i$$

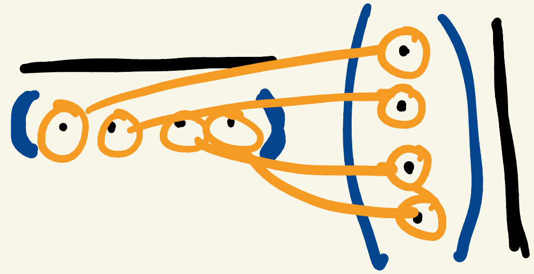
INTRO TO MATRICES

Step into...

① do vectors have to have the same length for the inner product? YES YES YES!

$\vec{v}^T\vec{v}$ ~ column

$\vec{v}\vec{v}^T$ ~ row



$(1 \times n)\ (n \times 1)$

$\downarrow$

(1×1) SCALAR

② $\langle \vec{v}, \vec{w} \rangle \overset{?}{=} \langle \vec{w}, \vec{v} \rangle$

$\vec{v}, \vec{w}$ are REAL → YES!

$\langle \vec{a}, \vec{b} \rangle = \langle \vec{b}, \vec{a} \rangle$

$\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} b_i a_i$

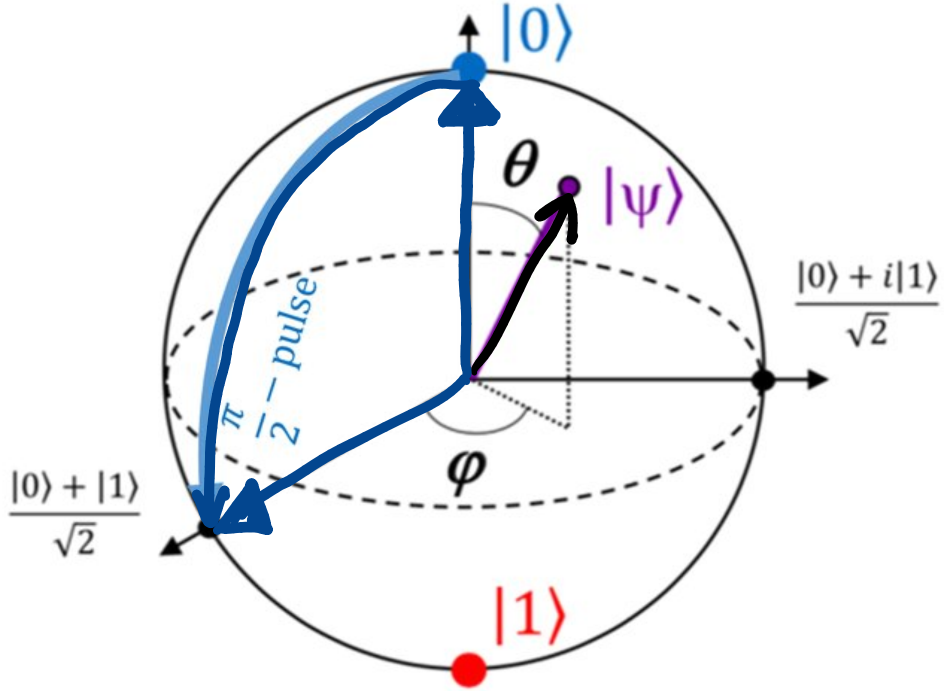
$\vec{v}, \vec{w}$ are COMPLEX → NO!

$\vec{v} + \vec{w} = \sum v_i^* w_i$

$\vec{w} + \vec{v} = \sum w_i^* v_i$

$\neq$

WHAT DO MATRICES MEAN FOR QUANTUM COMPUTING?



$$|\psi\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\varphi}sin\frac{\theta}{2}|1\rangle$$

As we saw, ***quantum states*** are represented as vectors.

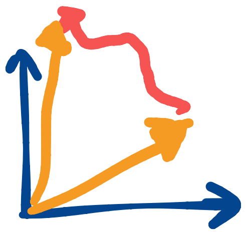
However, in quantum computing, our goal is to manipulate these states, to run quantum algorithms.

To do so, we use **quantum gates**, which are represented as matrices!

MATRIX

You can think of a ***matrix*** as a collection of *row vectors* or a collection of *column vectors*.

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix} \qquad \boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix}$$



Geometrically, matrices are transformations that allow us to both rotate and scale vectors.

