INTRO TO QUANTUM COMPUTING

LECTURE #4

INTRO TO VECTORS & COMPLEX NUMBERS

FRANCISCA VASCONCELOS

11/1/2020

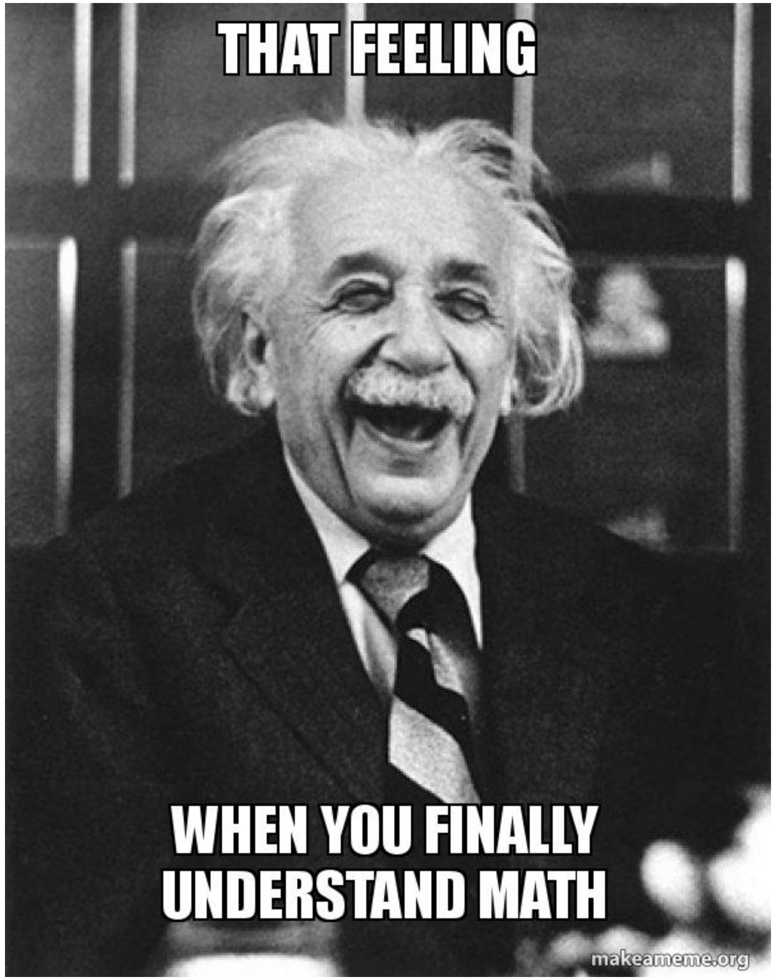
WHY ALL THE MATH?

WHY ALL THE MATH?

1. The math is a ***necessary*** foundation and toolset for the quantum to come!

2. Linear algebra, probability, and complex numbers are extremely useful for ***every field of STEM***! You will be better prepared for college coursework, even if you don't end up doing quantum computing.

3. The math is hard and you probably won't fully understand everything this first time. ***That is OK!*** Practice makes perfect and you will need to be exposed to these concepts several times in order to master them.



Fran's Hot Take:
Most important of all, you can understand the memes!!

WHY ALL THE MATH?

Quantum mechanics is a beautiful generalization of the laws of probability: a generalization based on the 2-norm rather than the 1-norm, and on complex numbers rather than nonnegative real numbers. It can be studied completely separately from its applications to physics (and indeed, doing so provides a good starting point for learning the physical applications later). This generalized probability theory leads naturally to a new model of computation – the quantum computing model – that challenges ideas about computation once considered a priori, and that theoretical computer scientists might have been driven to invent for their own purposes, even if there were no relation to physics. In short, while quantum mechanics was invented a century ago to solve technical problems in physics, today it can be fruitfully explained from an extremely different perspective: as part of the history of ideas, in math, logic, computation, and philosophy, about the limits of the knowable.

- Professor Scott Aaronson (UT Austin)

(excerpt from *Quantum Computing Since Democritus*)

Image Source: ETH Zurich

TODAY'S LECTURE

1. Intro to Vectors
   a) Scalars vs. Vectors
   b) Vector Representation
   c) Vector Properties
   d) Vector Operations
      a) Vector Addition
      b) Vector-Scalar Multiplication
   e) Vector Decomposition
   f) Vector Generalization

2. Intro to Complex Numbers
   a) Why Complex Numbers?
   b) Complex Numbers Definition
   c) The Complex Plane
      a) Vector Representation
      b) Polar Form
   d) Euler's Identity
   e) Complex Number Operations

INTRO TO VECTORS

SCALARS vs VECTORS

Scalar

- **A quantity having only magnitude (no direction).**
- What we typically think of when we think of numbers.
- Written as: $a \in \mathbb{R}$
- Some examples in physics:
  - Distance (m)
  - Speed (m/s)
  - Time (s)
  - Mass (kg)

Vector

- **A quantity with both magnitude and direction.**
- Can be described by a list of scalars (Cartesian form) or a radius and an angle (Polar form).
- Written as: $\vec{v} \in \mathbb{R}^n$
- Some examples in physics:
  - Displacement (m)
  - Velocity (m/s)
  - Acceleration (m/s^2)
  - Weight (N)

SCALAR EXAMPLES

Source: PickPic (CC)

30 M&M’s

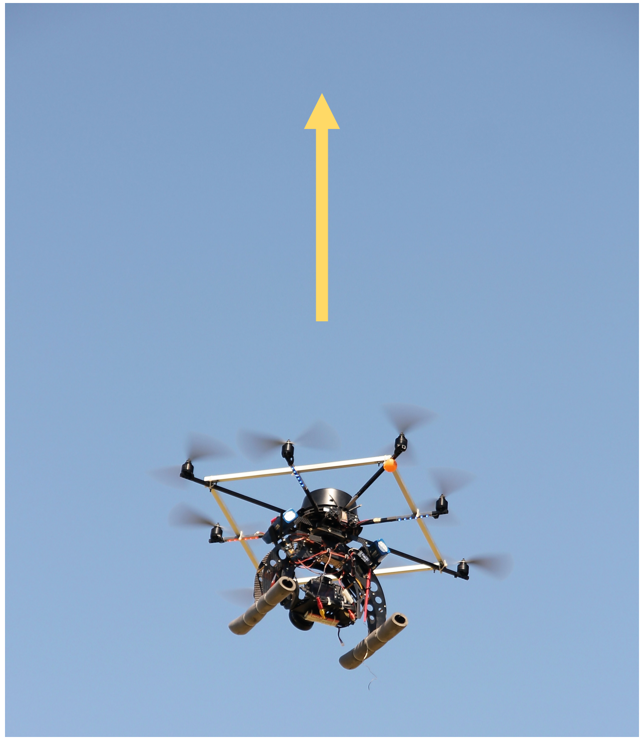
Source: Wikimedia (CC)

1000ºC

Source: freeimageslive (CC)

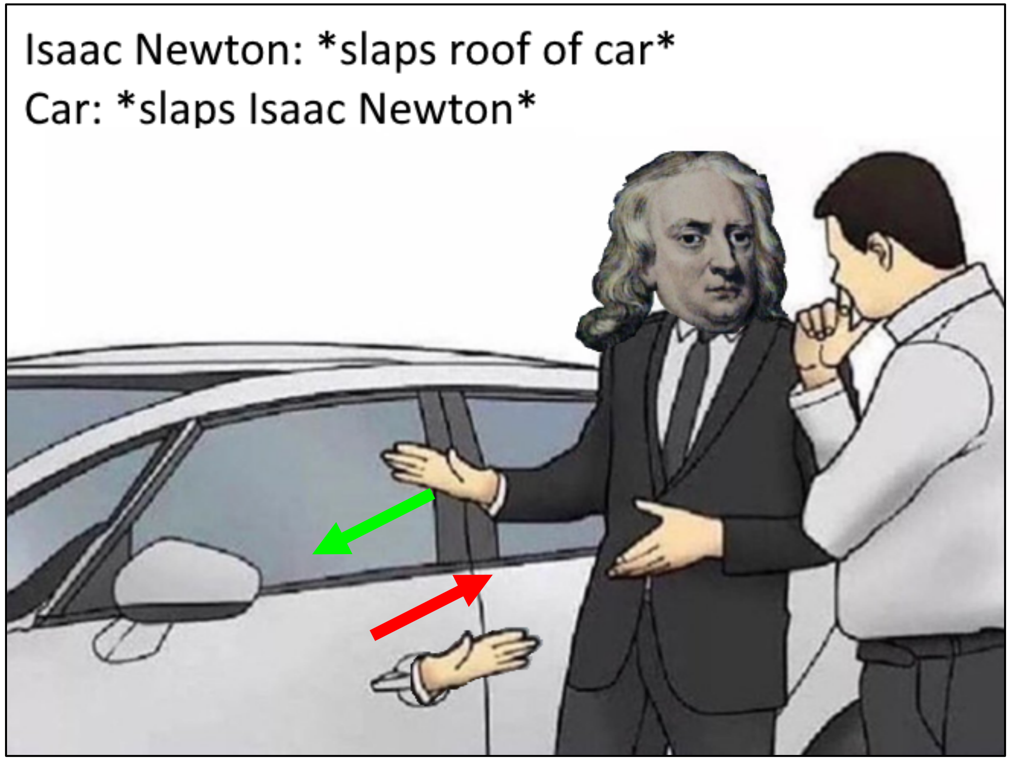
1TB

VECTOR EXAMPLES



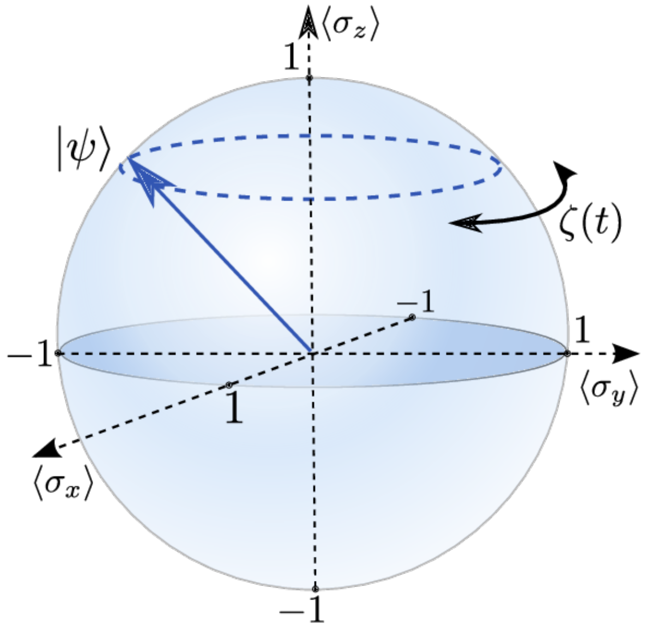
Source: Marko Milivojevic photography (CC)

Flying **up** at **10m/s**



Force of **20N** at **-20º**

Force of **-20N** at **-20º**



Source: Researchgate

Quantum states....

QUANTUM QUIZ TIME!

Determine whether each of the following is a scalar or vector...
(Some are tricky! Read the wording carefully...)

(1)

Source: pixabyay,com (CC)

The fish is swimming at 1m/s.

(2)

Source: pixabyay,com (CC)

He kicked the soccer ball at a 40º angle with a force of 12N.

(3)

Source: Wikimedia (CC)

The piano has a mass of 160kg.

(4)

Source: Wikimedia (CC)

The piano has a weight of 9000N.

QUANTUM QUIZ SOLUTION

Determine whether each of the following is a scalar or vector…
(Some of these are tricky! Read the wording carefully…)

(1)

Source: pixabyay,com (CC)

The fish is swimming at 1m/s.

SCALAR

(2)

Source: pixabyay,com (CC)

He kicked the soccer ball at a 40º angle with a force of 12N.

VECTOR

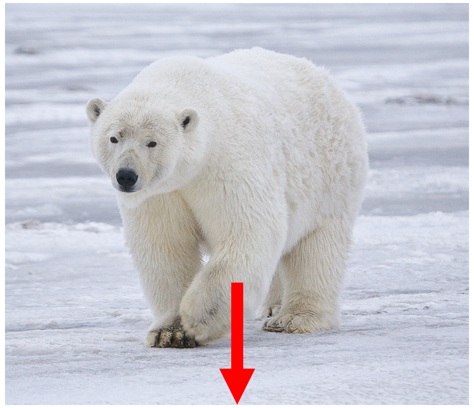
(3)

Source: Wikimedia (CC)

The piano has a mass of 160kg.

