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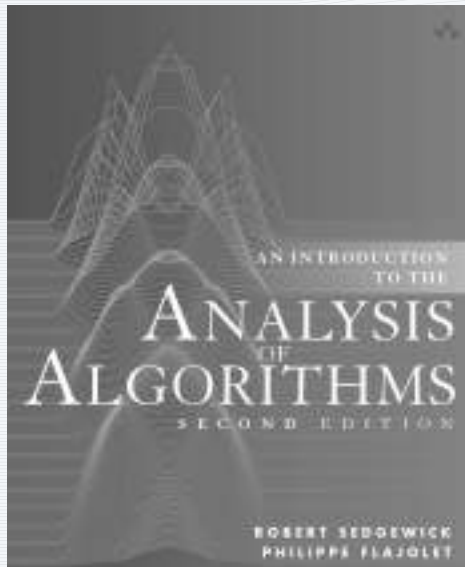# 3. Generating Functions

3. Generating Functions

- **OGFs**
- Solving recurrences
- Catalan numbers
- EGFs
- Counting with GFs

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Ordinary generating functions

Definition.

$$A(z) = \sum_{k \geq 0} a_k z^k \qquad \text{is the ordinary generating function (OGF)}$$

of the sequence $\quad a_0, a_1, a_2, \ldots, a_k, \ldots$

Notation. $[z^N]A(z)$ is "the coefficient of z^N in $A(z)$"

sequence	OGF
1, 1, 1, 1, 1, ...	$\displaystyle\sum_{N \geq 0} z^N = \frac{1}{1-z}$
1, 1/2, 1/6, 1/24, ...	$\displaystyle\sum_{N \geq 0} \frac{z^N}{N!} = e^z \qquad \longleftarrow \quad [z^N]e^z = 1/N!$

Significance. Can represent an entire sequence with a single function.

Operations on OGFs: Scaling

If $\quad A(z) = \sum_{k \geq 0} a_k z^k \quad$ is the OGF of $\quad a_0, a_1, a_2, \ldots, a_k, \ldots$

then $\quad A(cz) = \sum_{k \geq 0} a_k c^k z^k \quad$ is the OGF of $\quad a_0, ca_1, c^2 a_2, c^3 a_3, \ldots$

sequence	OGF
1, 1, 1, 1, 1, ...	$\displaystyle\sum_{N \geq 0} z^N = \frac{1}{1-z}$
1, 2, 4, 8, 16, 32, ...	$\displaystyle\sum_{N \geq 0} 2^N z^N = \frac{1}{1-2z}$

$$[z^N]\frac{1}{1-2z} = 2^N$$

Operations on OGFs: Addition

If $A(z) = \sum_{k \geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \ldots, a_k, \ldots$

and $B(z) = \sum_{k \geq 0} b_k z^k$ is the OGF of $b_0, b_1, b_2, \ldots, b_k, \ldots$

then $A(z) + B(z)$ is the OGF of $a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots, a_k + b_k \ldots$

Example:

sequence	OGF
1, 1, 1, 1, 1, $\ldots$	$\displaystyle\sum_{N \geq 0} z^N = \frac{1}{1 - z}$
1, 2, 4, 8, 16, 32, $\ldots$	$\displaystyle\sum_{N \geq 0} 2^N z^N = \frac{1}{1 - 2z}$
0, 1, 3, 7, 15, 31, $\ldots$	$\displaystyle\frac{1}{1 - 2z} - \frac{1}{1 - z}$

Operations on OGFs: Differentiation

If $A(z) = \sum_{k \geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \ldots, a_k, \ldots$

then $zA'(z) = \sum_{k \geq 1} k a_k z^k$ is the OGF of $0, a_1, 2a_2, 3a_3, \ldots, ka_k, \ldots$

OGF	sequence
$\dfrac{1}{1-z} = \sum_{N \geq 0} z^N$	1, 1, 1, 1, 1, ...
$\dfrac{z}{(1-z)^2} = \sum_{N \geq 1} N z^N$	0, 1, 2, 3, 4, 5, ...
$\dfrac{z^2}{(1-z)^3} = \sum_{N \geq 2} \binom{N}{2} z^N$	0, 0, 1, 3, 6, 10, ...
$\dfrac{z^M}{(1-z)^{M+1}} = \sum_{N \geq M} \binom{N}{M} z^N$	0, ..., 1, M+1, (M+2)(M+1)/2, ...
$\dfrac{1}{(1-z)^{M+1}} = \sum_{N \geq 0} \binom{N+M}{M} z^N$	1, M+1, (M+2)(M+1)/2, ...

Operations on OGFs: Integration

If $A(z) = \sum_{k \geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \ldots, a_k, \ldots$

then $\int_0^z A(t)dt = \sum_{n \geq 1} \frac{a_{n-1}}{n} z^n$ is the OGF of $0, a_0, \frac{a_1}{2}, \frac{a_2}{3}, \ldots, \frac{a_{k-1}}{k}, \ldots$

Example:

OGF	sequence
$\dfrac{1}{1-z} = \sum_{N \geq 0} z^N$	1, 1, 1, 1, 1, ...
$\ln \dfrac{1}{1-z} = \sum_{N \geq 1} \dfrac{z^N}{N}$	0, 1, 1/2, 1/3, 1/4, 1/5, ...

Operations on OGFs: Partial sum

If $A(z) = \sum_{k \geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \ldots, a_k, \ldots$

then $\dfrac{1}{1-z} A(z)$ is the OGF of $a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots$

Proof.

$$\frac{1}{1-z} A(z) = \sum_{k \geq 0} z^k \sum_{n \geq 0} a_n z^n$$

Distribute
$$= \sum_{k \geq 0} \sum_{n \geq 0} a_n z^{n+k}$$

Change n to $n-k$
$$= \sum_{k \geq 0} \sum_{n \geq k} a_{n-k} z^n$$

Switch order of summation.
$$= \sum_{n \geq 0} \left(\sum_{0 \leq k \leq n} a_{n-k} \right) z^n$$

Change k to $n-k$
$$= \sum_{n \geq 0} \left(\sum_{0 \leq k \leq n} a_k \right) z^n$$

Operations on OGFs: Partial sum

If $A(z) = \sum_{k \geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \ldots, a_k, \ldots$

then $\dfrac{1}{1-z} A(z)$ is the OGF of $a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots$

Example:

OGF	sequence
$\dfrac{1}{1-z} = \sum_{N \geq 0} z^N$	1, 1, 1, 1, 1, ...
$\ln \dfrac{1}{1-z} = \sum_{N \geq 1} \dfrac{z^N}{N}$	0, 1, 1/2, 1/3, 1/4, 1/5, ...
$\dfrac{1}{1-z} \ln \dfrac{1}{1-z} = \sum_{N \geq 1} H_N z^N$	1, 1 + 1/2, 1 + 1/2 + 1/3, ...