MATRIX NOTATION AND SHAPE

An $(\mathrm{n} \times \mathrm{m})$ matrix is written as,

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

3×2

MATRIX SHAPE:

$$\underbrace{(\,\#\ \text{rows} \times \#\ \text{cols}\,)}_{(m \times n)}$$

QUANTUM PRACTICE TIME!

(1) State whether the following are vectors or matrices and (2) state their shapes.

(1)

$$\begin{pmatrix} 1 & -3 & 1 \\ 3 & -4 & 1 \\ 0 & 2 & -1 \end{pmatrix}$$

(2)

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(3)

$$\begin{pmatrix} 2 & 3 \end{pmatrix}$$

(4)

$$\begin{pmatrix} 1 & 1 \\ 6 & 4 \\ 7 & 8 \\ 9 & 9 \end{pmatrix}$$

QUANTUM PRACTICE SOLUTION

(1) State whether the following are vectors or matrices and (2) state their shapes.

(1) $\begin{pmatrix} 1 & -3 & 1 \\ 3 & -4 & 1 \\ 0 & 2 & -1 \end{pmatrix}$ 3x3 Matrix

(2) $\underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{1}$ 3x1 Vector

SHAPE:
(# ROWS, # COLS)

(3) $\begin{pmatrix} 2 & 3 \end{pmatrix}$ 1x2 Vector

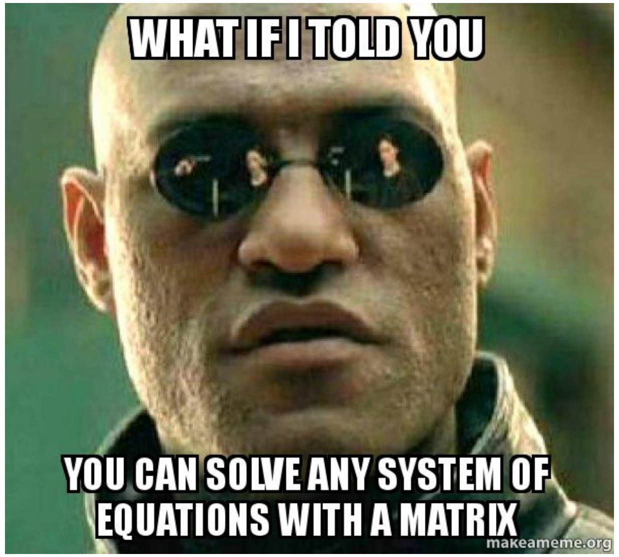
(4) $\begin{pmatrix} 1 & 1 \\ 6 & 4 \\ 7 & 8 \\ 9 & 9 \end{pmatrix}$ 4x2 Matrix

SOLVING LINEAR SYSTEMS OF EQUATIONS

$$\begin{cases} x - 3y + z = 2 \\ 3x - 4y + z = 0 \\ 2y - z = 1 \end{cases}$$

2* 4

Systems of equations like these can be represented and solved more formulaically with vectors and matrices!



$$\begin{pmatrix} 1 & -3 & 1 \\ 3 & -4 & 1 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ -8 \\ -17 \end{pmatrix}$$

But first, we need to learn some matrix operations….

SOLVING LINEAR SYSTEMS OF EQUATIONS

Let's write the following system of equations in matrix and vector form...

$$\begin{cases} 7x + 8y = 1 \\ -4y + 10z = -3 \\ x + 2y + z = 4 \end{cases}$$

3x3

$$-\begin{pmatrix} 7 & 8 & 0 \\ 0 & -4 & 10 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$$

MATRIX ADDITION

Let's work through an example…

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 1+1 & 2+1 \\ 3+1 & 4+2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$$

MATRIX ADDITION

General Equation:

$$\boldsymbol{A}+\boldsymbol{B}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2m}+b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nm}+b_{nm} \end{pmatrix}$$

** Note: You can only add matrices of the same shape!

QUANTUM PRACTICE TIME!

Perform the following matrix additions or state if they are not possible.

(1)

$$\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 2 & 1 \end{pmatrix}$$

(2)

$$\begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 5 & 6 \\ 2 & 1 \end{pmatrix}$$

QUANTUM PRACTICE SOLUTION

Perform the following matrix additions or state if they are not possible.

