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4.4 SHORTEST PATHS

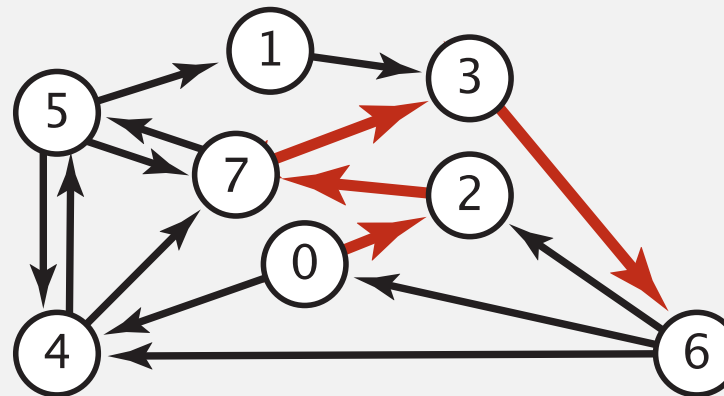
- ▶ *APIs*
- ▶ *shortest-paths properties*
- ▶ *Dijkstra's algorithm*
- ▶ *edge-weighted DAGs*
- ▶ *negative weights*

Shortest paths in an edge-weighted digraph

Given an edge-weighted digraph, find the shortest path from s to t .

edge-weighted digraph

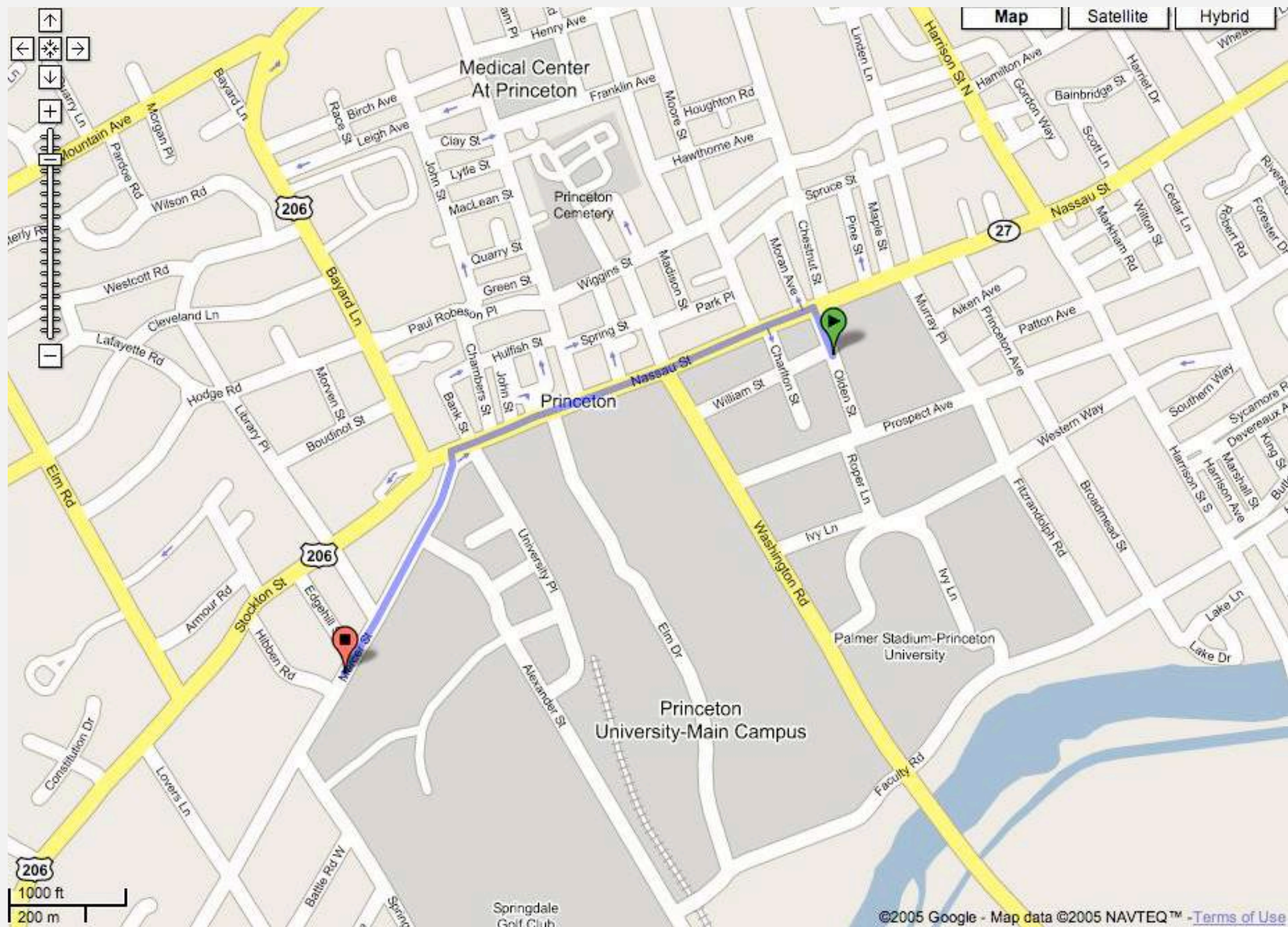
4→5	0.35
5→4	0.35
4→7	0.37
5→7	0.28
7→5	0.28
5→1	0.32
0→4	0.38
0→2	0.26
7→3	0.39
1→3	0.29
2→7	0.34
6→2	0.40
3→6	0.52
6→0	0.58
6→4	0.93



shortest path from 0 to 6

0→2	0.26
2→7	0.34
7→3	0.39
3→6	0.52

Google maps



Car navigation



Shortest path applications

- PERT/CPM.
- Map routing.
- Seam carving.
- Robot navigation.
- Texture mapping.
- Typesetting in TeX.
- Urban traffic planning.
- Optimal pipelining of VLSI chip.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Exploiting arbitrage opportunities in currency exchange.
- Optimal truck routing through given traffic congestion pattern.



http://en.wikipedia.org/wiki/Seam_carving



Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

Shortest path variants

Which vertices?

- **Single source:** from one vertex s to every other vertex.
- Source-sink: from one vertex s to another t .
- All pairs: between all pairs of vertices.

Restrictions on edge weights?

- Nonnegative weights.
- Euclidean weights.
- Arbitrary weights.

Cycles?

- No directed cycles.
- No "negative cycles."

