Lecture Transcript
Properties of Line Segment

Hello and welcome to this next session on data structures and algorithms. Starting today, we'll discuss some very simple yet interesting algorithms from the area of computational geometry. Today we'll discuss some properties associated with line segments for example, how would you know if given three points p0, p1 and p2 whether the angle suspended at p0 by the line segments p1, p0 and p2, p0 is clockwise or counter-clockwise that is how do you find out if p1 lies above or below the line segment p0, p2 and vice-a-versa. So, computational geometry happen to be a branch of computer science that studies algorithms for solving geometrical problems and it has applications in graphics, robotics, VLSI design, computer-aided design, molecular modeling etc. Basically, all areas that have something to do with shapes of objects and understanding properties associated with those shapes. So the general input is a set of geometric objects set of points, line segments, vertices of polygon etc. Output is either response to query about the objects, or some cases use synthesizes new geometric object.

So today's discussion will be largely about query about some objects, for example, do two line segments intersect. In the next class, we will discuss an instance of the second kind of output the synthesis of a new geometric object. We'll discuss the synthesis something called the hull or more specifically the convex hull of a set of points. You could think of the convex hull as the tightest enclosing polygon around the set of points for this you might have to drop some points from the boundary of the hull and let them lie in the interior of the hull. Coming back to line segments we will be interested in characterization of line segments as a convex combination of the two points that form the n points of the line segments. So let's call these two points p1 and p2, so any point p3 that lies between p1 and p2 is understood to be a convex combination of p1 and p2 and the coordinates of p3 are basically convex combinations of the coordinates of p1 and p2 which is alpha x1 + (1 - alpha x2) for some alpha between 0 and 1, y3 is alpha y1 +(1 - alpha y2). So, more generally we call the point itself to be a convex combination of the two points p1 and p2. So p3 is alpha p1 +(1 - alpha p2). So we will start with the simplest of inquiries given points p0, p1 and p2 and let's say for simplicity sake we let p0 be the (0,0) that is we set p0 to be the reference point origin and our question is whether p1, p0 is clockwise or counter-clockwise with respect to p2, p0 and this how you depict? As you can see in this figure p0p1 is counter-clockwise from p0p2, one might also be interested in whether any two given line segments intersect at all in which case the inquiry would be more like do p1, p2 and P3, P4 intersect at all.

Let's answer the first question direction of P1 from P2 with respect to P0. A very fundamental tool to answer this question is to look at the cross product, cross product is depicted as P1 X P2 and it is not commutative so P1 X P2 is not necessarily equal to P2 X P1. Now, one can understand this cross product as follows. So if P0 is here and we consider a parallelogram with corners at P0, P1 and P2 and the fourth corner being the sum of P1 and P2 which happens to be here. One can interpret this cross product as the area of this parallelogram. However, this area is not commutative with respect to P1 and P2, so we call this the signed area. So, the signed area P1 X P2 is not the same as a signed area P2 X P1. So one can recall that area of such a parallelogram can be obtained as the determinant of this matrix with P1 as a first column and P2 as a second column. This determinant is x1 y2 - x2 y1. Similarly, the cross product P2 X P1 can be written as this determinant x2 y2 x1 y1 and well this is nothing but x2 y2 - x1 y1. So one can see that P2 X P1 is exactly the opposite sign as P1 X P2 and that is why we refer to this area as a signed area.

Now, what does a sign actually denote, well it turns out that if P1 is clockwise from P2, which is a case here, then this signed area or the determinant of P1 X P2 is positive. On the other hand you can verify that the sign of P2 X P1 which is exactly the negative of P1 X P2 is negative. So, what this says is that while P1 is clockwise from P2, P2 is counter-clockwise from P1. So, continue with this same example where we assume that P0 is the origin, the cross product is computed as follows P1 X P2 is x1 * y2 - x2 * y1 and for this specific example P2 is (4,1) P1 is (2, 3) and this cross product is -10. Since the result P1 X P2 is negative, we have verified that P1 is counter-clockwise from P2 with respect to the origin P0. One can immediately see that P2 X P1 is +10 and therefore p2 is clockwise from p1 with respect to the origin p0. How about two arbitrary line segments p0p1 and p1p2? So, what does it mean? Well, this means p0p1 and p1p2.

Now, we'll not make any specific assumption about p1 or p0 or p2. Do they turn left or right at point p1? Now, it's similar to the previous question of whether p2 turns clockwise or counter clockwise from p0 with respect p1. So this will be equivalent to the question, does p2 turn counter clockwise or clockwise from p0 with respect to p1? And that's what we have summarized here. The approach will be to check whether the directed segment p0p2 is clockwise or counter-clockwise with respect to the directed segment p0p1, and that is through these cross product (p2 - p0) i.e. the first vector that we want (p2 - p0) i.e.the vector whose orientation needs to be check with respect to (p1 - p0). One can generalize the discussion so far to address the following question. We are interested in determining whether two line segments intersect or not. We will answer this question by looking at the direction of turn. We already seen that if our points were p0, p1 and p2 then the cross product (p1 - p0) x (p2 - p0)if it's positive then we know that p1 turns clockwise from p2 with respect to p0.