SCALAR

(4)

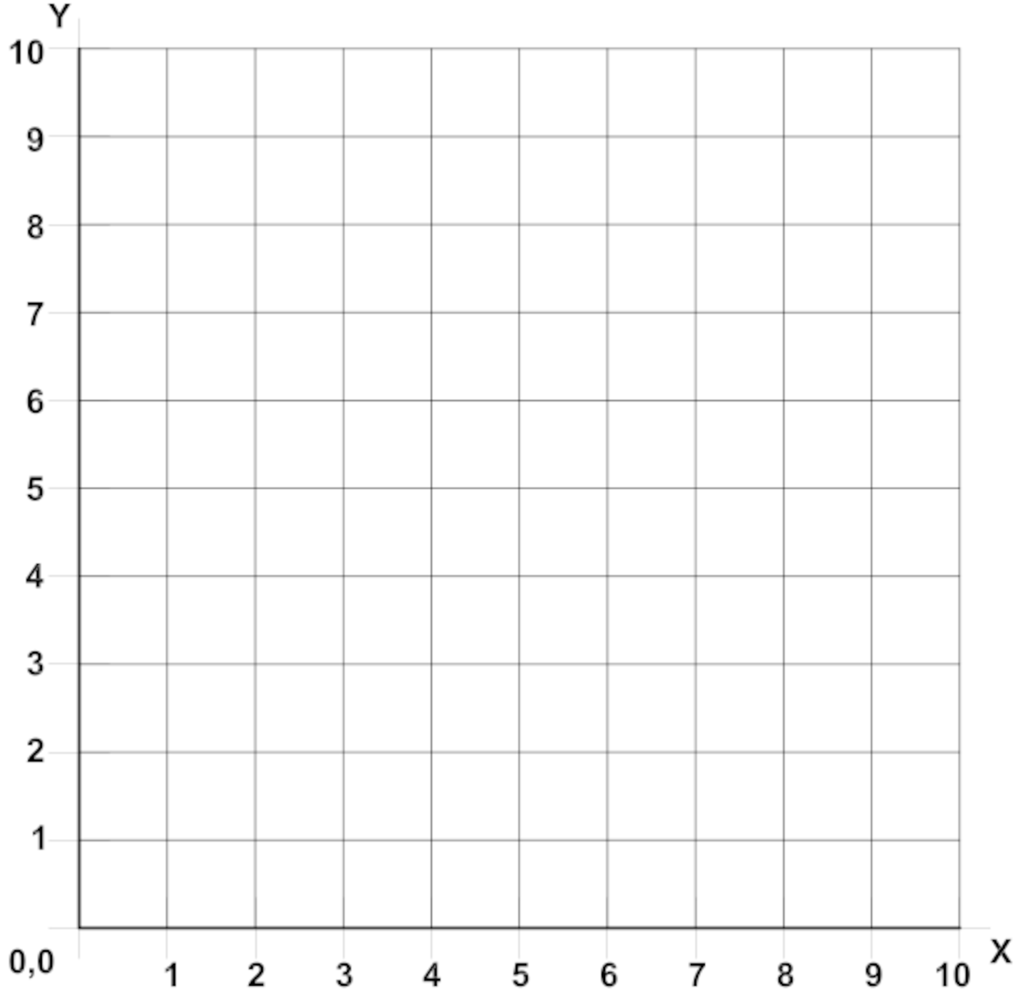


Source: Wikimedia (CC)

The piano has a weight of 9000N.

VECTOR

OK, SO HOW DO WE REPRESENT VECTORS?

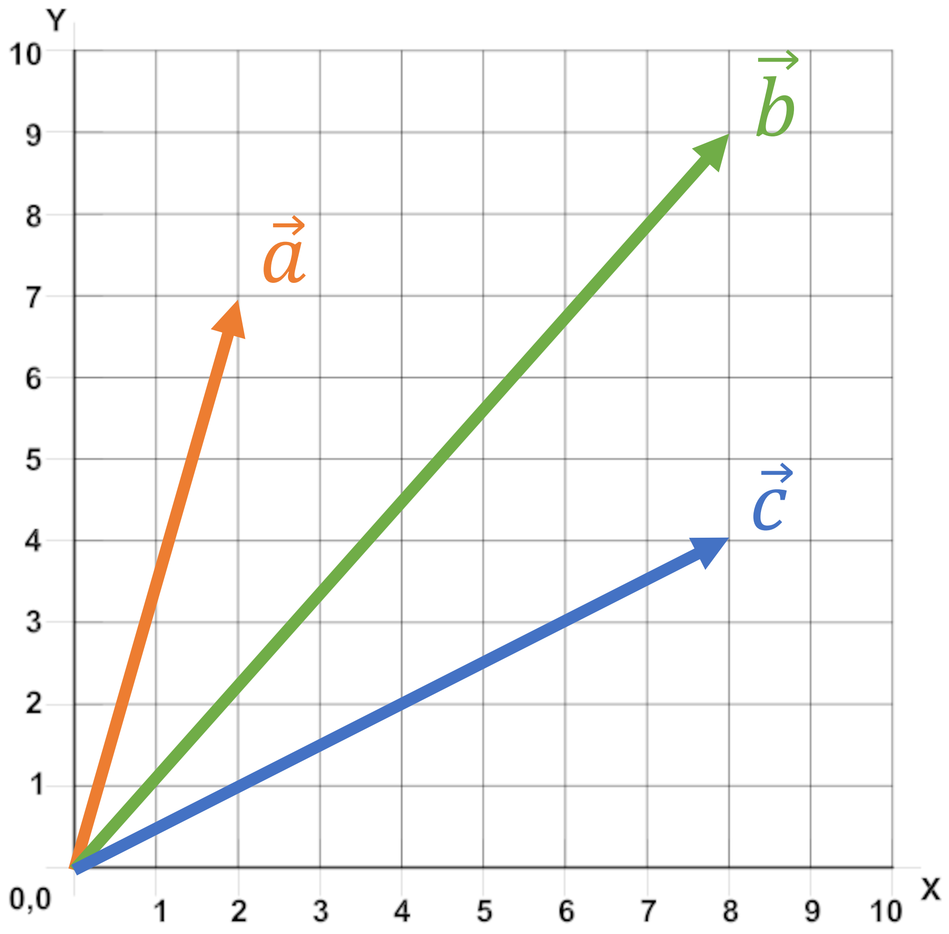


General 2D Vector Notation:

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

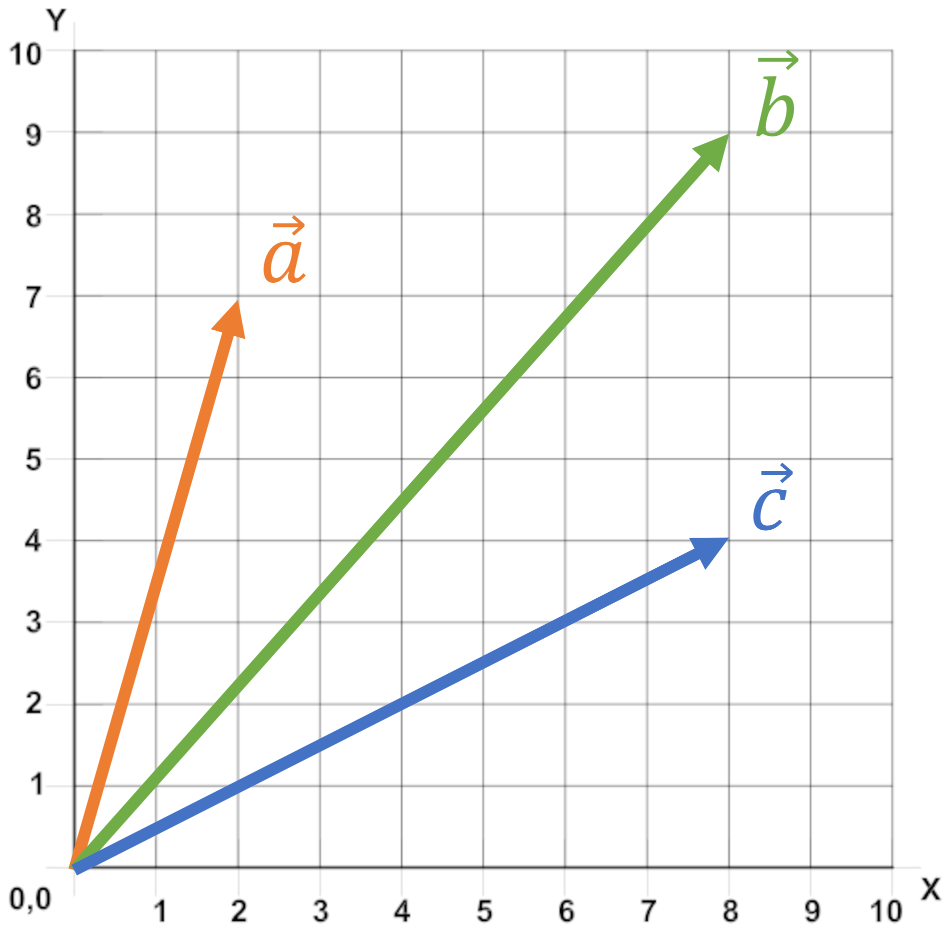
QUANTUM QUIZ TIME!

Write out the vector representation of the 3 following vectors.



QUANTUM QUIZ SOLUTION

Write out the vector representation of the 3 following vectors.



$$\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$$

$$\vec{c} = \begin{pmatrix} c_x \\ c_y \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

QUANTUM QUIZ TIME!

Sketch each of the 3 following vectors on a Cartesian plane.

$$\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

$$\vec{c} = \begin{pmatrix} c_x \\ c_y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

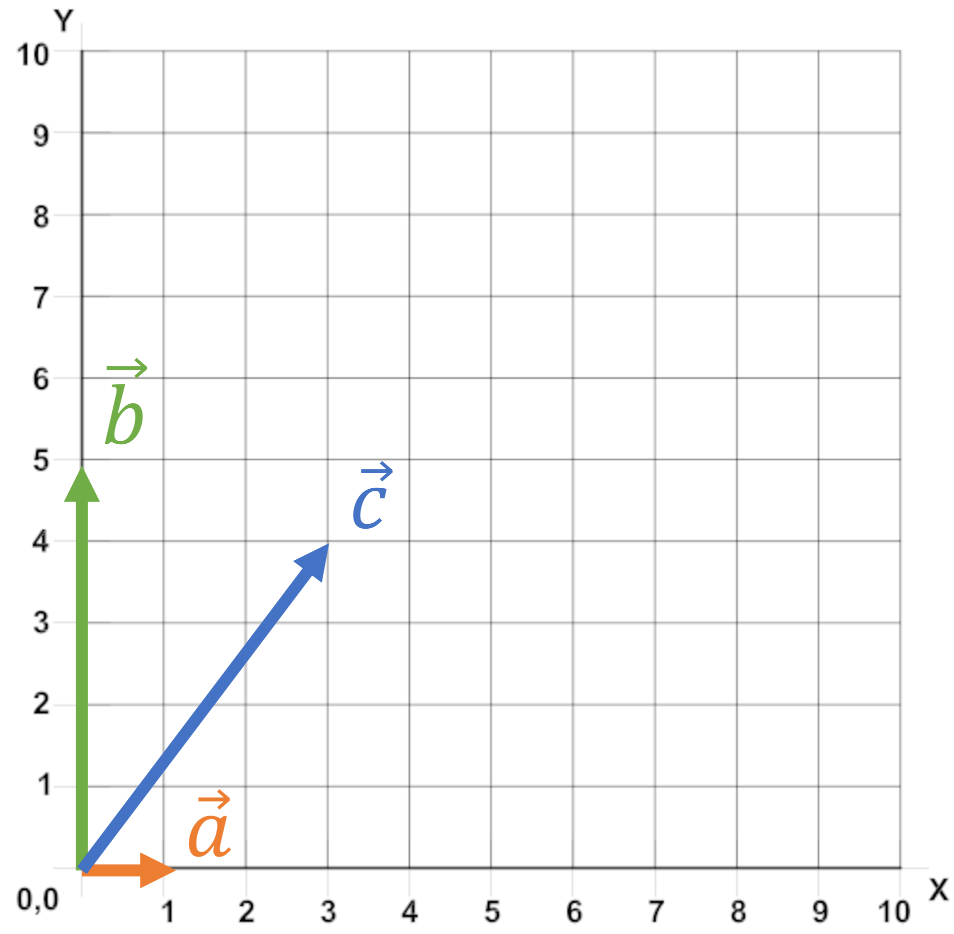
QUANTUM QUIZ SOLUTION

Sketch each of the 3 following vectors on a Cartesian plane.

$$\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

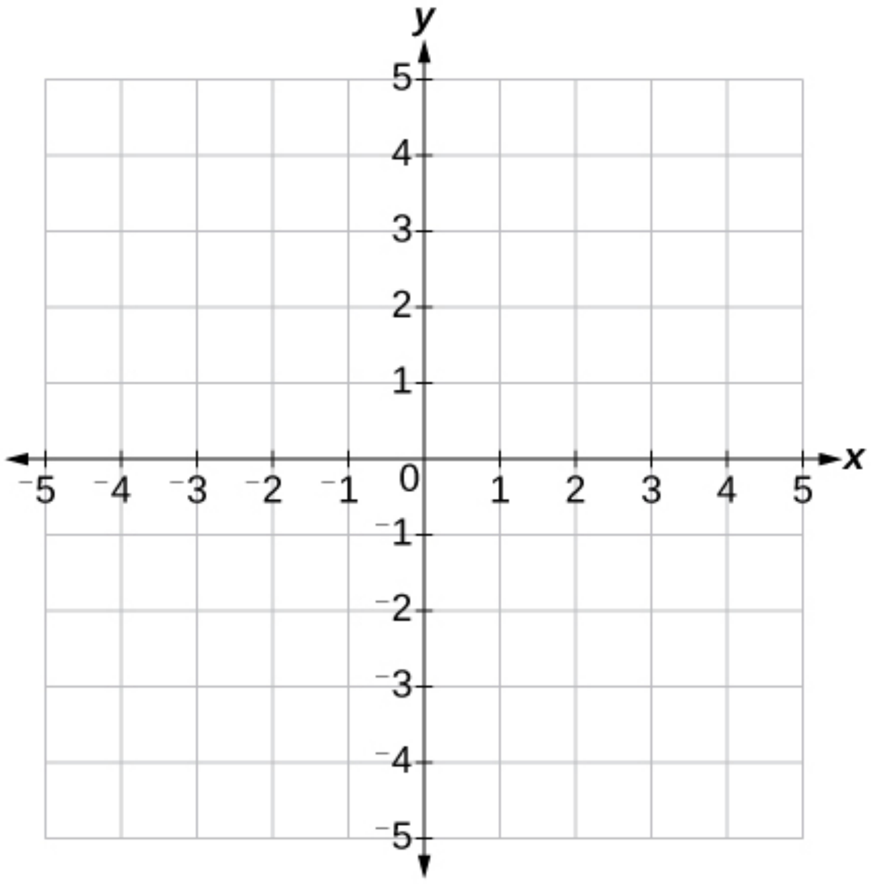
$$\vec{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

$$\vec{c} = \begin{pmatrix} c_x \\ c_y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$