Simplifying assumption. Shortest paths from s to each vertex v exist.



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Weighted directed edge API

```
public class DirectedEdge
```

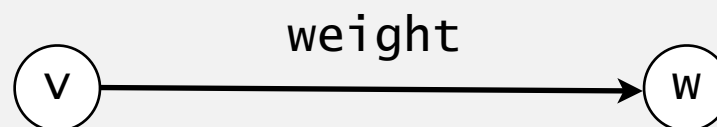
```
    DirectedEdge(int v, int w, double weight)    weighted edge  $v \rightarrow w$ 
```

```
    int from()                                    vertex  $v$ 
```

```
    int to()                                       vertex  $w$ 
```

```
    double weight()                              weight of this edge
```

```
    String toString()                            string representation
```



Idiom for processing an edge e : `int v = e.from(), w = e.to();`

Weighted directed edge: implementation in Java

Similar to Edge for undirected graphs, but a bit simpler.

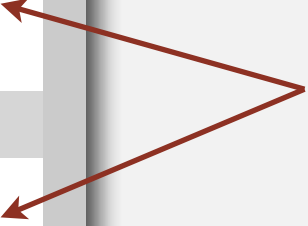
```
public class DirectedEdge
{
    private final int v, w;
    private final double weight;

    public DirectedEdge(int v, int w, double weight)
    {
        this.v = v;
        this.w = w;
        this.weight = weight;
    }

    public int from()
    { return v; }

    public int to()
    { return w; }

    public int weight()
    { return weight; }
}
```



from() and to() replace
either() and other()

Edge-weighted digraph API

```
public class EdgeWeightedDigraph
```

```
    EdgeWeightedDigraph(int V)    edge-weighted digraph with  $V$  vertices
```

```
    EdgeWeightedDigraph(In in)    edge-weighted digraph from input stream
```

```
    void addEdge(DirectedEdge e)    add weighted directed edge  $e$ 
```

```
    Iterable<DirectedEdge> adj(int v)    edges pointing from  $v$ 
```

```
    int V()    number of vertices
```

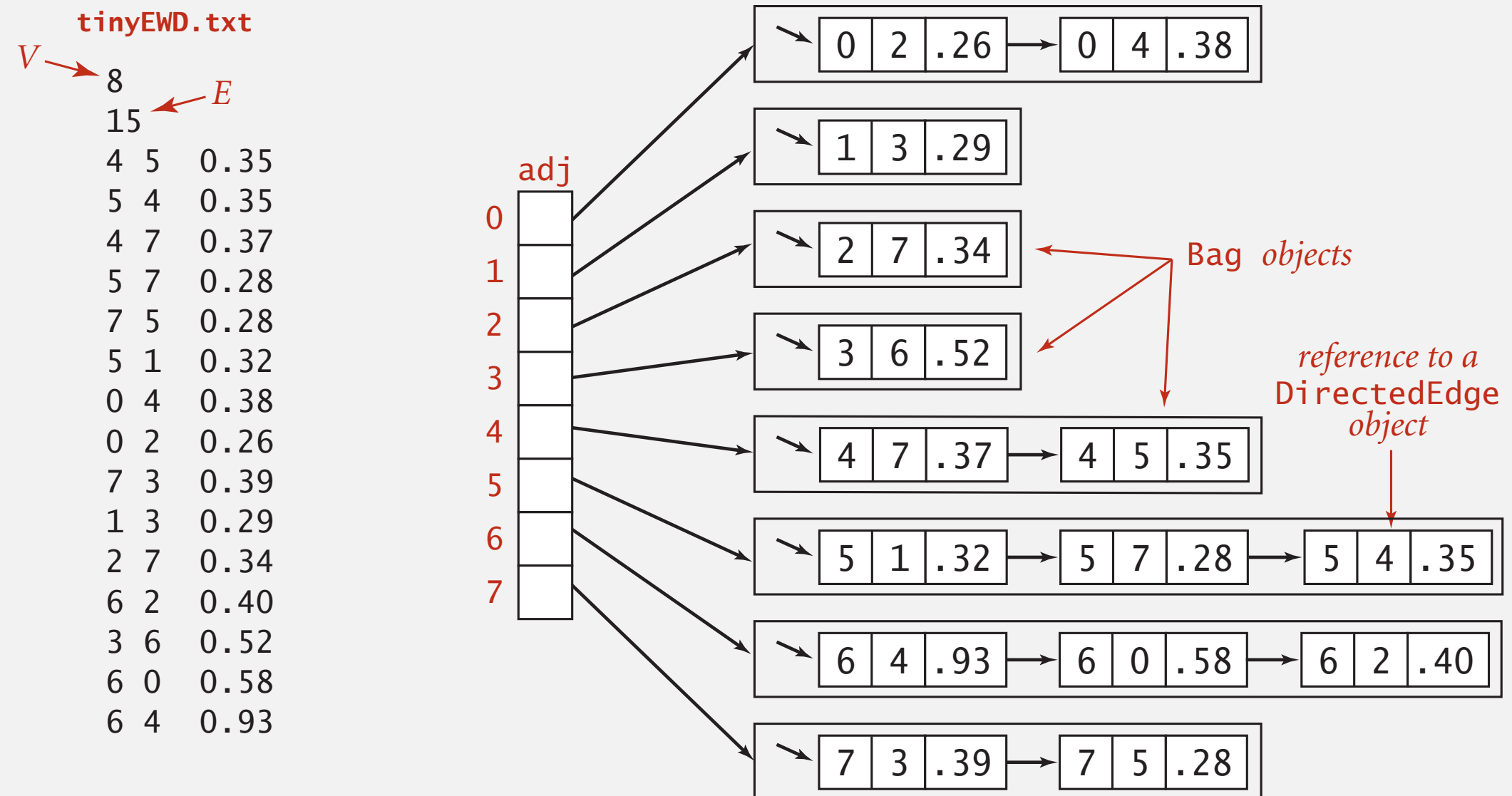
```
    int E()    number of edges
```

```
    Iterable<DirectedEdge> edges()    all edges
```

```
    String toString()    string representation
```

Conventions. Allow self-loops and parallel edges.