(1)

$$\underbrace{\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}}_{2\times 2} + \underbrace{\begin{pmatrix} 5 & 6 \\ 2 & 1 \end{pmatrix}}_{2\times 2} = \begin{pmatrix} 1+5 & 3+6 \\ 4+2 & 2+1 \end{pmatrix} = \underbrace{\begin{pmatrix} 6 & 9 \\ 6 & 3 \end{pmatrix}}_{2\times 2}$$

(2)

$$\underbrace{\begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \end{pmatrix}}_{2\times 3} + \underbrace{\begin{pmatrix} 3 & 1 \\ 5 & 6 \\ 2 & 1 \end{pmatrix}}_{3\times 2}$$

The matrices are of different shapes.

Not possible!

MATRIX-SCALAR MULTIPLICATION

Let's work through an example…

$$c = 2 \qquad A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$c * A = \begin{pmatrix} 2*1 & 2*2 \\ 2*3 & 2*4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

MATRIX-SCALAR MULTIPLICATION

General Equation:

$$c * \boldsymbol{A} = c * \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} c * a_{11} & c * a_{12} & \cdots & c * a_{1m} \\ c * a_{21} & c * a_{22} & \cdots & c * a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c * a_{n1} & c * a_{n2} & \cdots & c * a_{nm} \end{pmatrix}$$

MATRIX-VECTOR MULTIPLICATION

Let's work through an example…

$$(n \times m)\,(m \times 1) \qquad (1 \times k)\,(k \times n)$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A\vec{v} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1*1 + 2*0 \\ 3*1 + 4*0 \end{pmatrix} = \begin{pmatrix} 1+0 \\ 3+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

MATRIX-VECTOR MULTIPLICATION

General Equation:

$$\boldsymbol{A}\vec{x} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} a_{11} * x_1 + a_{12} * x_2 + \cdots + a_{1m} * x_m \\ a_{21} * x_1 + a_{22} * x_2 + \cdots + a_{2m} * x_m \\ \vdots \\ a_{n1} * x_1 + a_{n2} * x_2 + \cdots + a_{nm} * x_m \end{pmatrix} = \begin{pmatrix} \langle \vec{a}_1, \vec{x} \rangle \\ \langle \vec{a}_2, \vec{x} \rangle \\ \vdots \\ \langle \vec{a}_n, \vec{x} \rangle \end{pmatrix}$$

Note: The vector height must match the matrix width.

$$(n \times \cancel{m}) \times (\cancel{m} \times 1) \longrightarrow (n \times 1)$$

where $\vec{a}_i$ is the i^{th} ***row vector*** of $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{pmatrix}$$

$$\boldsymbol{A}\vec{x} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

MATRIX-MATRIX MULTIPLICATION

Let's work through an example…

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

A $\quad$ B

$2 \times \cancel{2} \quad \cancel{2} \times 2$

$\downarrow$

2×2

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1(1) + 2(0) & 1(2) + 2(1) \\ 3(1) + 4(0) & 3(2) + 4(1) \end{pmatrix}$$

$$= \begin{pmatrix} 1+0 & 2+2 \\ 3+0 & 6+4 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 4 \\ 3 & 10 \end{pmatrix}}$$

MATRIX-MATRIX MULTIPLICATION

General Equation:

$$\boldsymbol{AB} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mk} \end{pmatrix} = \begin{pmatrix} \langle \vec{a}_1, \vec{b}_1 \rangle & \langle \vec{a}_1, \vec{b}_2 \rangle & \cdots & \langle \vec{a}_1, \vec{b}_k \rangle \\ \langle \vec{a}_2, \vec{b}_1 \rangle & \langle \vec{a}_2, \vec{b}_2 \rangle & \cdots & \langle \vec{a}_2, \vec{b}_k \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \vec{a}_n, \vec{b}_1 \rangle & \langle \vec{a}_n, \vec{b}_2 \rangle & \cdots & \langle \vec{a}_n, \vec{b}_k \rangle \end{pmatrix}$$

Remember to always check your shapes! :

$$(n \times \cancel{m}) \times (\cancel{m} \times k) \longrightarrow (n \times k)$$

Note: The first matrix width must match the second matrix height!

where $\vec{a}_i$ is the i^{th} ***row vector*** of $\boldsymbol{A}$ and $\vec{b}_j$ is the j^{th} ***column vector*** of $\boldsymbol{B}$

$$\boldsymbol{AB} = \begin{pmatrix} \boxed{a_{11} \quad a_{12} \quad \cdots \quad a_{1m}} \\ \boxed{a_{21} \quad a_{22} \quad \cdots \quad a_{2m}} \\ \vdots \quad\; \vdots \quad\; \ddots \quad\; \vdots \\ \boxed{a_{n1} \quad a_{n2} \quad \cdots \quad a_{nm}} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mk} \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_k \end{pmatrix}$$

QUANTUM PRACTICE TIME!

