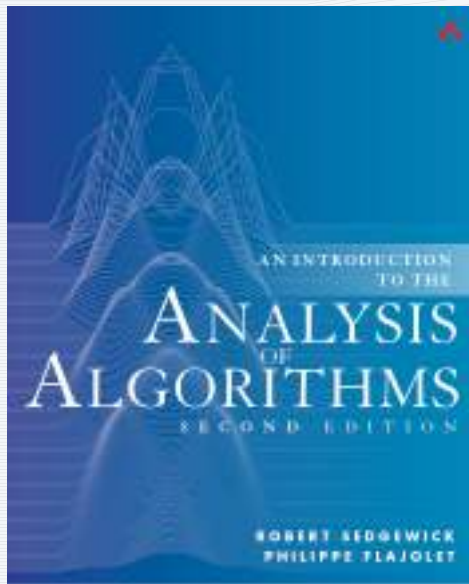


# ANALYTIC COMBINATORICS

## PART ONE

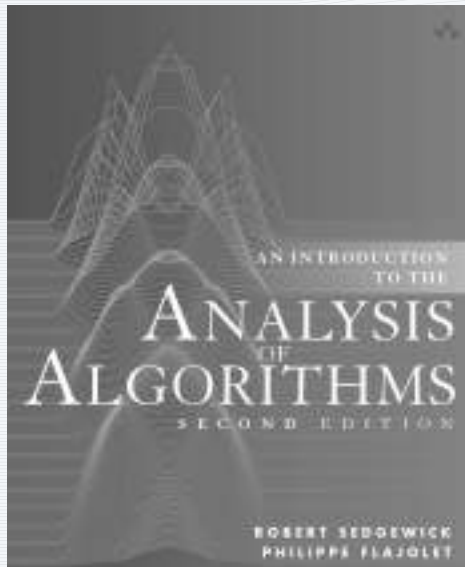


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## 4. Asymptotic Approximations

# ANALYTIC COMBINATORICS

## PART ONE



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## 4. Asymptotic Approximations

- Standard scale
- Manipulating expansions
- Asymptotics of finite sums
- Bivariate asymptotics

## Asymptotic approximations

---

Goal: Develop *accurate* and *concise* estimates of quantities of interest

$$O(\log N)$$

✗ *not accurate*

$$H_N = \sum_{0 \leq k \leq N} \frac{1}{k}$$

✗ *not concise*

$$\ln N + \gamma + O\left(\frac{1}{N}\right)$$

✓

Informal definition of *concise*:

“easy to compute with constants and standard functions”

## Notation (revisited)

---

“Big-Oh” notation for upper bounds

$$g(N) = O(f(N)) \text{ iff } |g(N)/f(N)| \text{ is bounded from above as } N \rightarrow \infty$$

“Little-oh” notation for lower bounds

$$g(N) = o(f(N)) \text{ iff } g(N)/f(N) \rightarrow 0 \text{ as } N \rightarrow \infty$$

“Tilde” notation for asymptotic equivalence

$$g(N) \sim f(N) \text{ iff } g(N)/f(N) \rightarrow 1 \text{ as } N \rightarrow \infty$$

## Notation for approximations

---

“Big-Oh” approximation

$$g(N) = f(N) + O(h(N))$$

Error will be at most **within a constant factor** of  $h(N)$  as  $N$  increases.

“Little-oh” approximation

$$g(N) = f(N) + o(h(N))$$

Error will **decrease** relative to  $h(N)$  as  $N$  increases.

“Tilde” approximation

$$g(N) \sim f(N)$$

Weakest nontrivial o-approximation.

## Standard asymptotic scale

---

**Definition.** A decreasing series  $g_k(N)$  with  $g_{k+1}(N) = o(g_k(N))$  is called an *asymptotic scale*. The series

$$f(N) \sim c_0 g_0(N) + c_1 g_1(N) + c_2 g_2(N) + \dots$$

is called an *asymptotic expansion* of  $f$ . The expansion represents the collection of formulae

$$f(N) = O(g_0(N))$$

$$f(N) = c_0 g_0(N) + O(g_1(N))$$

$$f(N) = c_0 g_0(N) + c_1 g_1(N) + O(g_2(N))$$

$$f(N) = c_0 g_0(N) + c_1 g_1(N) + c_2 g_2(N) + O(g_3(N))$$

$$\vdots$$

The **standard scale** is products of powers of  $N$ ,  $\log N$ , iterated logs and exponentials.

Typically, we:

- use only 2, 3, or 4 terms (continuing until unused terms are extremely small)
- use  $\sim$ -notation to *drop* information on unused terms.
- use  $O$ -notation or  $o$ -notation to *specify* information on unused terms.

Methods extend in principle to any desired precision.

## Example: Asymptotics of linear recurrences

**Theorem.** Assume that a rational GF  $f(z)/g(z)$  with  $f(z)$  and  $g(z)$  relatively prime and  $g(0) \neq 0$  has a unique pole  $1/\beta$  of smallest modulus and that the multiplicity of  $\beta$  is  $\nu$ . Then

$$[z^n] \frac{f(z)}{g(z)} \sim C \beta^n n^{\nu-1} \quad \text{where} \quad C = \nu \frac{(-\beta)^\nu f(1/\beta)}{g^{(\nu)}(1/\beta)}$$

Proof sketch

$$\sum_{0 \leq j < m_1} c_{1j} n^j \beta_1^n + \sum_{0 \leq j < m_2} c_{2j} n^j \beta_2^n + \dots + \sum_{0 \leq j < m_r} c_{rj} n^j \beta_r^n$$

Largest term dominates.

Ex.	$N$	$3^N$	$3^N + 2^N$
	7	2187	275
	8	6561	17811
	9	19683	20195
	10	59049	60073
	11	177147	179195

Notes:

- Pole of smallest modulus usually dominates.
- Easy to extend to cover multiple poles in neighborhood of pole of smallest modulus.

