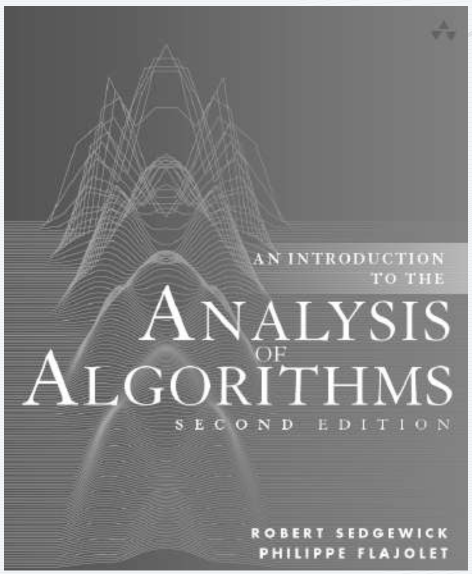
ANALYTIC COMBINATORICS

PART ONE



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7. Permutations

- Basics
- Sets of cycles
- Left-right-minima
- **Other parameters**
- BGFs and distributions

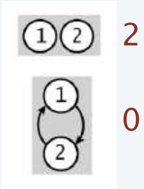
7d.Perms.Others

1-Cycles

Q. How many *1-cycles* in a random permutation of size N?

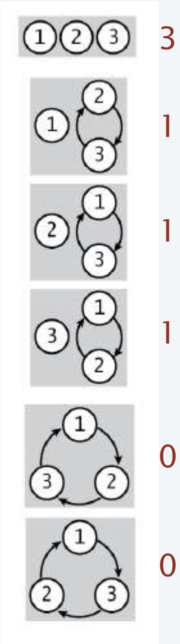
$B_1 = 1$

$B_1/P_1 = 1$



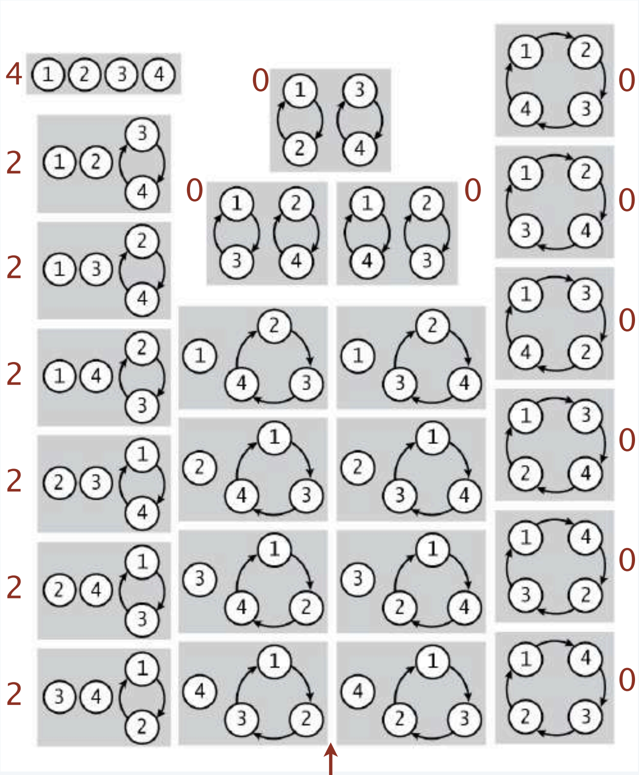
$B_2 = 0 + 2 = 2$

$B_2/P_2 = 2/2 = 1$



$B_3 = 2 \cdot 0 + 3 \cdot 1 + 1 \cdot 3 = 6$

$B_3/P_3 = 1$

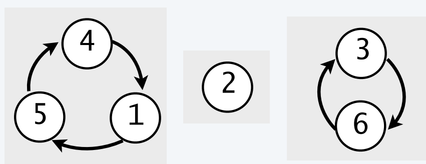


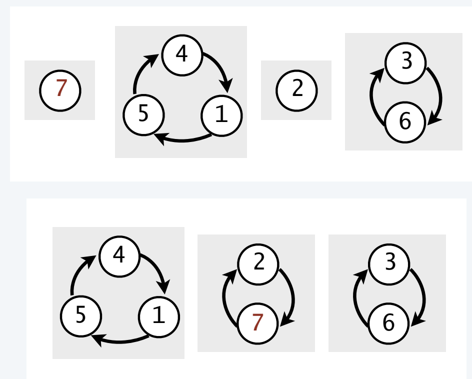
$B_4 = 9 \cdot 0 + 8 \cdot 1 + 6 \cdot 2 + 1 \cdot 4 = 24$

