Design and Analysis of Algorithms I

Probability Review

Part II

Topics Covered

- Conditional probability
- Independence of events and random variables

See also:

- Lehman-Leighton notes (free PDF)
- Wikibook on Discrete Probability

Concept #1 – Sample Spaces

Sample Space Ω : "all possible outcomes"
[in algorithms, Ω is usually finite]

Also : each outcome $i \in \Omega$ has a probability p(i) >= 0

Constraint : $\sum_{i \in \Omega} p(i) = 1$

An event is a subset $S \subseteq \Omega$

The probability of an event S is $\sum_{i \in S} p(i)$

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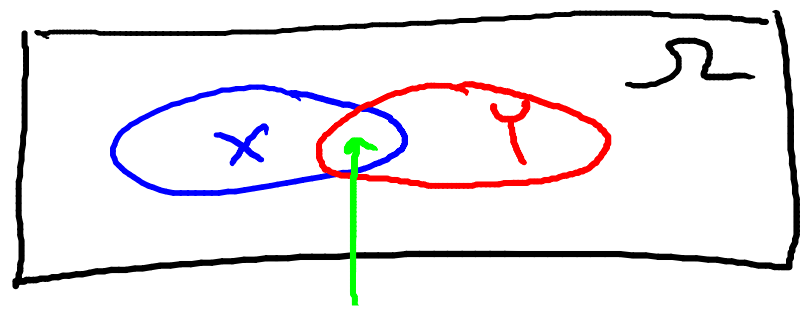
Concept #6 – Conditional Probability

$$Let \;\; X, Y \subseteq \Omega \;\; be \;\; events.$$



$$X \cap Y$$

$$Then \;\; Pr[X|Y] = \frac{Pr[X \cap Y]}{Pr[Y]}$$

("X given Y")

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Suppose you roll two fair dice. What is the probability that at least one die is a 1, given that the sum of the two dice is 7?

- $1/36$
- $1/6$
- $1/3$
- $1/2$

X = at least one die is a 1

Y = sum of two dice = 7

= {(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)}

$$=> X \cap Y = \{(1,6),(6,1)\}$$

$$Pr[X|Y] = \frac{Pr[X \cap Y]}{Pr[Y]} = \frac{(2/36)}{(6/36)} = \frac{1}{3}$$

Concept #7 – Independence (of Events)

Definition : Events $X, Y \subseteq \Omega$ are independent
if (and only if) $Pr[X \cap Y] = Pr[X] \cdot Pr[Y]$

You check : this holds if and only if Pr[X | Y] = Pr[X]
<==> Pr[Y|X] = Pr{Y]

WARNING : can be a very subtle concept.
(intuition is often incorrect!)

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Independence (of Random Variables)

Definition : random variables A, B (both defined on Ω)
are independent if and only if the events Pr[A=1], Pr[B=b] are independent for all a,b. [<==> Pr[A = a and B = b] = Pr[A=z]*Pr[B=b]]

Claim : if A,B are independent, then E[AB] = E[A]*E[B]

Proof :
$$E[AB] = \sum_{a,b} (a \cdot b) \cdot Pr[A = a \ and \ B = b]$$
$$= \sum_{a,b} (a \cdot b) \cdot Pr[A = a] \cdot Pr[B = b] \quad \text{(Since A,B independent)}$$
$$= \underbrace{\boxed{(\sum_a a \cdot Pr[A = a])}}_{E[A]} \underbrace{\boxed{(\sum_b b \cdot Pr[B = b])}}_{E[B]}$$

Q.E.D.

Example

Let $X_1, X_2 \in \{0,1\}$ be random, and $X_3 = X_1 \overset{\text{XOR}}{\oplus} X_2$

<u>formally</u> : $\Omega = \{000, 101, 011, 110\}$, each equally likely.

<u>Claim</u> : X_1 and X_3 are independent random variables (you check)

<u>Claim</u> : X_1X_3 and X_2 are not independent random variables.

<u>Proof</u> : suffices to show that

$$\underbrace{E[X_1X_2X_3]}_{=\,0} \neq \underbrace{E[X_1X_3]}_{=\,E[X1]E[X3]\,=\,1/4}\overbrace{E[X_2]}^{=\,1/2}$$

Since X_1 and X_3 independent