Operations on OGFs: Convolution

If $A(z) = \sum_{k \geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \ldots, a_k, \ldots$

and $B(z) = \sum_{k \geq 0} b_k z^k$ is the OGF of $b_0, b_1, b_2, \ldots, b_k, \ldots$

then $A(z)B(z)$ is the OGF of $a_0 b_0, a_1 b_0 + a_1 b_0, \ldots, \sum_{0 \leq k \leq n} a_k b_{n-k}, \ldots$

Proof.
$$A(z)B(z) = \sum_{k \geq 0} a_k z^k \sum_{n \geq 0} b_n z^n$$

Distribute
$$= \sum_{k \geq 0} \sum_{n \geq 0} a_k b_n z^{n+k}$$

Change n to $n{-}k$
$$= \sum_{k \geq 0} \sum_{n \geq k} a_k b_{n-k} z^n$$

Switch order of summation.
$$= \sum_{n \geq 0} \sum_{0 \leq k \leq n} a_k b_{n-k} \; z^n$$

Operations on OGFs: Convolution

If $\quad A(z) = \sum_{k \geq 0} a_k z^k \quad$ is the OGF of $\quad a_0, a_1, a_2, \ldots, a_k, \ldots$

and $\quad B(z) = \sum_{k \geq 0} b_k z^k \quad$ is the OGF of $\quad b_0, b_1, b_2, \ldots, b_k, \ldots$

then $\quad A(z)B(z) \quad$ is the OGF of $\quad a_0 b_0, a_1 b_0 + a_1 b_0, \ldots, \sum_{0 \leq k \leq n} a_k b_{n-k}, \ldots$

Example:

OGF	sequence
$\dfrac{1}{1-z} = \sum_{N \geq 0} z^N$	1, 1, 1, 1, 1, ...
$\dfrac{1}{(1-z)^2} = \sum_{N \geq 0} (N+1)z^N$	1, 2, 3, 4, 5, ...

Expanding a GF (summary)

The process of expressing an unknown GF as a power series
(finding the coefficients) is known as expanding the GF.

Techniques we have been using:

1. Taylor theorem:

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \frac{f''''(0)}{4!}z^4 + \ldots$$

Example.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots$$

2. Reduce to known GFs.

Example.

$$[z^N]\frac{1}{(1-z)}\ln\frac{1}{1-z} = \sum_{1 \leq k \leq N}\frac{1}{k} = H_N$$

Integrate $\dfrac{1}{1-z}$ to get $\ln\dfrac{1}{1-z}$

then convolve $\dfrac{1}{1-z}$ with $\ln\dfrac{1}{1-z}$

Exercise 3.4 Prove that
$$\sum_{1 \le k \le N} H_k = (N+1)(H_{N+1} - 1)$$

Operations on OGFs: Partial sum

If $A(z) = \sum_{k \ge 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \ldots, a_k, \ldots$

then

$\dfrac{1}{1-z} A(z)$ is the OGF of $a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots$

Example:

OGF	sequence
$\dfrac{1}{1-z} = \sum_{N \ge 0} z^N$	1, 1, 1, 1, 1, ...
$\ln \dfrac{1}{1-z} = \sum_{N \ge 1} \dfrac{z^N}{N}$	0, 1, 1/2, 1/3, 1/4, 1/5, ...
$\dfrac{1}{1-z} \ln \dfrac{1}{1-z} = \sum_{N \ge 1} H_N z^N$	1, 1 + 1/2, 1 + 1/2 + 1/3, ...

1. Find GF for LHS (convolve $\dfrac{1}{1-z}$ with $\dfrac{1}{1-z} \ln \dfrac{1}{1-z}$)

$$\frac{1}{(1-z)^2} \ln \frac{1}{1-z}$$

2. Expand GF to find RHS coefficients (convolve $\ln \dfrac{1}{1-z}$ with $\dfrac{1}{(1-z)^2}$)

$$[z^N] \frac{1}{(1-z)^2} \ln \frac{1}{1-z} = \sum_{1 \le k \le N} \frac{1}{k}(N+1-k)$$

3. Do some math
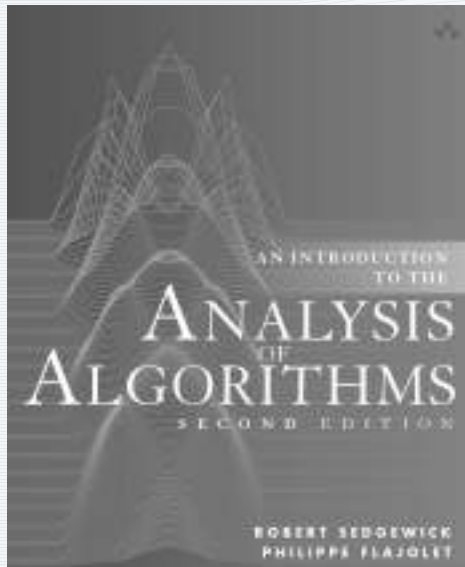
$$= (N+1)H_N - N$$
$$= (N+1)(H_{N+1} - \frac{1}{N+1}) - N$$
$$= (N+1)(H_{N+1} - 1)$$
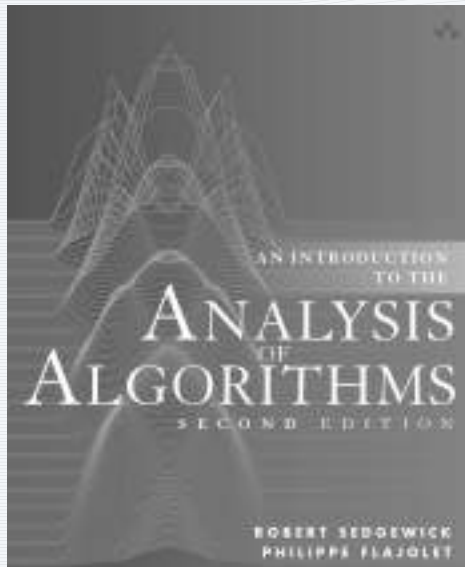
13

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Solving recurrences with OGFs

General procedure:

- Make recurrence valid for all n.

- Multiply both sides of the recurrence by z^n and sum on n.

- Evaluate the sums to derive an equation satisfied by the OGF.

- Solve the equation to derive an explicit formula for the OGF.
  (Use the initial conditions!)

- Expand the OGF to find coefficients.
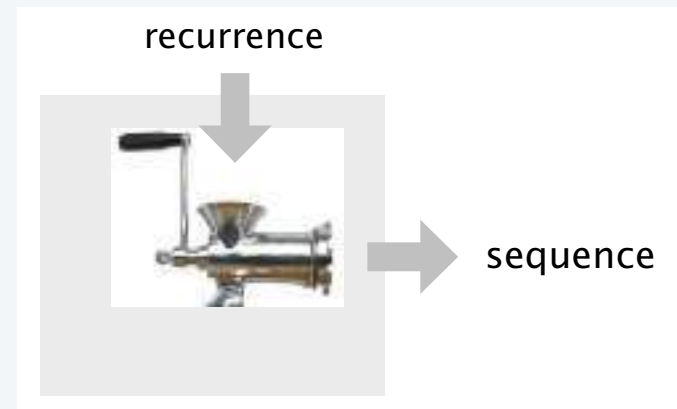
recurrence

sequence

Solving recurrences with GFs

For linear recurrences with constant coefficients, the GF equation is a polynomial, so the general procedure is a algorithm.

Example 4 from previous lecture.

$$a_n = 5a_{n-1} - 6a_{n-2} \quad \text{for } n \geq 2 \text{ with } a_0 = 0 \text{ and } a_1 = 1$$

Make recurrence valid for all n.

$$a_n = 5a_{n-1} - 6a_{n-2} + \delta_{n1}$$

Multiply by z^n and sum on n.

$$A(z) = 5zA(z) - 6z^2A(z) + z$$

Solve.

$$A(z) = \frac{z}{1 - 5z + 6z^2}$$

Use partial fractions:
  solution must be of the form

$$A(z) = \frac{c_0}{1 - 3z} + \frac{c_1}{1 - 2z}$$

Solve for coefficients.

$$c_0 + c_1 = 0$$
$$2c_0 + 3c_1 = -1$$

Solution is $c_0 = 1$ and $c_1 = -1$

$$A(z) = \frac{1}{1 - 3z} - \frac{1}{1 - 2z}$$

Expand.

$$a_n = 3^n - 2^n$$

Solving linear recurrences with GFs

For linear recurrences with constant coefficients, the GF equation is a polynomial, so the general procedure is a algorithm.