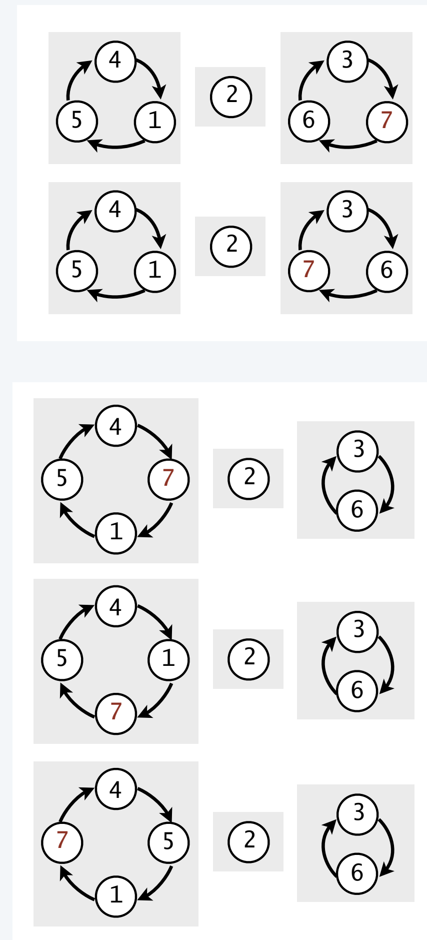
$B_4/P_4 = 24/24 = 1$

Construction for 1-cycles

Create $|p|+1$ perms from a perm p by inserting $|p|+1$ into every position in every cycle (including the null cycle)







Original perm has $\text{cyc}_1(p)$ 1-cycles.

Q. How many 1-cycles in the set of constructed perms?

A. $(|p| + 1)\, \text{cyc}_1(p) \;\; + 1 \;\; - \text{cyc}_1(p) \;\; = |p|\, \text{cyc}_1(p) + 1$

$|p| + 1$ copies of the original perm

from the null cycle

1-cycles changed to 2-cycles

Average number of 1-cycles in a random permutation

CGF.
$$B(z) = \sum_{p \in \mathcal{P}} \text{cyc}_1(p) \frac{z^{|p|}}{|p|!} = \sum_{N \ge 0} B_N \frac{z^N}{N!}$$

Apply construction.
$$= \sum_{p \in \mathcal{P}} (|p|\text{cyc}_1(p) + 1) \frac{z^{|p|+1}}{(|p|+1)!}$$

Differentiate.
$$B'(z) = \sum_{p \in \mathcal{P}} |p|\text{cyc}_1(p) \frac{z^{|p|}}{|p|!} + \sum_{p \in \mathcal{P}} \frac{z^{|p|}}{|p|!} \quad = zB'(z) + \frac{1}{1-z}$$

Solve.
$$B'(z) = \frac{1}{(1-z)^2}$$

Integrate.
$$B(z) = \frac{1}{1-z}$$

Expand.
$$[z^N]B(z) = \frac{B_N}{N!} = 1$$

cumulated cost

average # 1-cycles in a random permutation

Application: Students and rooms revisited

A group of N students who live in single rooms go to a party that leads to a state of inebriation. When returning, they each end up in a random room.

Q. What is the average number of students who wind up in their own room?

A. One (!)

In-class exercises

Q. How many *2-cycles* in a random permutation of size N?

A. 1/2

Q. How many *r-cycles* in a random permutation of size N?

A. $1/r$

Inversions

Def. An inversion in a permutation is the number of pairs (i)(j) with `i > j`.

