The Chinese University of Hong Kong, Shenzhen

PHY 1002

Physics Laboratory

Report of Large Amplitude Pendulum Experiment

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1 Introduction:

A pendulum is a weight suspended from a pivot so that it can swing freely. When a pendulum is displaced sideways from its resting, equilibrium position, it is subject to a restoring force due to gravity that will accelerate it back toward the equilibrium position. When released, the restoring force acting on the pendulum's mass causes it to oscillate about the equilibrium position, swinging back and forth. The time for one complete cycle, a left swing and a right swing, is called the period. The period depends on the length of the pendulum and also to a slight degree on the amplitude, the width of the pendulum's swing. Simple Pendulum is one of the most common phenomenon in our daily life. During the development of Physics, it has drawn a large amount of attention from physicists and been studied for a long time.

This is an experiment to figure out the difference between *Large Amplitude Pendulum* and *Small Amplitude Pendulum*. Therefore, a series of three experiments are conducted to explore the dependence of the period of a simple pendulum on the amplitude of the oscillation. Also, we manage to plot the graph of the angular displacement, angular velocity, and angular acceleration versus time for large amplitude to show the difference from the sinusoidal motion of low amplitude oscillations. We are going to use fundamental calculus to quantitatively figure out the relation between the angular displacement, angular velocity, and angular acceleration curves.

2 Theory:

For a small amplitude physical pendulum, its period is approximately given by

$$T_0 = 2\pi\sqrt{\frac{I}{mgd}} \tag{1}$$

where I is the rotational inertia of the pendulum about the pivot point, m is the total mass of the pendulum, and d is the distance from the pivot to the the center of the mass. The approximation becomes more exact as the amplitude goes to zero. With this limit, the motion is simple harmonic and the angular displacement is given by

$$\theta = \theta_0 \sin(At + \phi) \tag{2}$$

where θ_0 is the initial angular displacement, $A = \frac{2\pi}{T_0}$ is a constant, representing the angular frequency, and ϕ is the phase angle which is determined by the value of θ when $t = 0$. It is thereby a constant. Then, by calculus, we can point out the angular velocity and angular acceleration

$$\omega = \frac{d\theta}{dt} = A\theta_0 \cos(At + \phi) \tag{3}$$

$$\alpha = \frac{d\omega}{dt} = -A^2\theta_0 \sin(At + \phi) \tag{4}$$

However, things change a lot in large amplitude. If we continue to use the approximation equations above, we will miss in errors. So, we need a more accurate one. For larger amplitudes, the restoring torque is not linear and the period is given by an infinite series

$$T = T_0[1 + \sum_{n=1}^{\infty}(\frac{(2n-1)!!}{2^n n!})^2 \sin^{2n}(\frac{A}{2})] \tag{5}$$

where A (same as the θ_0 in Equation 2, Equation 3 and Equation 4) is the oscillation amplitude. Consider its first five terms and we have

$$T \approx T_0[1 + (\frac{1}{2})^2 \sin^2(\frac{A}{2}) + (\frac{3}{8})^2 \sin^4(\frac{A}{2}) + (\frac{15}{48})^2 \sin^6(\frac{A}{2}) + (\frac{105}{384})^2 \sin^8(\frac{A}{2})] \tag{6}$$

3 Experiment Process:

The following illustrates the facility of experiment and details of the experiment approaches.

3.1 Equipment Setup

First, we thread the metal rod into the iron base and attach the rotary motion sensor on it. Attach the rotary motion sensor to one *Paspot* input on the PASCO 850 Universal Interface. Then, we remove the silver thumbscrew to install the pulley to the rotary motion sensor. We make the large step of the pulley towards out on the rotary motion sensor and attach the pendulum rod at its *center* using another longer thumbscrew which matches the rod perfectly. The rod is positioned so the bumps on the pulley hold it in place. At last, we attach two brass masses on the each end of the rod. One is even with the end, while the other is left about 3cm to it. As shown in Figure 1.

Figure 1: Experiment 1 Setup

3.2 Qualitative Analysis

In this experiment, the rotary motion sensor is at 100Hz.

For *small pendulum*, we first make sure the pendulum rests at its equilibrium position. We click the *RECORD* button in PASCO Capstone™to initialize the rotary motion sensor, making its start point to origin. Then, we displace the pendulum at an angle less than 20° (about 0.35rad) from equilibrium. Release the pendulum and click *STOP* after a few oscillations. Then, we have the data shown in *Figure 2*, *Figure 3* and *Figure 4*.

For *large pendulum*, we repeat the previous steps but displace the pendulum nearly 180° from equilibrium.

3.3 Quantitative Analysis of Small Amplitude Oscillations

In this experiment, the rotary motion sensor is at 100Hz.

Before we start, we first slide down the upper mass until it is touching the pulley. And we use the thumbscrew to hold it at that place. Likely, we first click *RECORD* with the pendulum resting at its equilibrium position, initializing the sensor to start at the origin.