Example with multiple roots.

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3} \quad \text{for } n \geq 3 \text{ with } a_0 = 0, a_1 = 1 \text{ and } a_2 = 4$$

Make recurrence valid for all n.
$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3} + \delta_{n1} - \delta_{n2}$$

Multiply by z^n and sum on n.
$$A(z) = 5zA(z) - 8z^2A(z) + 4z^3A(z) + z - z^2$$

Solve.
$$A(z) = \frac{z - z^2}{1 - 5z + 8z^2 - 4z^3}$$

Simplify.
$$A(z) = \frac{z(1 - z)}{(1 - z)(1 - 2z)^2} = \frac{z}{(1 - 2z)^2}$$

Expand.
$$a_n = n2^{n-1}$$

multiplicity 3 gives terms of the form $n^2\beta^n$, etc.

18

Solving linear recurrences with GFs

For linear recurrences with constant coefficients, the GF equation is a polynomial, so the general procedure is a algorithm.

Example with complex roots.

$$a_n = 2a_{n-1} - a_{n-2} + 2a_{n-3} \quad \text{for } n \geq 3 \text{ with } a_0 = 1, a_1 = 0 \text{ and } a_2 = -1$$

Make recurrence valid for all n.
$$a_n = 2a_{n-1} - a_{n-2} + 2a_{n-3} + \delta_{n0} - 2\delta_{n1}$$

Multiply by z^n and sum on n.
$$A(z) = 2zA(z) - z^2 A(z) + 2z^3 A(z) + 1 - 2z$$

Solve.
$$A(z) = \frac{1 - 2z}{1 - 2z + z^2 - 2z^3}$$

Simplify.
$$A(z) = \frac{1 - 2z}{(1 - 2z)(1 + z^2)} = \frac{1}{(1 + z^2)}$$

Use partial fractions.
$$A(z) = \frac{1}{2}\left(\frac{1}{1 - iz} + \frac{1}{1 + iz}\right)$$

Expand.
$$a_n = \frac{1}{2}(i^n + (-i)^n) = \frac{1}{2}i^n(1 + (-1)^n)$$

$$1, \ 0, \ -1, \ 0, \ 1, \ 0, \ -1, \ 0, \ 1 \ldots$$

Solving linear recurrences with GFs (summary)

Solution to $\quad a_n = x_1 a_{n-1} + x_2 a_{n-2} + \ldots + x_t a_{n-t}$

is a linear combination of t terms.

Suppose the roots of the polynomial $\quad 1 - x_1 z + x_2 z^2 + \ldots + x_t z^t$

are $\beta_1, \beta_2, \ldots, \beta_r$ where the multiplicity of β_i is m_i so $m_1 + m_2 + \ldots + m_r = t$

Solution is
$$\sum_{0 \le j < m_1} c_{1j} n^j \beta_1^n + \sum_{0 \le j < m_2} c_{2j} n^j \beta_2^n + \ldots + \sum_{0 \le j < m_r} c_{rj} n^j \beta_r^n \quad \longleftarrow \text{ t terms}$$

The t constants c_{ij} are determined from the initial conditions.

Note: complex roots (and -1) introduce periodic behavior.

Solving the Quicksort recurrence with OGFs

$$C_N = N + 1 + \frac{2}{N} \sum_{1 \le k \le N} C_{k-1}$$

Multiply both sides by N.

$$NC_N = N(N+1) + 2 \sum_{1 \le k \le N} C_{k-1}$$

Multiply by z^N and sum.

$$\sum_{N \ge 1} NC_N z^N = \sum_{N \ge 1} N(N+1)z^N + 2 \sum_{N \ge 1} \sum_{1 \le k \le N} C_{k-1} z^N$$

Evaluate sums to get an *ordinary differential equation*

$$C'(z) = \frac{2}{(1-z)^3} + 2\frac{C(z)}{1-z}$$

homogeneous equation
$$\rho'(z) = 2\rho(z)/(1-z)$$
solution (integration factor)
$$\rho(z) = 1/(1-z)^2$$

Solve the ODE.

$$((1-z)^2 C(z))' = (1-z)^2 C'(z) - 2(1-z)C(z)$$

$$= (1-z)^2 \left(C'(z) - 2\frac{C(z)}{1-z} \right) = \frac{2}{1-z}$$

Integrate.

$$C(z) = \frac{2}{(1-z)^2} \ln \frac{1}{1-z}$$

Expand.

$$C_N = [z^N] \frac{2}{(1-z)^2} \ln \frac{1}{1-z} = 2(N+1)(H_{N+1} - 1)$$

21

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Catalan numbers

How many triangulations of an (N+2)-gon?

$T_0 = 1$

$T_1 = 1$

$T_2 = 2$

$T_3 = 5$

$T_4 = 14$

$$T_N = \sum_{0 \leq k < N} T_k T_{N-1-k} + \delta_{N0}$$

Catalan numbers

How many gambler's ruin sequences with N wins?
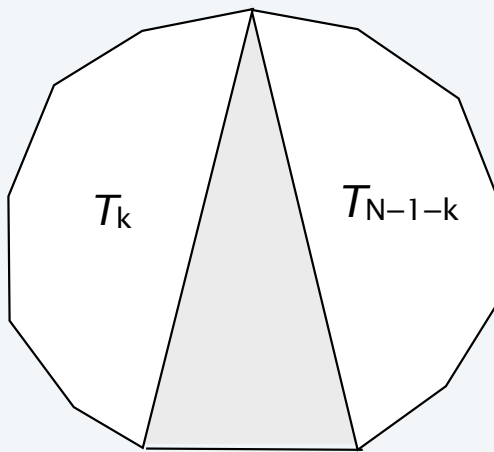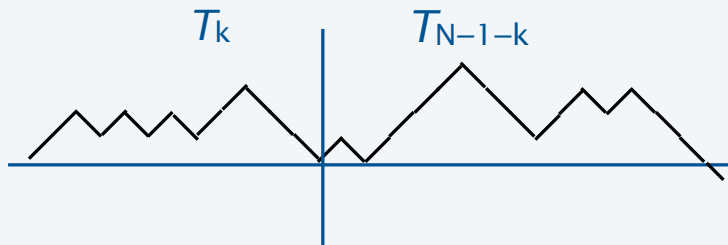
T_k T_{N-1-k}