Edge-weighted digraph: adjacency-lists representation



Edge-weighted digraph: adjacency-lists implementation in Java

Same as EdgeWeightedGraph except replace Graph with Digraph.

```
public class EdgeWeightedDigraph
{
    private final int V;
    private final Bag<DirectedEdge>[] adj;

    public EdgeWeightedDigraph(int V)
    {
        this.V = V;
        adj = (Bag<DirectedEdge>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<DirectedEdge>();
    }

    public void addEdge(DirectedEdge e)
    {
        int v = e.from();
        adj[v].add(e);
    }

    public Iterable<DirectedEdge> adj(int v)
    { return adj[v]; }
}
```

← add edge $e = v \rightarrow w$ to
only v 's adjacency list

Single-source shortest paths API

Goal. Find the shortest path from s to every other vertex.

```
public class SP
```

```
    SP(EdgeWeightedDigraph G, int s)    shortest paths from s in graph G
```

```
    double distTo(int v)                length of shortest path from s to v
```

```
    Iterable <DirectedEdge> pathTo(int v)    shortest path from s to v
```

```
    boolean hasPathTo(int v)            is there a path from s to v?
```

```
SP sp = new SP(G, s);
for (int v = 0; v < G.V(); v++)
{
    StdOut.printf("%d to %d (%.2f): ", s, v, sp.distTo(v));
    for (DirectedEdge e : sp.pathTo(v))
        StdOut.print(e + " ");
    StdOut.println();
}
```

Single-source shortest paths API

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```
    double distTo(int v)                length of shortest path from s to v
```

```
    Iterable <DirectedEdge> pathTo(int v)    shortest path from s to v
```

```
    boolean hasPathTo(int v)            is there a path from s to v?
```

```
% java SP tinyEWD.txt 0
0 to 0 (0.00):
0 to 1 (1.05): 0->4 0.38 4->5 0.35 5->1 0.32
0 to 2 (0.26): 0->2 0.26
0 to 3 (0.99): 0->2 0.26 2->7 0.34 7->3 0.39
0 to 4 (0.38): 0->4 0.38
0 to 5 (0.73): 0->4 0.38 4->5 0.35
0 to 6 (1.51): 0->2 0.26 2->7 0.34 7->3 0.39 3->6 0.52
0 to 7 (0.60): 0->2 0.26 2->7 0.34
```




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- ▶ *negative weights*



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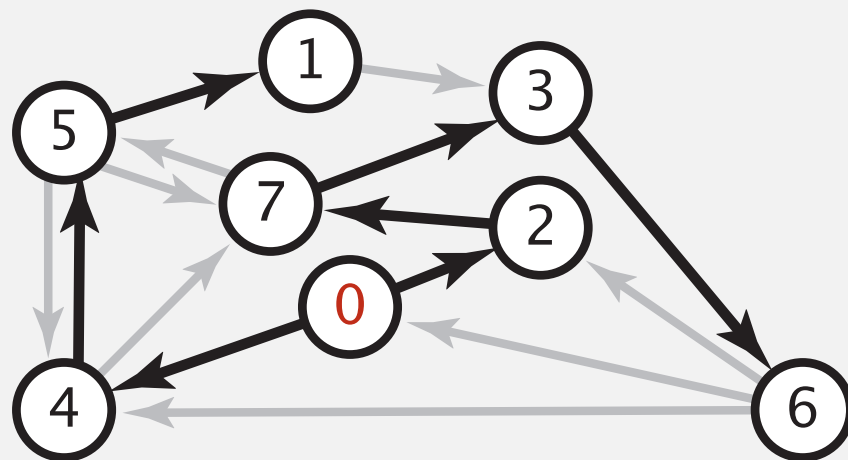
Data structures for single-source shortest paths

Goal. Find the shortest path from s to every other vertex.

Observation. A **shortest-paths tree** (SPT) solution exists. Why?

Consequence. Can represent the SPT with two vertex-indexed arrays:

- $\text{distTo}[v]$ is length of shortest path from s to v .
- $\text{edgeTo}[v]$ is last edge on shortest path from s to v .



shortest-paths tree from 0

	edgeTo[]	distTo[]
0	null	0
1	5->1 0.32	1.05
2	0->2 0.26	0.26
3	7->3 0.37	0.97
4	0->4 0.38	0.38
5	4->5 0.35	0.73
6	3->6 0.52	1.49
7	2->7 0.34	0.60

parent-link representation

Data structures for single-source shortest paths

Goal. Find the shortest path from s to every other vertex.

Observation. A **shortest-paths tree** (SPT) solution exists. Why?

Consequence. Can represent the SPT with two vertex-indexed arrays:

- `distTo[v]` is length of shortest path from s to v .
- `edgeTo[v]` is last edge on shortest path from s to v .