Equivalent: Sum number of entries larger and to the left of each entry.

Q. How many *inversions* in a random permutation of size N?

1	0

$P_1 = 1$

$B_1 = 0$

$B_1/P_1 = 0$

1 2	0
2 1	1

$P_2 = 2$

$B_2 = 0 + 1 = 1$

$B_2/P_2 = 1/2 = 0.5$

1 2 3	0
2 1 3	1
3 1 2	2
1 3 2	1
2 3 1	2
3 2 1	3

$P_3 = 6$

$B_3 = 2 \cdot 1 + 2 \cdot 2 + 1 \cdot 3 = 9$

$B_3/P_3 = 9/6 = 1.5$

0	1 2 3 4	1 2 4 3	1
1	2 1 3 4	2 1 4 3	2
2	3 1 2 4	3 1 4 2	3
2	4 1 2 3	4 1 3 2	4
1	1 3 2 4	1 3 4 2	2
2	2 3 1 4	2 3 4 1	4
3	3 2 1 4	3 2 4 1	4
3	4 2 1 3	4 2 3 1	4
2	1 4 2 3	1 4 3 2	3
3	2 4 1 3	2 4 3 1	4
4	3 4 1 2	3 4 2 1	5
5	4 3 1 2	4 3 2 1	6

$P_4 = 24$

$B_4 = 3 \cdot 1 + 7 \cdot 2 + 5 \cdot 3 + 6 \cdot 4 + 2 \cdot 5 + 1 \cdot 6 = 72$

$B_4/P_4 = 72/24 = 3$

Application: Insertion sort

```
public static void sort(Comparable[] a)
{
   int N = a.length;
   for (int i = 1; i < N; i++)
   {
      for (int j = i; j > 0; j--)
         if (less(a[j], a[j-1]))
            exch(a, j, j-1);
         else break;
   }
}
```

i = 10

sorted before i — untouched after i

A	E	G	I	N	O	R	S	T	X	M	P	L	E	D	Q	Z
A	E	G	I	N	O	R	S	T	M	X	P	L	E	D	Q	Z
A	E	G	I	N	O	R	S	M	T	X	P	L	E	D	Q	Z
A	E	G	I	N	O	R	M	S	T	X	P	L	E	D	Q	Z
A	E	G	I	N	O	M	R	S	T	X	P	L	E	D	Q	Z
A	E	G	I	N	M	O	R	S	T	X	P	L	E	D	Q	Z
A	E	G	I	M	N	O	R	S	T	X	P	L	E	D	Q	Z

exchanges put M in place among elements to its left

Q. How many exchanges during the sort?

A. The number of inversions in the permutation.

Q. How many inversions in a random permutation?

Section 2.1

Construction for inversions

Create $|p|+1$ perms from a perm p by "largest" construction.

	3 6 2 5 1 4 7	
	3 6 2 5 1 7 4	← adds 1 inversion
	3 6 2 5 7 1 4	← adds 2 inversions
3 6 2 5 1 4 ★ 1 =	3 6 2 7 5 1 4	← adds 3 inversions
	3 6 7 2 5 1 4	← adds 4 inversions
	3 7 6 2 5 1 4	← adds 5 inversions
	7 3 6 2 5 1 4	← adds 6 inversions

Original perm has inv(p) inversions.

Q. How many inversions in the set of constructed perms?

A. $\underbrace{(|p| + 1)\ \text{inv}(p)}_{|p|+1 \text{ copies of the original perm}} + \underbrace{(|p| + 1)\ |p|\ /\ 2}_{\text{all the inversions caused by } |p|+1}$

$|p| + 1$ copies of the original perm

all the inversions caused by $|p| + 1$

Average number of inversions in a random permutation

CGF.
$$B(z) = \sum_{p\in\mathcal{P}} \text{inv}(p)\frac{z^{|p|}}{|p|!} \qquad = \sum_{N\geq 0} B_N \frac{z^N}{N!}$$

Apply construction.
$$= \sum_{p\in\mathcal{P}} \big((|p|+1)\text{inv}(p) + (|p|+1)|p|/2\big)\frac{z^{|p|+1}}{(|p|+1)!}$$