Example 1.

$$3^N + 2^N \sim 3^N$$



Example 2.

$$2^N + 1.99999^N \sim 2^N$$



## Asymptotics of linear recurrences

**Theorem.** Assume that a rational GF  $f(z)/g(z)$  with  $f(z)$  and  $g(z)$  relatively prime and  $g(0) \neq 0$  has a unique pole  $1/\beta$  of smallest modulus and that the multiplicity of  $\beta$  is  $\nu$ . Then

$$[z^n] \frac{f(z)}{g(z)} \sim C \beta^n n^{\nu-1} \quad \text{where} \quad C = \nu \frac{(-\beta)^\nu f(1/\beta)}{g^{(\nu)}(1/\beta)}$$

Example from earlier lectures.

$$a_n = 5a_{n-1} - 6a_{n-2} \quad \text{for } n \geq 2 \text{ with } a_0 = 0 \text{ and } a_1 = 1$$

Make recurrence valid for all  $n$ .

$$a_n = 5a_{n-1} - 6a_{n-2} + \delta_{n1}$$

Multiply by  $z^n$  and sum on  $n$ .

$$A(z) = 5zA(z) - 6z^2A(z) + z$$

Solve.

$$A(z) = \frac{z}{1 - 5z + 6z^2}$$

Smallest root of denominator is  $1/3$ .

$$a_n \sim 3^n \quad C = 1 \frac{(-3)(1/3)}{-5 + 12/3} = 1$$



## Fundamental asymptotic expansions

---

are immediate from Taylor's theorem.

exponential	$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4)$
logarithmic	$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)$
binomial	$(1 + x)^k = 1 + kx + \binom{k}{2}x^2 + \binom{k}{3}x^3 + O(x^4)$
geometric	$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + O(x^4)$

as  $x \rightarrow 0$ .

## Fundamental asymptotic expansions

---

are immediate from Taylor's theorem.

Substitute  $x = 1/N$  to get expansions as  $N \rightarrow \infty$ .

exponential	$e^{1/N} = 1 + \frac{1}{N} + \frac{1}{2N^2} + \frac{1}{6N^3} + O\left(\frac{1}{N^4}\right)$
logarithmic	$\ln\left(1 + \frac{1}{N}\right) = \frac{1}{N} - \frac{1}{2N^2} + \frac{1}{3N^3} + O\left(\frac{1}{N^4}\right)$
binomial	$\left(1 + \frac{1}{N}\right)^k = 1 + \frac{k}{N} + \binom{k}{2} \frac{1}{N^2} + \binom{k}{3} \frac{1}{N^3} + O\left(\frac{1}{N^4}\right)$
geometric	$\frac{1}{N-1} = \frac{1}{N} + \frac{1}{N^2} + \frac{1}{N^3} + O\left(\frac{1}{N^4}\right)$

as  $N \rightarrow \infty$ .

## Inclass exercise

---

Develop the following asymptotic approximations

$$\ln\left(1 + \frac{1}{N}\right) + \ln\left(1 - \frac{1}{N}\right) \quad \text{to} \quad O(1/N^3)$$

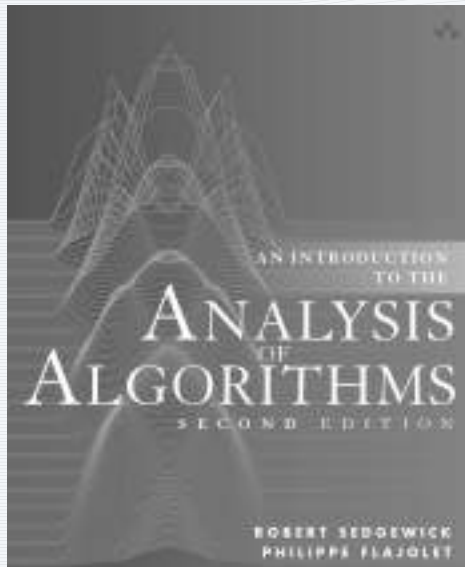
$$\begin{aligned} &= \frac{1}{N} - \frac{1}{2N^2} + O\left(\frac{1}{N^3}\right) - \frac{1}{N} - \frac{1}{2N^2} + O\left(\frac{1}{N^3}\right) \\ &= -\frac{1}{N^2} + O\left(\frac{1}{N^3}\right) \end{aligned}$$

$$\ln\left(1 + \frac{1}{N}\right) - \ln\left(1 - \frac{1}{N}\right) \quad \text{to} \quad O(1/N^3)$$

$$\begin{aligned} &= \frac{1}{N} - \frac{1}{2N^2} + O\left(\frac{1}{N^3}\right) + \frac{1}{N} + \frac{1}{2N^2} + O\left(\frac{1}{N^3}\right) \\ &= \frac{2}{N} + O\left(\frac{1}{N^3}\right) \end{aligned}$$

# ANALYTIC COMBINATORICS

## PART ONE



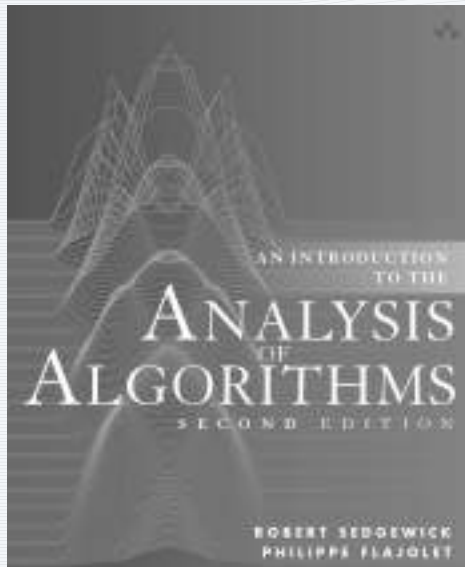
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### 4. Asymptotic Approximations

- Standard scale
- Manipulating expansions
- Asymptotics of finite sums
- Bivariate asymptotics

# ANALYTIC COMBINATORICS

## PART ONE



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### 4. Asymptotic Approximations

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## Manipulating asymptotic expansions

### Goal.

Develop expansion **on the standard scale** for any given expression.

$$\frac{1}{N^2 + N}$$

$$\frac{H_N}{\ln(N+1)}$$

$$e^{H_N}$$

$$\left(1 - \frac{1}{N}\right)^N$$

$$(H_N)^2$$

$$\binom{2N}{N}$$

### Techniques.

- simplification
- substitution
- factoring
- multiplication
- division
- composition
- exp/log

### Why?

Facilitate comparisons of different quantities.  
Simplify computations.