Then, we displace the pendulum less than 5° (about 0.10rad) from equilibrium. Release it and click *STOP* after 13 oscillations.

3.4 Quantitative Analysis of Large Amplitude Oscillations

In this experiment, the theory motion sensor is at 200Hz.

Similarly, we first click *RECORD* with the pendulum resting at its equilibrium position, initializing the sensor to start at the origin. Then, we displace the pendulum at about 10° (about 0.20rad) from equilibrium. Release it and click *STOP* after 3 oscillations.

4 Raw Data:

4.1 Raw Data Figures

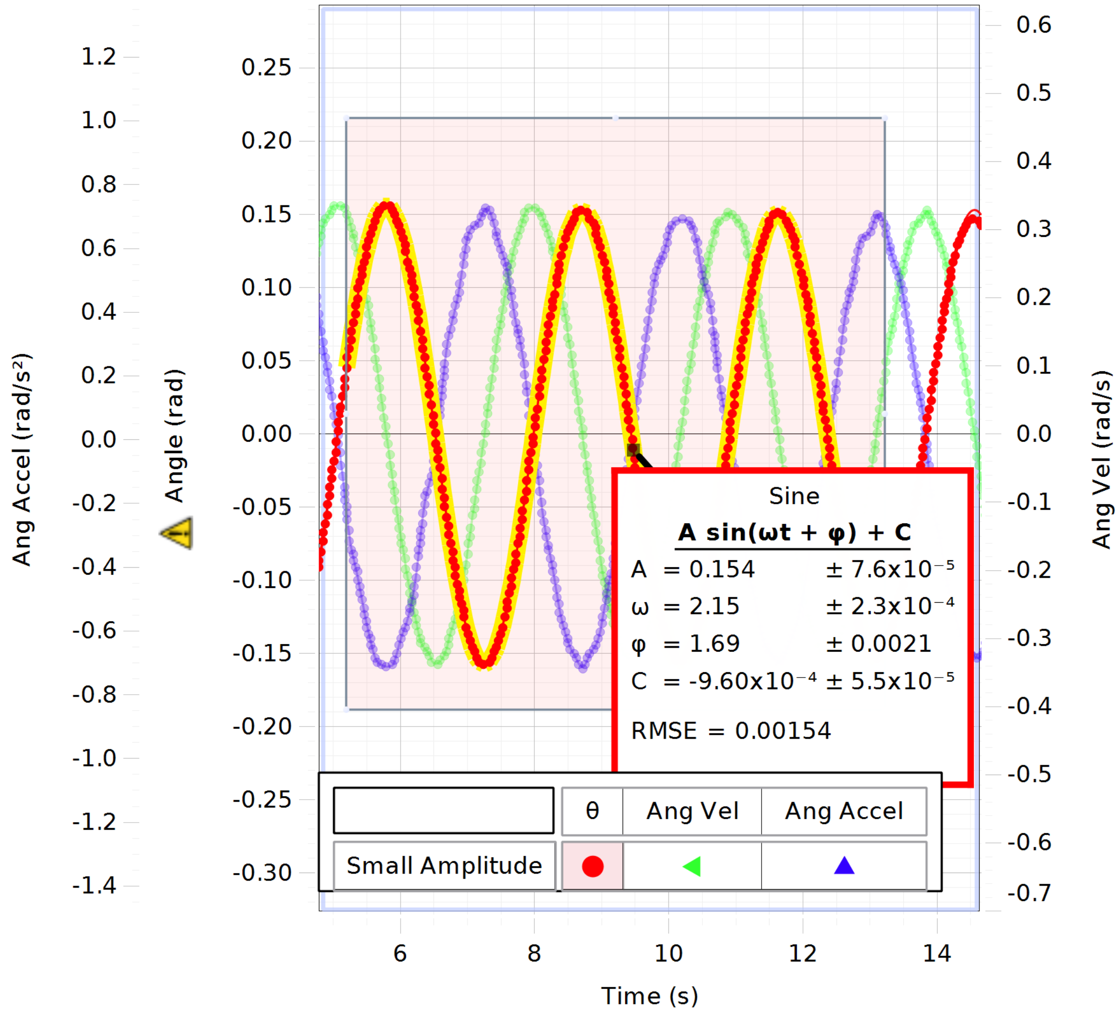

Figure 2: Sine Curve Approximation of Angular Magnitude in Qualitative Analysis for Small Amplitude