$$T_N = \sum_{0 \leq k < N} T_k T_{N-1-k} + \delta_{N0}$$

$T_0 = 1$ $T_1 = 1$

$T_2 = 2$

$T_3 = 5$
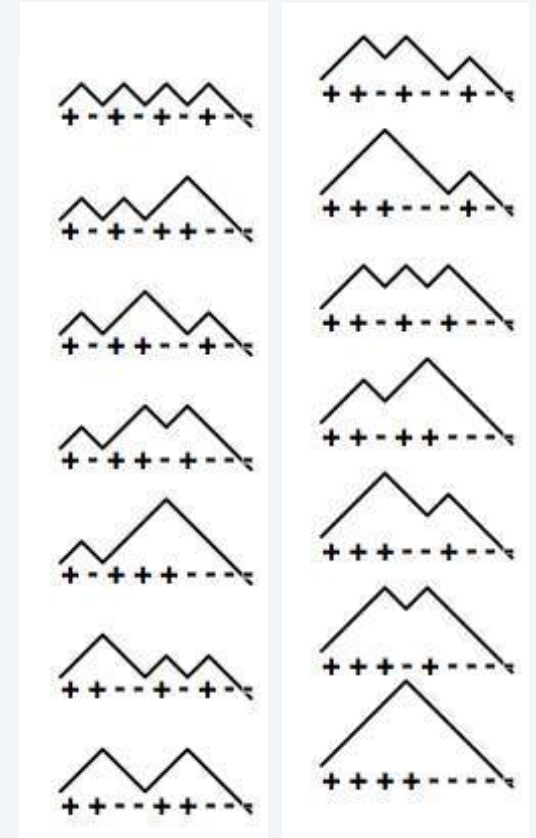
$T_4 = 14$

Catalan numbers

How many binary trees with N nodes?
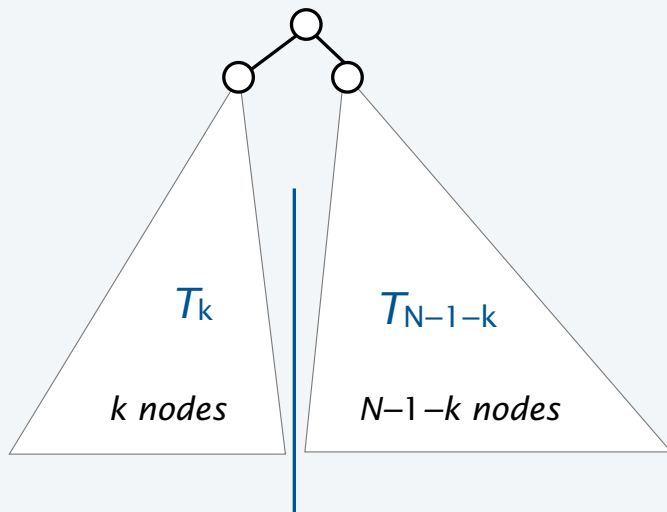
$$T_N = \sum_{0 \le k < N} T_k T_{N-1-k} + \delta_{N0}$$
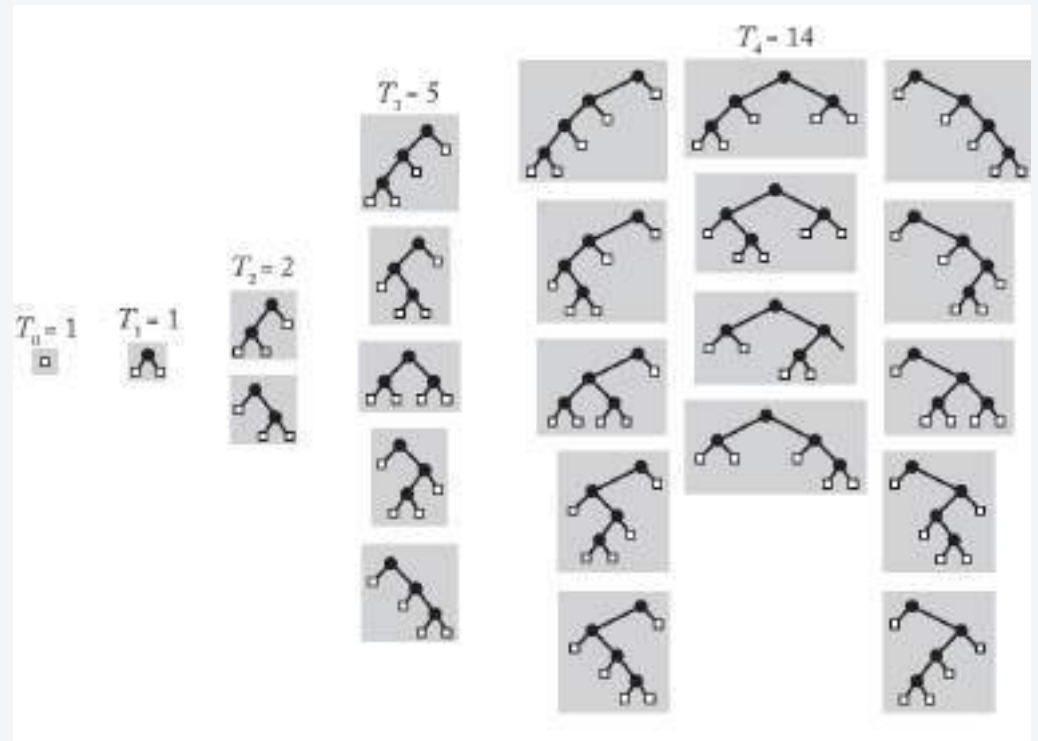
Catalan numbers

How many trees with $N+1$ nodes?
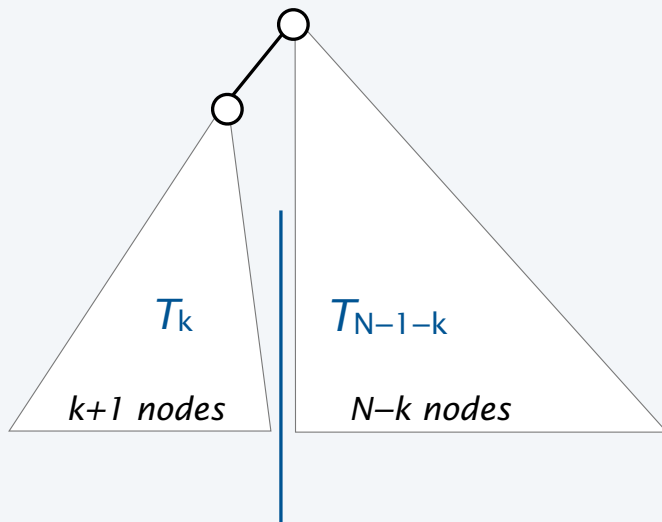
$$T_N = \sum_{0 \le k < N} T_k T_{N-1-k} + \delta_{N0}$$
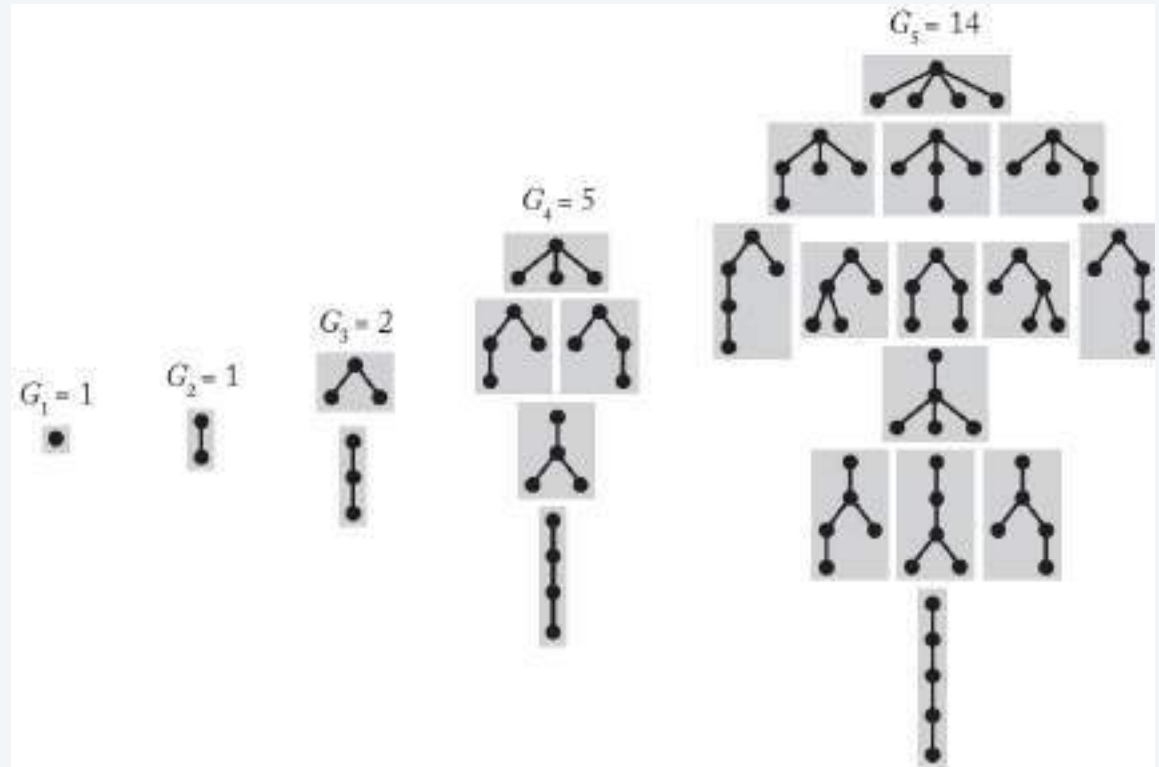
Solving the Catalan recurrence with GFs