Ex.  $\frac{1}{4^N} \binom{2N}{N}$

$N = 10^6 ??$



## Manipulating asymptotic expansions

---

**Simplification.** An asymptotic series is only as good as its O-term.

Discard smaller terms.

$$\ln N + \gamma + O(1) \quad \times$$

$$\ln N + O(1) \quad \checkmark$$

**Substitution.** Change variables in a known expansion.

Taylor series

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4) \quad \text{as } x \rightarrow 0$$

Substitute  $x = 1/N$

$$\ln\left(1 + \frac{1}{N}\right) = \frac{1}{N} - \frac{1}{2N^2} + \frac{1}{3N^3} + O\left(\frac{1}{N^4}\right) \quad \text{as } N \rightarrow \infty$$

## Manipulating asymptotic expansions

---

**Factoring.** Estimate the leading term, factor it out, expand the rest.

$$\frac{1}{N^2 + N}$$

Factor out  $1/N^2$ .

$$= \frac{1}{N^2} \frac{1}{1 + 1/N}$$

Expand the rest.

$$= \frac{1}{N^2} \left( 1 - \frac{1}{N} + O\left(\frac{1}{N^2}\right) \right)$$

Distribute.

$$= \frac{1}{N^2} - \frac{1}{N^3} + O\left(\frac{1}{N^4}\right)$$



## Manipulating asymptotic expansions

**Multiplication.** Do term-by-term multiplication, simplify, collect terms.

Ex.

$$(H_N)^2 = \left( \ln N + \gamma + O\left(\frac{1}{N}\right) \right) \left( \ln N + \gamma + O\left(\frac{1}{N}\right) \right)$$

Term-by-term  
multiplication.

$$= \left( (\ln N)^2 + \gamma \ln N + O\left(\frac{\log N}{N}\right) \right)$$

$$+ \left( \gamma \ln N + \gamma^2 + O\left(\frac{1}{N}\right) \right)$$

$$+ \left( O\left(\frac{\log N}{N}\right) + O\left(\frac{1}{N}\right) + O\left(\frac{1}{N^2}\right) \right)$$

slight improvement  
in precision

big improvement  
in precision

Collect terms.

$$= (\ln N)^2 + 2\gamma \ln N + \gamma^2 + O\left(\frac{\log N}{N}\right)$$

	$(H_N)^2$	$(\ln N)^2$	$+2\gamma \ln N$	$+\gamma^2$
1000	56.032	47.717	55.692	56.025
10000	95.797	84.830	95.463	95.796
100000	146.172	132.547	145.838	146.172

May need trial-and-error to get desired precision.



## Manipulating asymptotic expansions

**Division.** Expand, factor denominator, expand  $1/(1-x)$ , multiply.

Ex.

$$\frac{H_N}{\ln(N+1)}$$

Expand.

$$= \frac{\ln N + \gamma + O\left(\frac{1}{N}\right)}{\ln N + O\left(\frac{1}{N}\right)}$$

Factor denominator.

$$= \frac{1 + \frac{\gamma}{\ln N} + O\left(\frac{1}{N}\right)}{1 + O\left(\frac{1}{N}\right)}$$

OK to simplify by replacing  $O(1/N \log N)$  by  $O(1/N)$

Expand  $1/(1-x)$ .

$$= \left(1 + \frac{\gamma}{\ln N} + O\left(\frac{1}{N}\right)\right) \left(1 + O\left(\frac{1}{N}\right)\right)$$

Multiply.

$$= 1 + \frac{\gamma}{\ln N} + O\left(\frac{1}{N}\right)$$

## Manipulating asymptotic expansions

**Composition.** Substitute an expansion.

$$e^{H_N}$$

Substitute  $H_N$  expansion.

$$= e^{\ln N + \gamma + O(1/N)}$$

$$= Ne^\gamma e^{O(1/N)}$$

**Lemma.**

$$e^{O(1/N)} = 1 + O\left(\frac{1}{N}\right)$$

Expand  $e^x$ .

$$= Ne^\gamma \left(1 + O\left(\frac{1}{N}\right) + O\left(O\left(\frac{1}{N}\right)^2\right)\right)$$

Simplify.

$$= Ne^\gamma \left(1 + O\left(\frac{1}{N}\right)\right)$$

Distribute.

$$= Ne^\gamma + O(1)$$

big improvement  
in precision

$N$	$e^{H_N}$	$Ne^\gamma$
1000	1782	1781
10000	17812	17811
100000	178108	178107
1000000	1781073	1781072



## Manipulating asymptotic expansions

**Exp/log.** Start by writing  $f(x) = \exp(\ln(f(x)))$ .

$$\left(1 - \frac{1}{N}\right)^N$$

**Exp/log.**

$$= \exp\left\{\ln\left(\left(1 - \frac{1}{N}\right)^N\right)\right\}$$

$$= \exp\left\{N \ln\left(1 - \frac{1}{N}\right)\right\}$$

**Expand  $\ln(1+x)$**

$$= \exp\left\{N\left(-\frac{1}{N} + O\left(\frac{1}{N^2}\right)\right)\right\}$$

**Distribute.**

$$= \exp\left\{-1 + O\left(\frac{1}{N}\right)\right\}$$

$$= 1/e + O\left(\frac{1}{N}\right)$$

big improvement  
in precision

Q. How would you  
compute values  
for large N?

**Lemma.**

$$e^{O(1/N)} = 1 + O\left(\frac{1}{N}\right)$$

$N$	$\left(1 - \frac{1}{N}\right)^N$	$1/e$
10000	0.367861	0.367879
100000	0.367878	0.367879
1000000	0.367879	0.367879