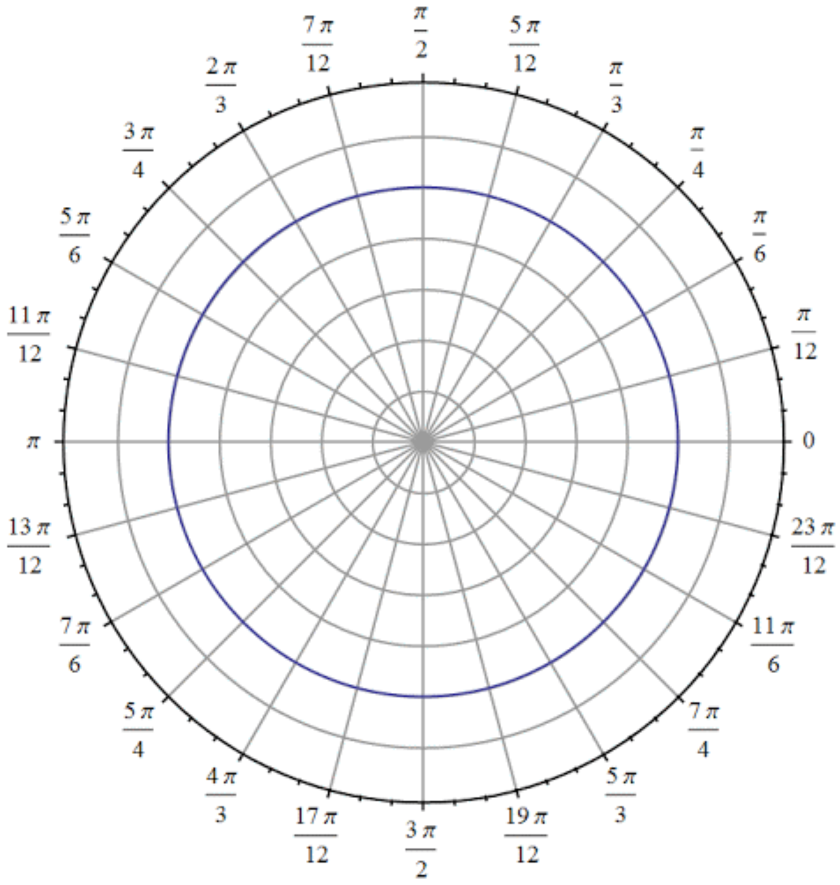
VECTOR PROPERTIES

Cartesian Form:



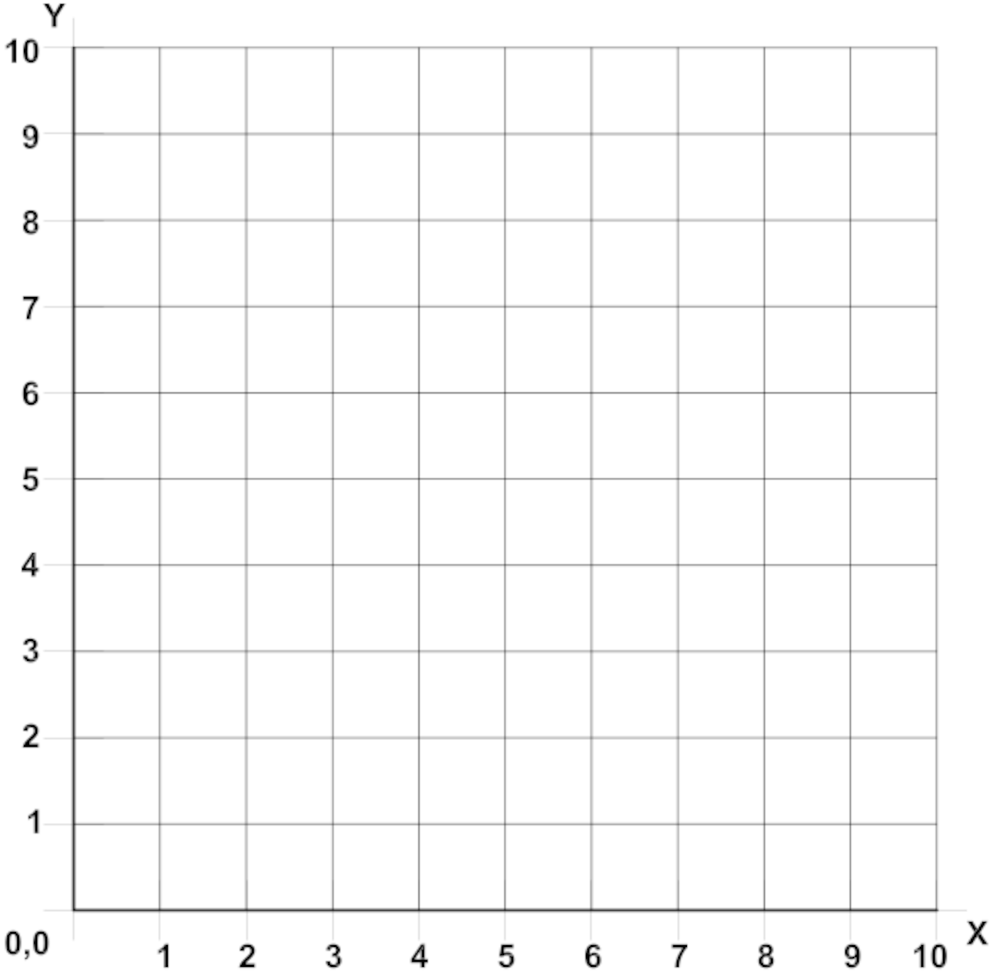
x and y components

Polar Form:



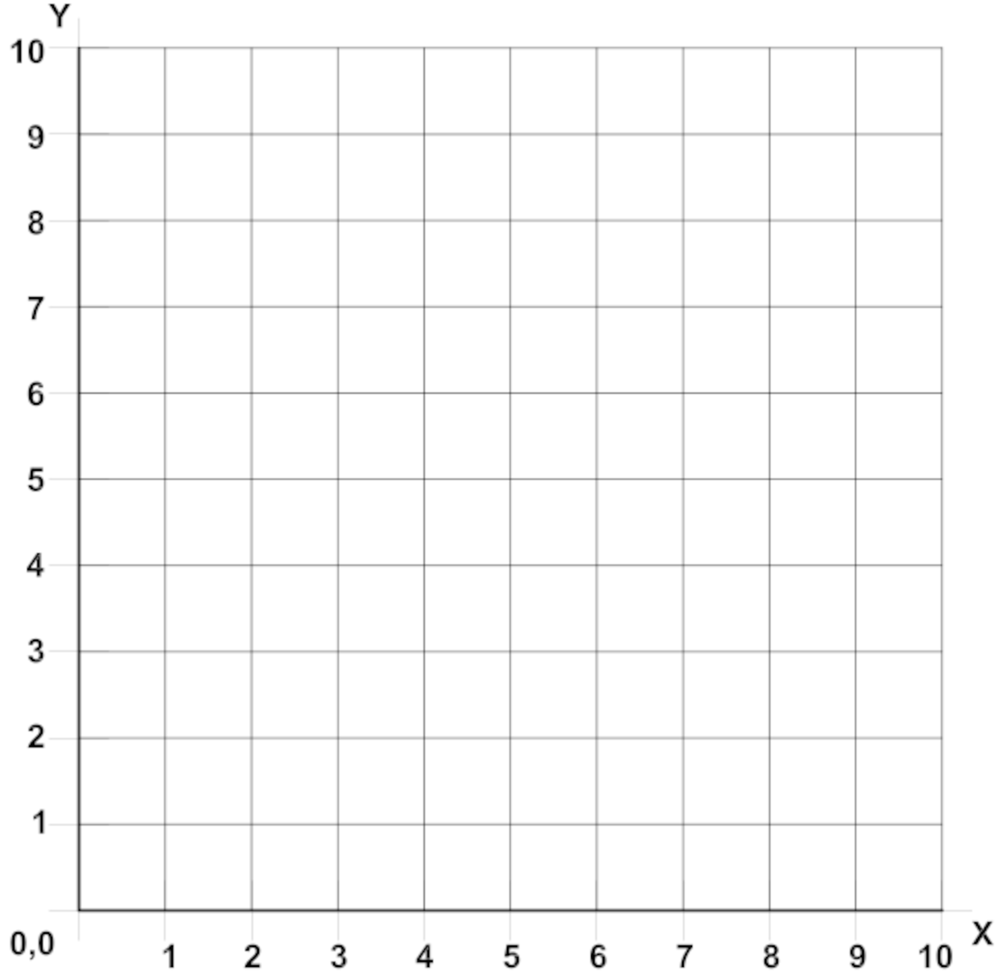
radius r and angle θ
(magnitude and direction)