Recurrence that holds for all N.

$$T_N = \sum_{0 \leq k < N} T_k T_{N-1-k} + \delta_{N0}$$

Multiply by z^N and sum.

$$T(z) \equiv \sum_{N \geq 0} T_N z^N = \sum_{N \geq 0} \sum_{0 \leq k < N} T_k T_{N-1-k} z^N + 1$$

Switch order of summation

$$T(z) = 1 + \sum_{k \geq 0} \sum_{N > k} T_k T_{N-1-k} z^N$$

convolution (backwards)

Change N to $N+k+1$
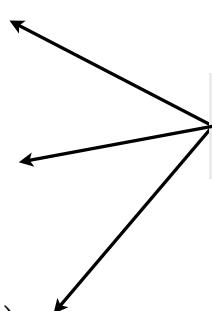
$$T(z) = 1 + \sum_{k \geq 0} \sum_{N \geq 0} T_k T_N z^{N+k+1}$$

Distribute.

$$T(z) = 1 + z \left(\sum_{k \geq 0} T_k z^k \right) \left(\sum_{N \geq 0} T_N z^N \right)$$

$$T(z) = 1 + z T(z)^2$$

Common-sense rule for working with GFs

It is always worthwhile to check your math with your computer.

Known from initial values:

$$T(z) = 1 + z + 2z^2 + 5z^3 + 14z^4 + \ldots$$

Check:

$$T(z) = 1 + zT(z)^2$$
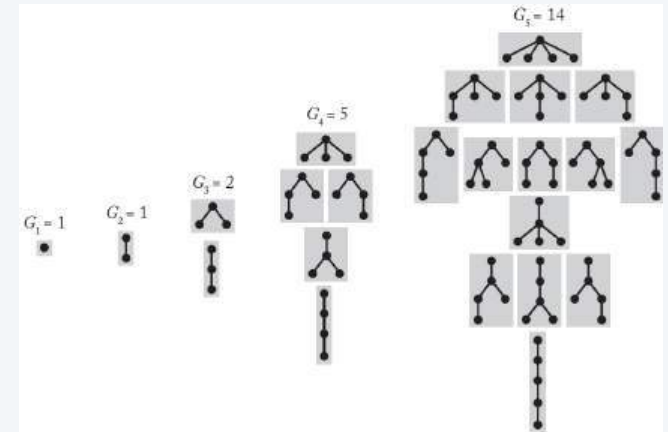

```
sage: ZP.<z> = ZZ[]
sage: 1 + z*(1+z+2*z^2+5*z^3+14*z^4)*(1+z+2*z^2+5*z^3+14*z^4)
196*z^9 + 140*z^8 + 81*z^7 + 48*z^6 + 42*z^5 + 14*z^4 + 5*z^3 + 2*z^2 + z + 1
```

not valid because
z^5 and beyond
missing in factors

✓

Solving the Catalan recurrence with GFs (continued)

Functional GF equation.

$$T(z) = 1 + zT(z)^2$$

Solve with quadratic formula.

$$zT(z) = \frac{1}{2}(1 \pm \sqrt{1 - 4z})$$

Expand via binomial theorem.

$$zT(z) = -\frac{1}{2} \sum_{N \geq 1} \binom{\frac{1}{2}}{N} (-4z)^N$$

Set coefficients equal

$$T_N = -\frac{1}{2} \binom{\frac{1}{2}}{N+1} (-4)^{N+1}$$

Expand via definition.

$$= -\frac{1}{2} \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\dots(\frac{1}{2}-N)(-4)^{N+1}}{(N+1)!}$$

Distribute $(-2)^N$ among factors.

$$= \frac{1 \cdot 3 \cdot 5 \cdots (2N-1) \cdot 2^N}{(N+1)!}$$

Substitute $(2/1)(4/2)(6/3)$... for 2^N.

$$= \frac{1}{N+1} \frac{1 \cdot 3 \cdot 5 \cdots (2N-1)}{N!} \frac{2 \cdot 4 \cdot 6 \cdots 2N}{1 \cdot 2 \cdot 3 \cdots N}$$

$$= \frac{1}{N+1} \binom{2N}{N}$$
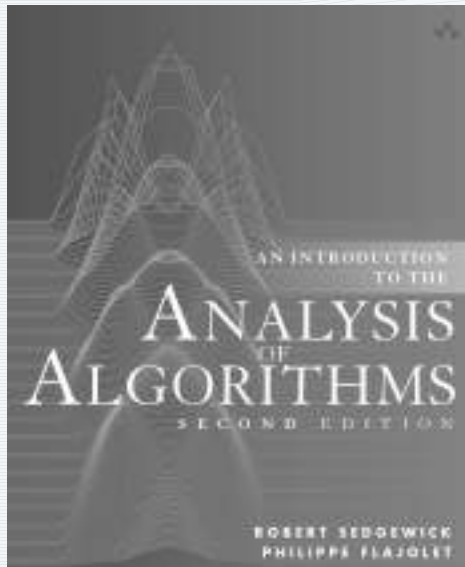
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Exponential generating functions (EGFs)

Definition.

$$A(z) = \sum_{k \geq 0} a_k \frac{z^k}{k!} \quad \text{is the exponential generating function (EGF)}$$

of the sequence $a_0, a_1, a_2, \ldots, a_k, \ldots$

sequence	EGF
1, 1, 1, 1, 1, ...	$\displaystyle\sum_{N \geq 0} \frac{z^N}{N!} = e^z$
1, 2, 4, 8, 16, 32, ...	$\displaystyle\sum_{N \geq 0} 2^N \frac{z^N}{N!} = e^{2z}$
1, 1, 2, 6, 24, 120 ...	$\displaystyle\sum_{N \geq 0} N! \frac{z^N}{N!} = \frac{1}{1-z}$