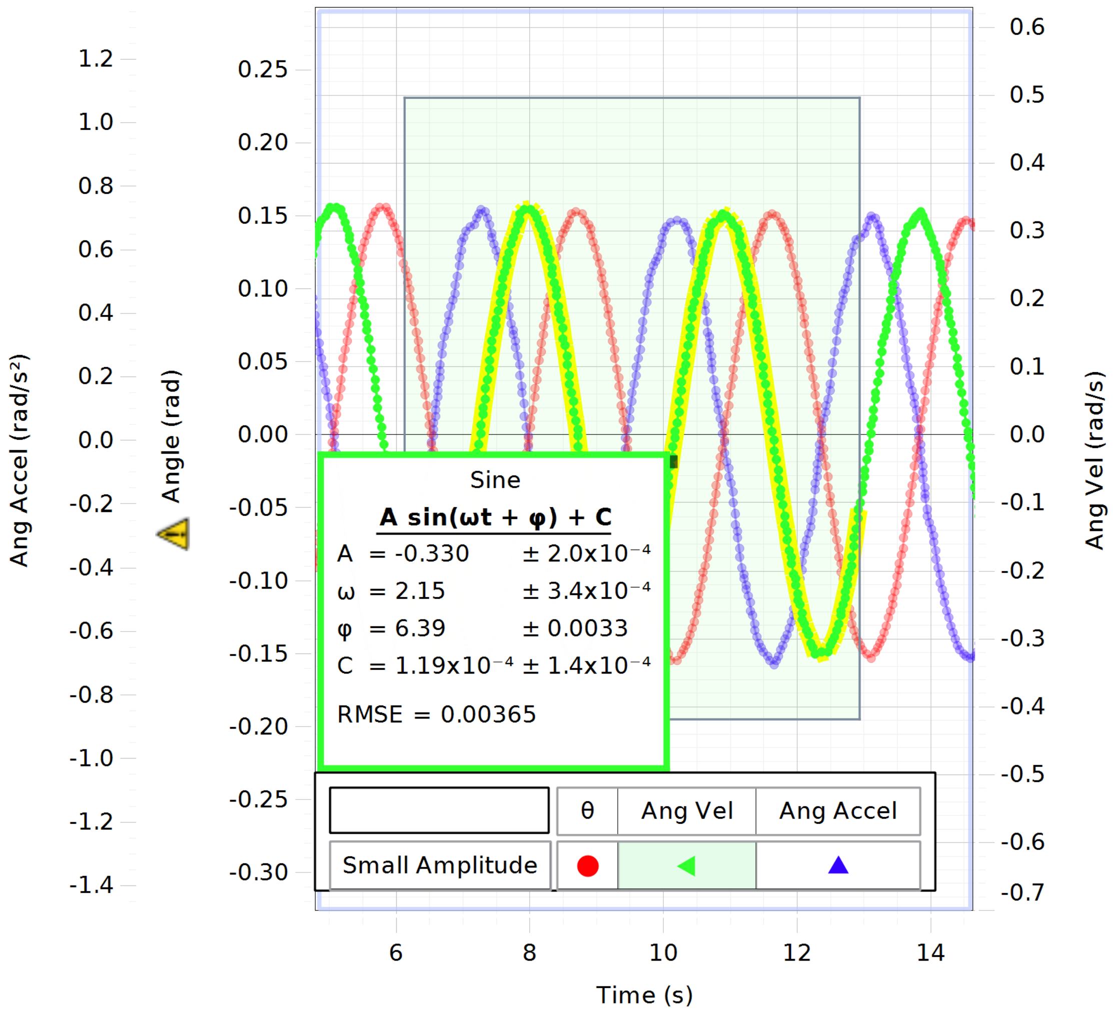

Figure 3: Sine Curve Approximation of Angular Velocity in Qualitative Analysis for Small Amplitude