VECTOR MAGNITUDE



$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2}$$

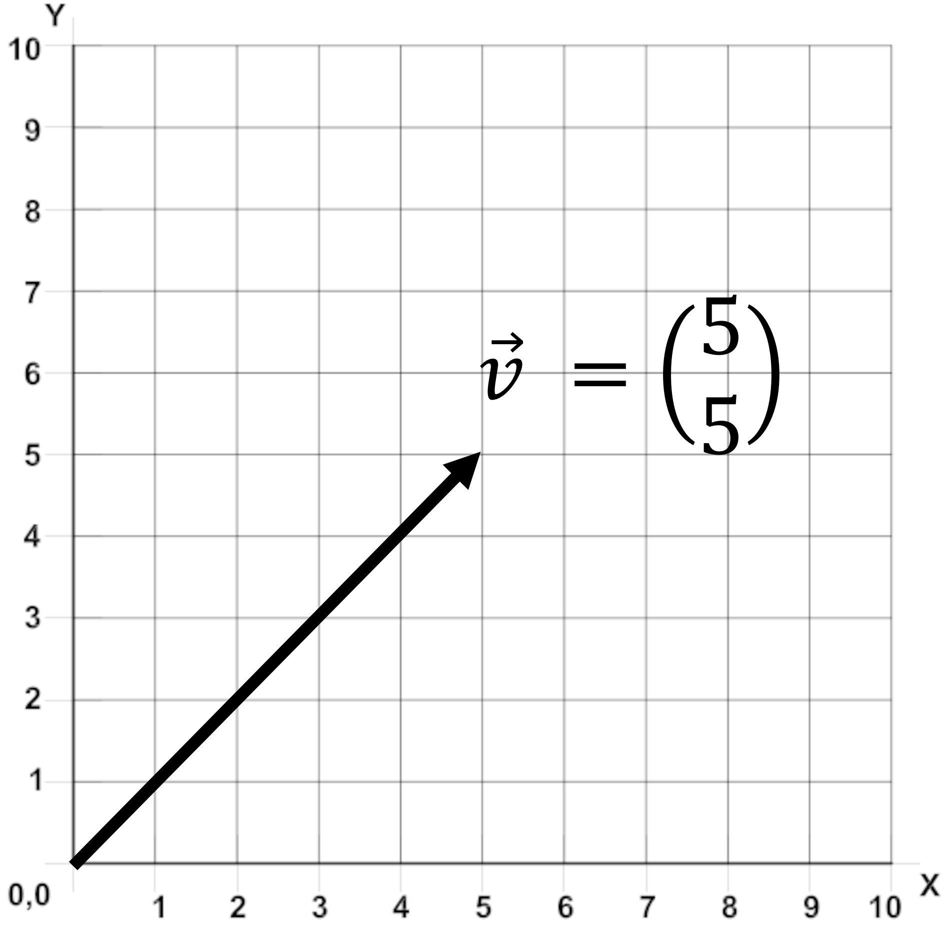
VECTOR DIRECTION



$$\angle \vec{v} = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

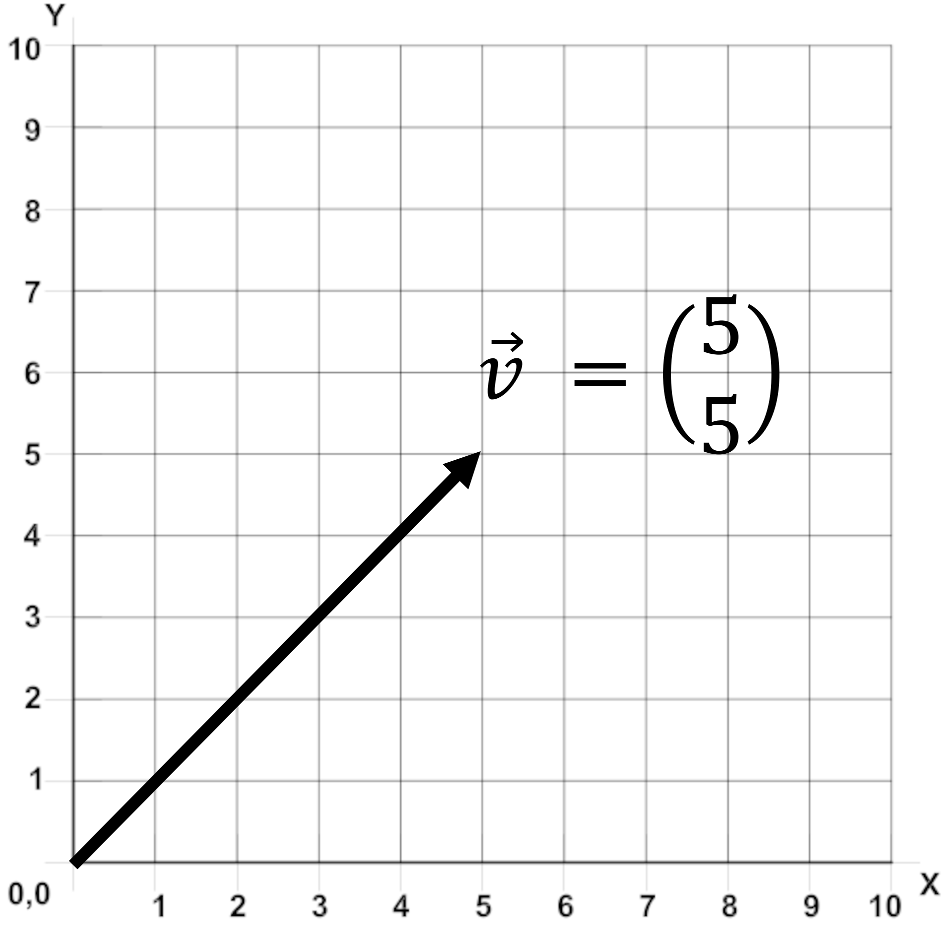
QUANTUM QUIZ TIME!

Calculate the magnitude and direction of the following vector.
(You can write out the expression. No need to calculate it out.)



QUANTUM QUIZ SOLUTION

Calculate the magnitude and direction of the following vector.
(You can write out the expression. No need to calculate it out.)



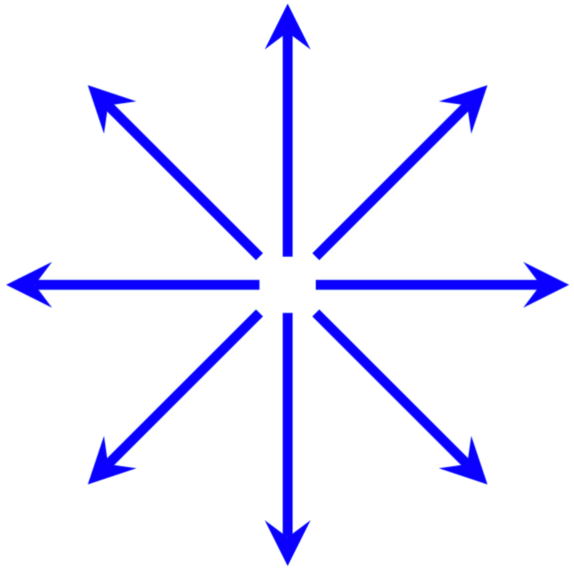
MAGNITUDE

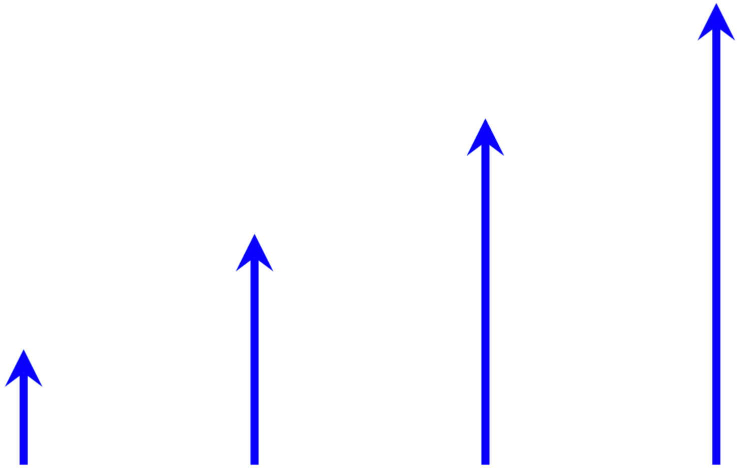
$$\|\vec{v}\| = \sqrt{5^2 + 5^2} = \sqrt{25 + 25} = \sqrt{50} \approx 7.07$$

DIRECTION

$$\angle \vec{v} = \tan^{-1}\left(\frac{5}{5}\right) = \tan^{-1}(1) = \frac{\pi}{4}\,\text{rad} = 45°$$