Simplify.
$$= \sum_{p\in\mathcal{P}} \text{inv}(p)\frac{z^{|p|+1}}{(|p|)!} + \frac{1}{2}\sum_{p\in\mathcal{P}} |p|\frac{z^{|p|+1}}{(|p|)!} \quad = zB(z) + \frac{z}{2}\sum_{k\geq 0} kz^k$$

Substitute.
$$= zB(z) + \frac{1}{2}\frac{z^2}{(1-z)^2}$$

Solve.
$$B(z) = \frac{1}{2}\frac{z^2}{(1-z)^3}$$

Expand.
$$[z^N]B(z) = \frac{B_N}{N!} = \frac{N(N-1)}{4}$$

cumulated cost

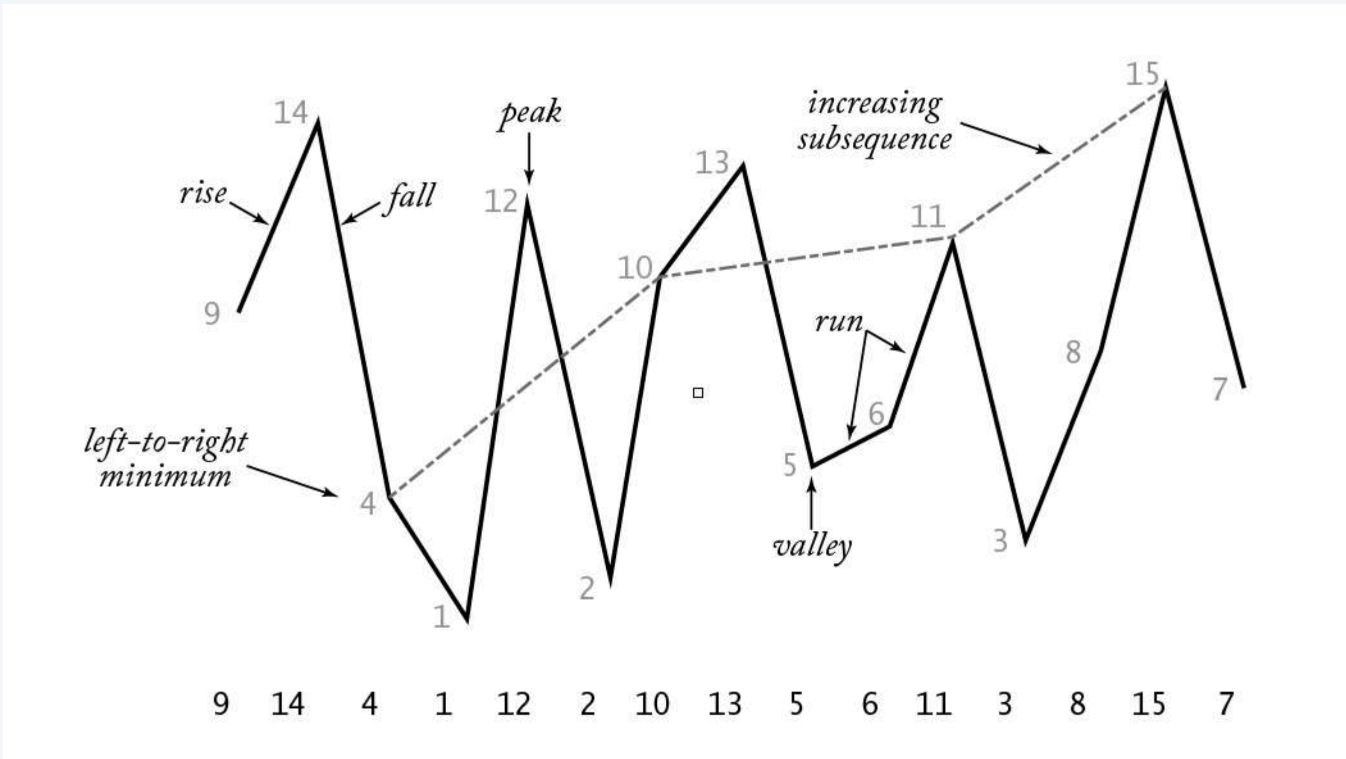
average # inversions in a random permutation