Now, can weuse this fact to answer the general question if two line segments p1p2 and p3p4 intersect? One would look at different possible angles here, but let me explain the main motivation. The motivation is to see if the points p1 and p2 lie on opposite sides of the line p3p4 and also simultaneously the points p3 and p4 lie on opposite sides of the line p1p2. So, you can convince yourself that if both occur that is p1 and p2 are on opposite sides of the line p3p4, and p3 and p4 are on opposite sides of the line p1 and p2 then indeed the line segments p1p2 and p3p4 intersect. There, of course, some corner cases that we need to handle. What is a point of intersection? What if the points actually meet at one of p1, p2, p3 or p4? But keeping that aside for the time being, we'll answer this question by looking at angles. So what we expect is that if p3 and p4 lie on opposite sides of the line p1p2, then the angle formed by p3 at p1 with respect to p2 should have the opposite sign as the angle formed by p4 at p1 with respect to p2. In other words, we would expect that on one hand if p3 was clockwise from p2 with respect to p1 then p4 should be anticlockwise from p2 with respect to p1 likewise you would expect p1 to be anticlockwise or counterclockwise from p4 with respect to p3 and p2 should be clockwise from p4 with respect to p3. In other words, we would expect the signs of cross product (p1-p3) with (p4-p3) to be exactly opposite to that of (p2-p3) with (p4-p3).

This has been depicted in this picture. We expect these line segment to intersect, if (p1-p3) cross product with (p4-p3) is negative, which means p1 turns counterclockwise from p4 with respect to p3 and on the other hand, p2 turns clockwise from p4 with respect to p3, this is indeed the case here. It's also possible that the p1 and p2 get exchanged, in which case all we expect is one to be clockwise and the other to be a counterclockwise. So what we are looking for in this entire process is the sign of (p1-p3) cross product with (p4-p3) to be exactly opposite of the sign of (p2-p3) cross product with (p4-p3). Similarly, you would expect the sign of (p4-p1) cross product with (p2-p1) to be exactly opposite of the sign of (p3-p1) cross product with (p2-p1). One can also consider the case where p3 and p4 are on opposite sides of the line p1,p2, whereas, p1 and p2 are on the same side of the line p3 and p4. In such a case, yes the lines do intersect, but the line segment do not.

This precisely is a significance of both the test. One needs to determine that the cross products have opposites signs for both the for both the pairs (p1, p2) and (p3, p4). It doesn't suffice to check for opposite signs only for one of them. Here is the corners case, what if one of the points say p3 happens to be collinear with p1 p2. In general, what do you expect if three points p1, p2 and p3 are collinear. Well, if they are collinear we would expect that p3 is neither counter clockwise nor clockwise from p2 with respect to p1. It can be neither positive nor negative. So we would expect this cross product of (p3-p1) with (p2-p1) to be 0. So the question is that if indeed p1 and p2 lie on opposite sides of the line p3 p4, but if between the cross products associated with p3 or p4, one of them turns out to be 0, then it is possible that the point with the zero cross product lies exactly on the line segment joining p1 and p2. To check if p3 lies on the line segment p1 p2, we are going to check if the x coordinate of p3 is between the x coordinates of p1 and p2. x3 should be exactly in the line segment or in the close interval [x1, x2] and check if y3 belongs to the close interval [y2, y1].

We have very implicitly referred to this specific orientation of p1 and p2 in defining these intervals [x1, x2] and [y2, y1]. We know that y2 is indeed smaller than y1 and x1 smaller than x2. In general, what you require is that x3 lies in the close interval min(x1, x2), max(x1, x2) and y3 should also lie in this close interval min(y1, y2), max(y1, y2). If this is not the case, it could mean that p3 is indeed collinear but it doesn't lie on the line segment. So let me present a case, p3 is collinear and here's p3, however we find that its x coordinate as well as the y coordinate lie outside the range close interval provided by p1 and p2. This is precisely the example, p3 is indeed collinear with p1 and p2 but unfortunately p3 doesn't lie on the line segment p1 and p2. So please note, how we've at times talked about lines and at another times talked about line segments. As per as points lying on opposite sides as concern, we talk of they are lying on opposite sides of a line. However, when I comes to collinearity we insist on line segments. So, we've put together our algorithm for detecting if two segments intersect. Initially, we compute 4 directions. Direction d1 is the direction of p1 from p4 with respect to p3 and to remind you what this means, this means that we are pivoted at p3, we look at p1-p3 and p4-p3, if the cross product p1-p3 times p4-p3 is positive it means that p1 is clockwise from p4 with respect to p3. So likewise, we look for the direction of p2 from p4 with respect to p3, direction of p3 from p2 with respect to p1 of p4 from p2 with respect to p1. Note that we have arbitrary chosen p3 and p1 from the line segments p3, p4 and p1, p2. As homework, you might want to look at the direction of p1 from p3 with respect to p4. Basically, swapping p3 and p4 here and similarly swapping p1 and p2 in the directions d3 and d4. You will find that not much will change at all. Now you find that d1 and d2 have opposite signs irrespective of whether the pivot is p3 or p4 and if directions d3 and d4 have opposite signs then you are done. If it turns out that any of the four that d1 to d4 are zero which means some three points are colinear then one checks if the collinearity if on the line segment itself.

So, if d1 is zero that is if the direction of p1 from p4 with respect to p3 is a zero then you check if p1 lies on the line segment p3, p4. If d2 zero, you check if p2 is on line segment p3, p4 and so on. In either of these four cases again, you know that the two line segments indeed intersect. If none of these are true which means the directions d1 and d2 and d3 and d4 don't have opposite signs, none of these three points are colinear, certainly, the line segments do not intersect at all and the case for this is shown here. As already indicated, direction pi, pj, pk looks at the direction of pk with respect to pj from pi. If the direction is positive it means that pk is clockwise from pj with respect to pi. And on the segment routine does exactly what we described earlier, you look at the min (xi and xj) and then max (xi and xj) and ensure that xk lies exactly between the two. You also need to ensure that yk lies in the closed interval determined by the min of yi and yj and max of yi and yj.

Thank you.