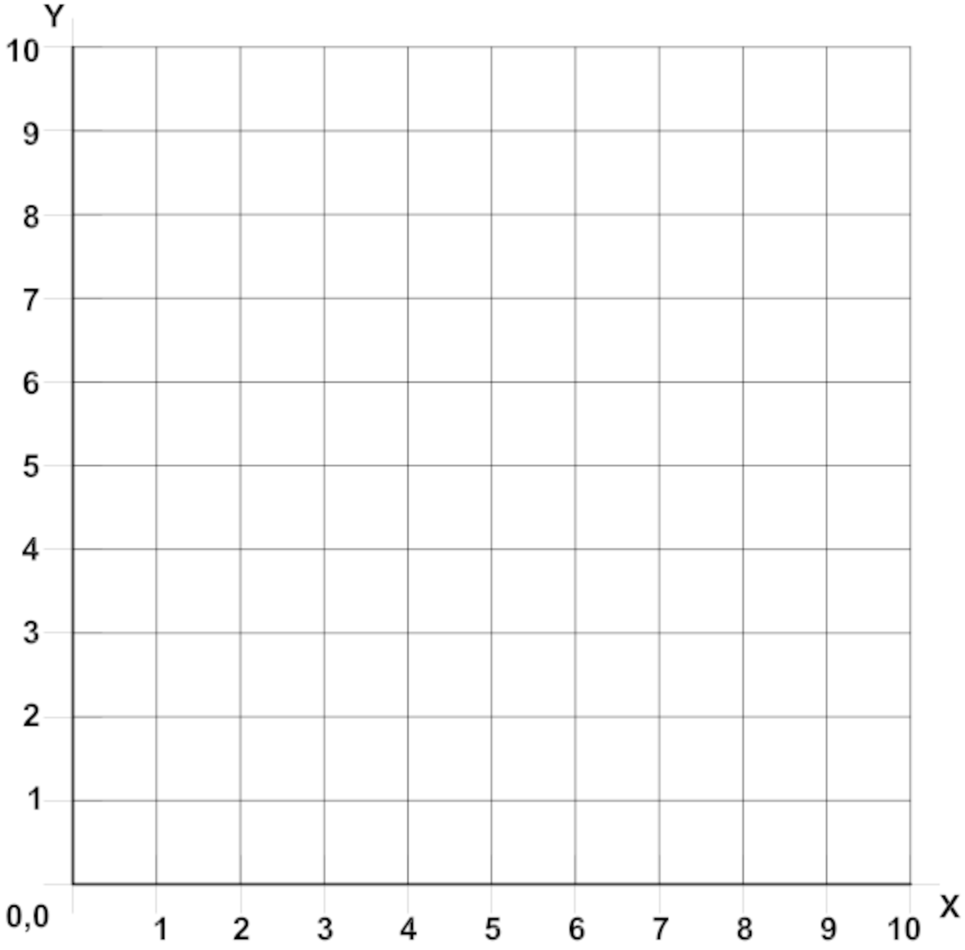
VECTOR PROPERTIES



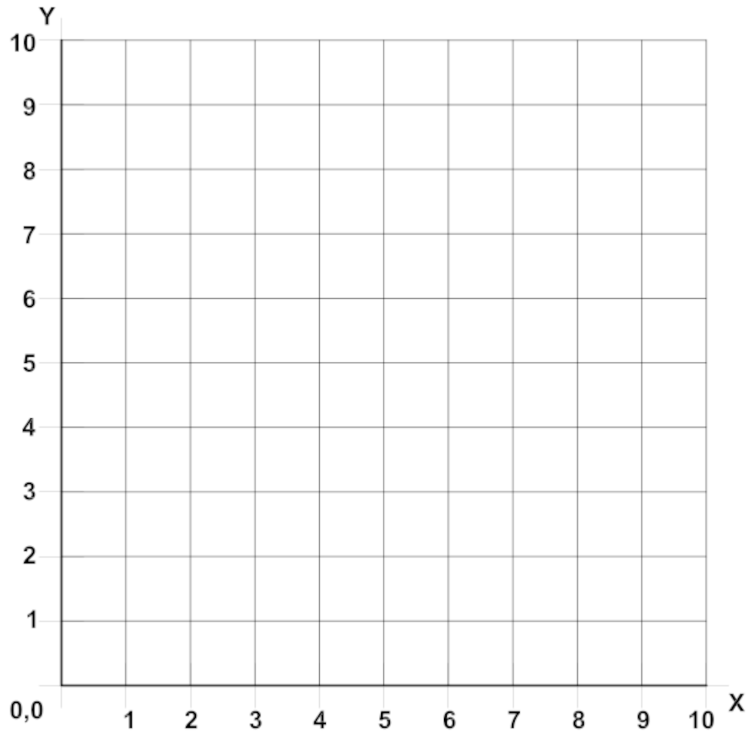


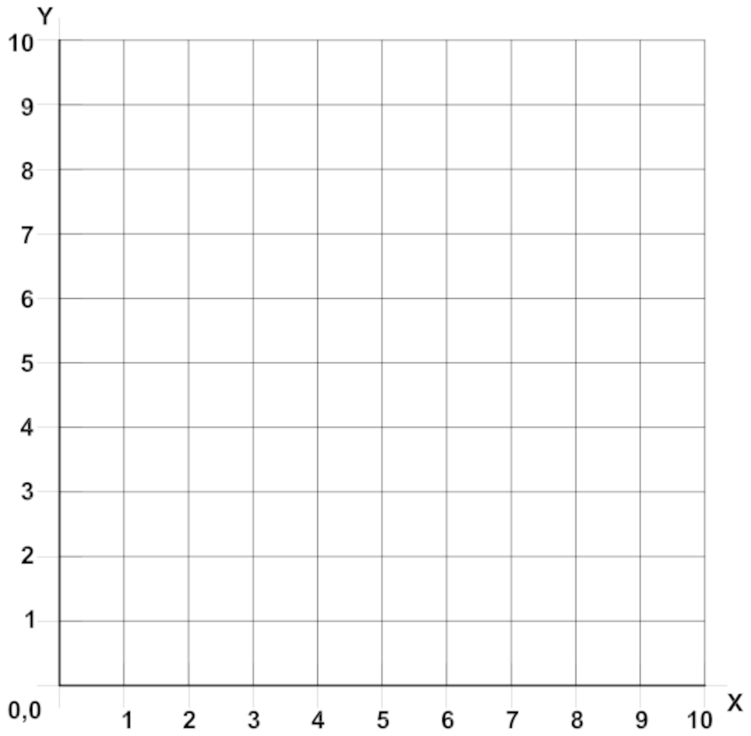
VECTOR ADDITION

$$\vec{a} + \vec{b} = \begin{pmatrix} a_x + b_x \\ a_y + b_y \end{pmatrix}$$



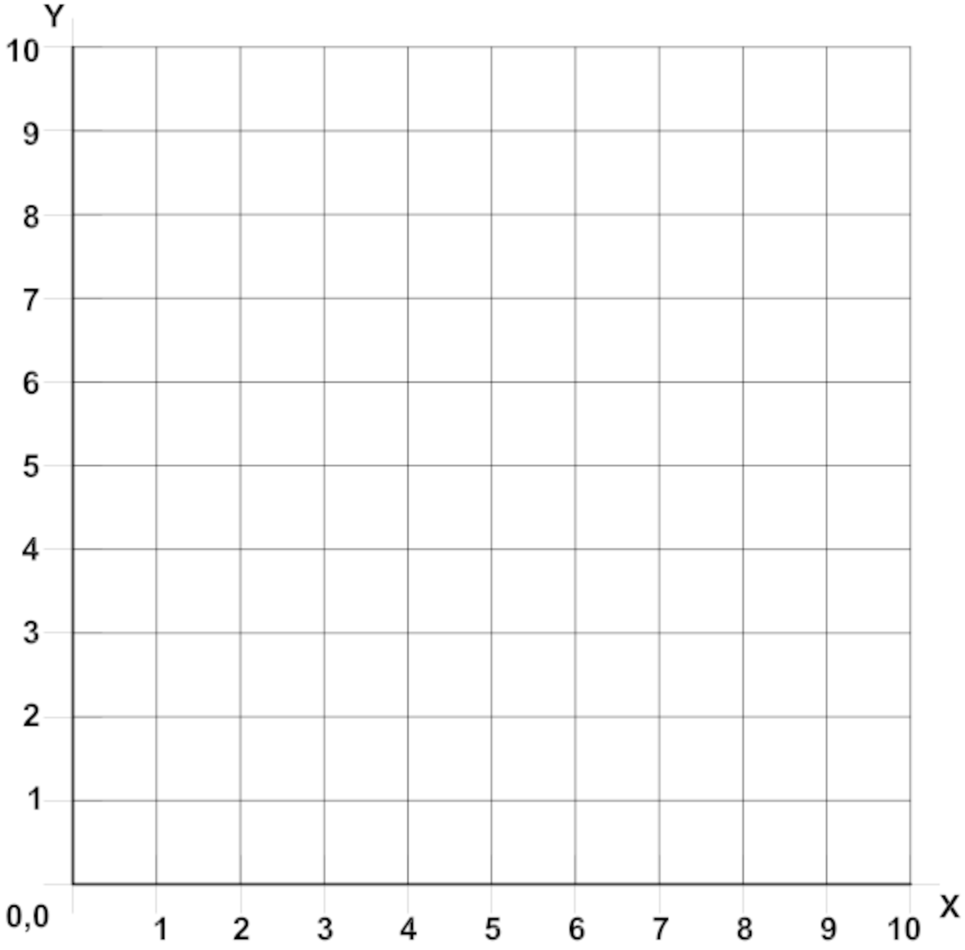
VECTOR ADDITION



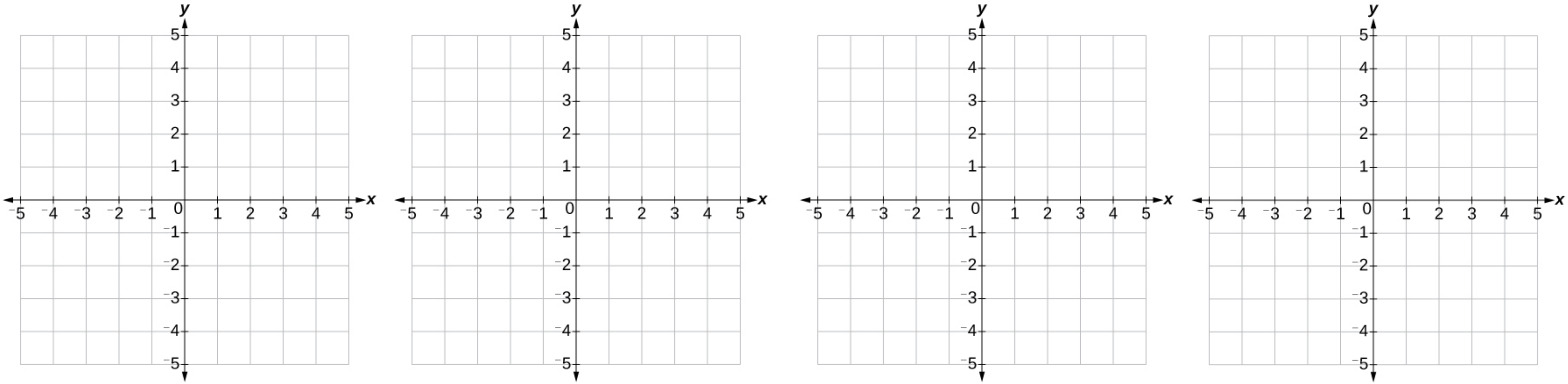


VECTOR-SCALAR MULTIPLICATION

$$c * \overrightarrow{w} = \begin{pmatrix} c * w_x \\ c * w_y \end{pmatrix}$$



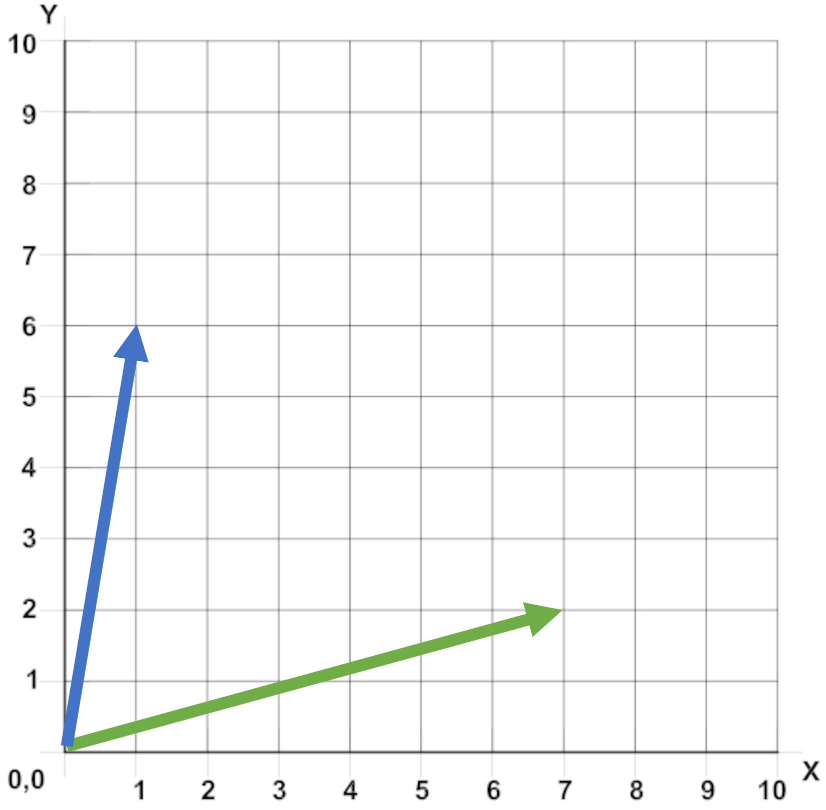
VECTOR-SCALAR MULTIPLICATION



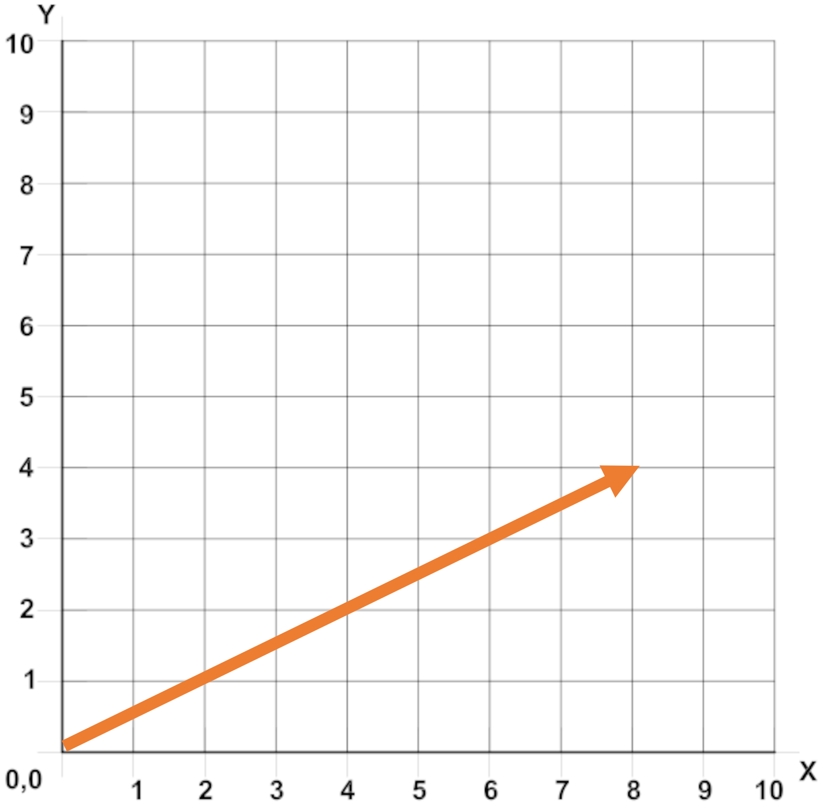
QUANTUM QUIZ TIME!

Calculate the following vector operations algebraically and plot the resulting vector alongside the original ones.

(1) $\vec{n} = \binom{7}{2} + \binom{1}{6}$



(2) $\vec{m} = 0.5 * \binom{8}{4}$

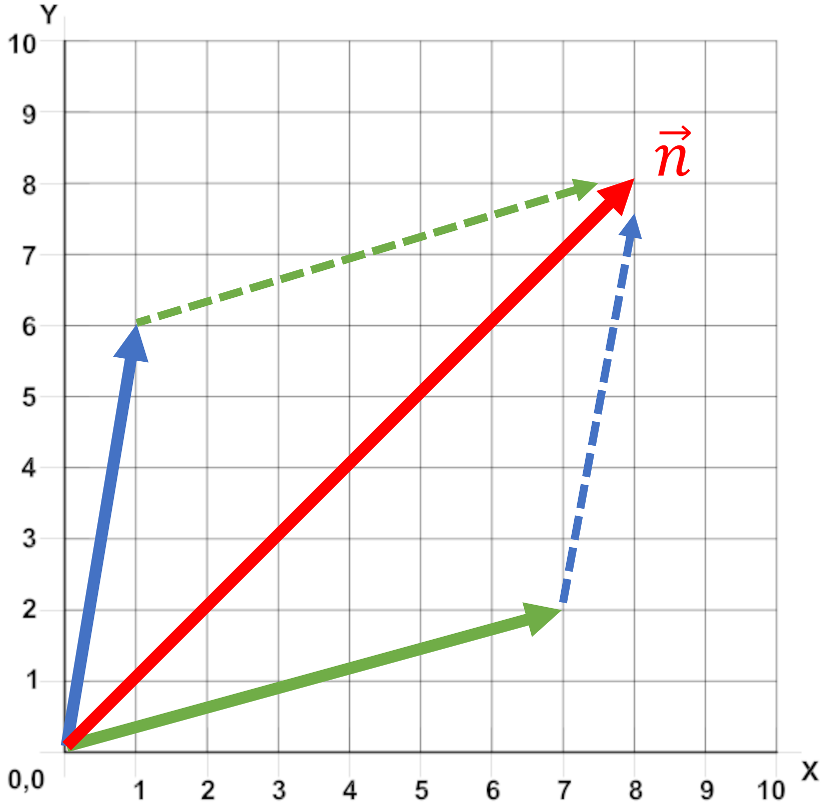


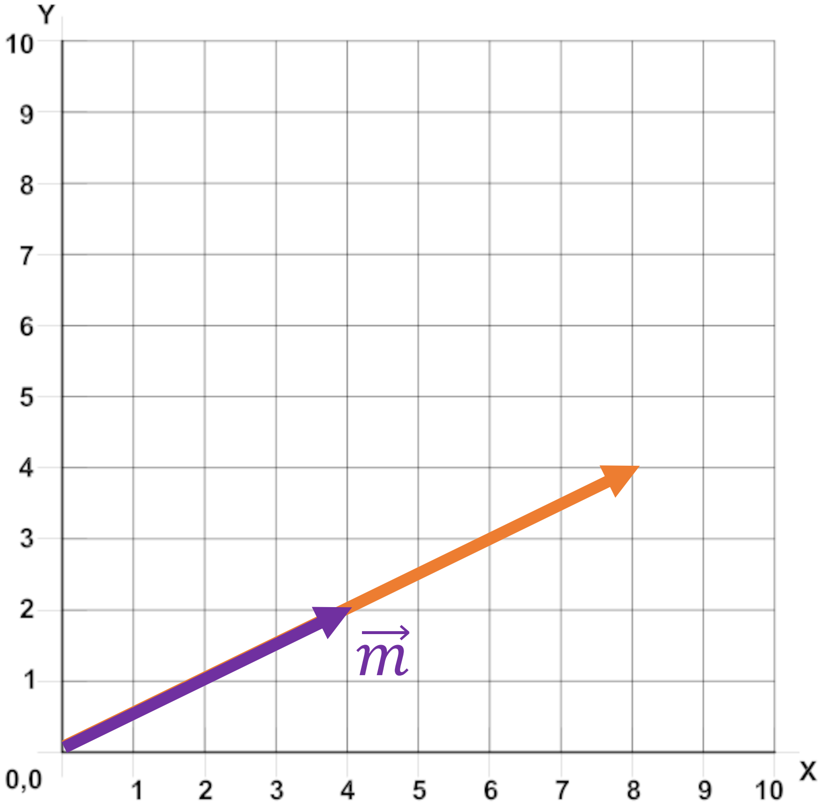
QUANTUM QUIZ SOLUTION

Calculate the following vector operations algebraically and plot the resulting vector alongside the original ones.