Is it possible to multiply the following matrices/vectors? If so, what is the dimension of the resultant matrix/vector? (Don't actually multiply them out.)

(1) $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 2 & 1 \end{pmatrix}$

(2) $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(3) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$

(4) $\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$

(5) $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 1 & 5 \\ 3 & 8 \end{pmatrix}$

(6) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \end{pmatrix}$

QUANTUM PRACTICE SOLUTIONS

Is it possible to multiply the following matrices/vectors? If so, what is the dimension of the resultant matrix/vector? (Don't actually multiply them out.)

(1) $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 2 & 1 \end{pmatrix}$ Yes, 2x2

$2\times\cancel{2} \quad \cancel{2}\times 2 \rightarrow 2\times 2$

(2) $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ Yes, 2x1

$2\times\cancel{2} \quad \cancel{2}\times 1 \rightarrow 2\times 1$

(3) $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ Not Possible

$2\times\cancel{1} \quad \cancel{2}\times 2$

(4) $\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ Yes, 1x2

$1\times\cancel{2} \quad \cancel{2}\times 2 \rightarrow 1\times 2$

(5) $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 4 & 5 \\ 1 & 5 \\ 3 & 8 \end{pmatrix}$ Yes, 2x2

$2\times\cancel{3} \quad \cancel{3}\times 2 \rightarrow 2\times 2$

(6) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \end{pmatrix}$ Not Possible

$3\times 3 \quad 2\times 3$

MATRIX-MATRIX MULTIPLICATION

Let's work through some more examples...

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$2 \times \cancel{3}$ $\qquad$ $\cancel{3} \times 2 \rightarrow 2 \times 2$

$$AB = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1(1) + 0(0) + 2(1) & 1(0) + 0(1) + 2(0) \\ 3(1) + 1(0) + 1(1) & 3(0) + 1(1) + 1(0) \end{pmatrix}$$

$$= \begin{pmatrix} 1+0+2 & 0+0+0 \\ 3+0+1 & 0+1+0 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 4 & 1 \end{pmatrix}$$

MATRIX TRANSPOSE

Let's see an example…

$$\underline{\underline{\boldsymbol{X}}} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{pmatrix} \quad 2 \times 3$$

$$\boldsymbol{X}^T = \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \\ x_{13} & x_{23} \end{pmatrix} \quad 3 \times 2$$

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \vec{v}^T = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

MATRIX TRANSPOSE

If $\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix}$, then

$$\boldsymbol{X}^T = \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ x_{12} & x_{22} & \cdots & x_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1m} & x_{2m} & \cdots & x_{nm} \end{pmatrix}$$

Remember to always check your shapes! : $(n \times m)^T \rightarrow (m \times n)$ **Flip the matrix about its diagonal!**

MATRIX CONJUGATE TRANSPOSE

If $\underline{\underline{X}} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix}$, then

$$X^{\dagger} = \begin{pmatrix} x_{11}^* & x_{21}^* & \cdots & x_{n1}^* \\ x_{12}^* & x_{22}^* & \cdots & x_{n2}^* \\ \vdots & \vdots & \ddots & \vdots \\ x_{1m}^* & x_{2m}^* & \cdots & x_{nm}^* \end{pmatrix}$$

The same as a matrix transpose, but conjugate any complex numbers in the matrix!

$\vec{v}^T \quad \vec{v}^\dagger$

$(X^T)^*$

$X = \begin{pmatrix} 3e^{i\theta} & 0 \\ 4+i & 0 \end{pmatrix} \quad X^\dagger = \begin{pmatrix} 3e^{-i\theta} & 4-i \\ 0 & 0 \end{pmatrix}$

QUANTUM PRACTICE TIME!