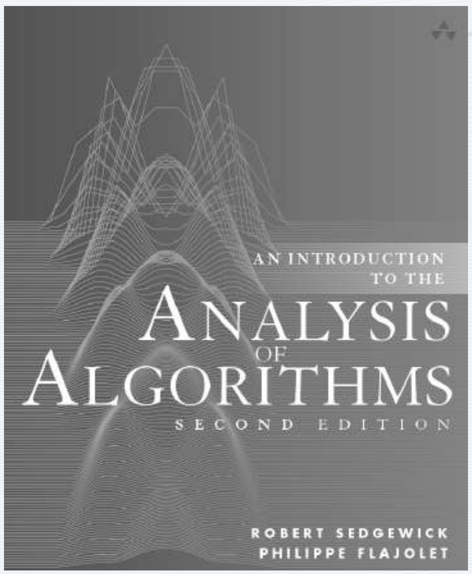
$$B_1/1! = \frac{1\cdot 0}{4} = 0$$
$$B_2/2! = \frac{2\cdot 1}{4} = 0.5$$
$$B_3/3! = \frac{3\cdot 2}{4} = 1.5$$
$$B_4/4! = \frac{4\cdot 3}{4} = 3$$

✓

Parameters of permutations

all can be handled in a similar manner





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7. Permutations

- Basics
- Sets of cycles
- Left-right-minima
- Other parameters
- **BGFs and distributions**

Bivariate generating functions

are the method of choice in analyzing combinatorial parameters.

Definition. A *combinatorial class* is a set of combinatorial objects and an associated size function that may have an associated parameter.

Definition. The *bivariate generating function* (BGF) associated with a class is the formal power series

$$A(z,u) = \sum_{a \in A} \frac{z^{|a|}}{|a|!} u^{cost(a)} \quad \text{(labelled)}$$

where |a| is the size and cost(a) is the value of the parameter.

Advantages of BGFs:

- Carry full information.
- Easy to compute counting sequence and CGF (see next slide).
- Full distribution often available via analytic combinatorics.

Basic BGF calculations

Definition. The *bivariate generating function* (BGF) associated with a labelled class is the formal power series $$A(z,u) = \sum_{a\in A} \frac{z^{|a|}}{|a|!} u^{cost(a)}$$

z marks size.
u marks the parameter.

Define A_{Nk} to be the number of elements of size N with parameter value k.

Fundamental (elementary) identity $$A(z,u) = \sum_{a\in A} \frac{z^{|a|}}{|a|!} u^{cost(a)} = \sum_{N\geq 0}\sum_{k\geq 0} A_{Nk}\frac{z^N}{N!}u^k$$

Q. How many objects of size N with value k ?

A. $N![z^N][u^k]A(z,u) = A_{Nk}$

Q. Average value of a parameter of a permutation ?

A. $[z^N]A_u(z,1) \equiv \frac{\partial}{\partial u}A(z,u)\Big|_{u=1}$

$$\frac{\partial}{\partial u}A(z,u) = \sum_{N\geq 0}\sum_{k\geq 0} kA_{Nk}\frac{z^N}{N!}u^{k-1}$$

$$A_u(z,1) \equiv \frac{\partial}{\partial u}A(z,u)\Big|_{u=1} = \sum_{N\geq 0}\sum_{k\geq 0} kA_{Nk}\frac{z^N}{N!}$$

$$[z^N]A_u(z,1) = \frac{\partial}{\partial u}A(z,u)\Big|_{u=1} = \sum_{k\geq 0} k\frac{A_{Nk}}{N!}$$

Review: Average number of cycles in a random permutation with CGFs

CGF. $$B(z) = \sum_{p \in \mathcal{P}} \text{cycles}(p) \frac{z^{|p|}}{|p|!} = \sum_{N \geq 0} B_N \frac{z^N}{N!}$$

