Data Structures and Algorithms

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Session: Convex Hull

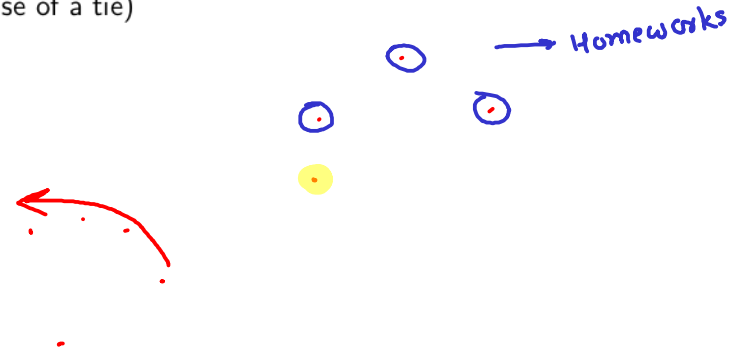
Introduction [1]

- The **convex hull** of a set Q of points, denoted $CH(Q)$, is the smallest convex polygon P for which each point in Q is either on the boundary of P or in its interior.
- Convex hull may be visualized as the shape enclosed by a rubber band stretched around all the points in Q
- Can be computed using a technique called rotational sweep

[1] Chapter 33, CLRS, Third Edition

Computing convex hull: Graham's scan

- Input: set of points Q
- Choose point p_0 as the point with the lowest y-coordinate (or leftmost such point in case of a tie)



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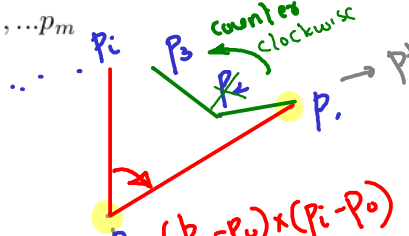
- Sort the remaining points of Q by polar angle relative to p_0 in counterclockwise order, by comparing the cross products. Designate the sorted sequence as $p_0, p_1, \ldots p_m$



p_1 → pt with smallest x coordinate

$(p_1 - p_0) \times (p_2 - p_0)$

$(p_1 - p_0) \times (p_3 - p_0)$

→ larger the magnitude, more is p_i counterclockwise rotated from p_1 w.r.t p_0

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- Push three points p_0, p_1, and p_2 on to the stack S → subset of points which could potentially belong to the CH(Q)

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- If the angle formed by points p_i (initially $i = 3$) and the first 2 points on a stack (in that order) is in the clockwise direction, then pop the topmost element from the stack, else push p_i on the stack

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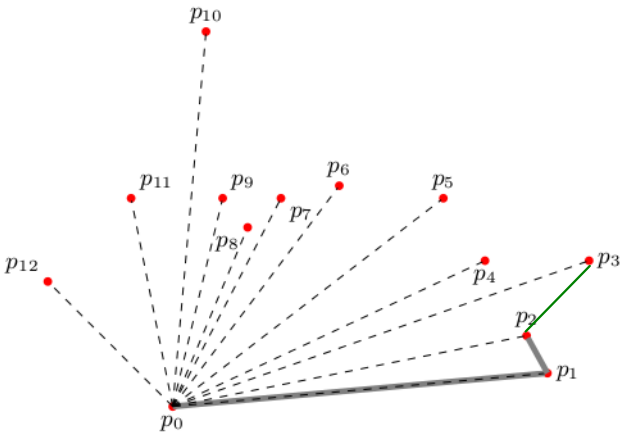
- If the angle formed by points p_i (initially $i = 3$) and the first 2 points on a stack (in that order) is in the clockwise direction, then pop the topmost element from the stack, else push p_i on the stack
- Finally, the stack S contains all the points, from bottom to top, exactly the vertices of $CH(Q)$

Illustration



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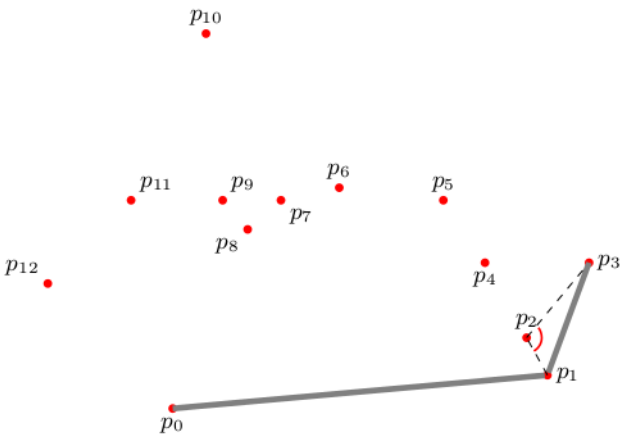