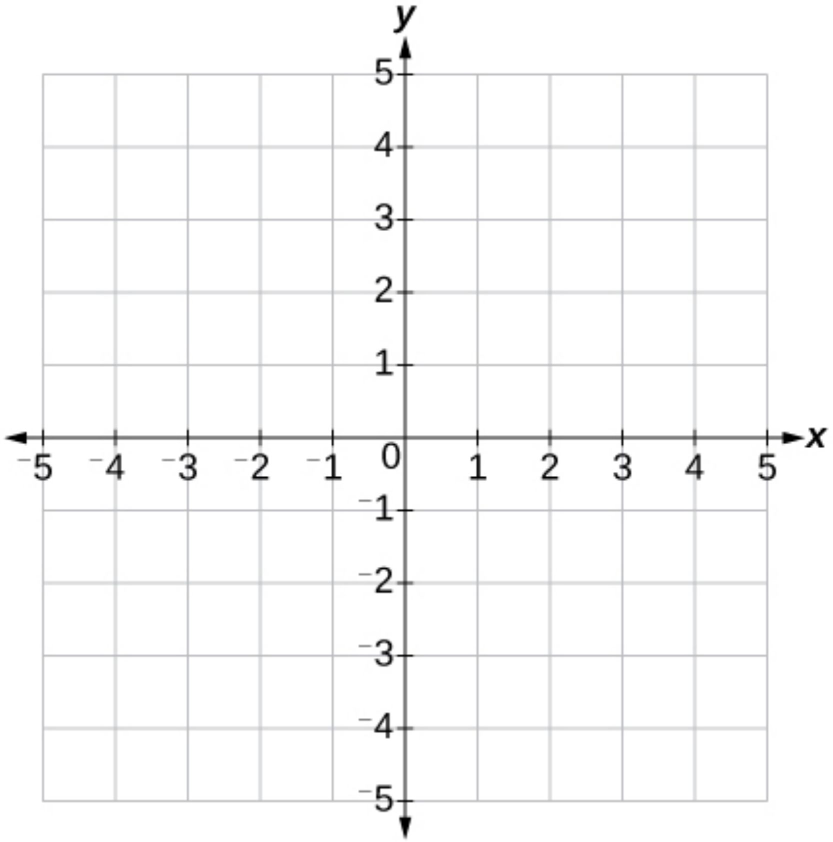
(1) $\vec{n} = \begin{pmatrix} 7 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 7+1 \\ 2+6 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \end{pmatrix}$

(2) $\vec{m} = 0.5 * \begin{pmatrix} 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 0.5 * 8 \\ 0.5 * 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$





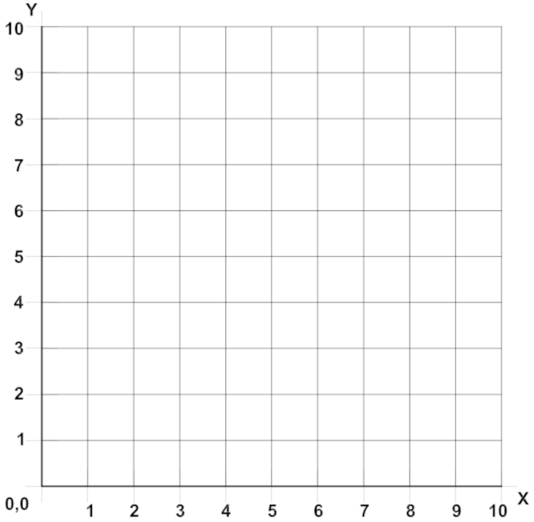
VECTOR DECOMPOSITION

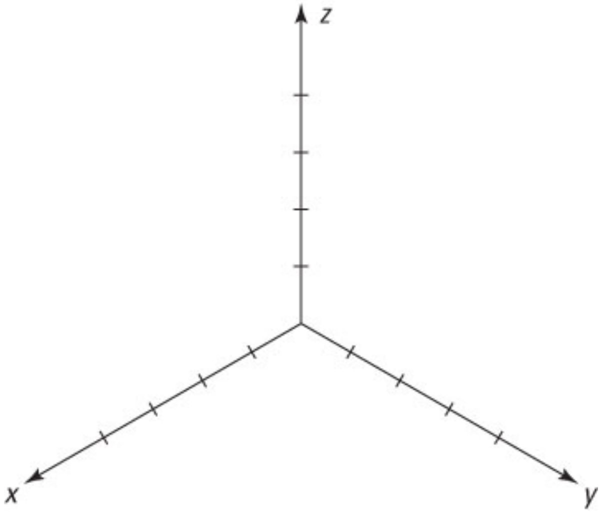


Define vectors: $\hat{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\hat{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Every vector in $\mathbb{R}^2$ can be expressed as a ***linear combination*** of $\hat{x}$ and $\hat{y}$!

VECTOR GENERALIZATION





VECTOR GENERALIZATION

Vector Representation:

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Vector Addition:

$$\vec{a} + \vec{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

Vector-Scalar Multiplication:

$$c * \vec{v} = \begin{pmatrix} c * v_1 \\ c * v_2 \\ \vdots \\ c * v_n \end{pmatrix}$$

Vector Magnitude:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

INTRO TO COMPLEX NUMBERS

(A NOT-SO-COMPLEX APPLICATION OF VECTORS!)

WHY COMPLEX NUMBERS?

Solving for roots of quadratic equations:

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\sqrt{100} = \pm 10$$

$$\sqrt{25} = \pm 5$$

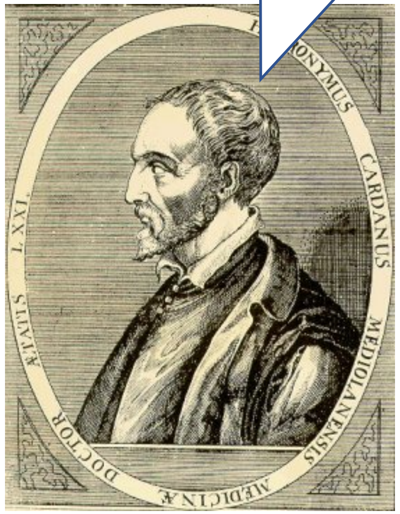
$$\vdots$$

$$\sqrt{-1} = ???$$

This can't be real?!?

Looks useless …
or better yet, ***imaginary***!

- Jerome Cardan
(16th century Renaissance polymath)

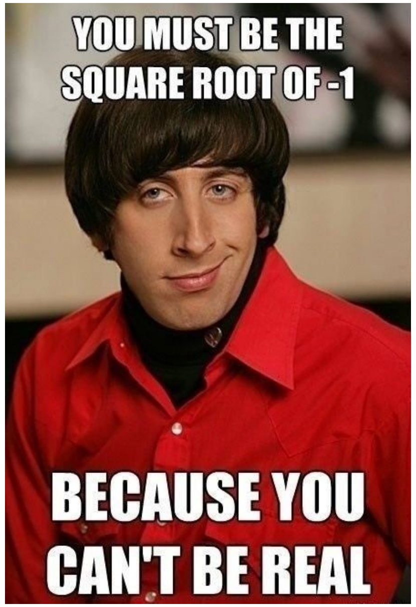


Source: Wikipedia (CC)

AY-AY-i (IMAGINARY NUMBERS)

The Imaginary Unit: $$i = \sqrt{-1}$$

$$i^2 = -1$$



AY-AY-i (IMAGINARY NUMBERS)

The Imaginary Unit: $\boxed{i = \sqrt{-1}}$ $\boxed{i^2 = -1}$

WHAT'S SO COMPLEX ABOUT COMPLEX NUMBERS?

A complex number consists of both a ***real*** and ***imaginary*** component.

$$z = a + i\,b$$

Complex Number

Real Component

Imaginary Unit

Imaginary Coefficient

Imaginary Component

QUANTUM QUIZ TIME!

State whether the following numbers are ***complex***, ***purely real***, or ***purely imaginary***.

(1) $3 + 2i$

(2) 5

(3) $-7i$

(4) $4 * i$

(5) $9 * i^2 * -1$

(6) $2 + 5 * \sqrt{-1}$

(7) $11 * i^3$

(8) $1 - i$

QUANTUM QUIZ SOLUTIONS

State whether the following numbers are ***complex***, ***real***, or ***imaginary***.

(1)	$3 + 2i$	**COMPLEX**	**(5)**	$9 * i^2 * -1 = 9 * -1 * -1 = 9$	**REAL**
(2)	5	**REAL**	**(6)**	$2 + 5 * \sqrt{-1} = 2 + 5i$	**COMPLEX**
(3)	$-7i$	**IMAGINARY**	**(7)**	$11 * i^3 = 11 * i^2 * i = -11i$	**IMAGINARY**
(4)	$4 * i$	**IMAGINARY**	**(8)**	$1 - i$	**COMPLEX**

WHAT'S SO COMPLEX ABOUT COMPLEX NUMBERS?

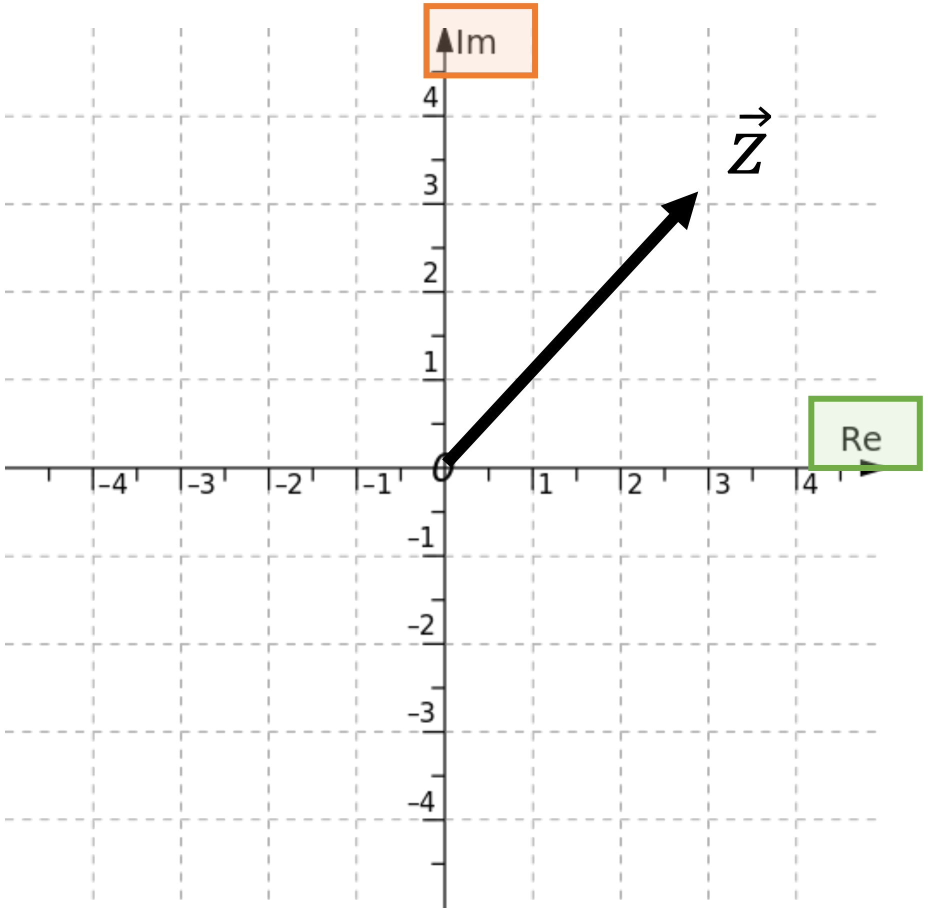
Complex numbers can additionally be represented as ***vectors***, in the 2D ***complex plane***!