## Inclass exercises

---

Develop asymptotic approximations for

$$\ln(N-2) \quad \text{to} \quad O(1/N^2)$$

$$= \ln N + \ln\left(1 - \frac{2}{N}\right) \quad \text{Factor out } \ln N.$$

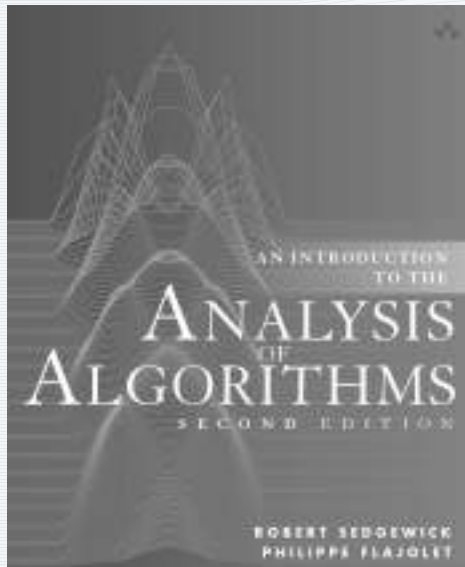
$$= \ln N - \frac{2}{N} + O\left(\frac{1}{N^2}\right) \quad \text{Expand the rest.}$$

$$(H_N)^2 \quad \text{to} \quad O(1/N)$$

$$\begin{aligned} (H_N)^2 &= \left(\ln N + \gamma + \frac{1}{2N} + O\left(\frac{1}{N^2}\right)\right) \left(\ln N + \gamma + \frac{1}{2N} + O\left(\frac{1}{N^2}\right)\right) \\ &= (\ln N)^2 + 2\gamma \ln N + \gamma^2 + \frac{\ln N}{N} + O\left(\frac{1}{N}\right) \end{aligned}$$

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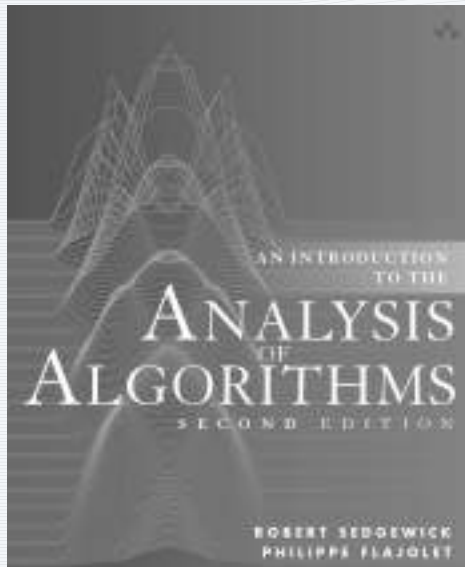
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## 4. Asymptotic Approximations

- Standard scale
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## Asymptotics of finite sums

**Bounding the tail.** Make a **rapidly decreasing** sum infinite.

$$N! \sum_{0 \leq k \leq N} \frac{(-1)^k}{k!} = N!e^{-1} - R_N \quad \text{where} \quad R_N = N! \sum_{k > N} \frac{(-1)^k}{k!}$$
$$= \frac{N!}{e} + O\left(\frac{1}{N}\right)$$
$$|R_N| < \frac{1}{N+1} + \frac{1}{(N+1)^2} + \frac{1}{(N+1)^3} + \dots = \frac{1}{N}$$

**Using the tail.** The last term of a **rapidly increasing** sum may dominate.

$$\sum_{0 \leq k \leq N} k! = N! \left( 1 + \frac{1}{N} + \sum_{0 \leq k \leq N-2} \frac{k!}{N!} \right) = N! \left( 1 + O\left(\frac{1}{N}\right) \right)$$

$N-1$  terms, each less than  $1/(N(N-1))$

**Approximating with an integral.**

$$H_N = \sum_{1 \leq k \leq N} \frac{1}{k} \sim \int_1^N \frac{1}{x} dx = \ln N$$
$$\ln N! = \sum_{1 \leq k \leq N} \ln k \sim \int_1^N \ln x dx = N \ln N - N + 1$$

see text for proofs;  
stay tuned for  
better approximations



## Euler-Maclaurin Summation

---

is a classic formula for estimating sums with integrals.

**Theorem.** (Euler-Maclaurin summation). Let  $f$  be a function defined on  $[1, \infty)$  whose derivatives exist and are absolutely integrable. Then

$$\sum_{1 \leq k \leq N} f(k) = \int_1^N f(x) dx + \frac{1}{2}f(N) + C_f + \frac{1}{12}f'(N) - \frac{1}{720}f'''(N) + \dots$$

Asymptotic series diverges; need to check bound on last term (see text for many details).  
BUT this form is useful for many applications.

Classic example 1. 
$$H_N = \ln N + \gamma + \frac{1}{2N} - \frac{1}{12N^2} + O\left(\frac{1}{N^4}\right)$$

Classic example 2. 
$$\ln N! = N \ln N - N + \ln \sqrt{2\pi N} + \frac{1}{12N} + O\left(\frac{1}{N^3}\right)$$

## Inclass exercise

---

Given Stirling's approximation  $\ln N! = N \ln N - N + \ln \sqrt{2\pi N} + O(\frac{1}{N})$

Develop an asymptotic approximation for  $\binom{2N}{N}$  to  $O(1/N)$  (relative error)

$$\begin{aligned}\binom{2N}{N} &= \exp(\ln(2N!) - 2 \ln N!) \\ &= \exp(2N \ln(2N) - 2N + \ln \sqrt{4\pi N} + O(1/N) \\ &\quad - 2(N \ln(N) - N + \ln \sqrt{2\pi N} + O(1/N))) \\ &= \exp(2N \ln 2 - \ln \sqrt{\pi N} + O(1/N)) \\ &= \frac{4^N}{\sqrt{\pi N}} (1 + O(\frac{1}{N}))\end{aligned}$$

$$\begin{aligned}\ln \sqrt{4\pi N} - 2 \ln \sqrt{2\pi N} &= \ln 2 - 2 \ln \sqrt{2} - \ln \sqrt{\pi N} \\ &= -\ln \sqrt{\pi N}\end{aligned}$$