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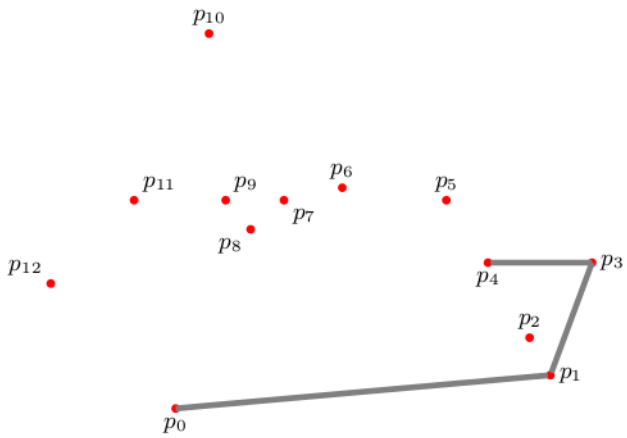

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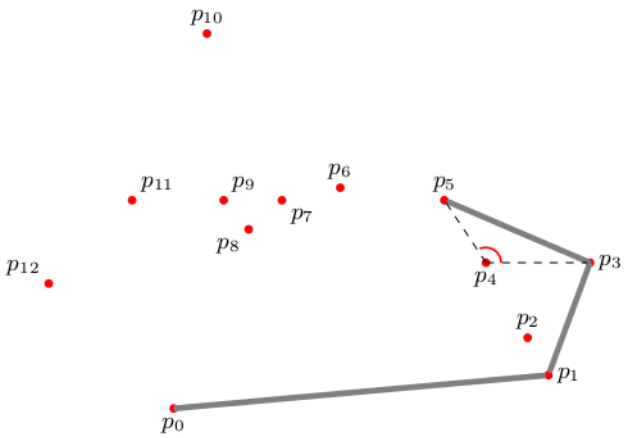

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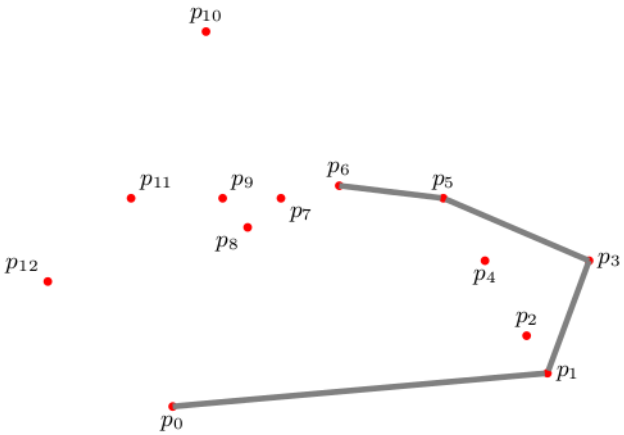

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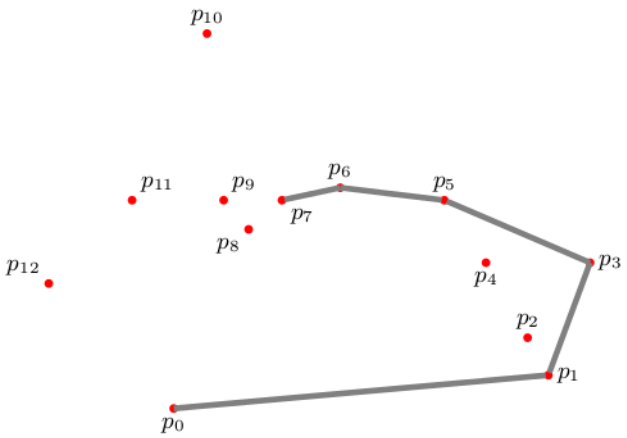

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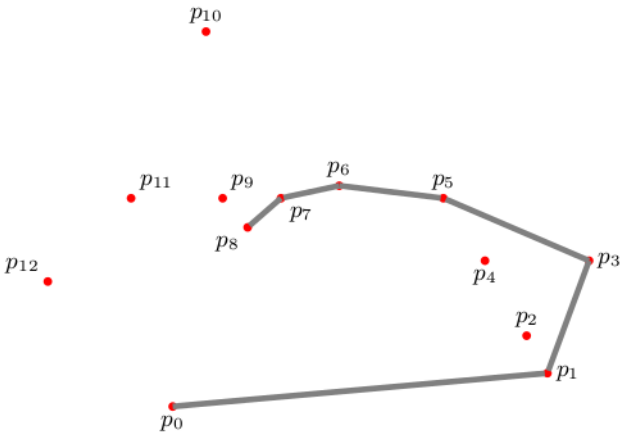

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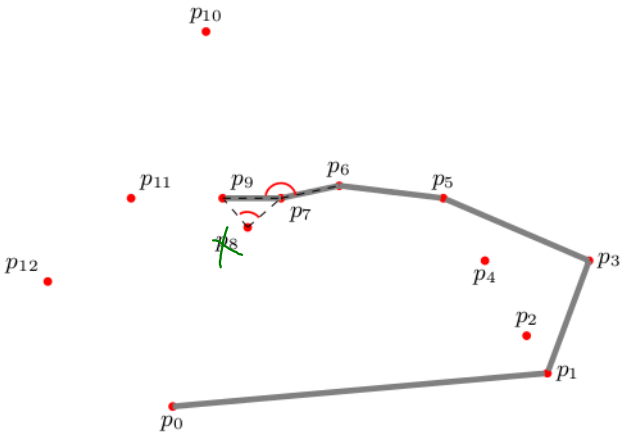