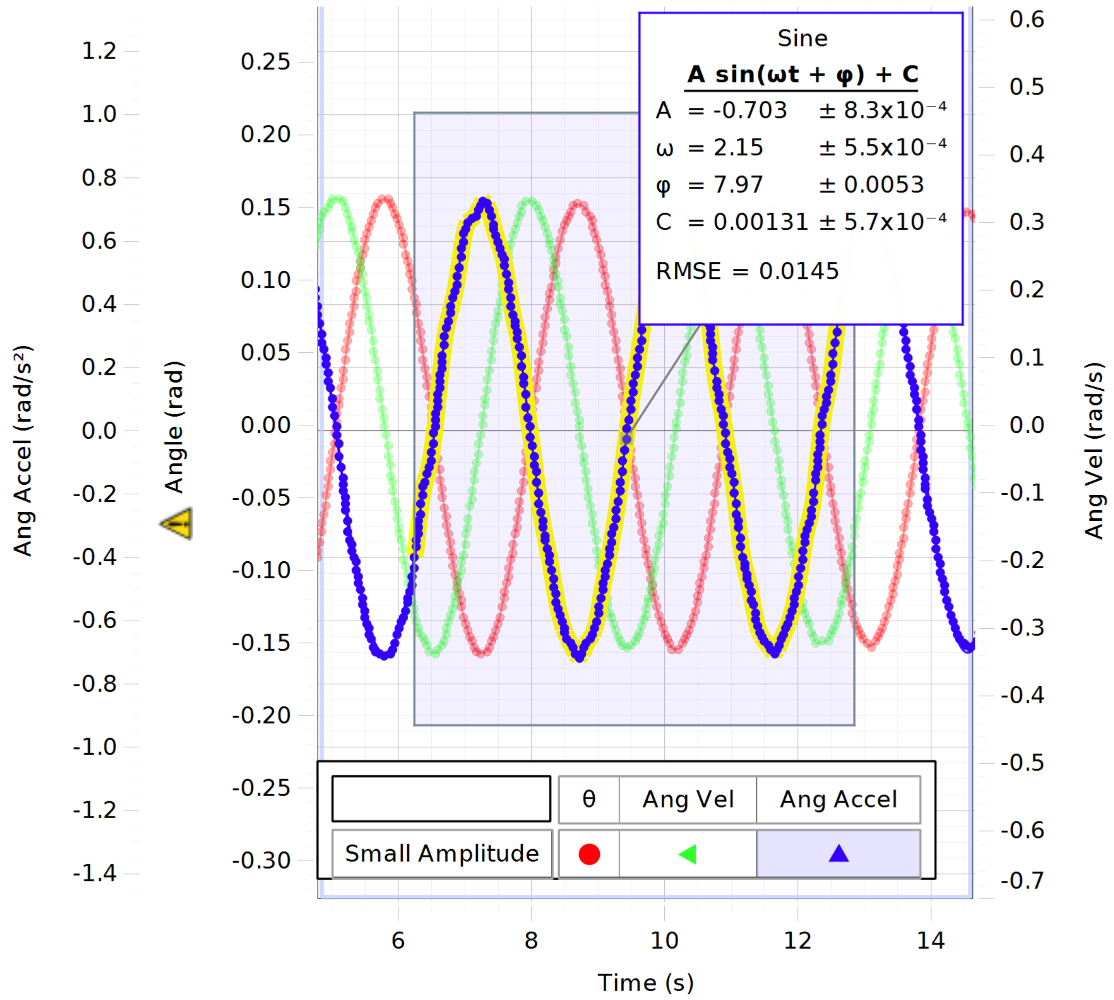

Figure 4: Sine Curve Approximation of Angular Acceleration in Qualitative Analysis for Small Amplitude

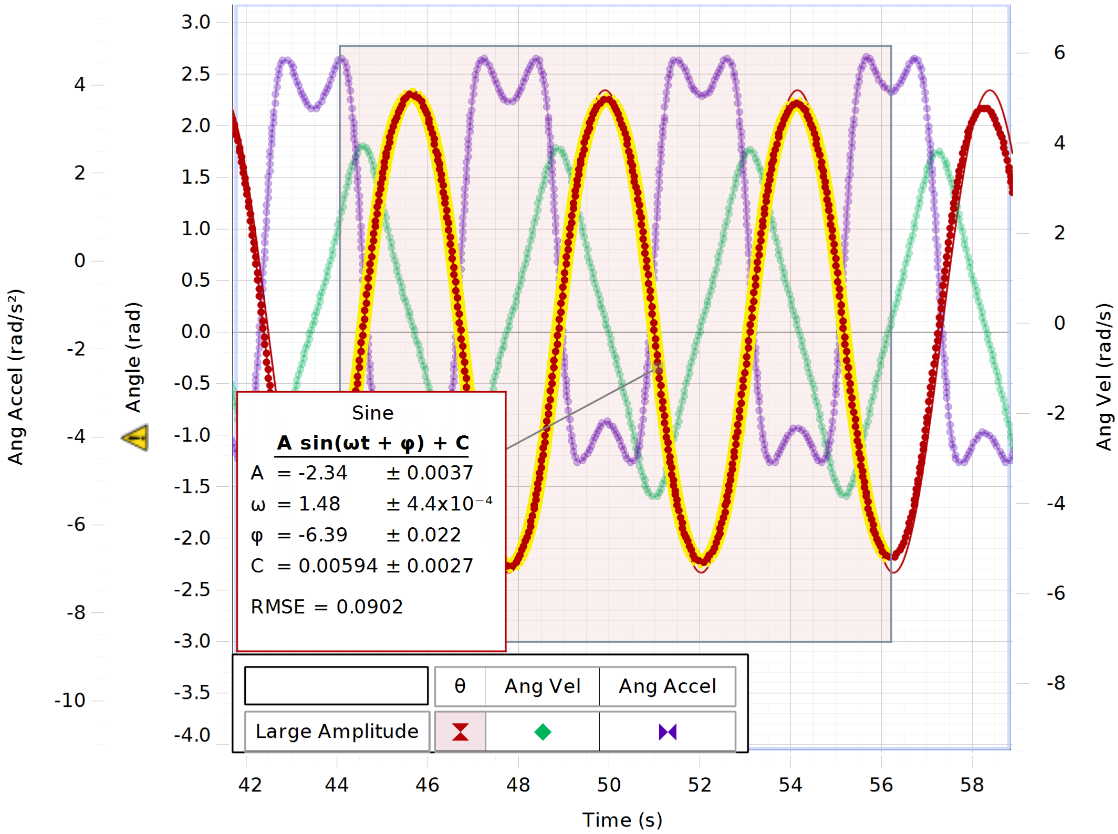

Figure 5: Sine Curve Approximation of Angular Magnitude in Qualitative Analysis for Large Amplitude