$$z = a + i\,b$$

$$\vec{z} = \begin{pmatrix} a \\ b \end{pmatrix}$$

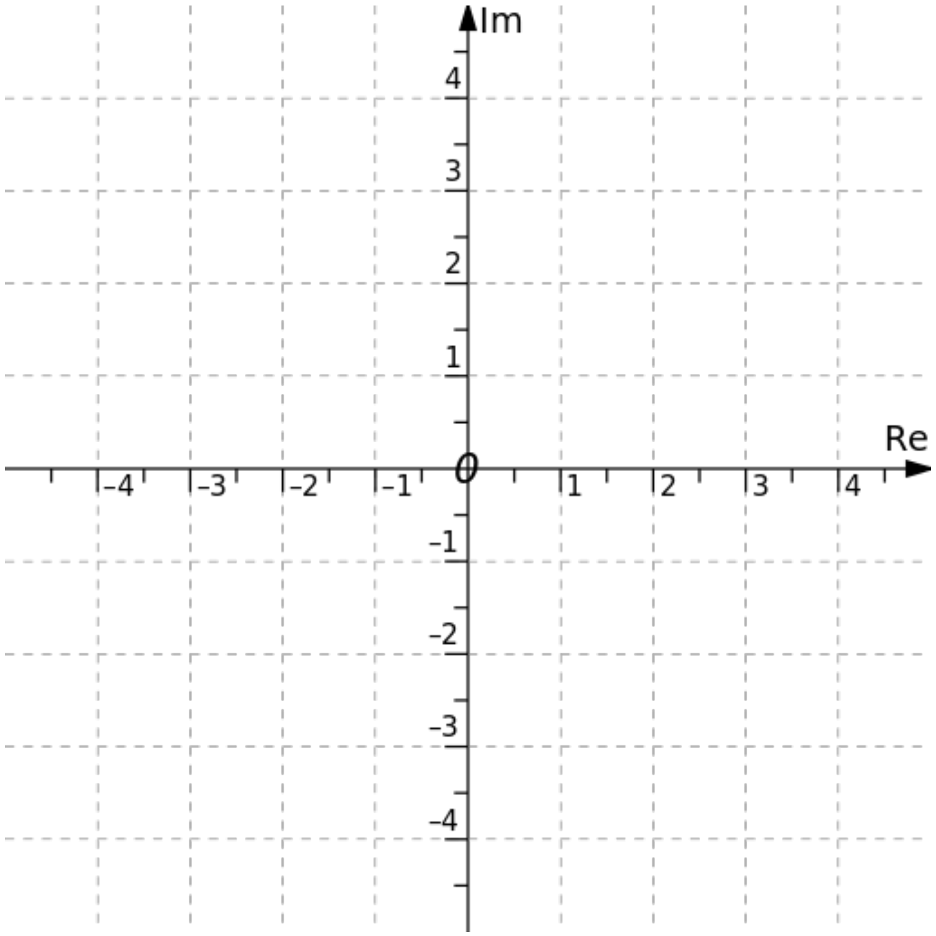
Real Component

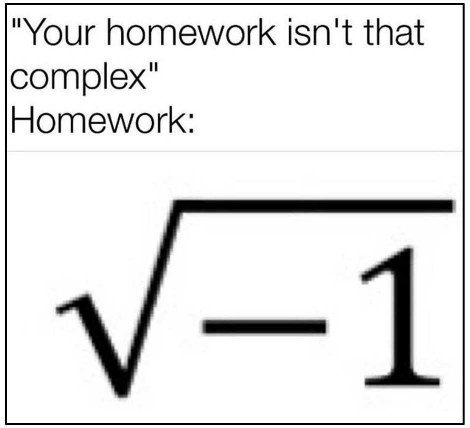
Imag. Component



$$a + i\,b$$

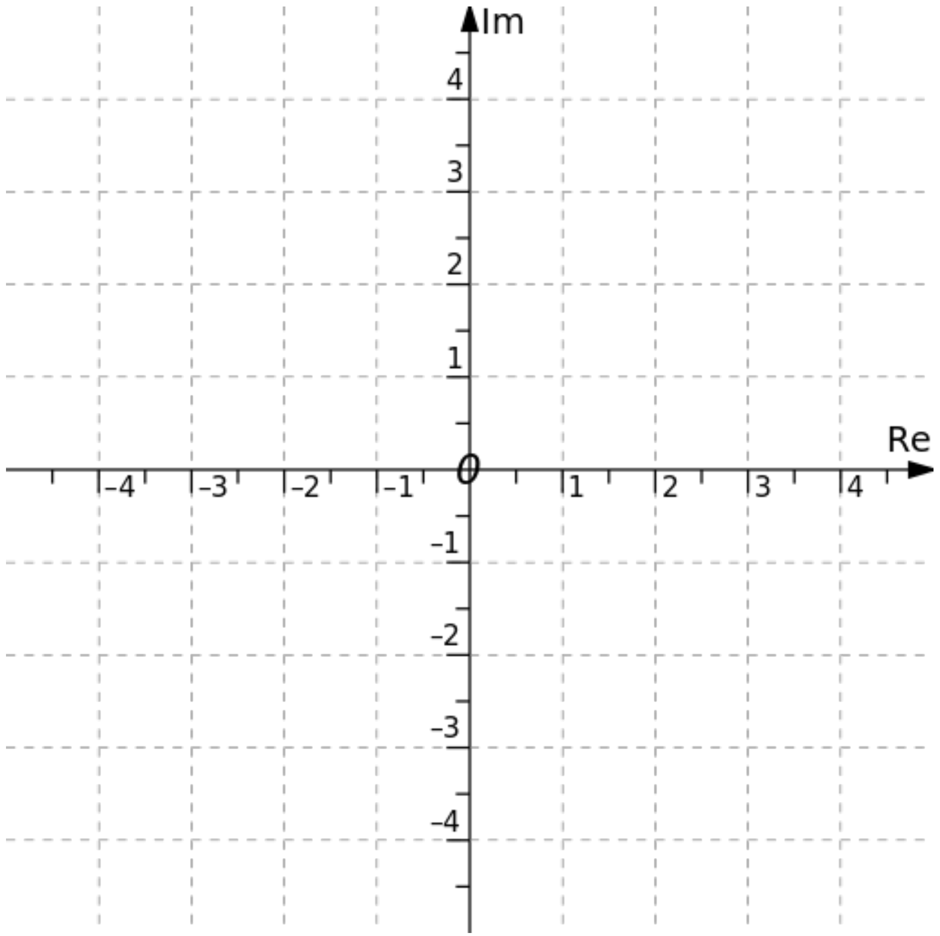
Complex Plane:





COMPLEX # ADDITION

$$(a + i\,b) + (c + id) = (a + c) + i(b + d)$$

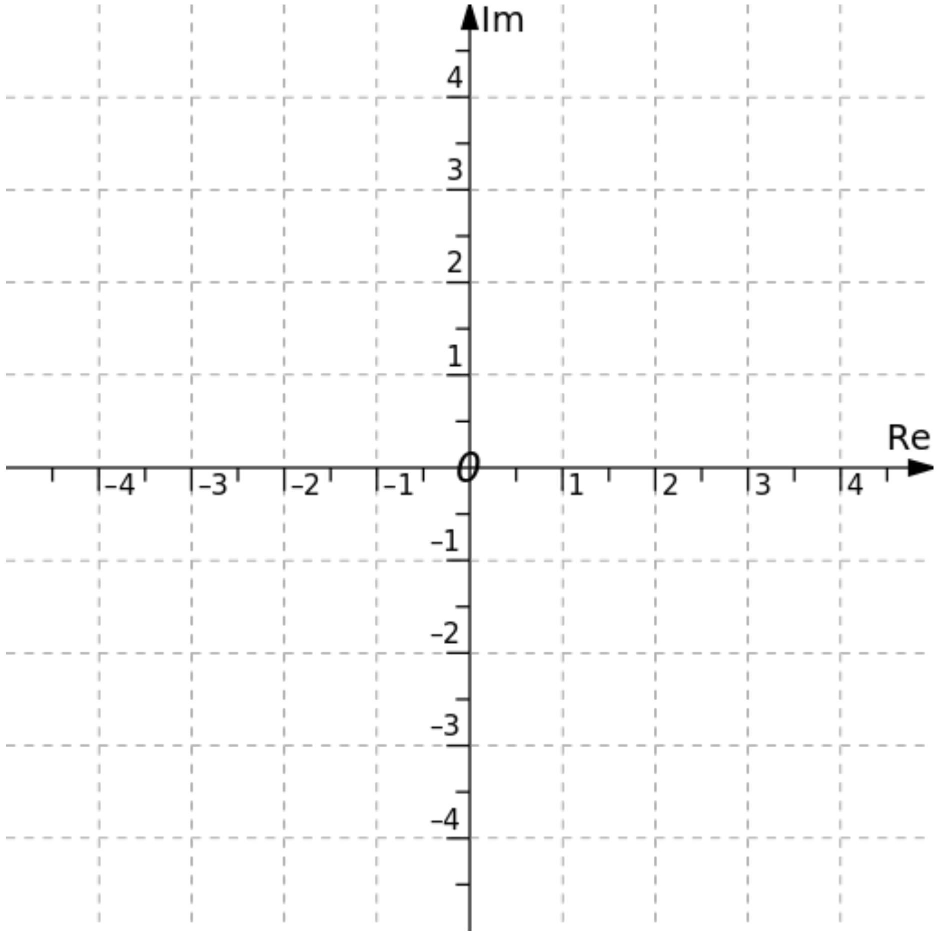


COMPLEX # MULTIPLICATION

$$(a + i\,b) * (c + id) = (ac - bd) + i(ad + bc)$$

COMPLEX # MULTIPLICATION

$$(a + i\,b) * (c + id) = (ac - bd) + i(ad + bc)$$



QUANTUM QUIZ TIME!

Perform the following complex number operations.

(1) $(3 + 2i) * (1 - 2i)$

(2) $(8 + 5i) + (2 - 7i)$

(3) $(3 + 2i) * (3 - 2i)$

QUANTUM SOLUTIONS

Perform the following complex number operations.

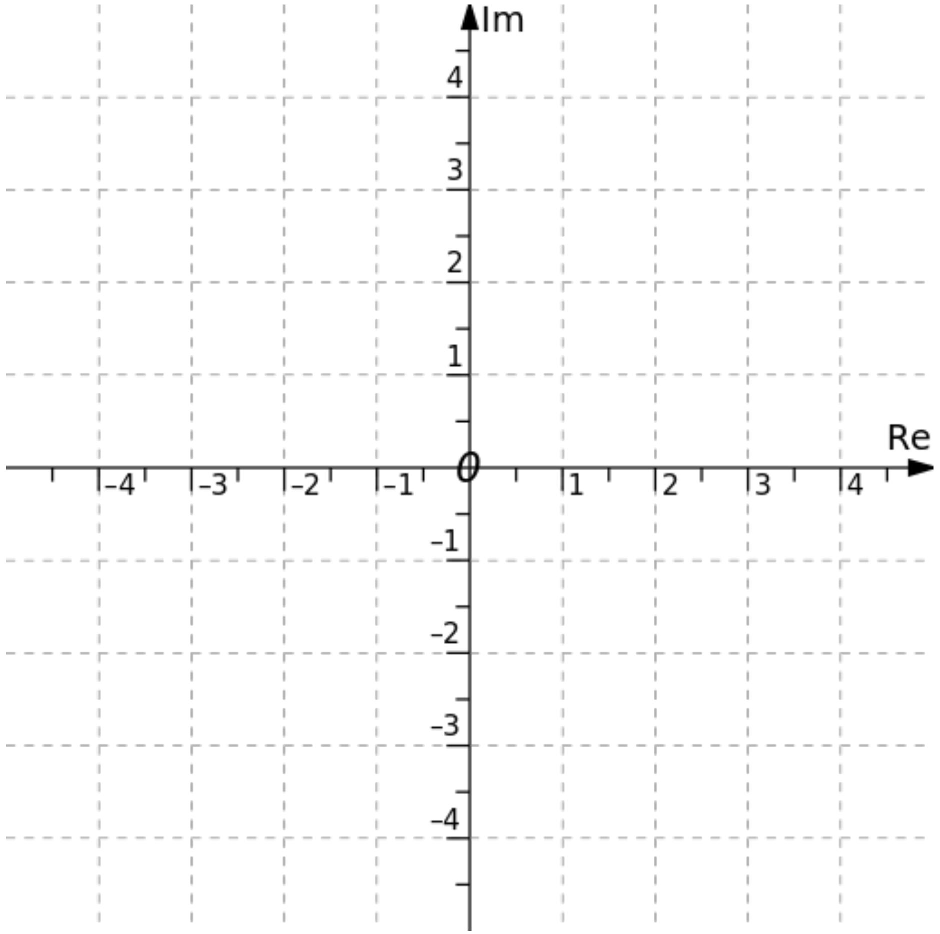
(1) $(3 + 2i) * (1 - 2i) = 3 - 6i + 2i - 4i^2 = 3 - 4i + 4 = 7 - 4i$

(2) $(8 + 5i) + (2 - 7i) = (8 + 2) + (5 - 7)i = 10 - 2i$

(3) $(3 + 2i) * (3 - 2i) = 9 + 6i - 6i - 4i^2 = 9 + 4 = 13$

COMPLEX # CONJUGATION

$$\overline{(a + i\,b)} = (a - ib)$$



COMPLEX # MODULUS

$$|a + ib| = \sqrt{a^2 + b^2}$$

$$|a + ib|^2 = a^2 + b^2$$

QUANTUM QUIZ TIME!

Perform the following complex number operations.

(1) $\overline{(3+2i)}$

(2) $|(3+2i)|$

(3) $|(3+2i)|^2$

(4) $(3+2i) * \overline{(3+2i)}$

QUANTUM QUIZ SOLUTIONS

Perform the following complex number operations.

(1) $\overline{(3-2i)} = 3+2i$

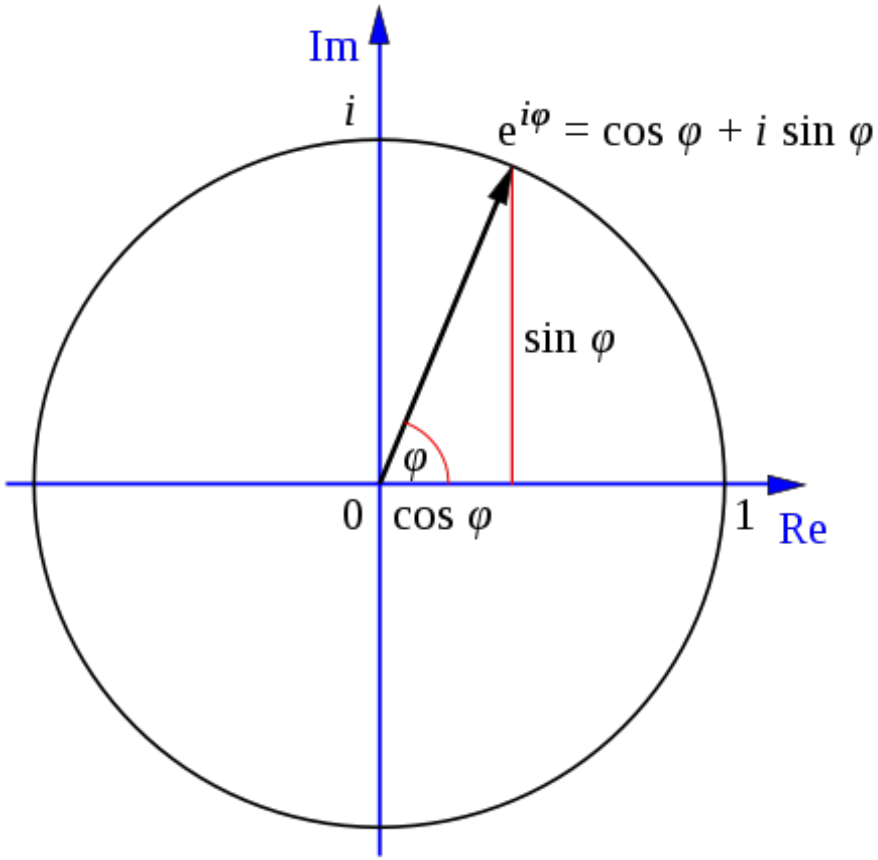
(2) $|(3-2i)| = \sqrt{3^2+(-2)^2} = \sqrt{9+4} = \sqrt{13}$

$$|z|^2 = z * \bar{z} = \bar{z} * z$$

(3) $|(3+2i)|^2 = 3^2+(-2)^2 = 9+4 = 13$

(4) $(3+2i) * \overline{(3+2i)} = (3+2i) * (3-2i) = 9+6i-6i-4i^2 = 9+4 = 13$

EULER'S FORMULA & COMPLEX EXPONENTIALS



Source: Wikipedia (CC)

