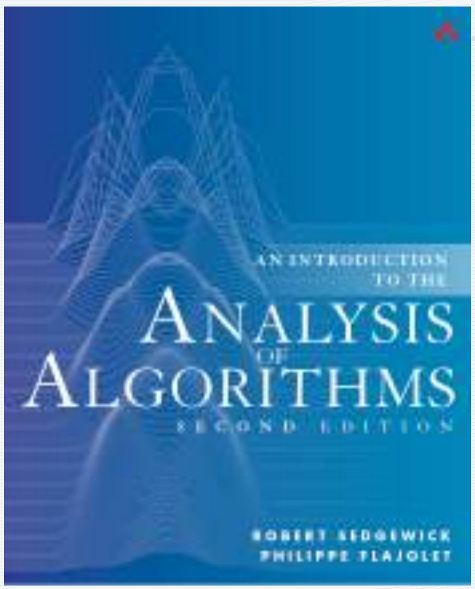
ANALYTIC COMBINATORICS
PART ONE

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# 5. Analytic Combinatorics

## Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Features:

- Analysis begins with formal *combinatorial constructions*.
- The *generating function* is the central object of study.
- *Transfer theorems* can immediately provide results from formal descriptions.
- Results extend, in principle, to any desired precision on the standard scale.
- Variations on fundamental constructions are easily handled.

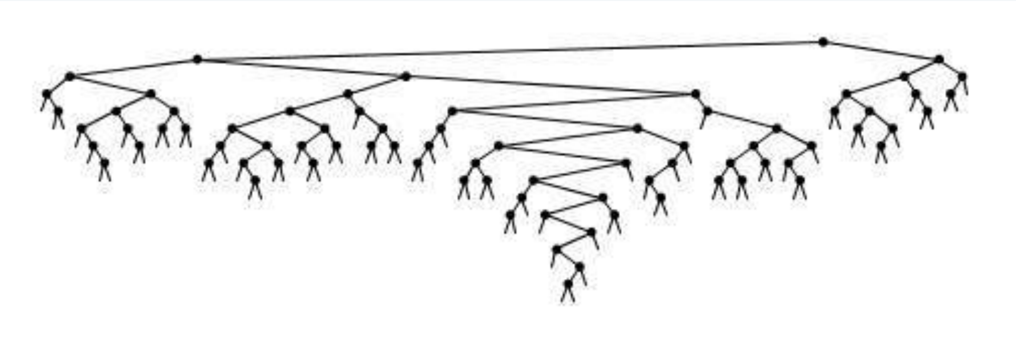

combinatorial constructions $\xrightarrow[\text{transfer theorem}]{\textit{symbolic}}$ generating function equation $\xrightarrow[\text{transfer theorem}]{\textit{analytic}}$ coefficient asymptotics

the "symbolic method"

# Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Ex: How many binary trees with $N$ nodes?



$$T = E + Z \times T \times T$$

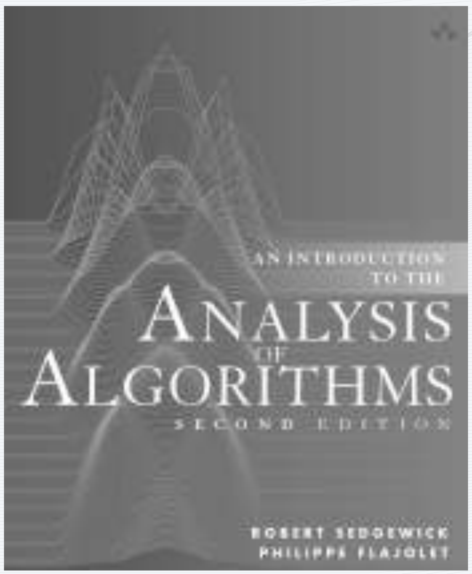
combinatorial construction

$$T(z) = 1 + zT(z)^2$$

GF equation

$$T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$$

coefficient asymptotics

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# 5. Analytic Combinatorics

- **The symbolic method**
- Labelled objects
- Coefficient asymptotics
- Perspective

## The symbolic method

is an approach for translating *combinatorial constructions* to *GF equations*

- Define a *class* of combinatorial objects.
- Define a notion of *size.*
- Define a *GF* whose coefficients count objects of the same size.
- Define *operations* suitable for constructive definitions of objects.
- Develop *translations* from constructions to operations on GFs.

| Examples |
| --- |
| $A$, $B$, $Z$ |
| $\mid b \mid$ |
| $A(z)$ |
| $\mathrm{A} \times \mathrm{B}$ |
| $A(z)B(z)$ |

Formal basis:

- A *combinatorial class* is a set of objects and a *size function.*
- An *atom* is an object of size 1.
- An *neutral object* is an atom of size 0.
- *A combinatorial construction* uses the union, product, and sequence operations to define a class in terms of atoms and other classes.

Building blocks

| *notation* | *denotes* | *contains* |
| --- | --- | --- |
| $Z$ | atomic class | an atom |
| $E$ | neutral class | neutral object |
| $\Phi$ | empty class | nothing |

## Unlabelled class example 1: natural numbers

**Def.** A *natural number* is a set (or a sequence) of atoms.

| ● | ●● | ●●● | ●●●● | ●●●●● |
|---|---|---|---|---|
| $I_1 = 1$ | $I_2 = 1$ | $I_3 = 1$ | $I_4 = 1$ | $I_5 = 1$ |

*unary* notation

| *counting sequence* | *OGF* |
|---|---|
| $I_N = 1$ | $\dfrac{1}{1-z}$ |

$$\sum_{N\geq 0} z^N = \frac{1}{1-z}$$

## Unlabelled class example 2: bitstrings

**Def.** A *bitstring* is a sequence of 0 or 1 bits.

(empty)

$B_0 = 1$

0
1

$B_1 = 2$

00
01
10
11

$B_2 = 4$

000
001
010
011
100
101
110
111

$B_3 = 8$

0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110
1111

$B_4 = 16$

| *counting sequence* | *OGF* |
| --- | --- |
| $B_N = 2^N$ | $\dfrac{1}{1-2z}$ |

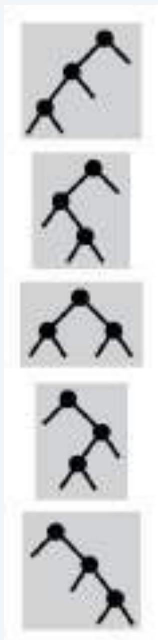
$$\sum_{N\geq 0} 2^N z^N = \sum_{N\geq 0} (2z)^N = \frac{1}{1-2z}$$

## Unlabelled class example 3: binary trees

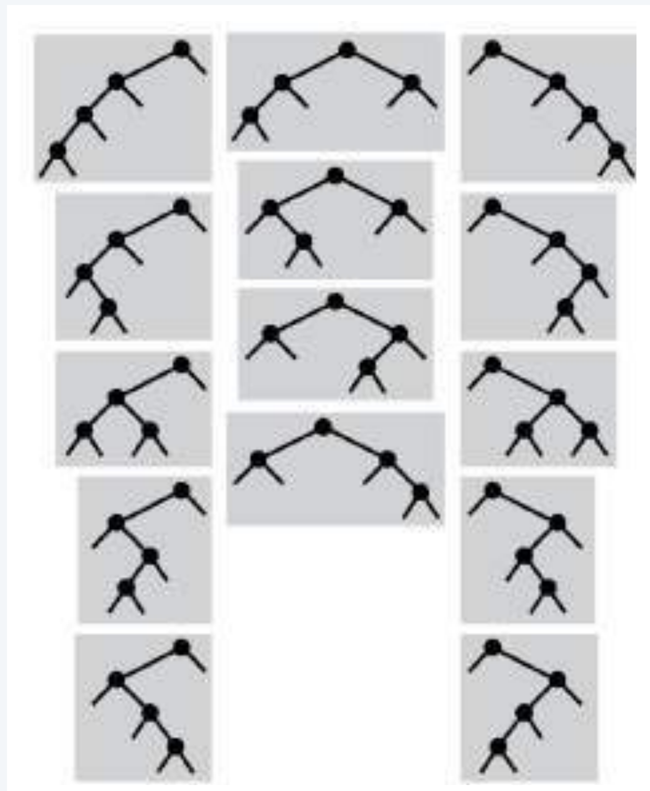
**Def.** A *binary tree* is empty or a sequence of a node and two binary trees

$T_1 = 1$

$T_2 = 2$



$T_3 = 5$



$T_4 = 14$

| *counting sequence* | *OGF* |
|---|---|
| $T_N = \frac{1}{N+1}\binom{2N}{N}$ | $\frac{1}{2z}(1 - \sqrt{1-4z})$ |

Catalan numbers (see Lecture 3)

$$T(z) = 1 + zT(z)^2$$

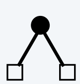
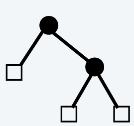
## Combinatorial constructions for unlabelled classes

| *construction* | *notation* | *semantics* |
|---|---|---|
| disjoint union | $A + B$ | disjoint copies of objects from $A$ and $B$ |
| Cartesian product | $A \times B$ | ordered pairs of copies of objects, one from $A$ and one from $B$ |
| sequence | $SEQ(A)$ | sequences of objects from $A$ |

$A$ and $B$ are combinatorial classes of unlabelled objects

Ex 1. ( 00 + 01 ) × ( 101 + 110 + 111 ) = 00101 00110 00111 01101 01111

Ex 2. $\bullet \times SEQ(\bullet) = \bullet\ \ \bullet\bullet\ \ \bullet\bullet\bullet\ \ \bullet\bullet\bullet\bullet\ \ \bullet\bullet\bullet\bullet\bullet\ \ \bullet\bullet\bullet\bullet\bullet\bullet\ \ \bullet\bullet\bullet\bullet\bullet\bullet\bullet\ \ \ldots$