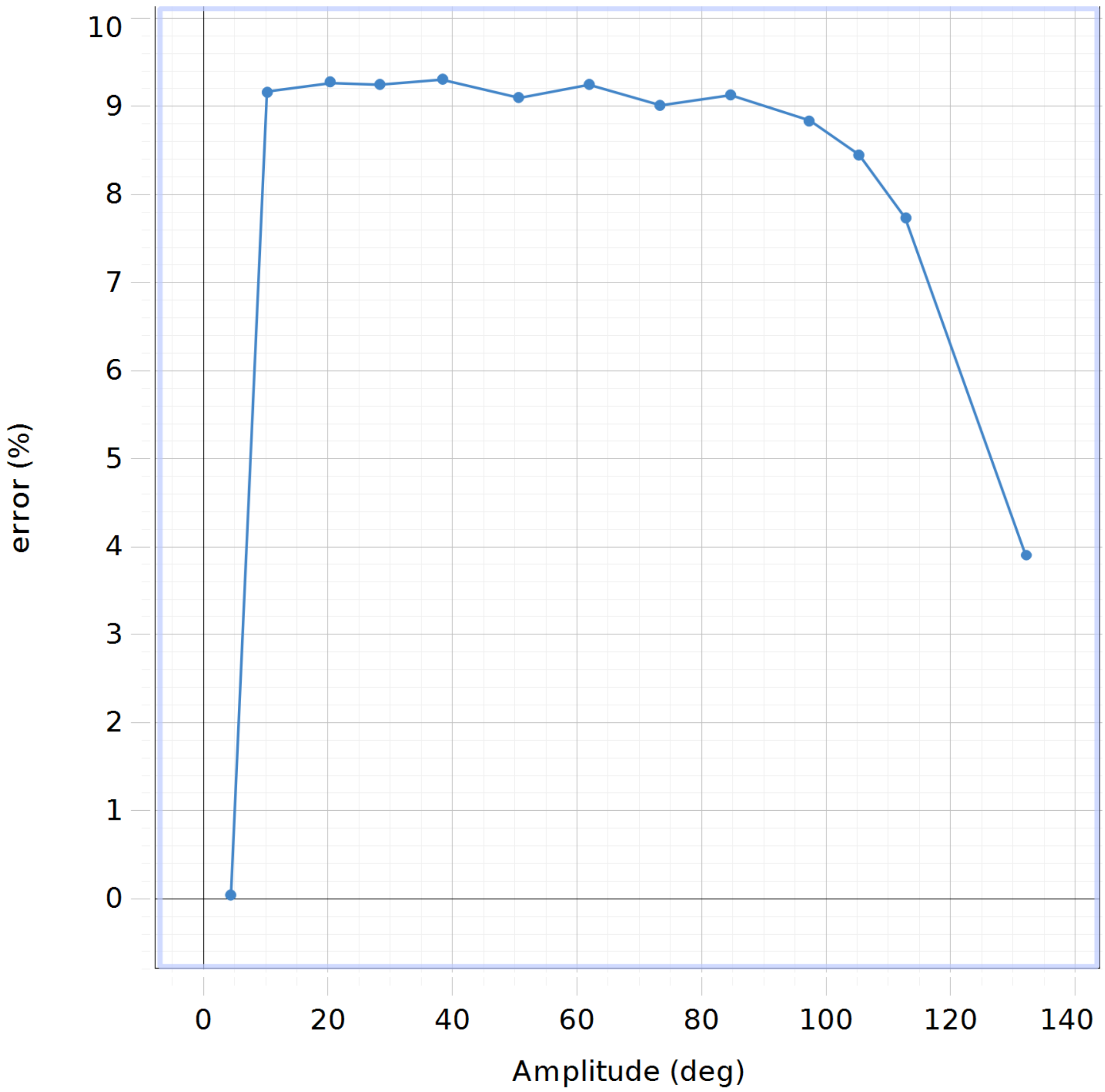

Figure 6: The Error Percentage between Actual and Calculated by Equation 6 Versus Amplitude Degree