```
public double distTo(int v)
{ return distTo[v]; }
```

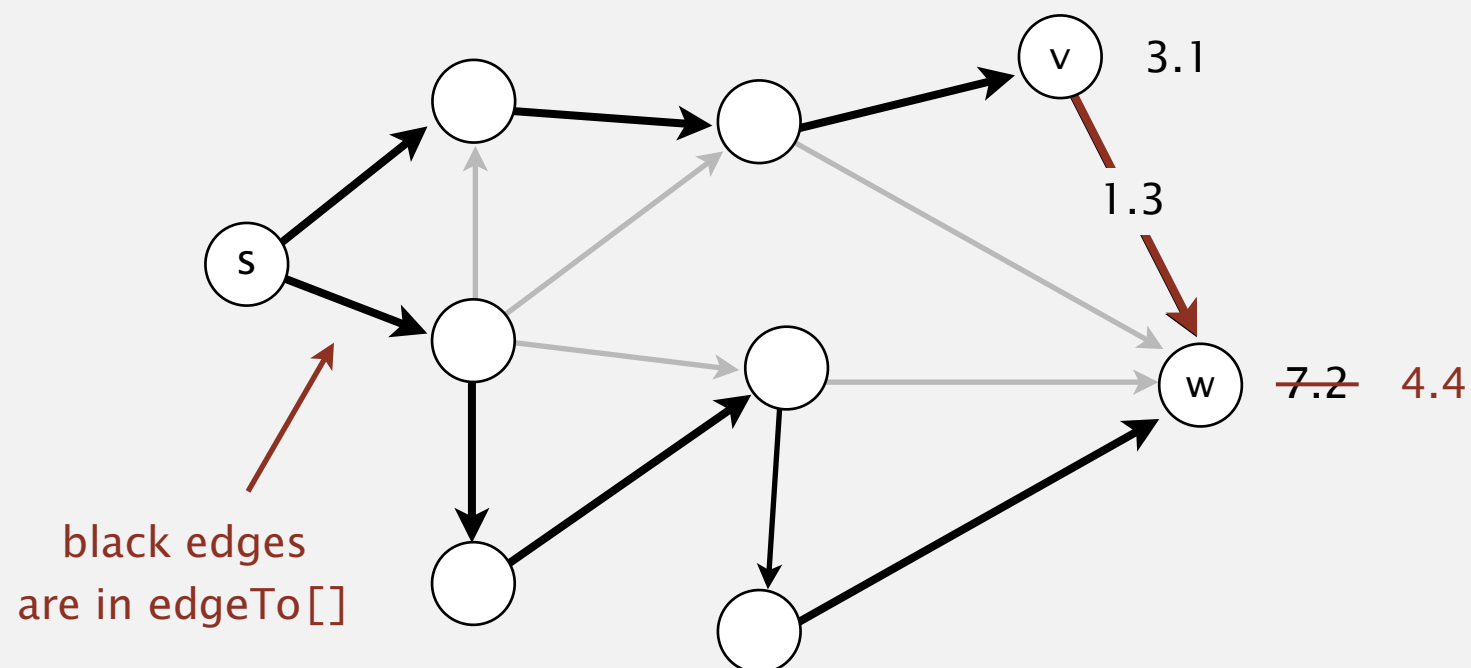
```
public Iterable<DirectedEdge> pathTo(int v)
{
    Stack<DirectedEdge> path = new Stack<DirectedEdge>();
    for (DirectedEdge e = edgeTo[v]; e != null; e = edgeTo[e.from()])
        path.push(e);
    return path;
}
```

Edge relaxation

Relax edge $e = v \rightarrow w$.

- $\text{distTo}[v]$ is length of shortest **known** path from s to v .
- $\text{distTo}[w]$ is length of shortest **known** path from s to w .
- $\text{edgeTo}[w]$ is last edge on shortest **known** path from s to w .
- If $e = v \rightarrow w$ gives shorter path to w through v ,
update both $\text{distTo}[w]$ and $\text{edgeTo}[w]$.

$v \rightarrow w$ successfully relaxes



Edge relaxation

Relax edge $e = v \rightarrow w$.

- `distTo[v]` is length of shortest **known** path from s to v .
- `distTo[w]` is length of shortest **known** path from s to w .
- `edgeTo[w]` is last edge on shortest **known** path from s to w .
- If $e = v \rightarrow w$ gives shorter path to w through v ,
update both `distTo[w]` and `edgeTo[w]`.