Ex 3. $\square \times \bullet \times$  = 

*"unlabelled" ?? Stay tuned.*

## The symbolic method for unlabelled classes (transfer theorem)

Theorem. Let $A$ and $B$ be combinatorial classes of unlabelled objects with OGFs $A(z)$ and $B(z)$. Then

| *construction* | *notation* | *semantics* | *OGF* |
|---|---|---|---|
| disjoint union | $A + B$ | disjoint copies of objects from $A$ and $B$ | $A(z) + B(z)$ |
| Cartesian product | $A \times B$ | ordered pairs of copies of objects, one from $A$ and one from $B$ | $A(z)B(z)$ |
| sequence | $SEQ(A)$ | sequences of objects from $A$ | $\dfrac{1}{1 - A(z)}$ |

## Proofs of transfers

are immediate from GF counting

*A* + *B*

$$\sum_{\gamma\in A+B} z^{|\gamma|} = \sum_{\alpha\in A} z^{|\alpha|} + \sum_{\beta\in B} z^{|\beta|} = A(z) + B(z)$$

*A* × *B*

$$\sum_{\gamma\in A\times B} z^{|\gamma|} = \sum_{\alpha\in A}\sum_{\beta\in B} z^{|\alpha|+|\beta|} = \Big(\sum_{\alpha\in A} z^{|\alpha|}\Big)\Big(\sum_{\beta\in B} z^{|\beta|}\Big) = A(z)B(z)$$

*SEQ*( *A* )

$$SEQ(A) \equiv \epsilon + A + A^2 + A^3 + A^4 + \ldots$$

$$1 + A(z) + A(z)^2 + A(z)^3 + A(z)^4 + \ldots = \frac{1}{1 - A(z)}$$

## Symbolic method: binary trees

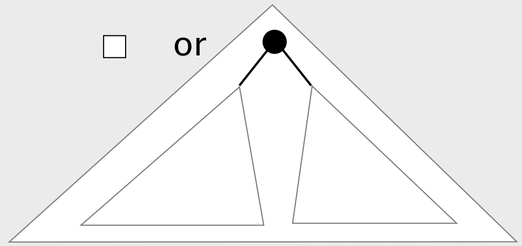
How many binary trees with $N$ nodes?

| | |
|---|---|
| *Class* | $T$, the class of all binary trees |
| *Size* | $\lvert t\rvert$, the number of internal nodes in $t$ |
| *OGF* | $T(z) = \sum_{t \in T} z^{\lvert t\rvert} = \sum_{N \geq 0} T_N z^N$ |

*Atoms*

| *type* | *class* | *size* | *GF* |
|---|---|---|---|
| external node | $Z_{\square}$ | 0 | 1 |
| internal node | $Z_{\bullet}$ | 1 | $z$ |

"a binary tree is an external node or an internal node connected to two binary trees"




Construction $$T = Z_{\square} + T \times Z_{\bullet} \times T$$

OGF equation $$T(z) = 1 + zT(z)^2$$

$$[z^N]T(z) = \frac{1}{N+1}\binom{2N}{N} \sim \frac{4^N}{\sqrt{\pi N^3}}$$

see Lecture 3 and stay tuned.

# Symbolic method: binary trees

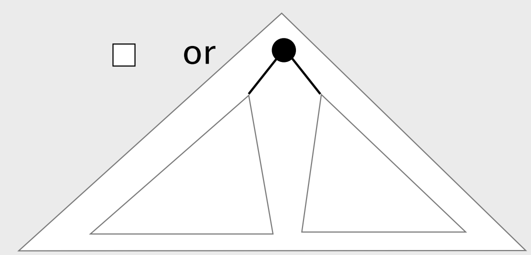
How many binary trees with $N$ *external* nodes?

| | |
|---|---|
| *Class* | $T$, the class of all binary trees |
| *Size* | $\boxed{t}$ , the number of *external* nodes in $t$ |
| *OGF* | $T^{\square}(z) = \sum_{t \in T} z^{\boxed{t}}$ |

*Atoms*

| *type* | *class* | *size* | *GF* |
|---|---|---|---|
| external node | $Z_{\square}$ | 1 | $z$ |
| internal node | $Z_{\bullet}$ | 0 | 1 |

"a binary tree is an external node or an internal node connected to two binary trees"




| | |
|---|---|
| Construction | $T = Z_{\square} + T \times Z_{\bullet} \times T$ |
| OGF equation | $T^{\square}(z) = z + T^{\square}(z)^2$ |

$$T^{\square}(z) = zT(z)$$

$$[z^N]T^{\square}(z) = [z^{N-1}]T(z) = \frac{1}{N}\binom{2N-2}{N-1}$$

← same as # binary trees with $N$–1 internal nodes

## Symbolic method: binary strings

Warmup: How many binary strings with *N* bits?

| | |
|---|---|
| *Class* | *B*, the class of all binary strings |
| *Size* | $\|b\|$, the number of bits in *b* |
| *OGF* | $B(z) = \sum_{b \in B} z^{\|b\|} = \sum_{N \geq 0} B_N z^N$ |

*Atoms*

| type | class | size | GF |
|---|---|---|---|
| 0 bit | $Z_0$ | 1 | $z$ |
| 1 bit | $Z_1$ | 1 | $z$ |

Construction $$B = SEQ(Z_0 + Z_1)$$

"a binary string is a sequence of 0 bits and 1 bits"

OGF equation $$B(z) = \frac{1}{1-2z}$$

$$[z^N]B(z) = 2^N \quad ✓$$

## Symbolic method: binary strings (alternate)

Warmup: How many binary strings with *N* bits?

| | |
|---|---|
| *Class* | $B$, the class of all binary strings |
| *Size* | $\lvert b \rvert$, the number of bits in $b$ |
| *OGF* | $B(z) = \sum_{b \in B} z^{\lvert b \rvert} = \sum_{N \geq 0} B_N z^N$ |

*Atoms*

| *type* | *class* | *size* | *GF* |
|---|---|---|---|
| 0 bit | $Z_0$ | 1 | $z$ |
| 1 bit | $Z_1$ | 1 | $z$ |

Construction $\quad B = E + (Z_0 + Z_1) \times B$

"a binary string is empty or a bit followed by a binary string"

OGF equation $\quad B(z) = 1 + 2zB(z)$

Solution $\quad B(z) = \dfrac{1}{1 - 2z}$

$$[z^N]B(z) = 2^N \quad \checkmark$$

## Symbolic method: binary strings with restrictions

Ex. How many $N$-bit binary strings have no two consecutive 0s?

| $T_0 = 1$ | $T_1 = 2$ | $T_2 = 3$ | $T_3 = 5$ | $T_4 = 8$ | $T_5 = 13$ |
|---|---|---|---|---|---|
| | 0 | 0 1 | 0 1 1 | 0 1 0 1 | 0 1 0 1 1 |
| | 1 | 1 0 | 0 1 0 | 0 1 1 0 | 0 1 0 1 0 |
| | | 1 1 | 1 0 1 | 0 1 1 1 | 0 1 1 0 1 |
| | | | 1 1 0 | 1 0 1 1 | 0 1 1 1 0 |
| | | | 1 1 1 | 1 0 1 0 | 0 1 1 1 1 |
| | | | | 1 1 0 1 | 1 0 1 0 1 |
| | | | | 1 1 1 0 | 1 0 1 1 0 |
| | | | | 1 1 1 1 | 1 0 1 1 1 |
| | | | | | 1 1 0 1 1 |
| | | | | | 1 1 0 1 0 |
| | | | | | 1 1 1 0 1 |
| | | | | | 1 1 1 1 0 |
| | | | | | 1 1 1 1 1 |

Stay tuned for general treatment (Chapter 8)

## Symbolic method: binary strings with restrictions

Ex. How many $N$-bit binary strings have no two consecutive 0s?

| | |
|---|---|
| *Class* | $B_{00}$, the class of binary strings with no 00 |
| *Size* | $\|b\|$, the number of bits in $b$ |
| *OGF* | $B_{00}(z) = \sum_{b \in B_{00}} z^{\|b\|}$ |

*Atoms*

| *type* | *class* | *size* | *GF* |
|---|---|---|---|
| 0 bit | $Z_0$ | 1 | $z$ |
| 1 bit | $Z_1$ | 1 | $z$ |

Construction $$B_{00} = E + Z_0 + (Z_1 + Z_0 \times Z_1) \times B_{00}$$

"a binary string with no 00 is either empty or 0 or it is 1 or 01 followed by a binary string with no 00"

OGF equation $$B_{00}(z) = 1 + z + (z + z^2)B_{00}(z)$$

solution $$B_{00}(z) = \frac{1+z}{1-z-z^2}$$

$$[z^N]B_{00}(z) = F_N + F_{N+1} = F_{N+2}$$

1, 2, 5, 8, 13, ... ✓

## Symbolic method: many, many examples to follow

How many ... with ... ?

| | |
|---|---|
| *Class* | |
| *Size* | |
| *OGF* | |

*Atoms*

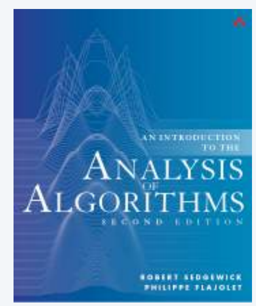
| *type* | *class* | *size* | *GF* |
|---|---|---|---|
| | | | |
| | | | |

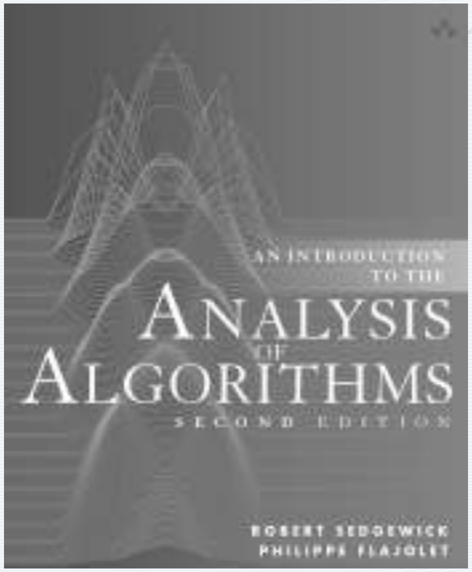
Construction

“a ... is either ... or ... and ...”