Operations on EGFs: Binomial convolution

If $A(z) = \sum_{k \geq 0} a_k \frac{z^k}{k!}$ is the EGF of $a_0, a_1, a_2, \ldots, a_k, \ldots$

and $B(z) = \sum_{k \geq 0} b_k \frac{z^k}{k!}$ is the EGF of $b_0, b_1, b_2, \ldots, b_k, \ldots$

then $A(z)B(z)$ is the EGF of $a_0 b_0, a_0 b_1 + a_1 b_0, \ldots, \binom{n}{k} a_k b_{n-k}, \ldots$

Proof.

$$A(z)B(z) = \sum_{k \geq 0} a_k \frac{z^k}{k!} \sum_{n \geq 0} b_n \frac{z^n}{n!}$$

Distribute.

$$= \sum_{k \geq 0} \sum_{n \geq 0} \frac{a_k}{k!} \frac{b_n}{n!} z^{n+k}$$

Change n to $n-k$

$$= \sum_{k \geq 0} \sum_{n \geq k} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} z^n$$

Multiply and divide by $n!$

$$= \sum_{k \geq 0} \sum_{n \geq k} \binom{n}{k} a_k b_{n-k} \frac{z^n}{n!}$$

Switch order of summation.

$$= \sum_{n \geq 0} \left(\sum_{0 \leq k \leq n} \binom{n}{k} a_k b_{n-k} \right) \frac{z^n}{n!}$$

Solving recurrences with EGFs

Choice of EGF vs. OGF is typically dictated naturally from the problem.

Example.
$$f_n = \sum_k \binom{n}{k} \frac{f_k}{2^k}$$

Multiply by $z^n/n!$ and sum on n.
$$f(z) = \sum_{n \geq 0} \sum_k \binom{n}{k} \frac{f_k}{2^k} \frac{z^n}{n!}$$

Switch order of summation.
$$f(z) = \sum_{k \geq 0} \sum_{n \geq k} \binom{n}{k} \frac{f_k}{2^k} \frac{z^n}{n!}$$

Change n to $n+k$.
$$f(z) = \sum_{k \geq 0} \sum_{n \geq 0} \binom{n+k}{k} \frac{f_k}{2^k} \frac{z^{n+k}}{(n+k)!}$$

binomial convolution (backwards)

Simplify.
$$f(z) = \sum_{k \geq 0} \sum_{n \geq 0} f_k \frac{(z/2)^k}{k!} \frac{z^n}{n!}$$

Distribute.
$$f(z) = \left(\sum_{k \geq 0} f_k \frac{(z/2)^k}{k!} \right) \left(\sum_{n \geq 0} \frac{z^n}{n!} \right)$$

convergence not assured
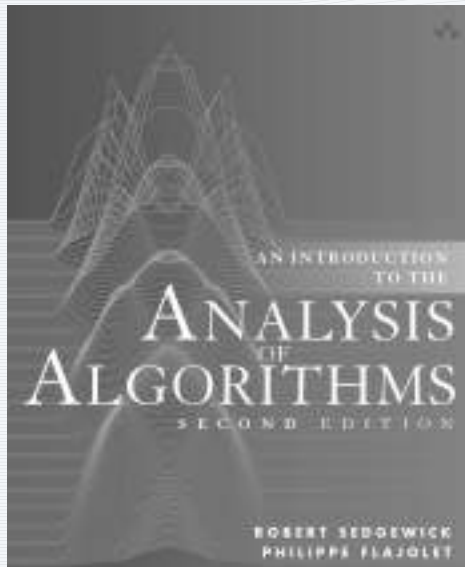
Evaluate and telescope.
$$f(z) = e^z f(z/2) = e^{z+z/2+z/4+z/8+\cdots} = e^{2z}$$

Expand.
$$f_n = 2^n$$

Check.
$$2^n = \sum_k \binom{n}{k} \frac{2^k}{2^k}$$

35

3. Generating Functions

- OGFs
- Solving recurrences
- Catalan numbers
- **EGFs**
- Counting with GFs
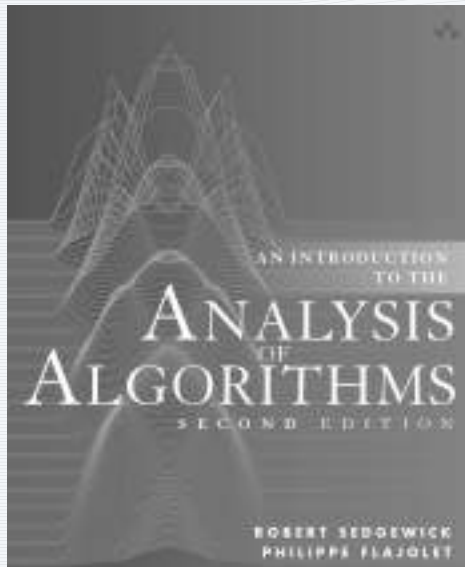
3. Generating Functions

- OGFs
- Solving recurrences
- Catalan numbers
- EGFs
- **Counting with GFs**

Counting with generating functions

An alternative (combinatorial) view of GFs
- Define a *class* of combinatorial obects with associated *size* function.
- GF is sum over all members of the class.

Example.

$T \equiv$ set of all binary trees

$|t| \equiv$ number of *internal* nodes in t $\in T$

$T_N \equiv$ number of t $\in T$ with $|t| = N$

$$T(z) \equiv \sum_{t \in \mathcal{T}} z^{|t|} = \sum_{N \geq 0} T_N z^N$$

Decompose from definition

$$T(z) = 1 + \sum_{t_L \in T} \sum_{t_R \in T} z^{|t_L| + |t_R| + 1}$$

Distribute

$$= 1 + z \left(\sum_{t_L \in T} z^{|t_L|} \right) \left(\sum_{t_R \in T} z^{|t_R|} \right)$$

$$= 1 + zT(z)^2$$

t_L t_R

$t_L \bullet nodes$ $t_R \bullet nodes$
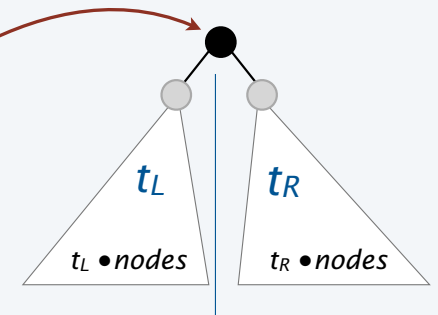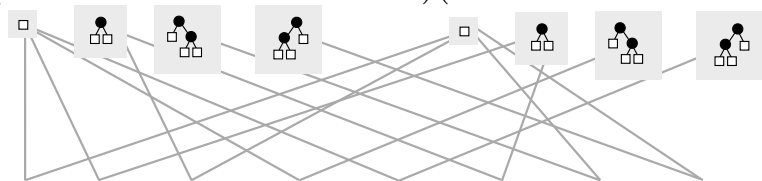
Combinatorial view of Catalan GF

Each term z^N in the GF corresponds to an object of size N.

Collect all the terms with the same exponent to expose counts.