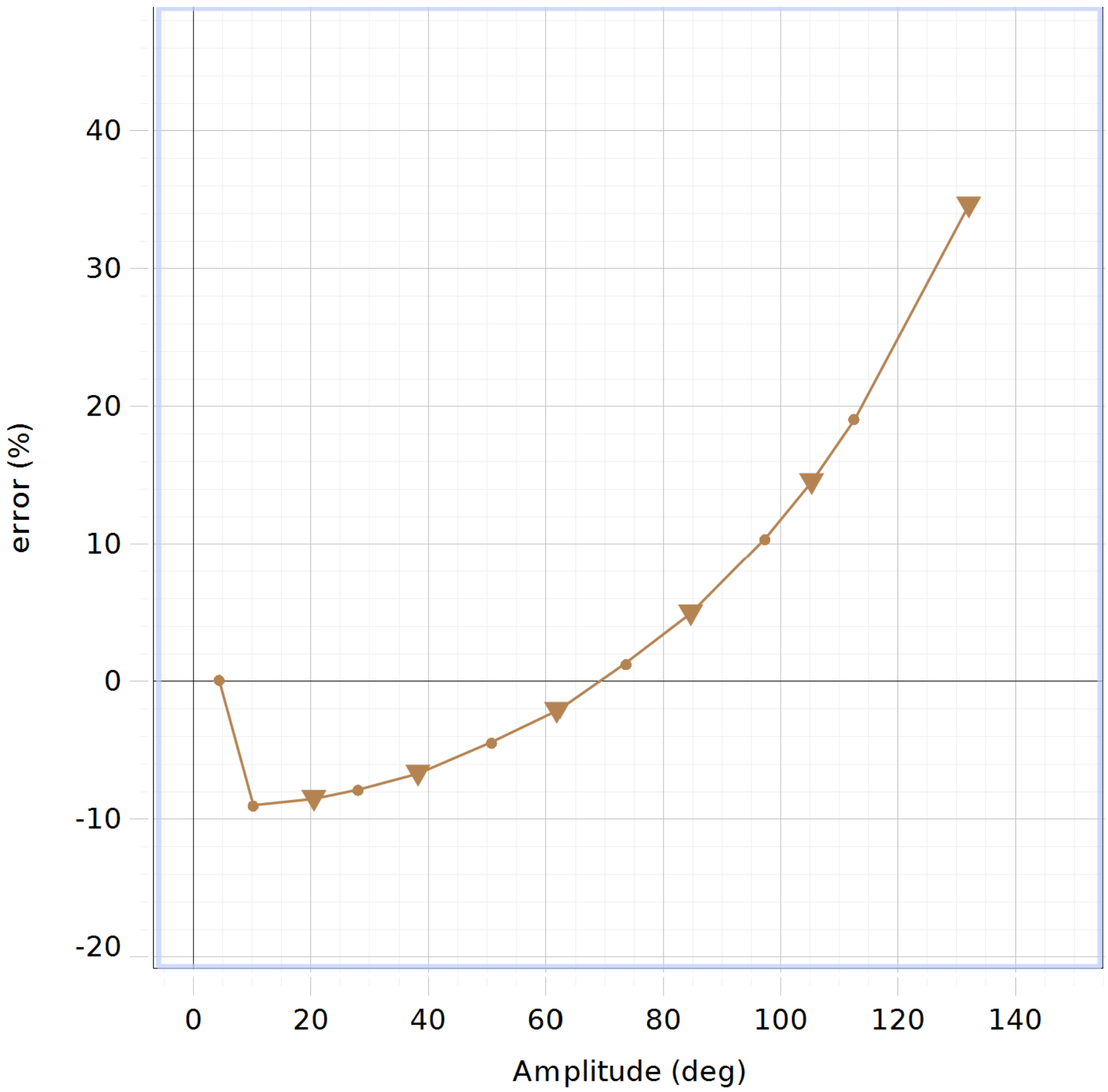

Figure 7: The Error Percentage between Actual and Calculated by Equation 1 Versus Amplitude Degree

4.2 Raw Data Tables

t_1(s)	t_2(s)	Period Zero(s)	Amplitude(rad)
4.471 ± 0.005	15.432 ± 0.005	1.096 ± 0.001	0.077 ± 0.002

Table 1: Low Amplitude Period

	t_1(s)	t_2(s)	Period(s)	Amplitude(rad)
1	-1.096 ± 0.005	1.096 ± 0.005	1.096 ± 0.004	0.077 ± 0.002
2	5.610 ± 0.005	7.605 ± 0.005	0.998 ± 0.004	0.178 ± 0.002
3	4.035 ± 0.005	6.040 ± 0.005	1.002 ± 0.004	0.358 ± 0.002
4	3.440 ± 0.005	5.460 ± 0.005	1.010 ± 0.004	0.493 ± 0.002
5	4.850 ± 0.005	6.895 ± 0.005	1.022 ± 0.004	0.668 ± 0.002
6	4.660 ± 0.005	6.755 ± 0.005	1.048 ± 0.004	0.886 ± 0.002
7	5.730 ± 0.005	7.875 ± 0.005	1.072 ± 0.004	1.081 ± 0.002
8	5.105 ± 0.005	7.325 ± 0.005	1.110 ± 0.004	1.282 ± 0.002
9	7.960 ± 0.005	10.260 ± 0.005	1.150 ± 0.004	1.478 ± 0.002
10	7.275 ± 0.005	9.695 ± 0.005	1.210 ± 0.004	1.700 ± 0.002
11	5.875 ± 0.005	8.385 ± 0.005	1.255 ± 0.004	1.838 ± 0.002
12	8.565 ± 0.005	11.175 ± 0.005	1.305 ± 0.004	1.967 ± 0.002
13	6.300 ± 0.005	9.250 ± 0.005	1.475 ± 0.004	2.306 ± 0.002

Table 2: Large Amplitude Period

5 Analysis & Questions:

5.1 Error Analysis

Time In all these experiments, we use the embedded clock in PASCO Capstone™to measure the time. According to the manual of PASCO Capstone™and PASCO 850 Universal Interface, the estimated error of time should be ± 0.005s.