OGF equation

solution

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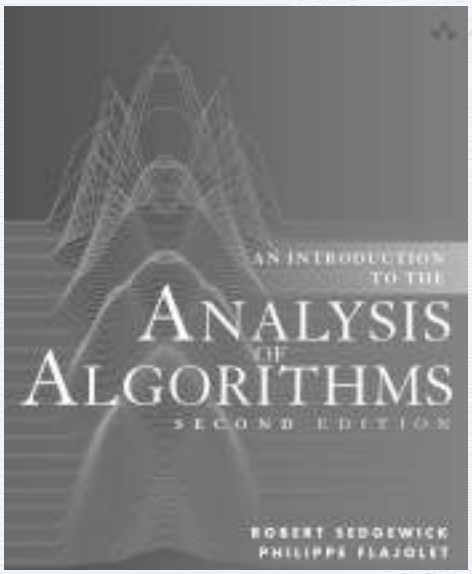
# 5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- Coefficient asymptotics
- Perspective

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# 5. Analytic Combinatorics

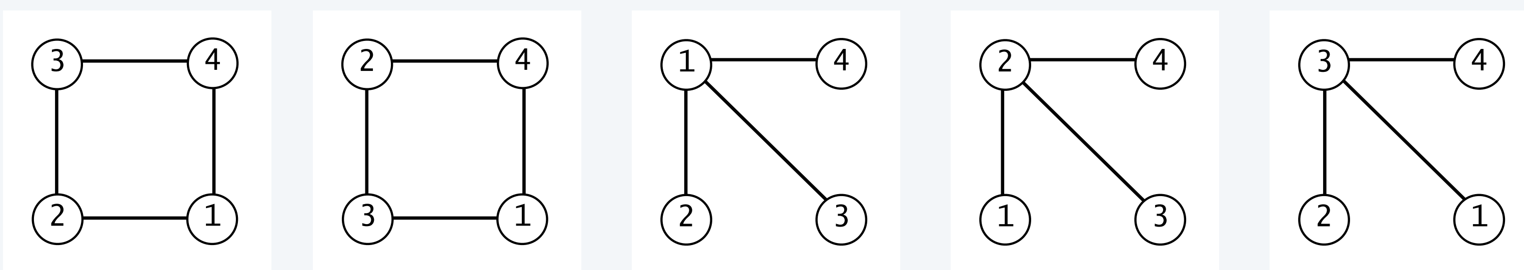
- The symbolic method
- Labelled objects
- Coefficient asymptotics
- Perspective

## Labelled combinatorial classes

have objects composed of *N* atoms, labelled with the integers 1 through *N*.

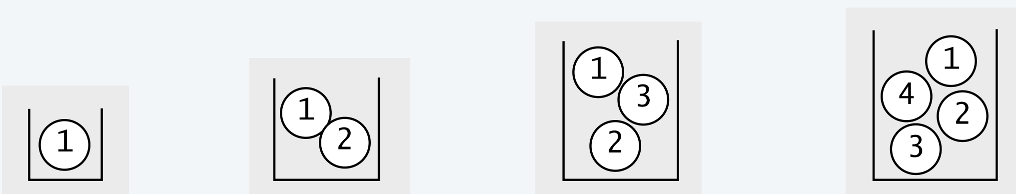
Ex. Different unlabelled objects

Ex. Different labelled objects



## Labelled class example 1: urns

**Def.** An *urn* is a set of labelled atoms.




$U_1 = 1$ $U_2 = 1$ $U_3 = 1$ $U_4 = 1$

| *counting sequence* | *EGF* |
| --- | --- |
| $U_N = 1$ | $e^z$ |

$$\sum_{N \geq 0} \frac{z^N}{N!} = e^z$$

## Labelled class example 2: permutations

**Def.** A *permutation* is a sequence of labelled atoms.

1

$P_1 = 1$

1 2
2 1

$P_2 = 1$

1 2 3
2 1 3
3 1 2
1 3 2
2 3 1
3 2 1

$P_3 = 2$

| | |
|---|---|
| 1 2 3 4 | 1 2 4 3 |
| 2 1 3 4 | 2 1 4 3 |
| 3 1 2 4 | 3 1 4 2 |
| 4 1 2 3 | 4 1 3 2 |
| 1 3 2 4 | 1 3 4 2 |
| 2 3 1 4 | 2 3 4 1 |
| 3 2 1 4 | 3 2 4 1 |
| 4 2 1 3 | 4 2 3 1 |
| 1 4 2 3 | 1 4 3 2 |
| 2 4 1 3 | 2 4 3 1 |
| 3 4 1 2 | 3 4 2 1 |
| 4 3 1 2 | 4 3 2 1 |

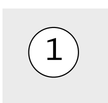
$P_4 = 6$

| *counting sequence* | *EGF* |
|---|---|
| $P_N = N!$ | $\dfrac{1}{1-z}$ |

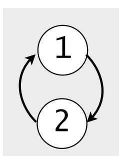
$$\sum_{N\geq 0} \frac{N!z^N}{N!} = \sum_{N\geq 0} z^N = \frac{1}{1-z}$$

## Labelled class example 3: cycles

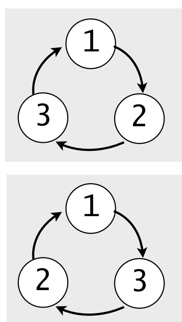
**Def.** A *cycle* is a cyclic sequence of labelled atoms



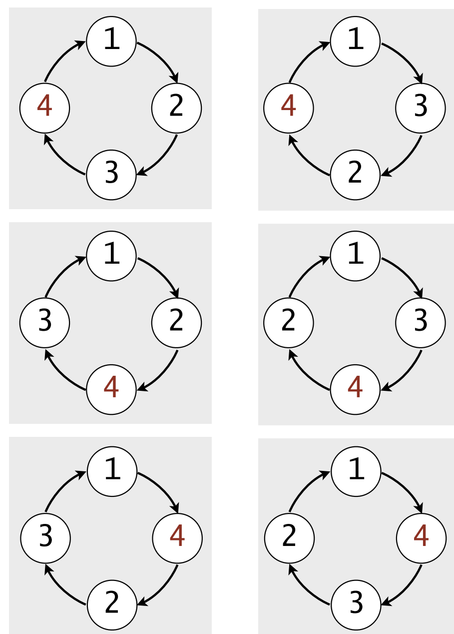
$C_1 = 1$



$C_2 = 1$



$C_3 = 2$



$C_4 = 6$

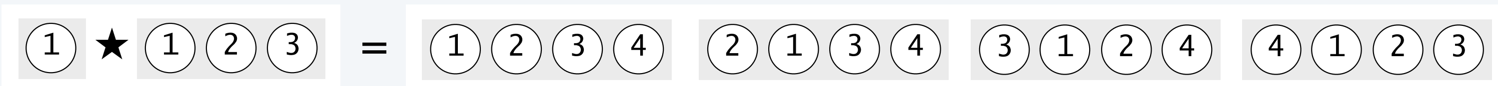
| *counting sequence* | *EGF* |
|---|---|
| $C_N = (N-1)!$ | $\ln \frac{1}{1-z}$ |

$$\sum_{N\geq 1} \frac{(N-1)!z^N}{N!} = \sum_{N\geq 1} \frac{z^N}{N} = \ln \frac{1}{1-z}$$