Given $\boldsymbol{X} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}$ solve for $\boldsymbol{X}^T\boldsymbol{X}$ and $\boldsymbol{X}\boldsymbol{X}^T$

QUANTUM PRACTICE SOLUTIONS

Given $\quad \boldsymbol{X} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix} \quad$ solve for $\quad \boldsymbol{X}^T\boldsymbol{X} \quad$ and $\quad \boldsymbol{X}\boldsymbol{X}^T$

$$\boldsymbol{X}^T\boldsymbol{X} = \underset{2\times\cancel{3}}{\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}} \underset{\cancel{3}\times 2}{\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}} \rightarrow 2\times 2$$

$$= \begin{pmatrix} 1(1)+2(2)+0(0) & 1(0)+2(1)+(0)(1) \\ 0(1)+2(1)+1(0) & 0(0)+1(1)+1(1) \end{pmatrix} = \begin{pmatrix} 1+4 & 2 \\ 2 & 1+1 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$$

$$\boldsymbol{X}\boldsymbol{X}^T = \underset{3\times\cancel{2}}{\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}} \underset{\cancel{2}\times 3}{\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}} \rightarrow \underline{3\times 3}$$

$$= \underbrace{\begin{pmatrix} 1(1)+0(0) & 1(2)+0(1) & 1(0)+0(1) \\ 2(1)+1(0) & 2(2)+1(1) & 2(0)+1(1) \\ 0(1)+1(0) & 0(2)+1(1) & 0(0)+1(1) \end{pmatrix}}_{3\times 3\ \checkmark}$$

THE IDENTITY MATRIX

The identity matrix is defined as:

[1s along the diagonal and 0s on off-diagonals.]

$$\mathbb{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$(n \times n)$

$a * 1 = a$

$A\,\mathbb{I} = \mathbb{I}A = A$

$\vec{v}^T\mathbb{I} = \vec{v}^T \qquad \mathbb{I}\vec{v} = \vec{v}$

What's so special about the identity matrix?

$$X\,\mathbb{I} = \mathbb{I}\,X = X$$

$$\cancel{\vec{x}\,\mathbb{I} = \mathbb{I}\,\vec{x} = \vec{x}}$$

$\vec{x}\,\mathbb{I} = \vec{x} \qquad \mathbb{I}\vec{y} = \vec{y}$

Multiplication by the identity matrix is analogous to scalar multiplication by 1!

MATRIX INVERSION

Now, what if for a given matrix $\boldsymbol{X}$ there existed some matrix, $\boldsymbol{X}^{-1}$, such that

$$\boxed{\boldsymbol{X}\boldsymbol{X}^{-1} = \boldsymbol{X}^{-1}\boldsymbol{X} = \mathbb{I}} \quad ?$$

$\boldsymbol{X}^{-1}$ is thus the ***inverse*** of matrix $\boldsymbol{X}$

**** Note: Many matrices do not have inverses!**

There are many ways to solve for a matrix inverse, which you will learn about in lab…
For now you can use the fact that for a 2x2 matrix:

If $\boldsymbol{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\boldsymbol{X}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

MATRIX INVERSION

Let's solve for $\vec{x}$.

(1) $A^{-1}(A\vec{x}) = A^{-1}(\vec{b}+\vec{c}) \rightarrow \cancel{(A^{-1}A)}^{I}\vec{x} = A^{-1}(\vec{b}+\vec{c}) \rightarrow \vec{x} = A^{-1}(\vec{b}+\vec{c})$

(2) $A^{-1}(AB\vec{x}) = A^{-1}(\vec{c}) \rightarrow \cancel{(A^{-1}A)}^{I}B\vec{x} = A^{-1}\vec{c} \rightarrow B^{-1}(B\vec{x}) = B^{-1}(A^{-1}\vec{c}) \rightarrow \cancel{(B^{-1}B)}^{I}\vec{x} = B^{-1}A^{-1}\vec{c}$

$\rightarrow \vec{x} = B^{-1}A^{-1}\vec{c}$

(3) $\vec{x}(\cancel{D}\,\cancel{D^{-1}}) = \vec{f}D^{-1} \rightarrow \vec{x} = \vec{f}D^{-1}$

(4) $\cancel{x}\,E\vec{x}D + G\vec{y} = \vec{z} \rightarrow E\vec{x}D = \vec{z} - G\vec{y} \rightarrow \boxed{\vec{x} = E^{-1}(\vec{z}-G\vec{y})D^{-1}}$

sorry that is was a typo

SOLVING LINEAR SYSTEMS OF EQUATIONS

Now that we know about matrix inversion, solving linear systems of equations is easy!

$$\begin{cases} x - 3y + z = 2 \\ 3x - 4y + z = 0 \\ 2y - z = 1 \end{cases}$$

$$\underbrace{\begin{pmatrix} 1 & -3 & 1 \\ 3 & -4 & 1 \\ 0 & 2 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} \overset{?}{=} \underbrace{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}}_{\vec{b}}$$