$$\text{Ex. } \frac{1}{4^N} \binom{2N}{N} \sim \frac{1}{\sqrt{\pi N}}$$

## Asymptotics of the Catalan numbers: an application

Q. How many bits to represent a binary tree with  $N$  internal nodes?

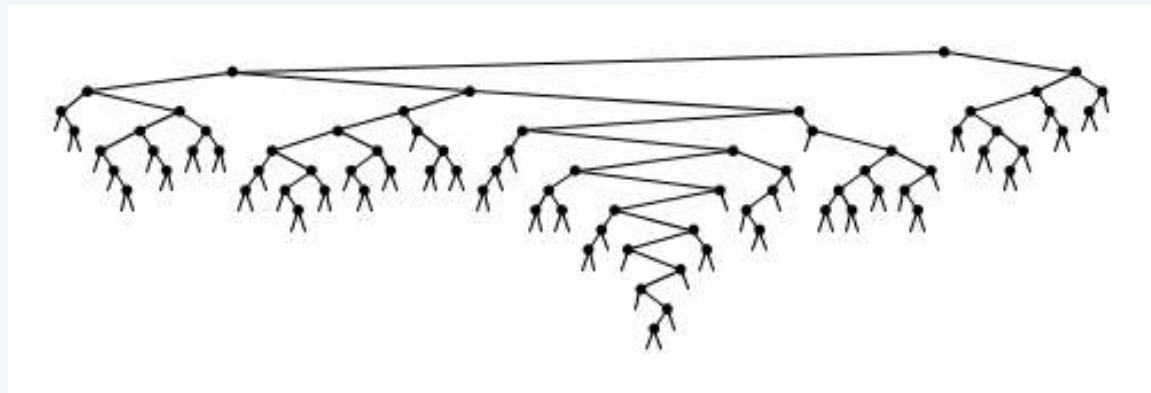
A. At least  $\lg \frac{1}{N+1} \binom{2N}{N}$  *✗ not concise*

$N = 10^6$  ??



A. At least  $\sim \lg \frac{4^N}{\sqrt{\pi N^3}} \sim 2N - 1.5 \lg N$  *✓*

Note: Can do it with  $2N$  bits

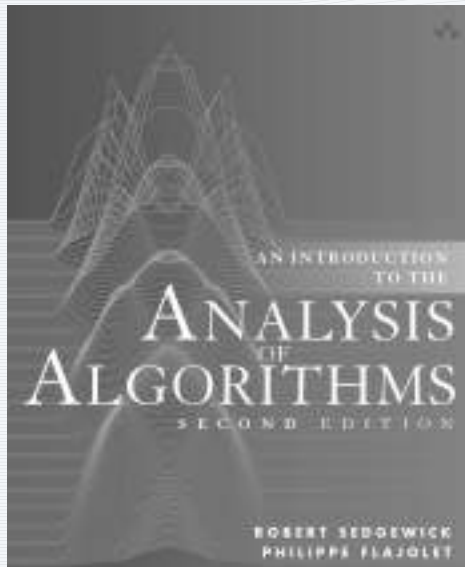


preorder traversal  
0 for internal nodes  
1 for external nodes

0000101100010101101011001101100000111001011011001011011010011011000001111000011  
0110000111001001001111011000101111010000110110110010110000110011001110101100111

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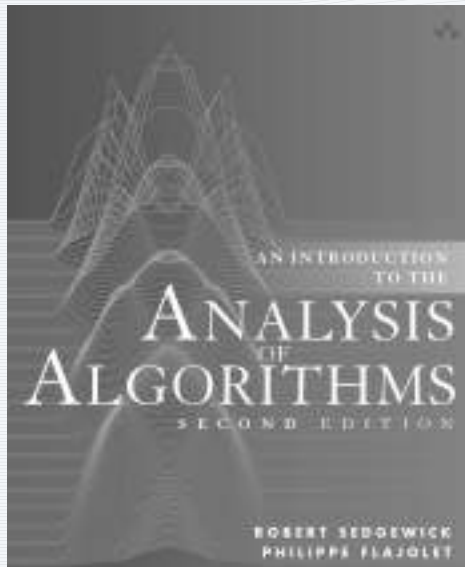
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## Bivariate asymptotics

---

is often required to analyze functions of **two** variables.

Ex. applications in analysis of algorithms involve

- $N$  (size)
- $k$  (cost)

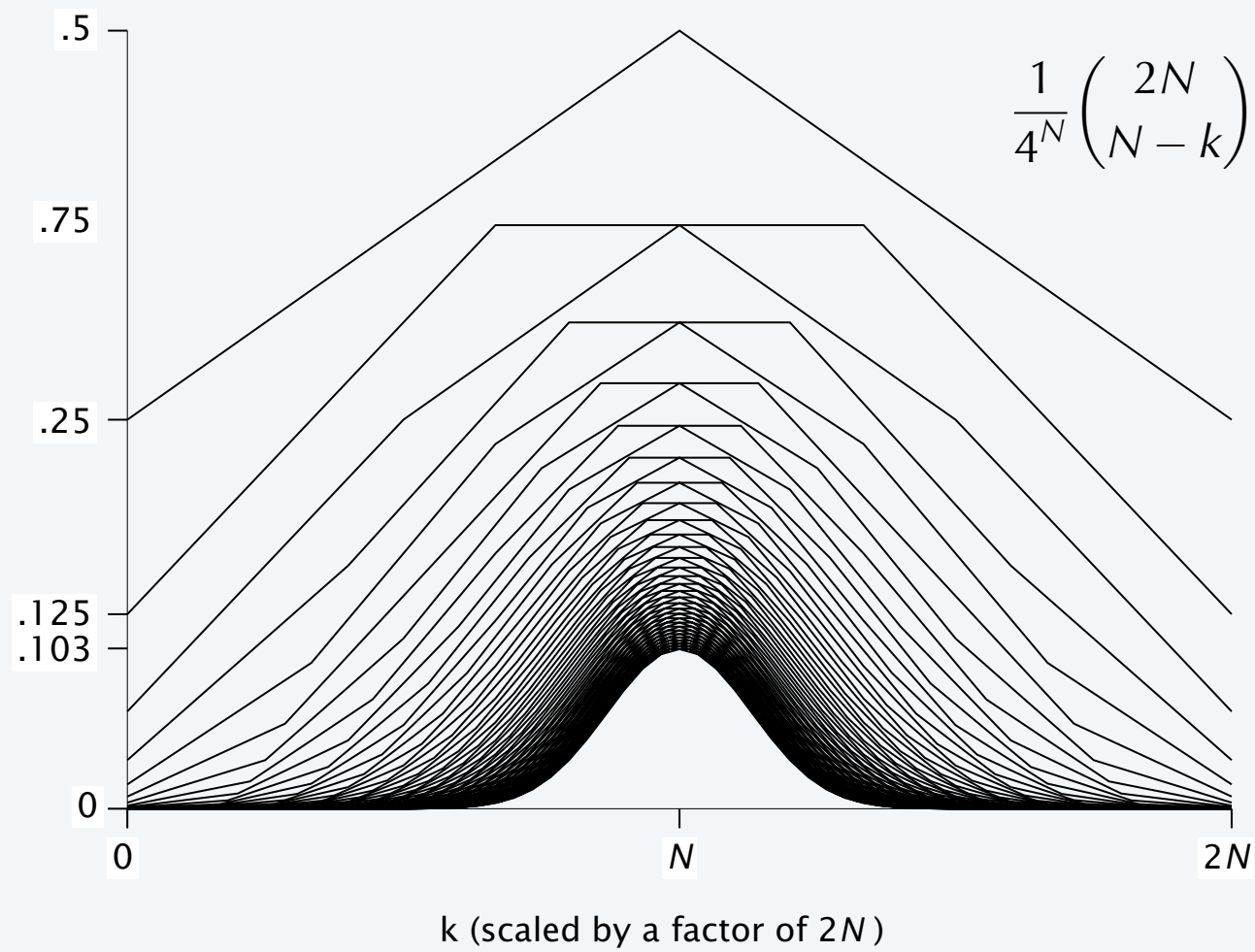
Challenges:

- asymptotics depends on **relative** values of variables
- may need to approximate **sums** over whole range of relative values.

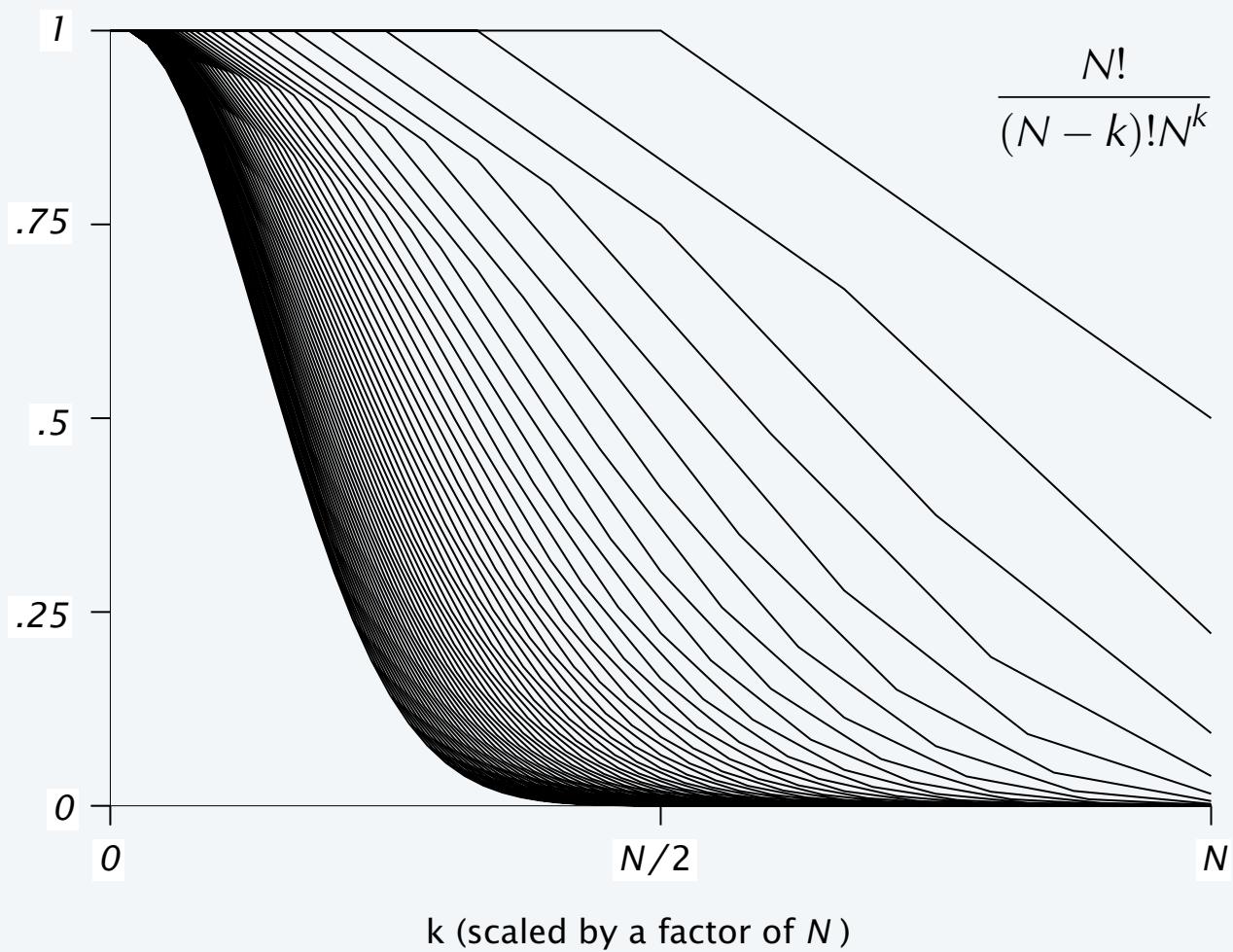
Example 1: Binomial distribution	$\frac{1}{4^N} \binom{2N}{N-k}$	$\sim \frac{1}{\sqrt{\pi N}}$ for $k = 0$ exponentially small for $k$ close to $N$
----------------------------------	---------------------------------	---

Example 2: Ramanujan Q-distribution	$\frac{N!}{(N-k)!N^k}$	1 for $k = 0$ exponentially small for $k$ close to $N$
-------------------------------------	------------------------	---