```
private void relax(DirectedEdge e)
{
    int v = e.from(), w = e.to();
    if (distTo[w] > distTo[v] + e.weight())
    {
        distTo[w] = distTo[v] + e.weight();
        edgeTo[w] = e;
    }
}
```


Shortest-paths optimality conditions

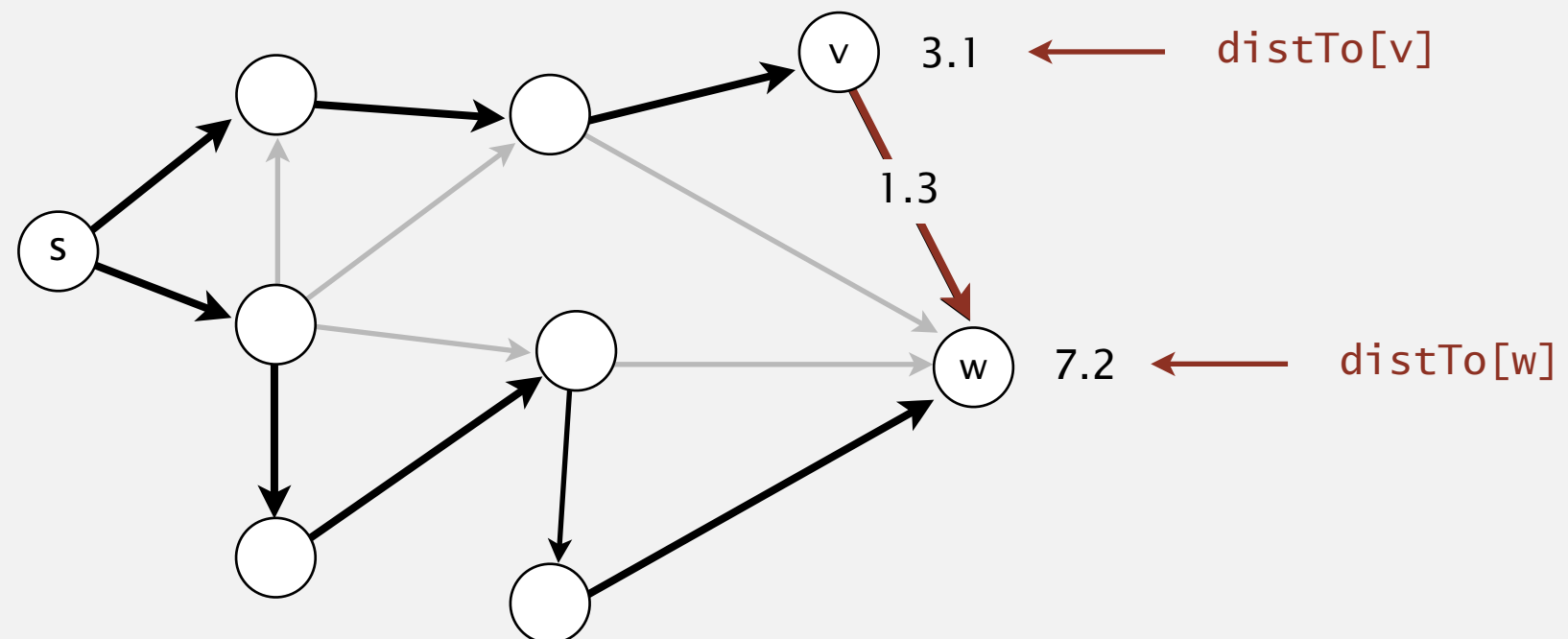
Proposition. Let G be an edge-weighted digraph.

Then $\text{distTo}[]$ are the shortest path distances from s iff:

- $\text{distTo}[s] = 0$.
- For each vertex v , $\text{distTo}[v]$ is the length of some path from s to v .
- For each edge $e = v \rightarrow w$, $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight}()$.

Pf. \Leftarrow [necessary]

- Suppose that $\text{distTo}[w] > \text{distTo}[v] + e.\text{weight}()$ for some edge $e = v \rightarrow w$.
- Then, e gives a path from s to w (through v) of length less than $\text{distTo}[w]$.



Shortest-paths optimality conditions


Proposition. Let G be an edge-weighted digraph.

Then $\text{distTo}[]$ are the shortest path distances from s iff:

- $\text{distTo}[s] = 0$.
- For each vertex v , $\text{distTo}[v]$ is the length of some path from s to v .
- For each edge $e = v \rightarrow w$, $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight}()$.

Pf. \Rightarrow [sufficient]

- Suppose that $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k = w$ is a shortest path from s to w .

- Then,
$$\begin{aligned} \text{distTo}[v_1] &\leq \text{distTo}[v_0] + e_1.\text{weight}() \\ \text{distTo}[v_2] &\leq \text{distTo}[v_1] + e_2.\text{weight}() \\ &\dots \\ \text{distTo}[v_k] &\leq \text{distTo}[v_{k-1}] + e_k.\text{weight}() \end{aligned}$$



$e_i = i^{\text{th}}$ edge on shortest path from s to w