Angular Magnitude In all these experiments, we use a rotary motion sensor to detect the angular magnitude and angular velocity. According to its official manual, the resolution of this sensor is 0.00157rad, so we consider its estimated error should be 0.002rad.

Therefore, the columns of estimated data in Table 1 and Table 2 have listed raw data with respective estimated error.

Standard Error If a function k is determined by n independent random variables, namely $p_1, p_2, ..., p_n$ and $p_i(i = 1, 2, ..., n) \sim N(\mu, \sigma^2)$; that is $k = f(p_1, p_2, ..., p_n)$, the standard error for k is given by

$$\delta k = \sqrt{\sum_{i=1}^{n} (\frac{\partial f}{\partial p_i})^2 (\delta p_i)^2} \tag{7}$$

where $\delta p_i(i = 1, 2, ..., n)$ are estimated errors.

In Table 1, $t_{\text{period\_zero}} = \frac{t_2 - t_1}{10}$, by Equation 7, we can point out that $\delta t_{\text{period\_zero}} = \sqrt{(\frac{1}{10})^2(0.005\text{s})^2 + (\frac{1}{10})^2(0.005\text{s})^2} \approx 0.001\text{s}$.

In Table 2, $t_{\text{period}} = \frac{t_2 - t_1}{2}$, by Equation 7, we can point out that $\delta t_{\text{period}} = \sqrt{(\frac{1}{2})^2(0.005\text{s})^2 + (\frac{1}{2})^2(0.005\text{s})^2} \approx 0.004\text{s}$

These calculated estimated standard errors are listed respectively in Table 1 and Table 2.

Error from Approximation of Infinite Series Let us recall Equation 5, which states the relationship between the period and the pendulum amplitude. Approximately, we usually take the first several terms as the result, such as Equation 6. Since each term of the infinite series is the product of a square and an even power, our approximation will always smaller than the actual period. If we denote the estimated period time as T_{est} and the calculated period time from pendulum amplitude as T_{cal}. We can use $\frac{T_{\text{est}} - T_{\text{cal}}}{T_{\text{cal}}}$ to represent the error percentage between theoretical result and actual result under a range of pendulum amplitude, as shown in Figure 6.

5.2 Data Analysis & Questions Answers

5.2.1 Qualitative Experiment for Small Angle

Does the θ fit a Sine curve as by Equation 2? We could figure out our answer directly from Figure 2, where RMSE, representing the approximation error, is only 0.00154. So, the θ fits the Sine curve as by Equation 2 perfectly.

Does the *angular velocity* fit a Cosine curve, or a Sine curve as required by Equation 3? We could figure out our answer directly from Figure 3, where RMSE, representing the approximation error, is only 0.00365. So, the *angular velocity* fits the Sine curve, or the Cosine curve as required in Equation 3 very well.

Does the *angular acceleration* fit a Sine curve as required by Equation 4? We could figure out our answer directly from Figure 4, where RMSE representing the approximation error, is 0.0145. So, the *angular acceleration* fits the Sine curve as required in Equation 4 quite well.

What does the minus sign in Equation 4 do? In fact, it is the gravity acting as resistance when the rod is raising up. By Newton's Second Law of Motion $F = ma$, considering the positive direction of *linear velocity*, gravity is at obtuse angle with it. So the gravity is acting as resistance and should put a minus sign on F to represent the opposite direction. This leads to a acting as minus *linear acceleration*. As *linear* ones are corresponding to *angular* ones, so there is a minus sign in Equation 4.

Are the periods of the three curves the same? Yes, they are. Theoretically, from Equation 2, Equation 3 and Equation 4, we can point out that the periods are all determined by ω and exactly same. From the perspective of experiment, we could compare Figure 2, Figure 3 and Figure 4. We could state that for their respective approximate Sine curves, ω is at the same value. So, the period determined by ω is the same.

5.2.2 Qualitative Experiment for Large Angle

Do the *angular velocity* and *angular acceleration* data fit a Sine curve as required by Equation 3 & 4 from Theory? No. Compare Equation 1 and Equation 5, we will find out the former one is the approximation of the latter one under circumstance of small pendulum amplitude. Equation 2, 3 and 4 is the corollary from Equation 1. Therefore, they do not meet with the circumstance of large pendulum amplitude.

Why do the places where the speed is zero correspond to turning points where the displacement is maximum? When the displacement is maximum, the mass is at its highest place, where its speed is changing from towards up to towards down, that is, the speed of raising part is descending until it reaches its highest position and begin to fall. Mathematically, speed is the derivative of displacement over time, when Sine curve comes to its maximum or minimum, its derivative, the Cosine curve should be zero.