## Binomial distribution



## Ramanujan Q-distribution





## Ramanujan Q-distribution

$$\frac{N!}{(N-k)!N^k} = \exp(\ln N! - \ln(N-k)! - k \ln N)$$

$$\ln N! = (N + \frac{1}{2}) \ln N - N + \ln \sqrt{2\pi} + O(\frac{1}{N})$$

Stirling's approximation.

$$= \exp\left((N + \frac{1}{2}) \ln N - N + \ln \sqrt{2\pi} - (N-k + \frac{1}{2}) \ln(N-k) + N-k - \ln \sqrt{2\pi} - k \ln N + O(\frac{1}{N})\right)$$

Collect terms

$$= \exp\left(-(N-k + \frac{1}{2}) \ln(1 - \frac{k}{N}) - k + O(\frac{1}{N})\right)$$

$$\ln(1 - \frac{k}{N}) = -\frac{k}{N} - \frac{k^2}{2N^2} + O(\frac{k^3}{N^3})$$

Expand  $\ln(1 - k/N)$

$$= \exp\left(k + \frac{k^2}{2N} - \frac{k^2}{N} - k + O(\frac{k^3}{N^2}) + O(\frac{k}{N})\right)$$

Simplify.

$$= e^{-k^2/2N} \left(1 + O(\frac{k^3}{N^2}) + O(\frac{k}{N})\right)$$

$k$	$k/N$	$k^3/N^2$
$N^{2/5}$	$1/N^{3/5}$	$1/N^{4/5}$
$N^{1/2}$	$1/N^{1/2}$	$1/N^{1/2}$
$N^{3/5}$	$1/N^{2/5}$	$1/N^{1/5}$

## Normal approximation to the binomial distribution

$$\binom{2N}{N-k} = \exp(\ln(2N!) - \ln(N-k)! - \ln(N+k)!)$$

$$\ln N! = (N + \frac{1}{2}) \ln N - N + \ln \sqrt{2\pi} + O(\frac{1}{N})$$

Stirling's  
approximation.

$$\begin{aligned} &= \exp\left((2N + \frac{1}{2}) \ln(2N) - 2N + \ln \sqrt{2\pi} + O(1/N)\right. \\ &\quad - (N - k + \frac{1}{2}) \ln(N - k) - N + k - \ln \sqrt{2\pi} + O(1/N) \\ &\quad \left. - (N + k + \frac{1}{2}) \ln(N + k) - N - k - \ln \sqrt{2\pi} + O(1/N)\right) \end{aligned}$$

Collect terms

$$= \exp\left((2N) \ln 2 - \ln \sqrt{\pi N} - (N - k + \frac{1}{2}) \ln(1 - \frac{k}{N}) - (N + k + \frac{1}{2}) \ln(1 + \frac{k}{N}) + O(1/N)\right)$$

Rearrange terms

$$\begin{aligned} &= \exp\left(2N \ln 2 - \ln \sqrt{\pi N}\right. \\ &\quad \left. - (N + \frac{1}{2})(\ln(1 - \frac{k}{N}) + \ln(1 + \frac{k}{N})) + k(\ln(1 - \frac{k}{N}) - \ln(1 + \frac{k}{N})) + O(1/N)\right) \end{aligned}$$

Expand  $\ln(1 - k/N)$   
and  $\ln(1 + k/N)$ .

$$= \exp\left((2N) \ln 2 - \ln \sqrt{\pi N} - \frac{k^2}{N} + O(\frac{k^4}{N^3}) + O(\frac{1}{N})\right)$$

$$\begin{aligned} \ln(1 - \frac{k}{N}) + \ln(1 + \frac{k}{N}) &= -\frac{k^2}{N^2} + O(\frac{k^3}{N^3}) \\ \ln(1 - \frac{k}{N}) - \ln(1 + \frac{k}{N}) &= -\frac{2k}{N} + O(\frac{k^3}{N^3}) \end{aligned}$$

$$\frac{1}{4^N} \binom{2N}{N-k} = \frac{e^{-k^2/N}}{\sqrt{\pi N}} \left(1 + O(\frac{k^4}{N^3}) + O(\frac{1}{N})\right)$$

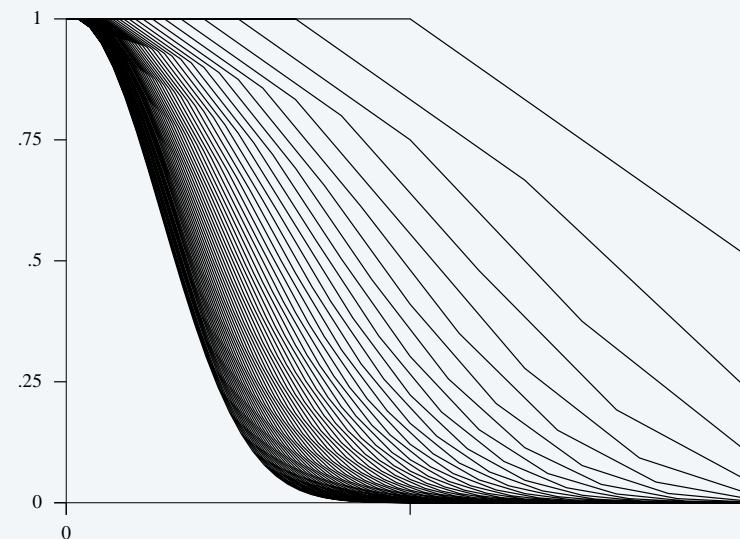
## Fundamental bivariate approximations

		uniform	central
normal	$\binom{2N}{N-k}$	$\frac{e^{-k^2/N}}{\sqrt{\pi N}} + O(\frac{1}{N^{3/2}})$	$\frac{e^{-k^2/N}}{\sqrt{\pi N}} \left(1 + O(\frac{1}{N}) + O(\frac{k^4}{N^3})\right)$
Poisson	$\binom{N}{k} \left(\frac{\lambda}{N}\right)^k \left(1 - \frac{\lambda}{N}\right)^{N-k}$	$\frac{\lambda^k e^{-\lambda}}{k!} + o(1)$	$\frac{\lambda^k e^{-\lambda}}{k!} \left(1 + O(\frac{1}{N}) + O(\frac{k}{N})\right)$
Q	$\frac{N!}{(N-k)!N^k}$	$e^{-k^2/(2N)} + O(\frac{1}{\sqrt{N}})$	$e^{-k^2/(2N)} \left(1 + O(\frac{k}{N}) + O(\frac{k^3}{N^2})\right)$

## Next challenge: Approximating sums via bivariate asymptotics

Example: Ramanujan Q-function

$$Q(N) \equiv \sum_{1 \leq k \leq N} \frac{N!}{(N-k)!N^k}$$



What is the area under this curve?