- Add inequalities; simplify; and substitute $\text{distTo}[v_0] = \text{distTo}[s] = 0$:

$$\text{distTo}[w] = \text{distTo}[v_k] \leq \underline{e_1.\text{weight}() + e_2.\text{weight}() + \dots + e_k.\text{weight}()}$$

weight of shortest path from s to w

- Thus, $\text{distTo}[w]$ is the weight of shortest path to w . ■

 weight of some path from s to w

Generic shortest-paths algorithm

Generic algorithm (to compute SPT from s)

Initialize $\text{distTo}[s] = 0$ and $\text{distTo}[v] = \infty$ for all other vertices.

Repeat until optimality conditions are satisfied:

- Relax any edge.
-

Proposition. Generic algorithm computes SPT (if it exists) from s .

Pf sketch.

- Throughout algorithm, $\text{distTo}[v]$ is the length of a simple path from s to v (and $\text{edgeTo}[v]$ is last edge on path).
- Each successful relaxation decreases $\text{distTo}[v]$ for some v .
- The entry $\text{distTo}[v]$ can decrease at most a finite number of times. ■

Generic shortest-paths algorithm

Generic algorithm (to compute SPT from s)

Initialize $\text{distTo}[s] = 0$ and $\text{distTo}[v] = \infty$ for all other vertices.

Repeat until optimality conditions are satisfied:

- Relax any edge.
-

Efficient implementations. How to choose which edge to relax?

Ex 1. Dijkstra's algorithm (nonnegative weights).

Ex 2. Topological sort algorithm (no directed cycles).

Ex 3. Bellman-Ford algorithm (no negative cycles).



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Edsger W. Dijkstra: select quotes

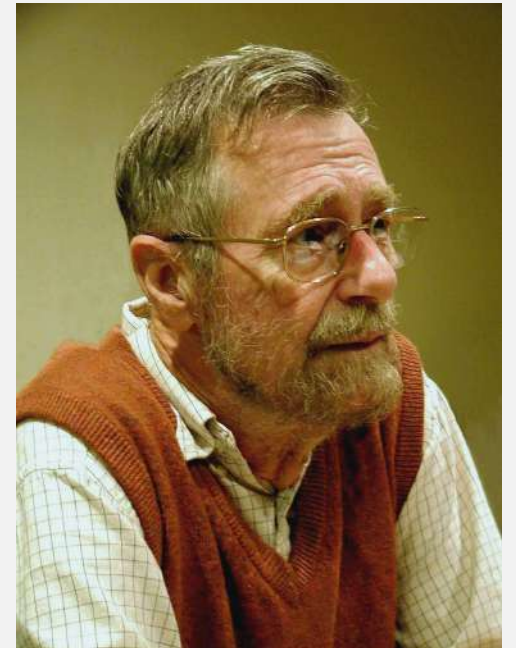
“ Do only what only you can do. ”

“ In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind. ”

“ The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence. ”