Each term $z^i z^j$ in a product corresponds to an object of size $i + j$.

$$T(z) = 1 + z + z^2 + z^2 + z^3 + z^3 + z^3 + z^3 + z^3 + \ldots$$

$$= 1 + z + 2z^2 + 5z^3 + \ldots$$

$$T(z) = 1 + zT(z)^2$$

$$= 1 + z(1 + z + z^2 + z^2 + \ldots)(1 + z + z^2 + z^2 + \ldots)$$

$$= 1 + z + z^2 + z^2 + z^3 + z^3 + z^3 + z^3 + z^3 + \ldots$$

$z \times 1 \times 1$

$z \times 1 \times z \quad z \times z \times 1 \quad z \times 1 \times z^2 \quad z \times 1 \times z^2 \quad z \times z \times z \quad z \times z^2 \times 1 \quad z \times z^2 \times 1$
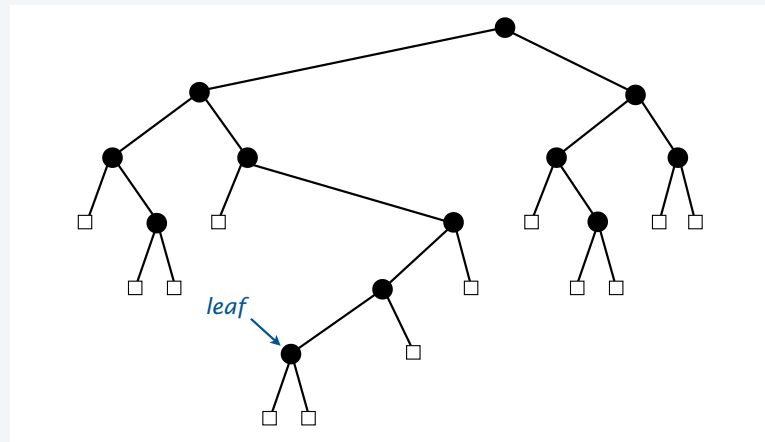
Values of parameters ("costs")

are often the object of study in the analysis of algorithms.

How many 1 bits in a random bitstring? (Easy)

0111010010001000111010100001010000

How many leaves in a random binary tree? (Not so easy)

leaf

Computing expected costs by counting

An alternative (combinatorial) view of probability

- Define a *class* of combinatorial obects.
- Model: All objects of size N are equally likely

$$\mathcal{P} \equiv \text{ set of all objects in the class}$$

$$|p| \equiv \text{ size of } p \in \mathcal{P}$$

$$P_N \equiv \text{ number of } p \in \mathcal{P} \text{ with } |p| = N$$

$$\text{cost}(p) \equiv \text{ cost associated with } p$$

$$P_{Nk} \equiv \text{ number of } p \in \mathcal{P} \text{ with } |p| = N \text{ and cost(p) } = k$$

Expected cost of an object of size N

$$C_N \equiv \sum_{k \geq 0} k \frac{P_{Nk}}{P_N}$$

P_{Nk}/P_N is the probability that the cost of on object of size N is k

$$= \frac{\sum_{k \geq 0} k P_{Nk}}{P_N}$$

"cumulated cost"

Def. *Cumulated cost* is total cost of all objects of a given size.

Expected cost is cumulated cost divided by number of objects.

Counting with generating functions: cumulative costs

An alternative (combinatorial) view of GFs

- Define a *class* of combinatorial obects.
- Model: All objects of size *N* are equally likely
- GF is sum over all members of the class.

$$
\begin{aligned}
\mathcal{P} &\equiv \text{set of all objects in the class} \\
|p| &\equiv \text{size of } p \in \mathcal{P} \\
P_N &\equiv \text{number of } p \in \mathcal{P} \text{ with } |p| = N \\
\text{cost}(p) &\equiv \text{cost associated with } p
\end{aligned}
$$

Counting GF $\qquad\qquad\qquad\qquad P(z) \equiv \sum_{p \in \mathcal{P}} z^{|p|} = \sum_{N \geq 0} P_N z^N$

Cumulative cost GF $\qquad\qquad C(z) \equiv \sum_{p \in \mathcal{P}} \text{cost}(p) z^{|p|} = \sum_{N \geq 0} \sum_{k \geq 0} k P_{Nk} z^N$

Average cost $\qquad\qquad\qquad [z^N]C(z)/[z^N]P(z)$

Bottom line: Reduces computing expectation to GF counting

Warmup: How many 1 bits in a random bitstring?

B is the set of all bitstrings.

$|b|$ is the number of bits in b.

ones(b) is the number of 1 bits in b.

B_N is the # of bitstrings of size N (2^N).

C_N is the total number of 1 bits in all bitstrings of size N.

Counting GF.

$$B(z) = \sum_{b \in B} z^{|b|} = \sum_{N \geq 0} 2^N z^N = \frac{1}{1 - 2z}$$

Cumulative cost GF.

$$C(z) = \sum_{b \in B} \text{ones}(b) z^{|b|}$$

$$= \sum_{b' \in B} (1 + 2 \cdot \text{ones}(b')) z^{|b'|+1}$$

0	b'
1	b'

$$= zB(z) + 2zC(z)$$

$$= \frac{z}{(1 - 2z)^2}$$

$$\frac{2z}{(1 - 2z)^2} = \sum_{N \geq 1} N(2z)^N$$

Average # 1 bits in a random bitstring of length N.

$$\frac{[z^N]C(z)}{[z^N]B(z)} = \frac{N2^{N-1}}{2^N} = \frac{N}{2} \quad \checkmark$$
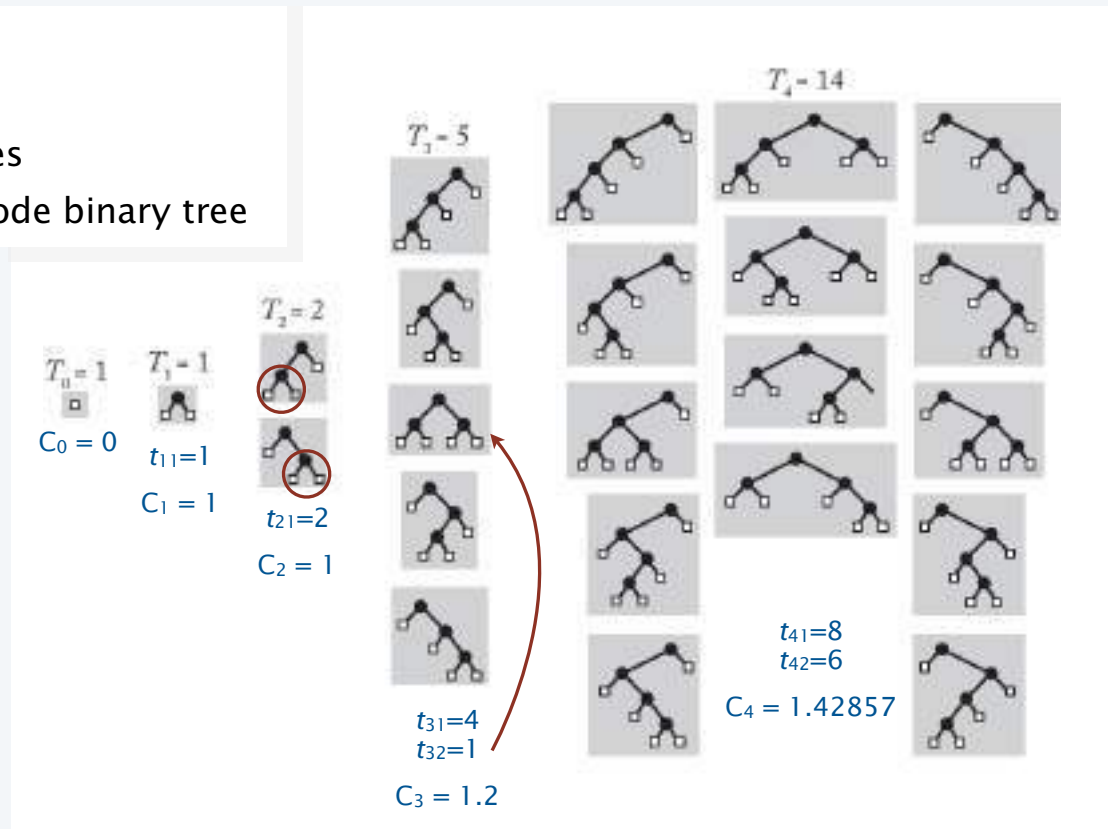
Leaves in binary trees

are internal nodes whose children are both external.