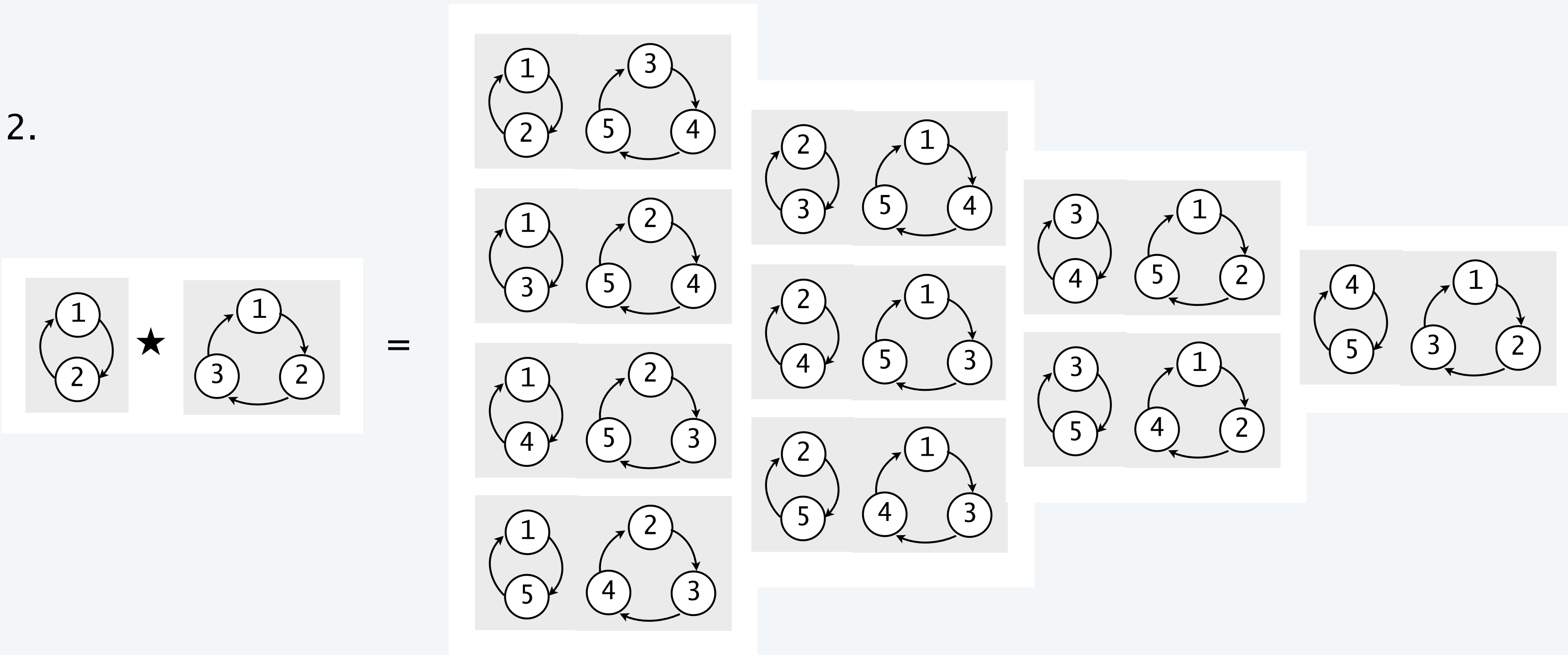
## Star product operation

Analog to Cartesian product requires *relabelling in all consistent ways.*

Ex 1.




Ex 2.



## Combinatorial constructions for labelled classes

| *construction* | *notation* | *semantics* |
|---|---|---|
| disjoint union | $A + B$ | disjoint copies of objects from $A$ and $B$ |
| labelled product | $A \star B$ | ordered pairs of copies of objects, one from $A$ and one from $B$ |
| sequence | $SEQ(A)$ | sequences of objects from $A$ |
| set | $SET(A)$ | sets of objects from $A$ |
| cycle | $CYC(A)$ | cyclic sequences of objects from $A$ |

$A$ and $B$ are combinatorial classes of labelled objects

## The symbolic method for labelled classes (transfer theorem)

Theorem. Let $A$ and $B$ be combinatorial classes of labelled objects with EGFs $A(z)$ and $B(z)$. Then

| *construction* | *notation* | *semantics* | *EGF* |
|---|---|---|---|
| disjoint union | $A + B$ | disjoint copies of objects from $A$ and $B$ | $A(z) + B(z)$ |
| labelled product | $A \star B$ | ordered pairs of copies of objects, one from $A$ and one from $B$ | $A(z)B(z)$ |
| sequence | $SEQ_k(A)$ | $k$- sequences of objects from $A$ | $A(z)^k$ |
| | $SEQ(A)$ | sequences of objects from $A$ | $\dfrac{1}{1-A(z)}$ |
| set | $SET_k(A)$ | $k$-sets of objects from $A$ | $A(z)^k/k!$ |
| | $SET(A)$ | sets of objects from $A$ | $e^{A(z)}$ |
| cycle | $CYC_k(A)$ | $k$-cycles of objects from $A$ | $A(z)^k/k$ |
| | $CYC(A)$ | cycles of objects from $A$ | $\ln \dfrac{1}{1-A(z)}$ |

# The symbolic method for labelled classes: basic constructions

| *class* | *construction* | *EGF* | *counting sequence* |
|---|---|---|---|
| urns | $U = SET(Z)$ | $U(z) = e^z$ | $U_N = 1$ |
| cycles | $C = CYC(Z)$ | $C(z) = \ln \frac{1}{1-z}$ | $C_N = (N-1)!$ |
| permutations | $P = SEQ(Z)$ <br> $P = E + Z \star P$ | $P(z) = \frac{1}{1-z}$ | $P_N = N!$ |

| *construction* | *notation* | *EGF* |
|---|---|---|
| disjoint union | $A + B$ | $A(z) + B(z)$ |
| labelled product | $A \star B$ | $A(z)B(z)$ |
| sequence | $SEQ_k(A)$ | $A(z)^k$ |
| | $SEQ(A)$ | $\frac{1}{1-A(z)}$ |
| set | $SET_k(A)$ | $A(z)^k/k!$ |
| | $SET(A)$ | $e^{A(z)}$ |
| cycle | $CYC_k(A)$ | $A(z)^k/k$ |
| | $CYC(A)$ | $\ln \frac{1}{1-A(z)}$ |

## Proofs of transfers

are immediate from GF counting

$A + B$

$$\sum_{\gamma\in A+B}\frac{z^{|\gamma|}}{|\gamma|!}=\sum_{\alpha\in A}\frac{z^{|\alpha|}}{|\alpha|!}+\sum_{\beta\in B}\frac{z^{|\beta|}}{|\beta|!}=A(z)+B(z)$$

$A \star B$

$$\sum_{\gamma\in \mathcal{A}\times\mathcal{B}}\frac{z^{|\gamma|}}{|\gamma|!}=\sum_{\alpha\in \mathcal{A}}\sum_{\beta\in \mathcal{B}}\binom{|\alpha|+|\beta|}{|\alpha|}\frac{z^{|\alpha|+|\beta|}}{(|\alpha|+|\beta|)!}=\Big(\sum_{\alpha\in \mathcal{A}}\frac{z^{|\alpha|}}{|\alpha|!}\Big)\Big(\sum_{\beta\in \mathcal{B}}\frac{z^{|\beta|}}{|\beta|!}\Big)=A(z)B(z)$$

Notation. We write $A^2$ for $A \star A$, $A^3$ for $A \star A \star A$, etc.

## Proofs of transfers

are immediate from GF counting

$$A(z)^k = \sum_{N\geq 0}\{\#k\text{-sequences of size } N\}\frac{z^N}{N!} = \sum_{N\geq 0} k\{\#k\text{-cycles of size } N\}\frac{z^N}{N!} = \sum_{N\geq 0} k!\{\#k\text{-sets of size } N\}\frac{z^N}{N!}$$

$$\frac{A(z)^k}{k} = \sum_{N\geq 0}\{\#k\text{-cycles of size } N\}\frac{z^N}{N!}$$

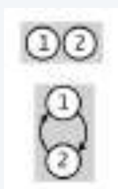
$$\frac{A(z)^k}{k!} = \sum_{N\geq 0}\{\#k\text{-sets of size } N\}\frac{z^N}{N!}$$

| *class* | *construction* | *EGF* |
|---|---|---|
| k-sequence | $SEQ_k(A)$ | $A(z)^k$ |
| sequence | $SEQ_k(A) = SEQ_0(A) + SEQ_1(A) + SEQ_2(A) + \ldots$ | $1 + A(z) + A(z)^2 + A(z)^3 + \ldots = \dfrac{1}{1 - A(z)}$ |
| k-cycle | $CYC_k(A)$ | $\dfrac{A(z)^k}{k}$ |
| cycle | $CYC_k(A) = CYC_0(A) + CYC_1(A) + CYC_2(A) + \ldots$ | $1 + \dfrac{A(z)}{1} + \dfrac{A(z)^2}{2} + \dfrac{A(z)^3}{3} + \ldots = \ln\dfrac{1}{1 - A(z)}$ |
| k-set | $SET_k(A)$ | $\dfrac{A(z)^k}{k!}$ |
| set | $SET_k(A) = SET_0(A) + SET_1(A) + SET_2(A) + \ldots$ | $1 + \dfrac{A(z)}{1!} + \dfrac{A(z)^2}{2!} + \dfrac{A(z)^3}{3!} + \ldots = e^{A(z)}$ |

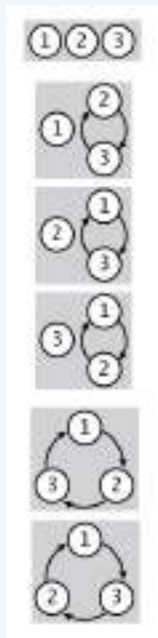
## Labelled class example 4: sets of cycles

Q. How many sets of cycles of labelled atoms?

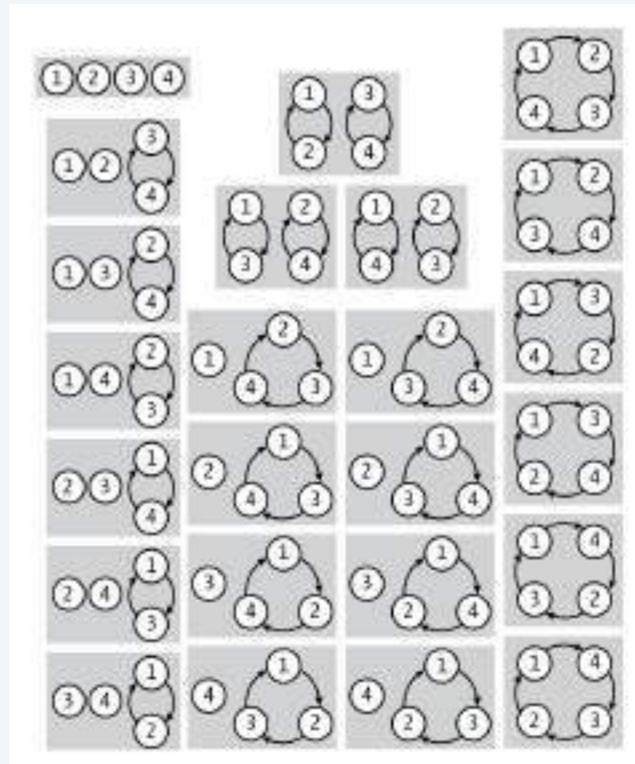
$P^*_1 = 1$



$P^*_2 = 2$



$P^*_3 = 6$



$P^*_4 = 24$

## Symbolic method: sets of cycles

How many sets of cycles of length $N$?