Why do places where the speed is maximum correspond to places where both the displacement and acceleration are zero? When the speed is maximum, the mass should be its lowest place, where it is falling to increase the speed and it is going to raise to decrease the speed, that is, the gravitational potential is fully transferred into kinetic energy. So, the displacement is zero. When the displacement is zero, the gravity is perpendicular to the *linear velocity*. Therefore, the gravity offers no acceleration on *linear velocity*'s direction. Then, the *angular acceleration* should be zero.

We observe double peaks with a valley in between in the acceleration curve. What does this correspond to in the velocity curve? When the gravity is perpendicular to the rod, the gravity is parallel to the *linear speed* and gives the largest *linear acceleration*, leading to the largest *angular acceleration*. However, under the circumstance of large amplitude pendulum, the rod may raise over 90°. Then, the gravity will perpendicular first and then cross as a acute angle, making the *sub-Force* acting on the direction of the *linear speed* descend. Therefore, the *linear acceleration* descend and so the *angular acceleration*.

5.2.3 Quantitative Experiment

Why is the expansion in Equation 5 including only first two terms enough to match the data well for angles which are less than 45° (about 0.8rad)? We can use $\frac{T_{\text{est}}-T_{\text{cal}}}{T_{\text{cal}}}$ to represent the error percentage between theoretical result and actual result. While the second term contributes $\frac{(\frac{1}{2})^2\sin^2(\frac{45^\circ}{2})}{1+(\frac{1}{2})^2\sin^2(\frac{45^\circ}{2})}\times 100\% \approx 4\%$, the third term contributes only $\frac{(\frac{3\times 1}{2^2\times 2\times 1})^2\sin^4(\frac{45^\circ}{2})}{1+(\frac{1}{2})^2\sin^2(\frac{45^\circ}{2})+(\frac{3\times 1}{2^2\times 2\times 1})^2\sin^4(\frac{45^\circ}{2})}\times 100\% \approx 0.302\%$. We can consider that the error cause from third term and so on are less crucial than the estimated error caused by equipment.

Why does the approximation begin to fail for angles above 90° (about 1.57rad) even when we include first five terms in the approximation? Similarly to the last question, consider the contribution of error diminution of the sixth term. It contributes $\frac{(\frac{63}{256})^2\sin^{10}(\frac{90^\circ}{2})}{1+(\frac{1}{2})^2\sin^2(\frac{90^\circ}{2})+(\frac{3}{8})^2\sin^4(\frac{90^\circ}{2})+(\frac{15}{48})^2\sin^6(\frac{90^\circ}{2})+(\frac{105}{384})^2\sin^8(\frac{90^\circ}{2})+(\frac{63}{256})^2\sin^{10}(\frac{90^\circ}{2})} \times 100\% > 1\%$. It exceeds our expected error 1%.

How many terms would be required for the approximation to give a good value for the case where the amplitude equals 3.0 rad? Since we do not collect the actually experimental data for 3.0rad, we use the difference of consecutive two terms to represent its accuracy. Using a Python program to calculate, then we find with the consecutive two terms different in 0.00001 we need at least 152 terms.

Assume the *Simple Harmonic Oscillation* period given by Equation 1 was applied to all amplitudes. How large is the error at an amplitude of 20° and 40°? We can take a careful observation on Figure 7. With amplitude$=20^\circ$, the error is about 8.6%, while it is about 6.5% when the amplitude goes 40°.

6 Conclusion

By conducting the experiments, we finally find out the the function curve of displacement, angular velocity and angular acceleration under small amplitude pendulum. Then, we also point out that these curves do not meet with the fact when it is under the circumstance of large amplitude pendulum. Therefore, we introduce the infinite series to manage to describe the fact more precisely. To examine its effect, we compare the actual data with these two kinds of theoretical data. The result is satisfied. While the calculated from Equation 1 lead to a big error, the new method, the infinite series matches the experimental data with small error only with first several terms. Therefore, we are now clear the mathematical model of both small and large amplitude pendulum.

Appendix

Listing 1: calculate.py

```
import math  # import math to support sine
import numpy as np  # import numpy to get more precise


# calculate double factorial
def doubleFactorial(n: np.longdouble):
    if n == 1:
        return 1
    else:
        return n*doubleFactorial(n-np.longdouble(2))


# the ith term of infinite series
def term(i: np.longdouble, rad=np.longdouble(3)):
    return (np.longdouble(doubleFactorial(2*i-1))/(np.longdouble(2)**i*
        np.longdouble(math.factorial(i))))**np.longdouble(2)*np.
        longdouble(math.sin(rad/np.longdouble(2))**(np.longdouble(2)*i)
        )


def main():
    count = 2
    error = np.longdouble(1.0)
    while (np.abs(error) > 0.00001):
        error = np.longdouble(np.longdouble(term(count)) -
                              np.longdouble(term(count-1)))
        count += 1
    print(count)
```

```


main()
```