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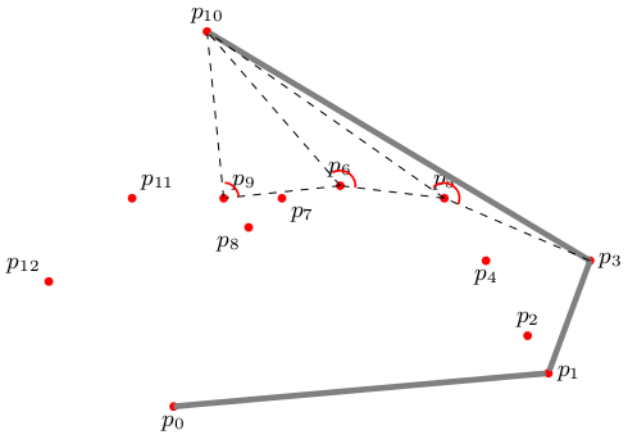

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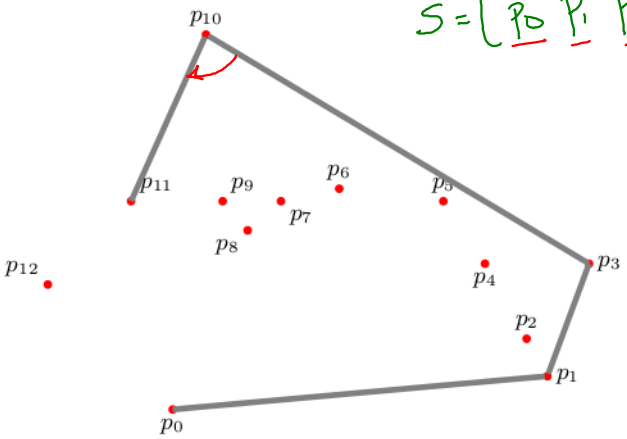
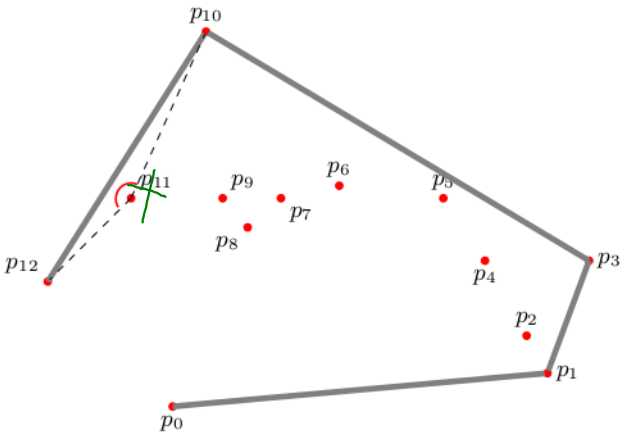


$S = [p_0\ p_1\ p_3\ p_{10}]\ p_{11}$

Illustration



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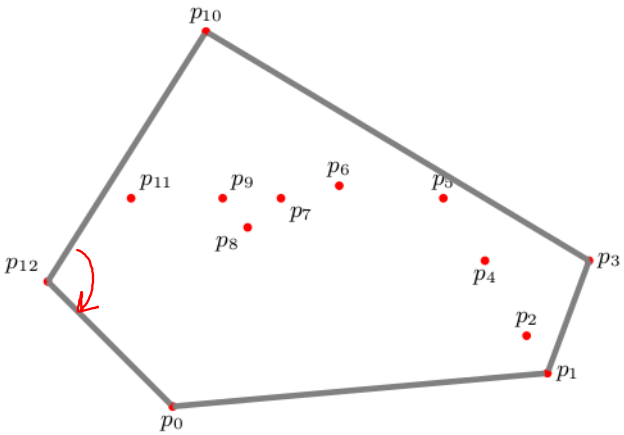
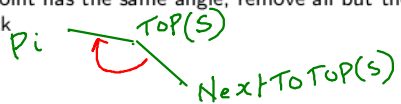


Algorithm for finding Convex Hull

Algorithm GrahamScanAlgorithm(Q)
Let p_0 be the point in Q with the minimum y-coordinate, or leftmost such point in case of tie
let $< p_1, p_2, ..., p_m >$ be the remaining points in Q, sorted by polar angle in counter-clockwise order around p_0 (if more than one point has the same angle, remove all but the one that is farthest from p_0
let S be an empty stack
PUSH(p_0, S)
PUSH(p_1, S)
PUSH(p_2, S)
for $i \in (3...m)$ **do**
  while the angle formed by points $NextToTOP(S), TOP(S)$, and p_i makes a non-left turn **do**
    $POP(S)$
  end while
  $PUSH(p_i, S)$
end for
return S



Until we encounter clockwise NextToTop(s) wrt Top(s)

P_i is oriented counterclockwise from NextToTop(s) wrt Top(s)

Figure: Graham-Scan-Algorithm

Complexity = $O(n \log n)$

Thank you