| | |
|---|---|
| *Class* | $P^*$, the class of all sets of cycles of atoms |
| *Size* | $\|p\|$, the number of atoms in $p$ |
| *EGF* | $P^*(z) = \sum_{p \in P^*} \frac{z^{\|p\|}}{\|p\|!} = \sum_{N \geq 0} P^*_N \frac{z^N}{N!}$ |

*Atom*

| *type* | *class* | *size* | *GF* |
|---|---|---|---|
| labelled atom | $Z$ | 1 | $z$ |

Construction $$P^* = SET(CYC(Z))$$

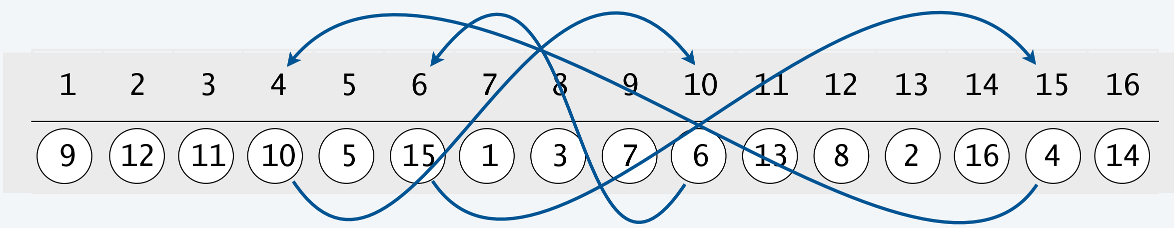
OGF equation $$P^*(z) = \exp\left(\ln \frac{1}{1-z}\right) = \frac{1}{1-z}$$

Counting sequence $$P^*_N = N![z^N]P^*(z) = N!$$

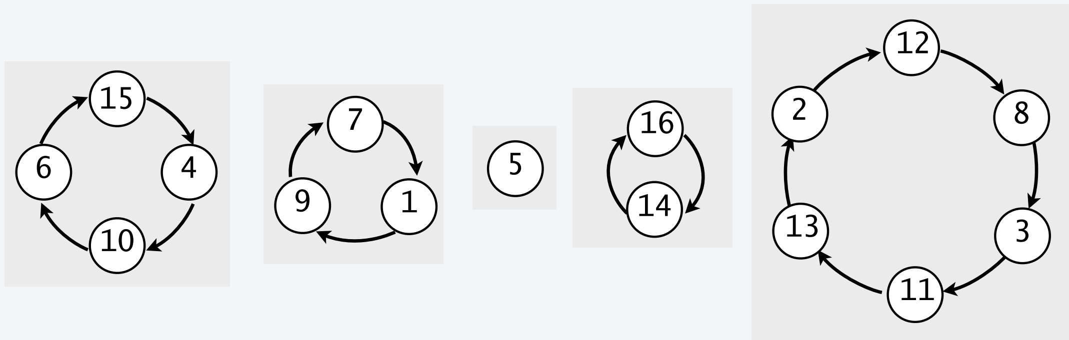
## Aside: A combinatorial bijection

A permutation is a set of cycles.

Standard representation




Set of cycles representation



## Derangements

*N* people go to the opera and leave their hats on a shelf in the cloakroom.
When leaving, they each grab a hat at random.

Q. *What is the probability that nobody gets their own hat ?*

Definition. A derangement is a permutation with no singleton cycles

## Derangements (various versions)

A group of $N$ people go to the opera and leave their hats in the cloakroom. When leaving, they each grab a hat at random.

Q. *What is the probability that nobody gets their own hat* ?

A professor returns exams to $N$ students by passing them out at random.

Q. *What is the probability that nobody gets their own exam* ?

A group of $N$ sailors go ashore for revelry that leads to a state of inebriation. When returning, they each end up sleeping in a random cabin.

Q. *What is the probability that nobody sleeps in their own cabin* ?

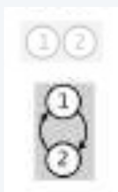
A group of $N$ students who live in single rooms go to a party that leads to a state of inebriation. When returning, they each end up in a random room.

Q. *What is the probability that nobody ends up in their own room* ?

## Derangements

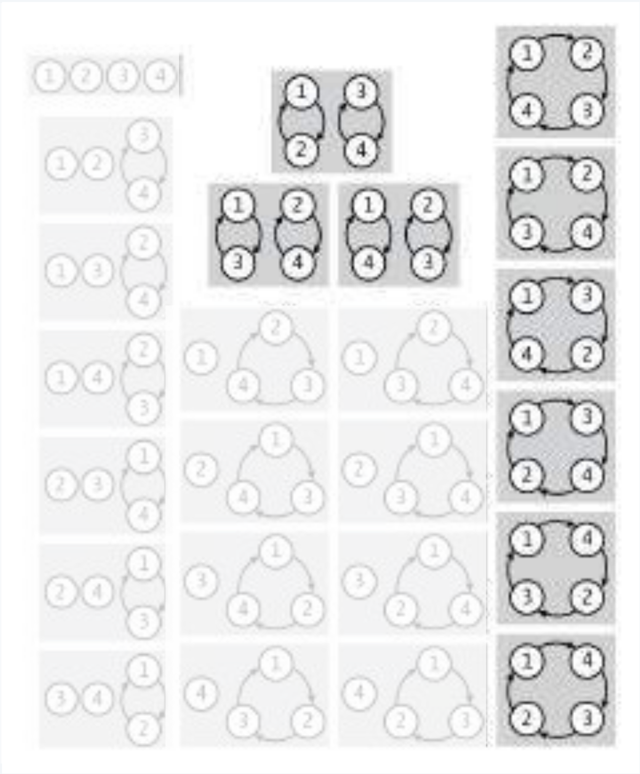
are permutations with no singleton cycles.

$D_1 = 0$



$D_2 = 1$

$D_3 = 2$



$D_4 = 9$

## Symbolic method: derangements

How many derangements of length $N$ ?

| | |
|---|---|
| *Class* | $D$, the class of all derangements |
| *Size* | $\lvert p \rvert$, the number of atoms in $p$ |
| *EGF* | $D(z) = \sum_{d \in D} \frac{z^{\lvert d \rvert}}{\lvert d \rvert !} = \sum_{N \geq 0} D_N \frac{z^N}{N!}$ |

*Atom*

| *type* | *class* | *size* | *GF* |
|---|---|---|---|
| labelled atom | $Z$ | 1 | $z$ |

Construction $D = SET(CYC_{>1}(Z))$

"Derangements are permutations with no singleton cycles"

OGF equation $D(z) = e^{z^2/2+z^3/3+z^4/4+\ldots} = \exp\left(\ln \frac{1}{1-z} - z\right) = \frac{e^{-z}}{1-z}$

Alternate derivation

$Set(Z) \star D = P$

$e^z D(z) = \frac{1}{1-z}$

Expansion $[z^N]D(z) \equiv \frac{D_N}{N!} = \sum_{0 \leq k \leq N} \frac{(-1)^k}{k!} \sim \frac{1}{e}$

probability that a random permutation is a derangement

simple convolution

see "Asymptotics" lecture

## Derangements

A group of $N$ students who live in single rooms go to a party that leads to a state of inebriation. When returning, they each end up in a random room.

Q. *What is the probability that nobody ends up in their own room?*

A. $\frac{1}{e} \doteq 0.36788$

## Derangements

A group of *N* graduating seniors each throw their hats in the air and each catch a random hat.

Q. *What is the probability that nobody gets their own hat back ?*

A. $\frac{1}{e} \doteq 0.36788$

## Generalized derangements

In the hats-in-the-air scenario, a student can get her hat back by "following the cycle".