“ It is practically impossible to teach good programming to students that have had a prior exposure to BASIC: as potential programmers they are mentally mutilated beyond hope of regeneration. ”

“ APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums. ”



Edsger W. Dijkstra
Turing award 1972

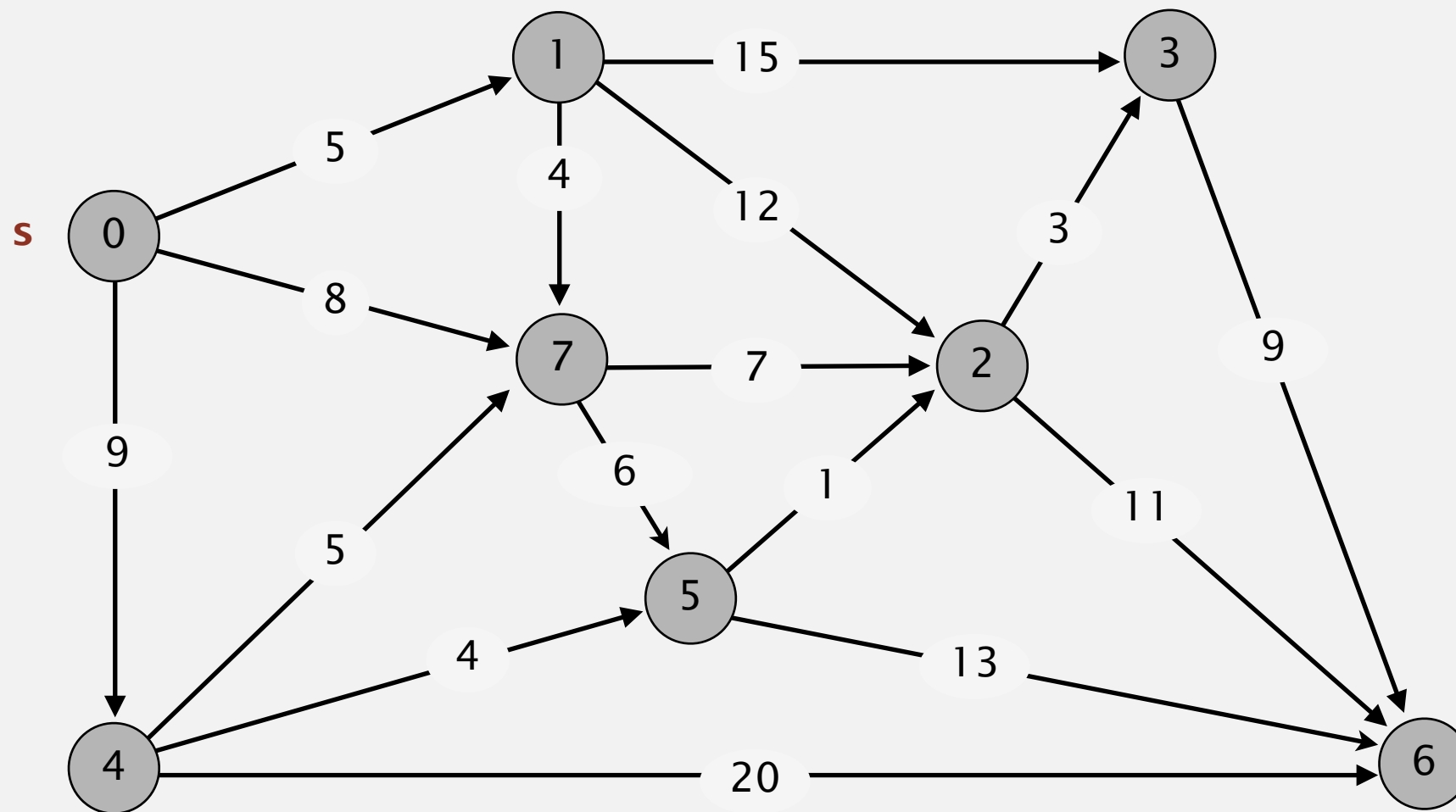
Edsger W. Dijkstra: select quotes



Dijkstra's algorithm demo



- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest $\text{distTo}[]$ value).
- Add vertex to tree and relax all edges pointing from that vertex.

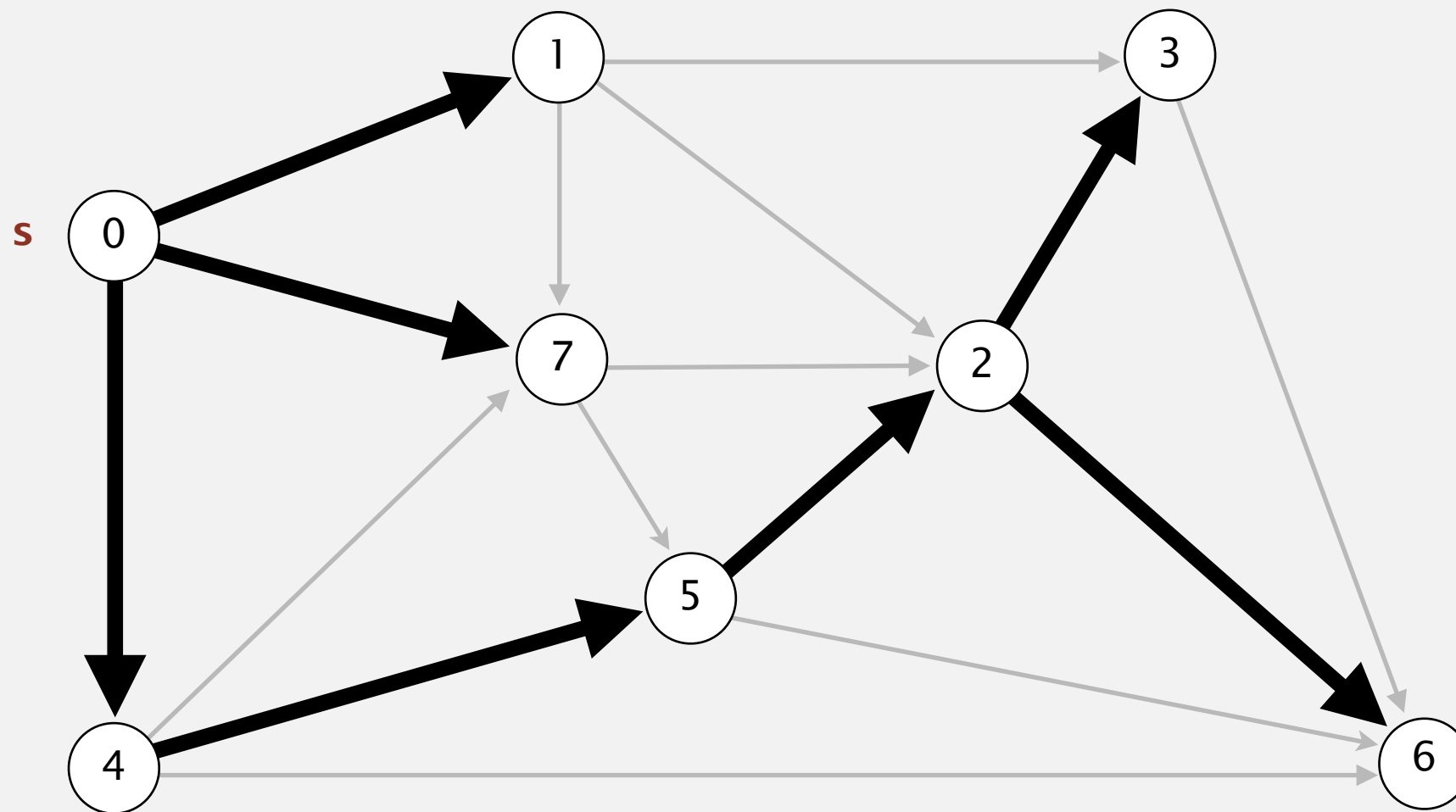


an edge-weighted digraph

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

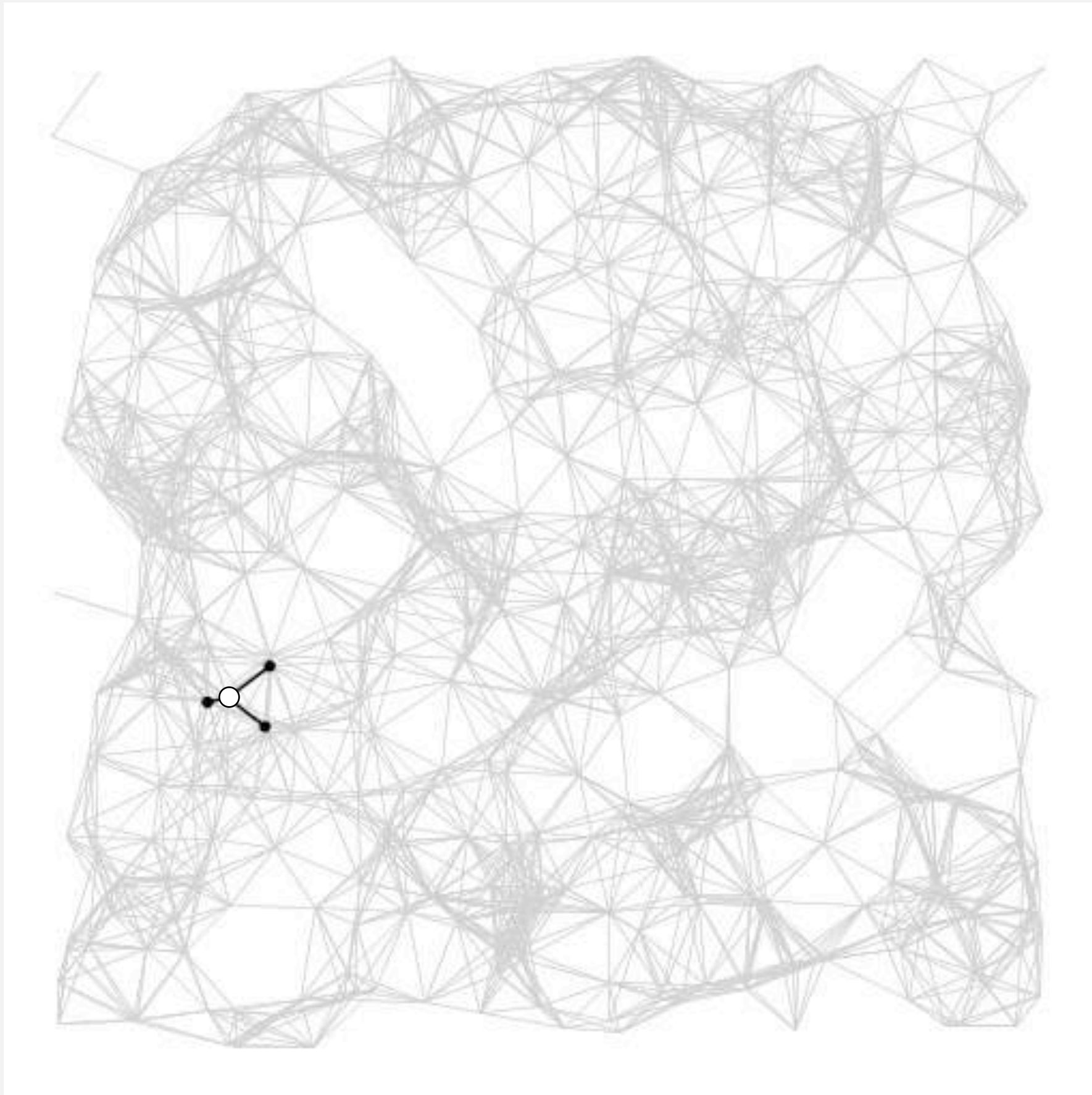
- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.



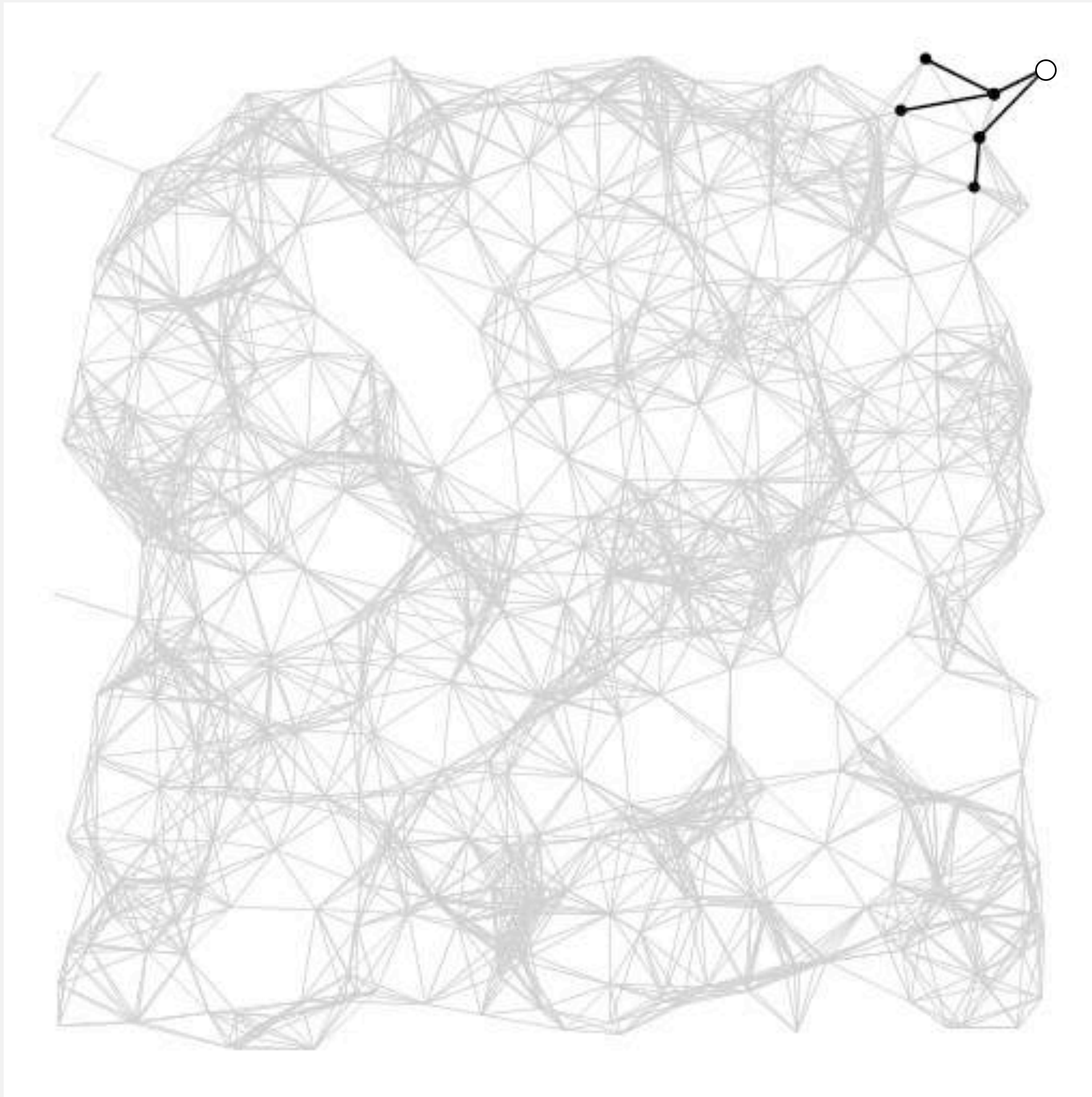
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

shortest-paths tree from vertex s

Dijkstra's algorithm visualization





Dijkstra's algorithm visualization



Dijkstra's algorithm: correctness proof

Proposition. Dijkstra's algorithm computes a SPT in any edge-weighted digraph with nonnegative weights.

Pf.

- Each edge $e = v \rightarrow w$ is relaxed exactly once (when v is relaxed), leaving $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight}()$.
- Inequality holds until algorithm terminates because:
 - $\text{distTo}[w]$ cannot increase  $\text{distTo}[]$ values are monotone decreasing
 - $\text{distTo}[v]$ will not change  we choose lowest $\text{distTo}[]$ value at each step (and edge weights are nonnegative)
- Thus, upon termination, shortest-paths optimality conditions hold. ■

Dijkstra's algorithm: Java implementation