Definitions:

T_N is the # of binary trees with N nodes.

t_{Nk} is the # of N-node binary trees with k leaves

C_N is the average # of leaves in a random N-node binary tree

$T_0 = 1$

$C_0 = 0$

$T_1 = 1$

$t_{11} = 1$

$C_1 = 1$

$T_2 = 2$

$t_{21} = 2$

$C_2 = 1$

$t_{31} = 4$
$t_{32} = 1$

$C_3 = 1.2$

$T_3 = 5$

$T_4 = 14$

$t_{41} = 8$
$t_{42} = 6$

$C_4 = 1.42857$

Q. How many leaves in a random binary tree?

How many leaves in a random binary tree?

T is the set of all binary trees.

$|t|$ is the number of internal nodes in t.

leaves(t) is the number of leaves in t.

T_N is the # of binary trees of size N (Catalan).

C_N is the total number of leaves in all binary trees of size N.

Counting GF.

$$T(z) = \sum_{t \in T} z^{|t|} = \sum_{N \geq 0} T_N z^N = \sum_{N \geq 0} \frac{1}{N+1} \binom{2N}{N} z^N$$

Cumulative cost GF.

$$C(z) = \sum_{t \in T} \text{leaves}(t) z^{|t|}$$

Average # leaves in a random N-node binary tree.

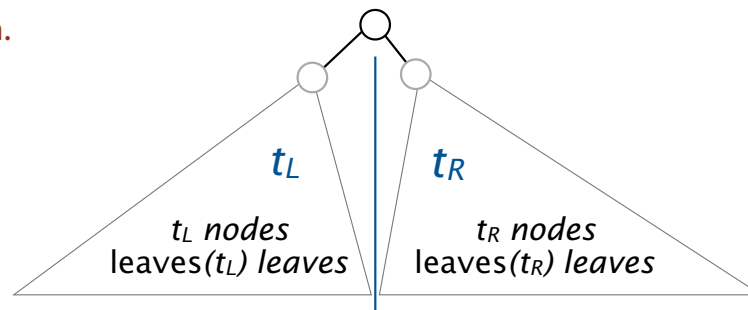$$\frac{[z^N]C(z)}{[z^N]T(z)} = \frac{[z^N]C(z)}{T_N}$$

Next: Derive a functional equation for the CGF.

CGF functional equation for leaves in binary trees

CGF.

$$C(z) = \sum_{t \in T} \text{leaves}(t) z^{|t|}$$

Decompose from definition.

$$C(z) = z + \sum_{t_L \in T} \sum_{t_R \in T} \left(\text{leaves}(t_L) + \text{leaves}(t_R) \right) z^{|t_L| + |t_R| + 1}$$

$$= z + z \sum_{t_L \in T} \text{leaves}(t_L) z^{|t_L|} \sum_{t_R \in T} z^{|t_R|} + z \sum_{t_L \in T} z^{|t_L|} \sum_{t_R \in T} \text{leaves}(t_R) z^{|t_R|}$$

$$= z + 2z C(z) T(z)$$

How many leaves in a random binary tree?

CGF.
$$C(z) = \sum_{t \in T} \text{leaves}(t) z^{|t|}$$

Decompose from definition.
$$C(z) = z + \sum_{t_L \in T} \sum_{t_R \in T} (\text{leaves}(t_L) + \text{leaves}(t_R)) z^{|t_L| + |t_R| + 1}$$

$$= z + 2zC(z)T(z)$$

Compute number of trees T_N.
Catalan numbers
$$T(z) = zT(z)^2 - z$$
$$= \frac{1}{2z}(1 - \sqrt{1 - 4z})$$

$$T_N = [z^N] \frac{1}{2z}(1 - \sqrt{1 - 4z})$$
$$= \frac{1}{N+1}\binom{2N}{N}$$

Compute cumulated cost C_N.
$$C(z) = z + 2zT(z)C(z)$$
$$= \frac{z}{1 - 2zT(z)} = \frac{z}{\sqrt{1 - 4z}}$$

$$C_N = [z^N] \frac{z}{\sqrt{1 - 4z}}$$
$$= \binom{2N - 2}{N - 1}$$

Compute *average number of leaves*.
$$C_N / T_N = \frac{\binom{2N-2}{N-1}}{\frac{1}{N+1}\binom{2N}{N}} = \frac{(N+1) \cdot N \cdot N}{2N(2N-1)} \sim \boxed{N/4}$$

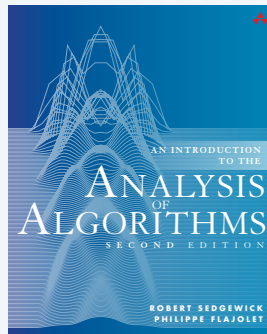http://aofa.cs.princeton.edu

3. Generating Functions

- OGFs
- Solving recurrences
- Catalan numbers
- EGFs
- **Counting with GFs**

Exercise 3.20

Solve a linear recurrence. Initial conditions matter.
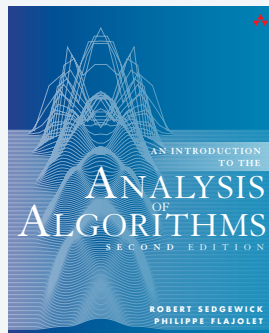
Exercise 3.20 Solve the recurrence

$$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} \qquad \text{for } n > 2 \text{ with } a_0 = a_1 = 0 \text{ and } a_2 = 1.$$

Solve the same recurrence with the initial condition on a_1 changed to $a_1 = 1$.

Exercise 3.28

The art of expanding GFs.

Exercise 3.28 Find an expression for

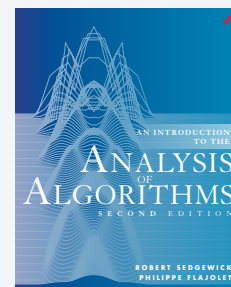$$[z^n] \frac{1}{\sqrt{1-z}} \ln \frac{1}{1-z}.$$

(*Hint*: Expand $(1-z)^{-\alpha}$ and differentiate with respect to α.)

Assignments for next lecture

1. Use a symbolic mathematics system
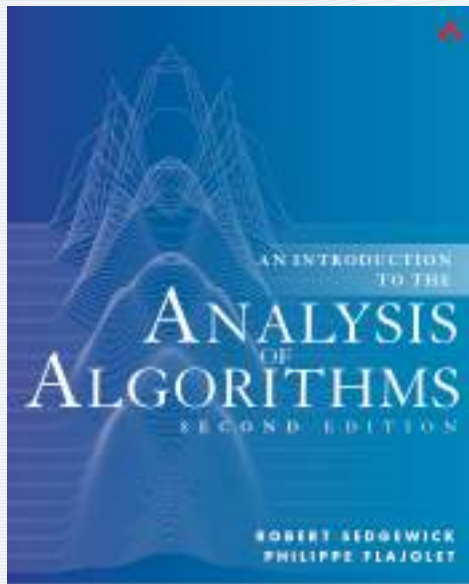   to check initial values for $C(z) = z + 2\,C(z)\,T(z)$.

2. Read pages 89-147 in text.

3. Write up solutions to Exercises 3.20 and 3.28.

http://aofa.cs.princeton.edu

3. Generating Functions