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 12 | 11 | 10 | 5 | 15 | 1 | 3 | 7 | 6 | 13 | 8 | 2 | 16 | 4 | 14 |

Q. *What is the probability that all cycles are of length > M ?*

## Symbolic method: generalized derangements

How many permutations of length $N$ have no cycles of length $\leq M$?

| | |
|---|---|
| *Class* | $D_M$, the class of all generalized derangements |
| *Size* | $|d|$, the number of atoms in $d$ |
| *EGF* | $D_M(z) = \sum_{d \in D_M} \frac{z^{|d|}}{|d|!} = \sum_{N \geq 0} D_{MN} \frac{z^N}{N!}$ |

*Atom*

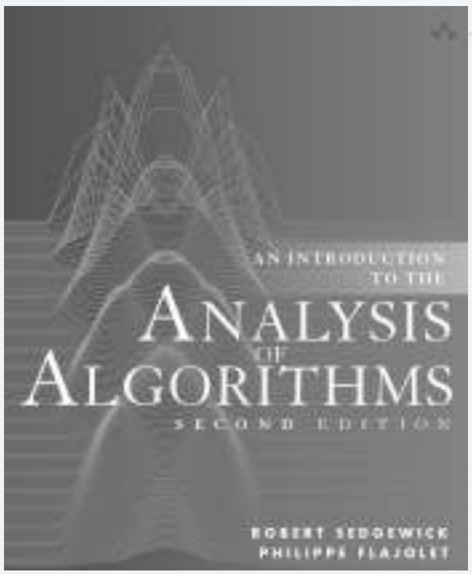
| *type* | *class* | *size* | *GF* |
|---|---|---|---|
| labelled atom | $Z$ | 1 | $z$ |

Construction $D_M = SET(CYC_{>M}(Z))$

OGF equation $D_M(z) = e^{\frac{z^{M+1}}{M+1} + \frac{z^{M+2}}{M+2} + \cdots} = \exp\left(\ln \frac{1}{1-z} - z - z^2/2 - \ldots - z^M/M\right)$

$$= \frac{e^{-z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots \frac{z^M}{M}}}{1-z}$$

Expansion $D_{MN} =$ ?? $M$-way convolution (stay tuned)

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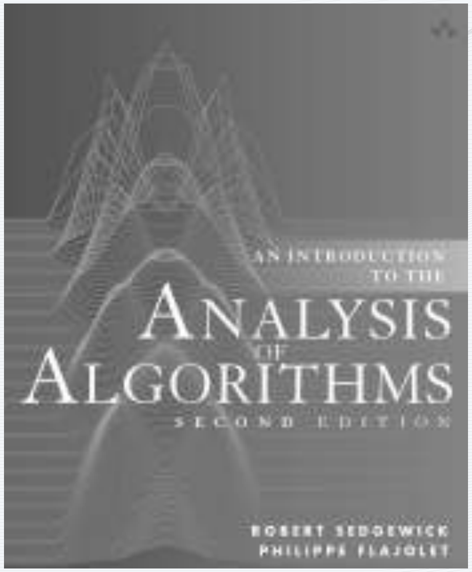
# 5. Analytic Combinatorics

- The symbolic method
- **Labelled objects**
- Coefficient asymptotics
- Perspective

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# 5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- **Coefficient asymptotics**
- Perspective

## Generating coefficient asymptotics

are often *immediately* derived via general "analytic" transfer theorems.

*Example* 1. Taylor's theorem

Theorem. If $f(z)$ has $N$ derivatives, then $[z^N]f(z) = f^{(N)}(0)/N!$

*Example* 2. Rational functions transfer theorem (see "Asymptotics" lecture)

Theorem. If $f(z)$ and $g(z)$ are polynomials, then

$$[z^n]\frac{f(z)}{g(z)} = -\frac{\beta f(1/\beta)}{g'(\beta)}\beta^n$$

where $1/\beta$ is the largest root of $g$ (provided that it has multiplicity 1).

see "Asymptotics" lecture for general case

*Example* 3. Radius-of-convergence transfer theorem

[see next slide]

Most are based on complex asymptotics. Stay tuned for Part 2

## Radius-of-convergence transfer theorem

Theorem. If $f(z)$ has radius of convergence >1 with $f(1) \neq 0$, then

$$[z^n]\frac{f(z)}{(1-z)^\alpha} \sim f(1)\binom{n+\alpha-1}{n} \sim \frac{f(1)}{\Gamma(\alpha)}n^{\alpha-1}$$

for any real $\alpha \notin 0, -1, -2, \ldots$

convolution,
$f_1 + f_2 + \ldots + f_n \sim f(1)$

standard asymptotics
with generalized
binomial coefficient

*Gamma function*
(*generalized factorial*)

| |
|---|
| $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ |
| $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ |
| $\Gamma(N+1) = N!$ |
| $\Gamma(1) = 1$ |
| $\Gamma(1/2) = \sqrt{\pi}$ |

Corollary. If $f(z)$ has radius of convergence $>\rho$ with $f(\rho) \neq 0$, then

$$[z^n]\frac{f(z)}{(1-z/\rho)^\alpha} \sim \frac{f(\rho)}{\Gamma(\alpha)}\rho^n n^{\alpha-1}$$

for any real $\alpha \notin 0, -1, -2, \ldots$

## Radius-of-convergence transfer theorem: applications

Corollary. If $f(z)$ has radius of convergence $>\rho$ with $f(\rho) \neq 0$, then

$$[z^n]\frac{f(z)}{(1 - z/\rho)^{\alpha}} \sim \frac{f(\rho)}{\Gamma(\alpha)}\rho^n n^{\alpha-1}$$

for any real $\alpha \notin 0, -1, -2, \ldots$

Ex 1: Catalan

$$T(z) = \frac{1}{2z}(1 - \sqrt{1-4z})$$

$$[z^N]T(z) \sim \frac{4^N}{\sqrt{\pi N^3}}$$

$\rho = 1/4 \qquad \alpha = -1/2 \qquad f(z) = -1/2$

$\Gamma(-1/2) = -2\Gamma(1/2) = -2\sqrt{\pi}$

Ex 2: Derangements

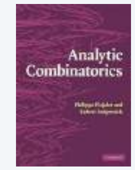
$$D_M(z) = \frac{e^{-z-z^2/2\ldots-z^M/M}}{1-z}$$

$$[z^N]D_M(z) \sim \frac{N!}{e^{H_M}}$$

$\rho = 1 \quad \alpha = 1 \quad f(z) = e^{-z-z^2/2\ldots-z^M/M}$

# Transfer theorems based on complex asymptotics

provide *universal laws* of sweeping generality




Example: Context-free constructions

A *system* of combinatorial constructions

$$
\begin{aligned}
<\mathbf{G}_0> &= OP_0(<\mathbf{G}_0>, <\mathbf{G}_1>, \ldots, <\mathbf{G_t}>)\\
<\mathbf{G}_1> &= OP_1(<\mathbf{G}_0>, <\mathbf{G}_1>, \ldots, <\mathbf{G_t}>)\\
&\ldots\\
<\mathbf{G_t}> &= OP_t(<\mathbf{G}_0>, <\mathbf{G}_1>, \ldots, <\mathbf{G_t}>)
\end{aligned}
$$

symbolic method

transfers to a system of GF equations

$$
\begin{aligned}
G_0(z) &= F_0(G_0(z), G_1(z), \ldots, G_t(z))\\
G_1(z) &= F_1(G_0(z), G_1(z), \ldots, G_t(z))\\
&\ldots\\
G_t(z) &= F_t(G_0(z), G_1(z), \ldots, G_t(z))
\end{aligned}
$$

Grobner basis elimination

that reduces to a *single* GF equation

$$G_0(z) = F(G_0(z), G_1(z), \ldots, G_t(z))$$

Drmota-Lalley-Woods theorem

that has an *explicit* solution

$$G(z) \sim c - a\sqrt{1 - bz}$$

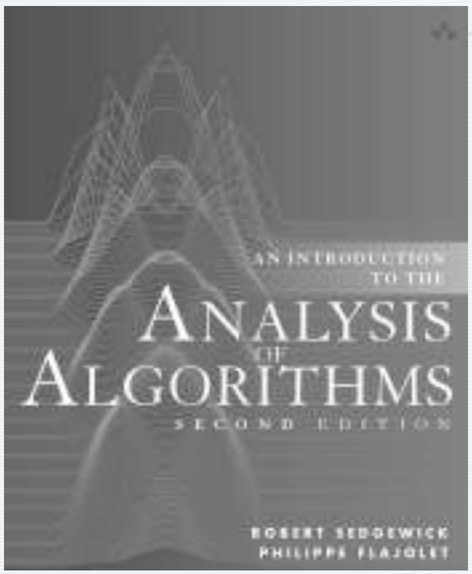
singularity analysis

that transfers to a *simple* asymptotic form

$$G_N \sim \frac{a}{2\sqrt{\pi N^3}} b^N$$

!!

Stay tuned for many more (in Part 2).

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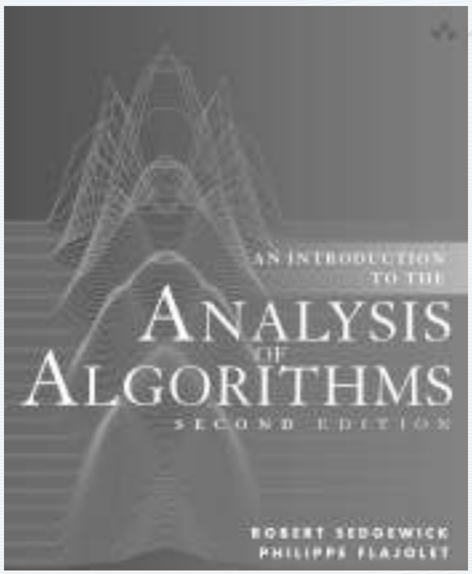
# 5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- **Coefficient asymptotics**
- Perspective

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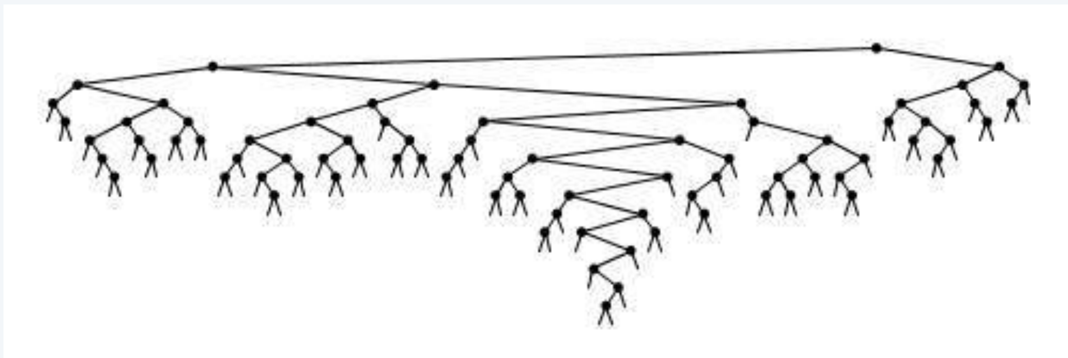
# 5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- Coefficient asymptotics
- **Perspective**

# Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Ex: How many binary trees with $N$ nodes?



$$T = E + Z \times T \times T$$

combinatorial construction

$$T(z) = \frac{1}{2z}(1 - \sqrt{1-4z})$$

GF

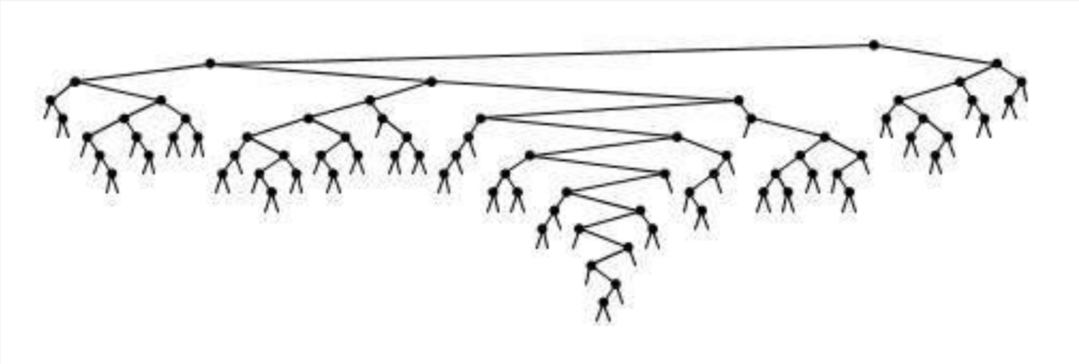
$$T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$$

coefficient asymptotics

## Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Ex: How many binary trees with $N$ nodes?



$$\boxed{\mathrm{T} = \mathrm{E} + \mathrm{Z} \times \mathrm{T} \times \mathrm{T}} \longrightarrow \boxed{T(z) = 1 + zT(z)^2} \longrightarrow \boxed{T_N \sim \frac{4^N}{\sqrt{\pi N^3}}}$$

combinatorial construction

GF equation

coefficient asymptotics

*Note*: With complex asymptotics, we can transfer directly from GF equation (no need to solve it). See Part 2.

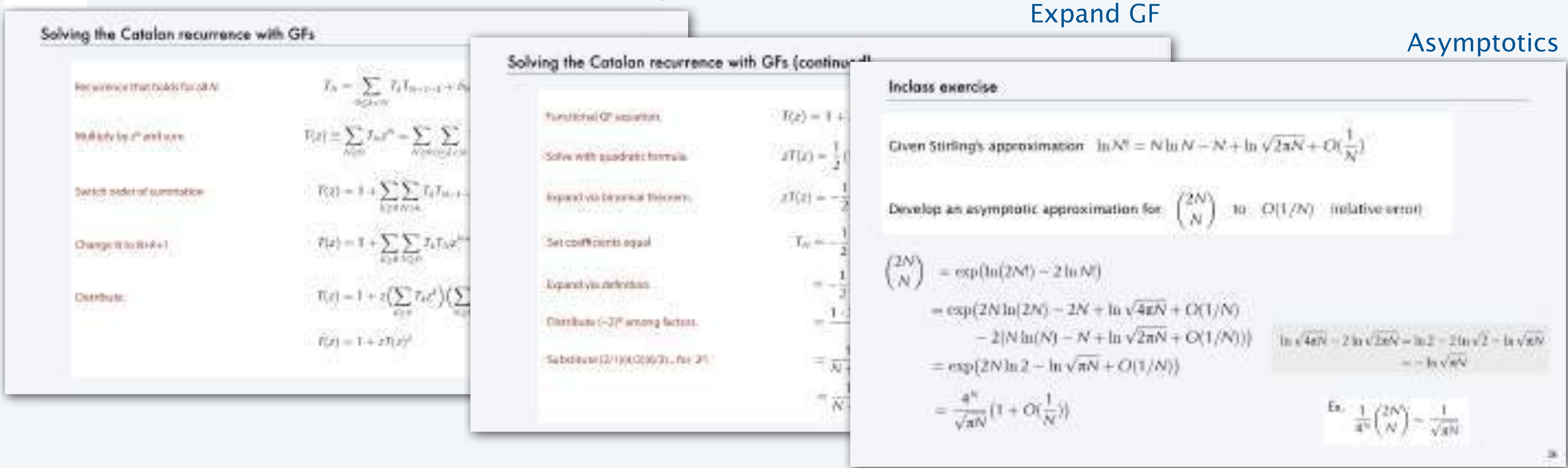
# Old vs. New: Two ways to count binary trees

Old

Recurrence → GF

Expand GF

Asymptotics




New

$$\mathrm{T} = \mathrm{E} + \mathrm{Z} \times \mathrm{T} \times \mathrm{T} \longrightarrow T(z) = 1 + zT(z)^2 \longrightarrow T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$$

## Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Ex: How many generalized derangements?

$$D_M = SET(CYC_{>M}(Z))$$

combinatorial construction

$$\frac{e^{-z-z^2/2\ldots-z^M/M}}{1-z}$$

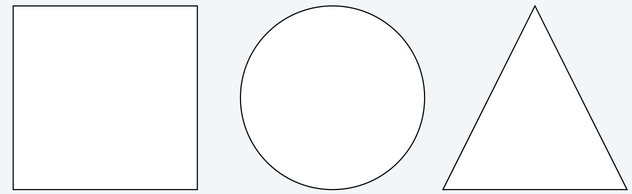
GF equation

$$\sim \frac{N!}{e^{H_M}}$$

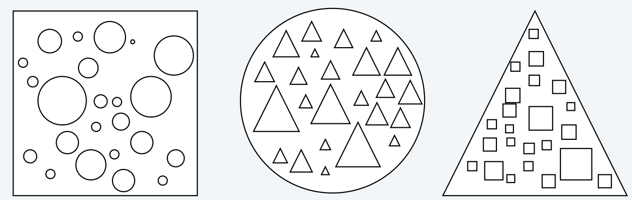
coefficient asymptotics

## A standard paradigm for analytic combinatorics

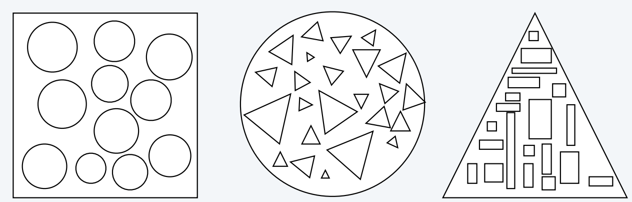
Fundamental constructs
- elementary or trivial
- confirm intuition



Compound constructs
- many possibilities
- classical combinatorial objects
- expose underlying structure

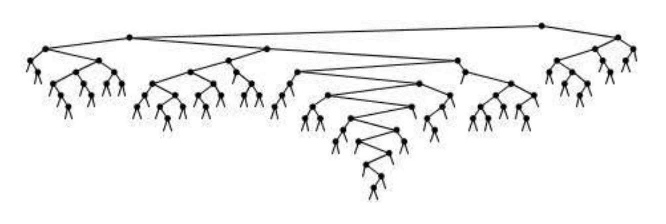


Variations
- unlimited possibilities
- *not* easily analyzed otherwise



## Combinatorial parameters

are handled as two counting problems via cumulated costs.



Ex: How many leaves in a random binary tree?

1. Count trees

T = E + Z × T × T → $T(z) = \frac{1}{2z}(1 - \sqrt{1-4z})$ → $T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$

2. Count leaves in all trees

T = E + Z × T × T → $T_u(1, z) = \frac{z}{\sqrt{1-4z}}$ → $C_N \sim \frac{4^{N-1}}{\sqrt{\pi N}}$

Symbolic method works for BGFs (see text)

3. Divide

$$\frac{C_N}{T_N} \sim \frac{N}{4}$$

## Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Features:

- Analysis begins with formal *combinatorial constructions*.
- The *generating function* is the central object of study.
- *Transfer theorems* can immediately provide results from formal descriptions.
- Results extend, in principle, to any desired precision on the standard scale.
- Variations on fundamental constructions are easily handled.