```
public class DijkstraSP
{
    private DirectedEdge[] edgeTo;
    private double[] distTo;
    private IndexMinPQ<Double> pq;

    public DijkstraSP(EdgeWeightedDigraph G, int s)
    {
        edgeTo = new DirectedEdge[G.V()];
        distTo = new double[G.V()];
        pq = new IndexMinPQ<Double>(G.V());

        for (int v = 0; v < G.V(); v++)
            distTo[v] = Double.POSITIVE_INFINITY;
        distTo[s] = 0.0;

        pq.insert(s, 0.0);
        while (!pq.isEmpty())
        {
            int v = pq.delMin();
            for (DirectedEdge e : G.adj(v))
                relax(e);
        }
    }
}
```

← relax vertices in order
of distance from s

Dijkstra's algorithm: Java implementation

```
private void relax(DirectedEdge e)
{
    int v = e.from(), w = e.to();
    if (distTo[w] > distTo[v] + e.weight())
    {
        distTo[w] = distTo[v] + e.weight();
        edgeTo[w] = e;
        if (pq.contains(w)) pq.decreaseKey(w, distTo[w]);
        else                pq.insert      (w, distTo[w]);
    }
}
```

← update PQ

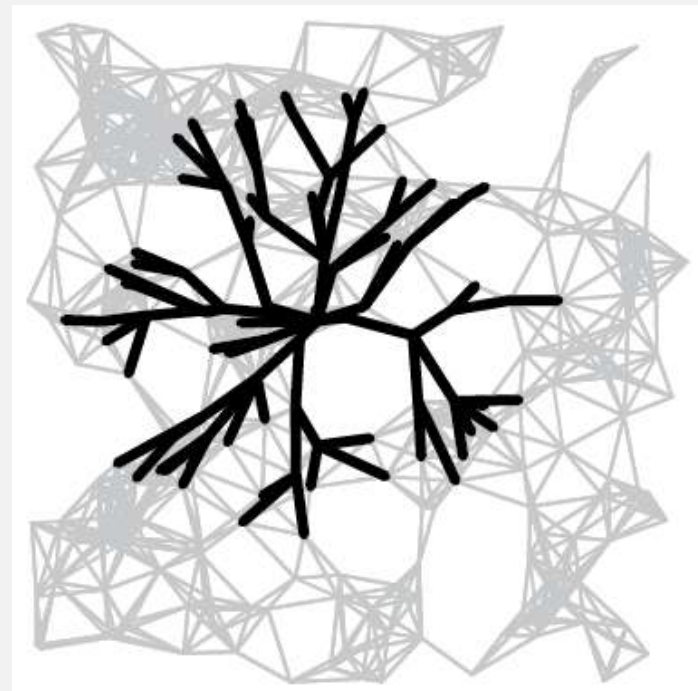
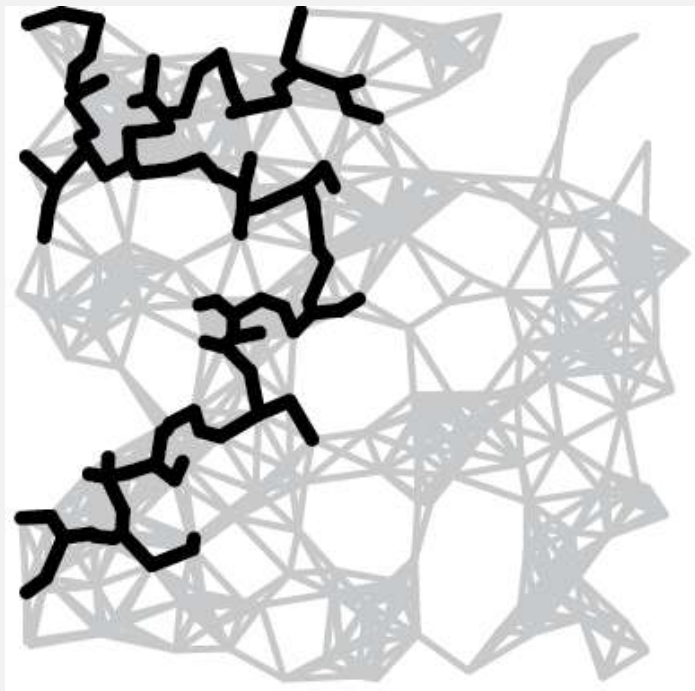
Computing spanning trees in graphs

Dijkstra's algorithm seem familiar?

- Prim's algorithm is essentially the same algorithm.
- Both are in a family of algorithms that compute a graph's spanning tree.

Main distinction: Rule used to choose next vertex for the tree.

- Prim's: Closest vertex to the **tree** (via an undirected edge).
- Dijkstra's: Closest vertex to the **source** (via a directed path).



Note: DFS and BFS are also in this family of algorithms.

Dijkstra's algorithm: which priority queue?

Depends on PQ implementation: V insert, V delete-min, E decrease-key.

PQ implementation	insert	delete-min	decrease-key	total
unordered array	1	V	1	V^2
binary heap	$\log V$	$\log V$	$\log V$	$E \log V$
d-way heap (Johnson 1975)	$\log_d V$	$d \log_d V$	$\log_d V$	$E \log_{E/V} V$
Fibonacci heap (Fredman–Tarjan 1984)	1 †	$\log V$ †	1 †	$E + V \log V$

† amortized

Bottom line.

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.



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