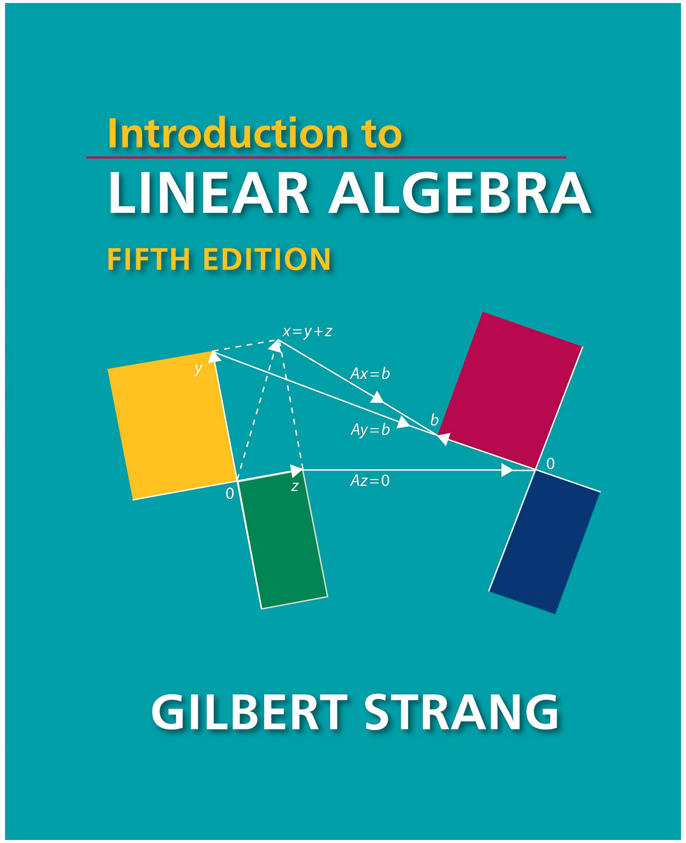
$$\begin{pmatrix} x - 3y + z \\ 3x - 4y + z \\ 2y - z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

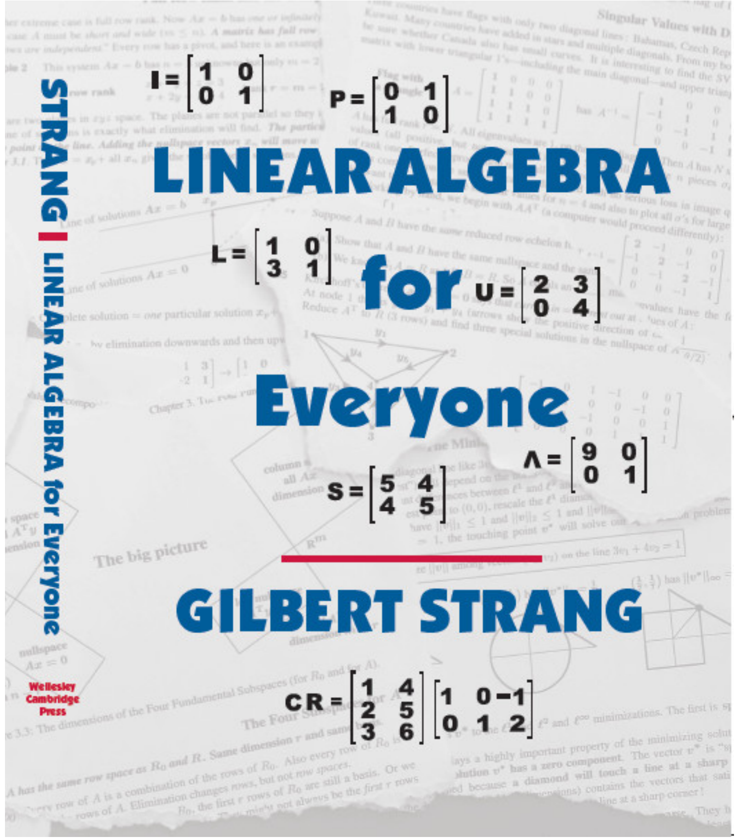
To solve for the values in vector $\vec{x}$:

$$(\underbrace{\cancel{A^{-1}A}}_{I})\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b} \rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \overset{3\times 3}{A^{-1}} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}^{3\times 1} \rightarrow 3\times 1$$

FURTHER LINEAR ALGEBRA RESOURCES





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