Euler's formula: $$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

Polar representation of complex numbers!

$$z = x + iy = |z|(\cos\varphi + i\sin\varphi) = re^{i\varphi}$$

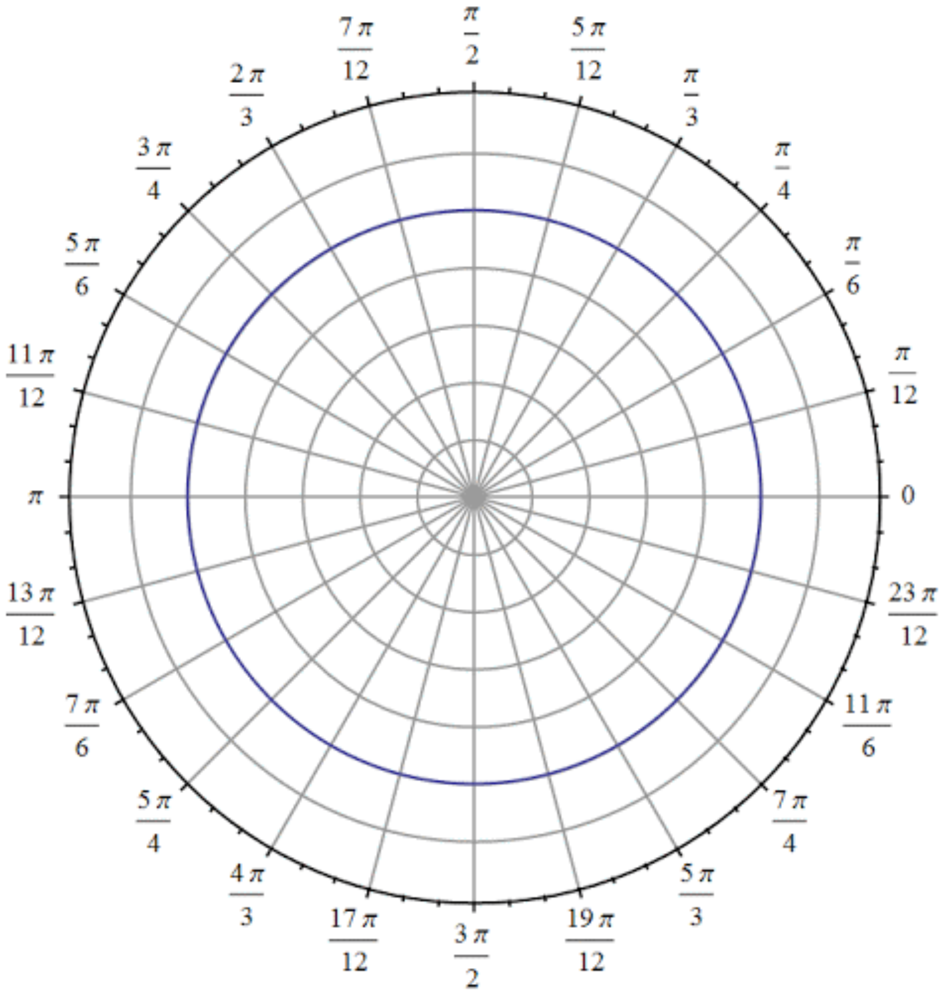
$$r = |z| = \sqrt{x^2 + y^2}$$

(vector radius)

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

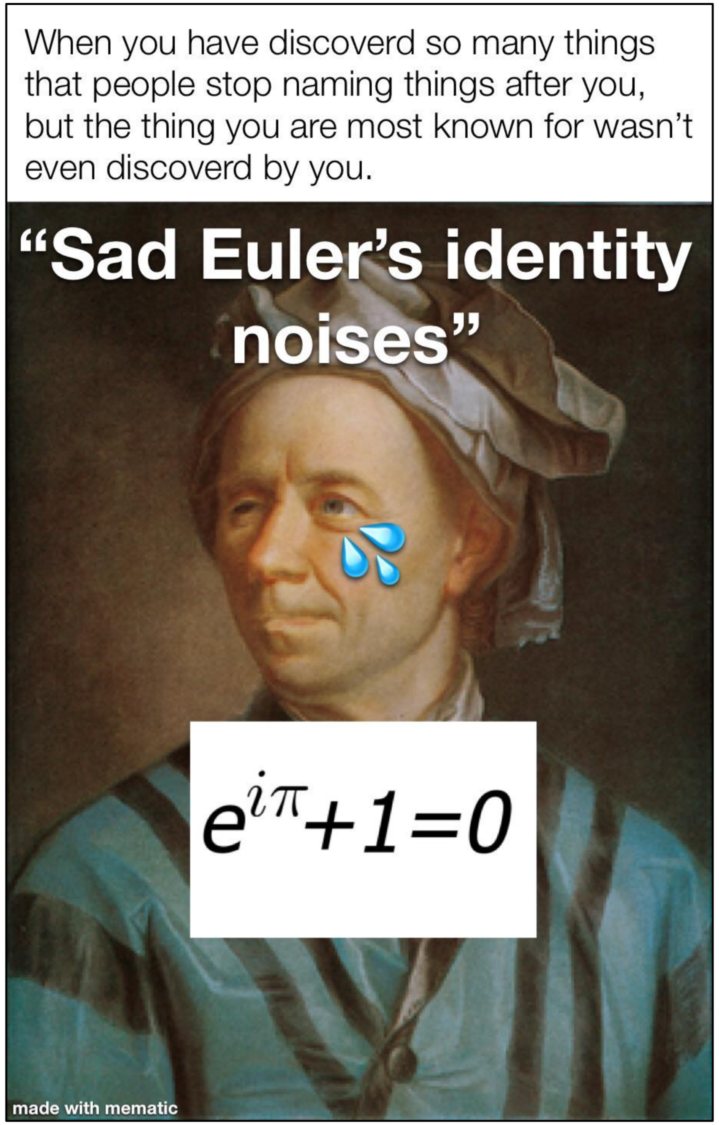
(vector angle)

EULER'S IDENTITY



$$e^{i\pi} + 1 = 0$$

HISTORY TANGENT



According to Wikipedia:

"Leonhard Euler's work touched upon so many fields that he is often the earliest written reference on a given matter. **In an effort to avoid naming everything after Euler, some discoveries and theorems are attributed to the first person to have proved them after Euler.**"

https://en.wikipedia.org/wiki/List_of_things_named_after_Leonhard_Euler

The English mathematician Roger Cotes (died 1716 when Euler was only 9 years old) was the first to know of Euler's Formula.

In 1714 he presented a geometrical argument that can be interpreted as:

$$ix = \ln(\cos x + i \sin x)$$

Exponentiating this equation yields Euler's formula. Note that the logarithmic statement is not universally correct for complex numbers, since a complex logarithm can have infinitely many values, differing by multiples of 2iπ.

Around 1740 Euler turned his attention to the exponential function instead of logarithms and obtained the formula that is named after him. He obtained the formula by comparing the series expansions of the exponential and trigonometric expressions.

COMPLEX EXP. ADDITION

Generally, additional of complex exponentials is challenging and the numbers need to be converted back to standard complex number notation ($a + ib$).

However, it is good to know the following two key identities!

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}$$

$$\sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$$

QUBIT X QUBIT

The <Coding School/>

COMPLEX EXP. MULTIPLICATION

$$e^{i\varphi}\, e^{i\theta} = e^{i(\varphi+\theta)}$$

DRAKE'S OPINION?

$$(a + i\,b) * (c + id) = (ac - bd) + i(ad + bc)$$

$$e^{i\varphi}\, e^{i\theta} = e^{i(\varphi+\theta)}$$

COMPLEX EXP. MODULUS

$$\boxed{|z| = \left|re^{i\varphi}\right| = r} \quad (r > 0)$$

COMPLEX EXP. CONJUGATION

$$z = x + iy = |z|(\cos\varphi + i\sin\varphi) = re^{i\varphi}$$

$$\boxed{\overline{e^{i\varphi}} = e^{-i\varphi}}$$

QUANTUM QUIZ TIME!

Perform the following complex exponential operations.

(1) $\overline{e^{-i\pi}}$

(2) $\left|4\ e^{-i*2.37}\right|$

(3) $2\ e^{-i*2.37} * \overline{2\ e^{-i*2.37}}$

(4) $e^{-i\frac{\pi}{2}} + e^{i\frac{\pi}{2}}$

(5) $e^{-i\frac{\pi}{3}} - e^{i\frac{\pi}{3}}$

QUBIT X QUBIT

QUANTUM SOLUTIONS

Perform the following complex exponential operations.

(1) $\overline{e^{-i\pi}} = e^{i\pi} = -1$

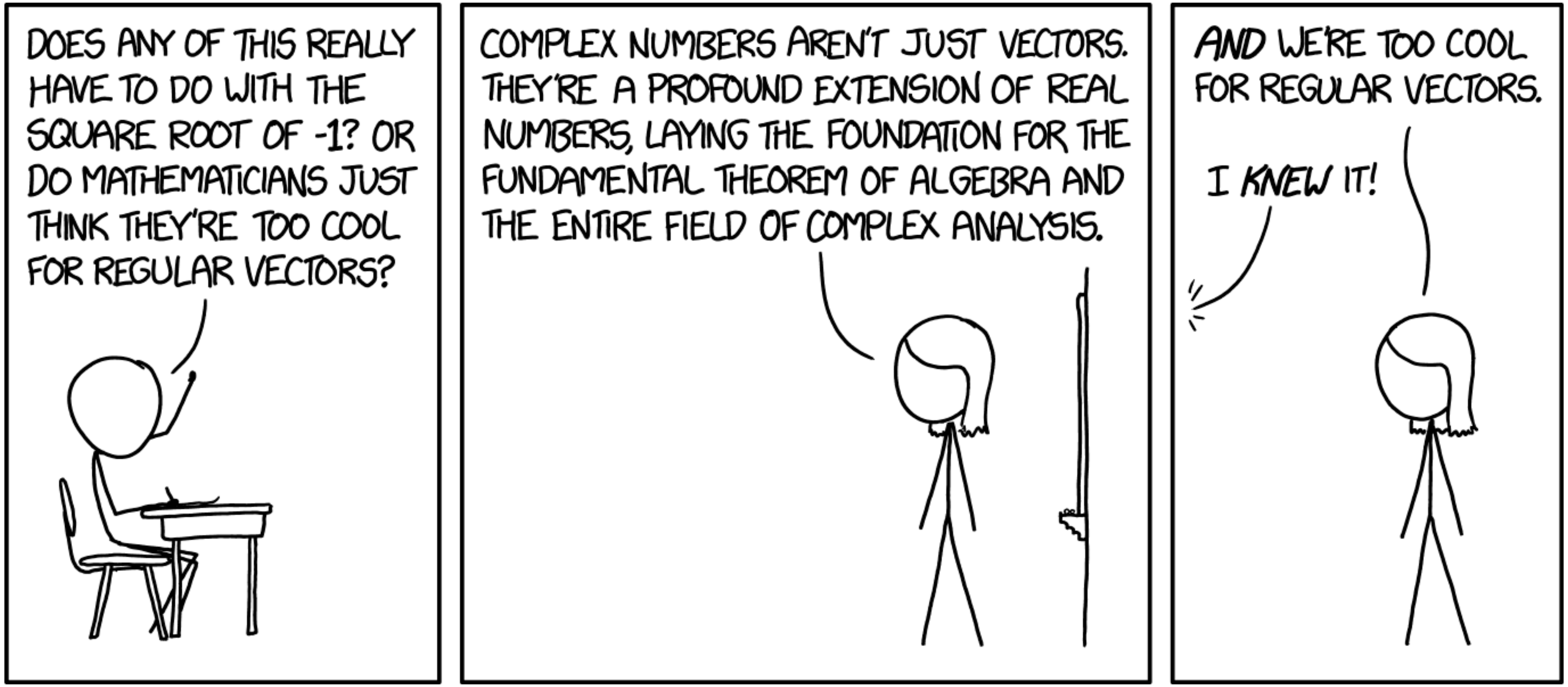
(2) $\left|4\, e^{-i*2.37}\right| = 4\left|\, e^{-i*2.37}\right| = 4$

(3) $2\, e^{-i*2.37} * \overline{2\, e^{-i*2.37}} = 2 * 2 * e^{-i*2.37} * \; e^{i*2.37} = 4\, e^{i*2.37-i*2.37} = 4\, e^{0} = 4$

(4) $e^{-i\frac{\pi}{2}} + e^{i\frac{\pi}{2}} = 2\cos\frac{\pi}{2} = 2 * 0 = 0$

(5) $e^{-i\frac{\pi}{3}} - e^{i\frac{\pi}{3}} = 2i\; \sin\frac{\pi}{2} = 2i * 1 = 2i$

NOW YOU UNDERSTAND!



Source: xkcd

ANNOUNCEMENTS