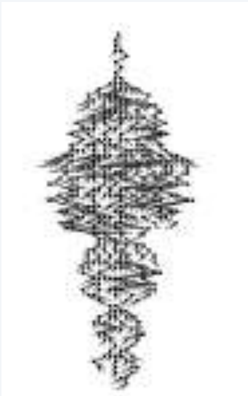

combinatorial constructions → (*symbolic* transfer theorem) → generating function equation → (*analytic* transfer theorem) → coefficient asymptotics

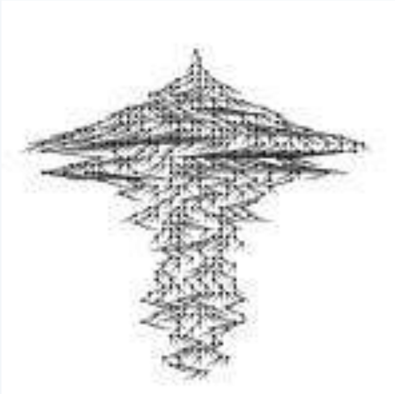
the "symbolic method"

## Stay tuned

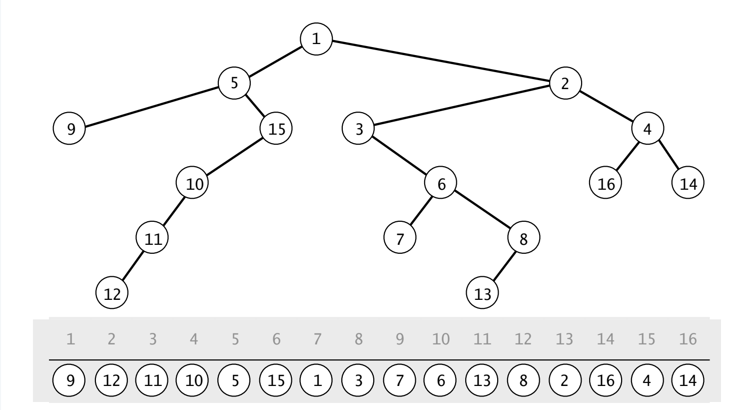
for many applications of analytic combinatorics *and applications to the analysis of algorithms*

### Trees

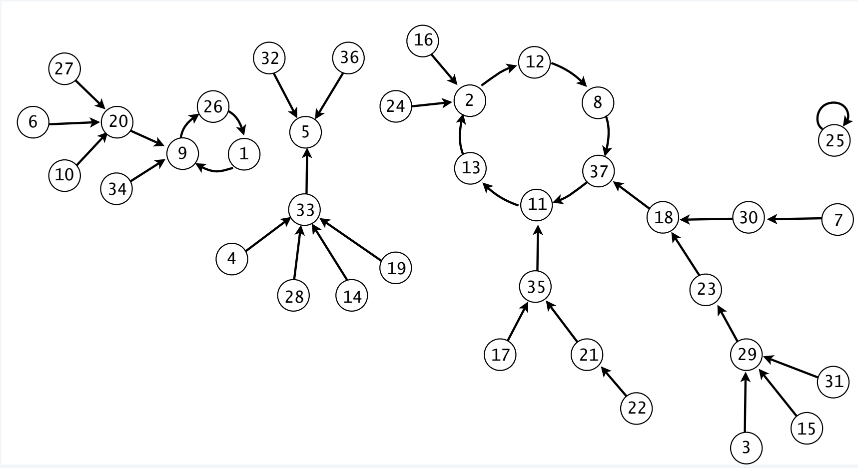




### Permutations

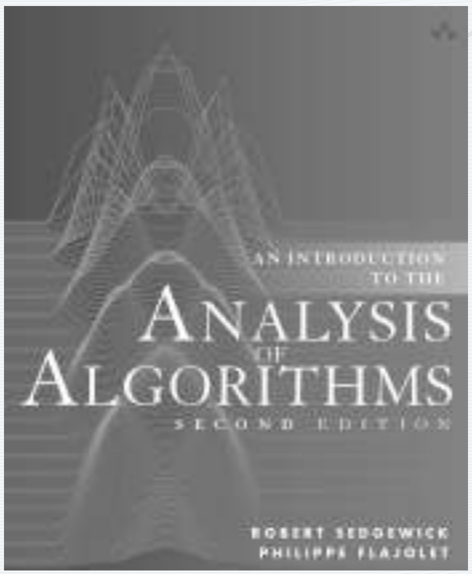



### Mappings



### Bitstrings

101111101001010011001110001001111101101101000001111000011001110111011111101011000
11010010100011110100111100110100111011010111111000000101101110011010000000111001110
11101110101100111010111001101000011000111001010111110011001000011001000101010010
10111000011011000110011101110011011011110111110011101011000011001100101000000110
10101100111010001101101110110010010110100101001101111110011000000111110100000 1111
10000010011000001100011000100001111001110011110000011001111110011011000100100111
10001010101110001110101100000110000011101010100010110001001101111110011110110010
00111011001011100100000110000100111110100100110011000011001110100110100001010 00111
001111111001101101110110111010100110110111000111111110101110100110000000100101110
101010001111000010100000110010000011010100101000110011001010101011101101111111 10
110000000101111011011000101011010110010010000011101110010000001101010000000101000
11101111011011111011111111110100111010010111111011101001110100011000100100010010
0011111110011101011011111000010001000111000011101011110010101111100111010101111 1

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# 5. Analytic Combinatorics

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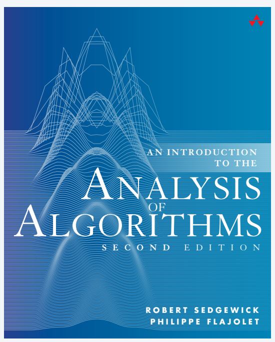
# Exercise 5.1

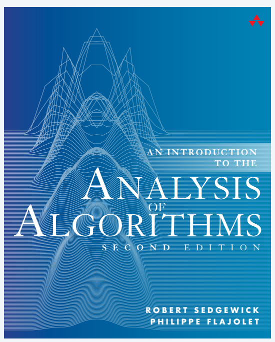
Practice with counting bitstrings.




**Exercise 5.1** How many bitstrings of length $N$ have no 000?

# Exercise 5.3

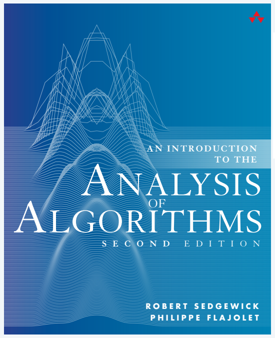
Practice with counting trees.




**Exercise 5.3** Let $\mathcal{U}$ be the set of binary trees with the size of a tree defined to be the total number of nodes (internal plus external), so that the generating function for its counting sequence is $U(z) = z + z^3 + 2z^5 + 5z^7 + 14z^9 + \ldots$. Derive an explicit expression for $U(z)$.

# Exercise 5.7

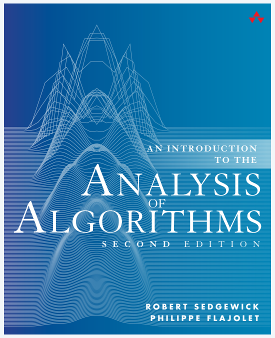
Practice with counting permutations.




**Exercise 5.7** Derive an EGF for the number of permutations whose cycles are all of odd length.

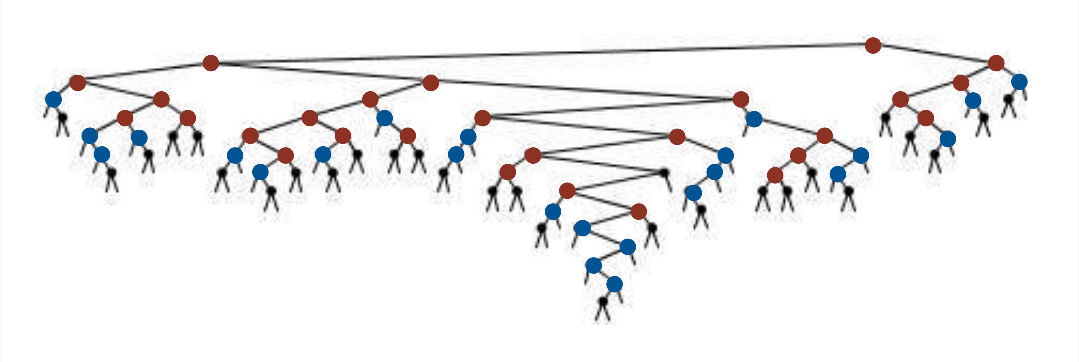
## Exercises 5.15 and 5.16

Practice with tree parameters.



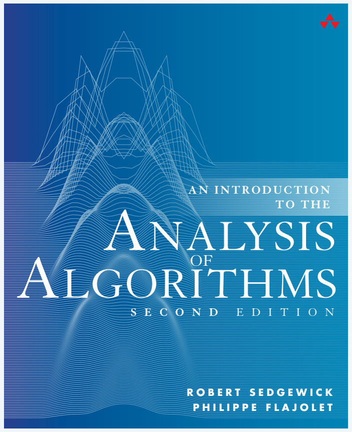

**Exercise 5.15** Find the average number of internal nodes in a binary tree of size $n$ with both children internal.

**Exercise 5.16** Find the average number of internal nodes in a binary tree of size $n$ with one child internal and one child external.



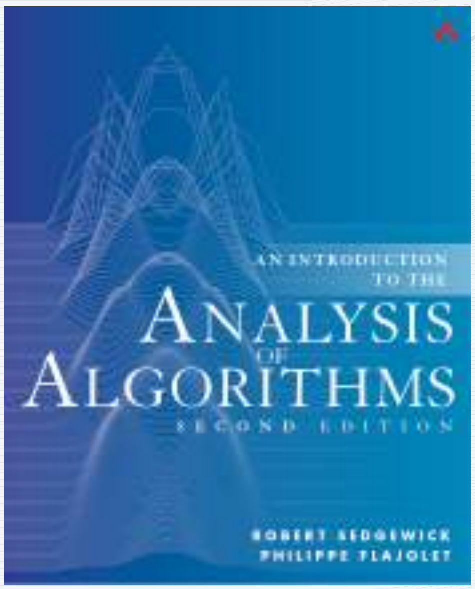
## Assignments for next lecture

1. Read pages 219-255 in text.




2. Write up solutions to Exercises 5.1, 5.3, 5.7, 5.15, and 5.16.

# ANALYTIC COMBINATORICS
# PART ONE

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## 5. Analytic Combinatorics