Observations:

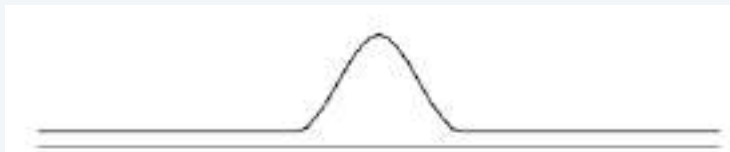
- nearly 1 for small k
- negligible for large k
- bivariate asymptotics needed to give different estimates in different ranges.

## Laplace method

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To approximate a sum:

- Restrict the range to an area that contains the largest summands.
- Approximate the summand.
- Extend the range by bounding the tails to get a simpler sum.
- Approximate the new sum with an integral.



## Laplace method for Ramanujan Q-function

$$Q(N) \equiv \sum_{1 \leq k \leq N} \frac{N!}{(N-k)!N^k}$$

Restrict the range to an area that contains the largest summands.

$$Q(N) = \sum_{1 \leq k \leq k_0} \frac{N!}{(N-k)!N^k} + \sum_{k_0 < k \leq N} \frac{N!}{(N-k)!N^k}$$

Take  $k_0 = o(N^{2/3})$  to make tail exponentially small.

Approximate the summand.

$$\sum_{1 \leq k \leq k_0} \frac{N!}{(N-k)!N^k} \sim \sum_{1 \leq k \leq k_0} e^{-k^2/2N}$$

Q-distribution approximation.

Extend the range by bounding the tails to get a simpler sum.

$$Q(N) \sim \sum_{k \geq 1} e^{-k^2/2N}$$

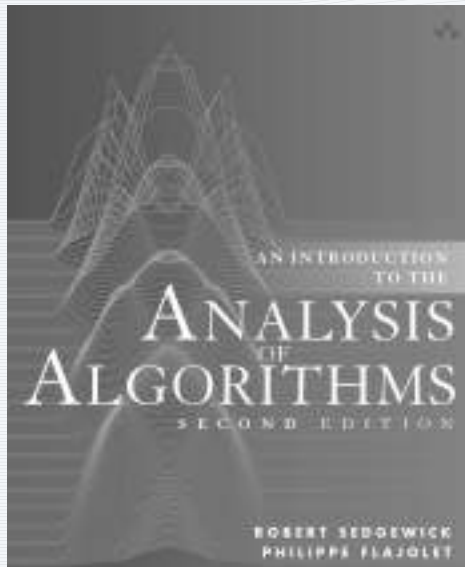
Tail is also exponentially small for this sum.

Approximate the new sum with an integral.

$$Q(N) \sim \sqrt{N} \int_0^\infty e^{-x^2/2} dx = \sqrt{\pi N/2}$$

# ANALYTIC COMBINATORICS

## PART ONE



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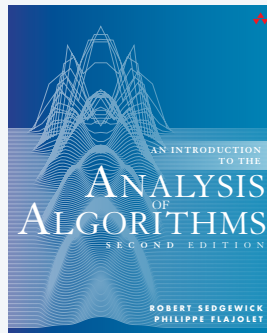
### 4. Asymptotic Approximations

- Standard scale
- Manipulating expansions
- Asymptotics of finite sums
- **Bivariate asymptotics**

## Exercise 4.9

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How small is "exponentially small"?



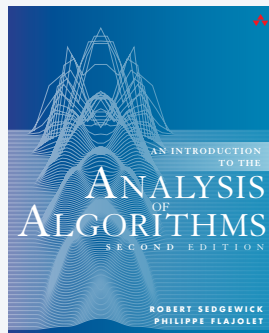
**Exercise 4.9** If  $\alpha < \beta$ , show that  $\alpha^N$  is exponentially small relative to  $\beta^N$ . For  $\beta = 1.2$  and  $\alpha = 1.1$ , find the absolute and relative errors when  $\alpha^N + \beta^N$  is approximated by  $\beta^N$ , for  $N = 10$  and  $N = 100$ .



## Exercise 4.71

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Asymptotics of another Ramanujan function.



**Exercise 4.71** Show that

$$P(N) = \sum_{k \geq 0} \frac{(N-k)^k (N-k)!}{N!} = \sqrt{\pi N/2} + O(1)$$

## Assignments for next lecture

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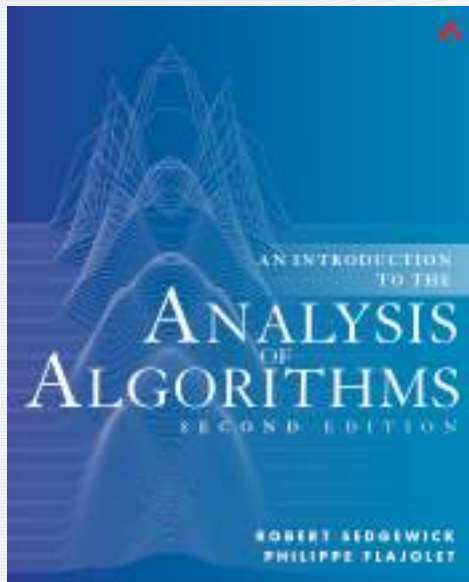
1. Write a program that takes  $N$  and  $k$  from the command line and prints

$$\lg \binom{N}{k}$$

2. Write up solutions to Exercises 4.9 and 4.70.
3. Read pages 149-215 (Asymptotic Approximations) in text.

# ANALYTIC COMBINATORICS

## PART ONE



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## 4. Asymptotic Approximations