Decompose. $$= \sum_{p \in \mathcal{P}} \big((|p|+1)\text{cycles}(p) + 1\big) \frac{z^{|p|+1}}{(|p|+1)!}$$

Simplify. $$= \sum_{p \in \mathcal{P}} \text{cycles}(p) \frac{z^{|p|+1}}{(|p|)!} + \sum_{p \in \mathcal{P}} \frac{z^{|p|+1}}{(|p|+1)!}$$

Substitute. $$= zB(z) + \sum_{k \geq 0} \frac{z^{k+1}}{(k+1)} = zB(z) + \ln \frac{1}{1-z}$$

Solve. $$B(z) = \frac{1}{1-z} \ln \frac{1}{1-z}$$

OGF for the Harmonic numbers

cumulated cost

Expand. $$[z^N]B(z) = \frac{B_N}{N!} = H_N$$

average # cycles in a random permutation

Average number of cycles in a random permutation with BGFs

BGF. $$B(z,u) = \sum_{p\in\mathcal{P}} \frac{z^{|p|}}{|p|!} u^{cycles(p)}$$

Apply construction . $$= \sum_{p\in\mathcal{P}} \frac{z^{|p|+1}}{(|p|+1)!} \left(u^{cycles(p)+1} + |p|u^{cycles(p)}\right)$$

Differentiate wrt z. $$B_z(z,u) = \sum_{p\in\mathcal{P}} \frac{z^{|p|}}{(|p|)!} u^{cycles(p)+1} + \sum_{p\in\mathcal{P}} \frac{z^{|p|}}{(|p|)!} |p| u^{cycles(p)}$$

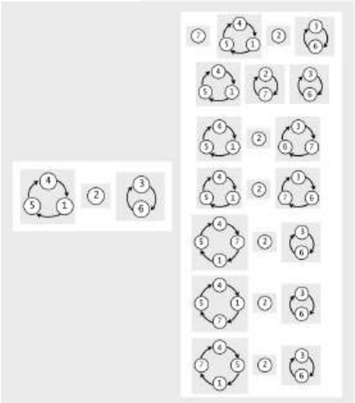
Substitute. $$= uB(z,u) + zB_z(z,u)$$

Solve for $B_z(z, u)$. $$B_z(z,u) = \frac{u}{1-z} B(z,u)$$

Solve ODE. $$B(z,u) = \frac{1}{(1-z)^u}$$

Average number of cycles. $$B_u(z,1) = \frac{1}{1-z} \ln \frac{1}{1-z}$$

$$[z^N]B_u(z,1) = H_N \quad \checkmark$$



Average number of cycles in a random permutation with BGFs and the symbolic method

Combinatorial class. P, the class of all permutations

Construction. $$P = SET(uCYC(Z))$$

BGF equation $$P(z,u) = \exp\Big(u \ln \frac{1}{1-z}\Big) = \frac{1}{(1-z)^u}$$

immediate from transfer theorem.

Average number of cycles. $$P_u(z,1) = \frac{1}{1-z} \ln \frac{1}{1-z}$$

$$[z^N]P_u(z,1) = H_N \checkmark$$

Bottom line: BGFs are the *method of choice* in analyzing parameters

Average number of cycles of a given size in a random permutation

Combinatorial class. P, the class of all permutations

Construction. $$P = SET(CYC_{\neq r} + uCYC_r(Z))$$

BGF equation $$P(z,u) = e^{\ln\frac{1}{1-z} - \frac{z^r}{r} + \frac{uz^r}{r}}$$

immediate from transfer theorem.

Average number of cycles. $$P_u(z,1) = \frac{z^r}{r}\frac{1}{1-z}$$

$$[z^N]P_u(z,1) = \frac{1}{r} \quad \text{for} \quad N \geq r \ ✓$$

BGFs are the *method of choice* in analyzing parameters.

Many, many examples to follow.

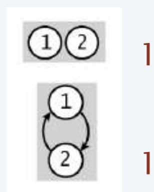
Stay tuned for Part 2

Number of permutations of size *N* with *k* cycles

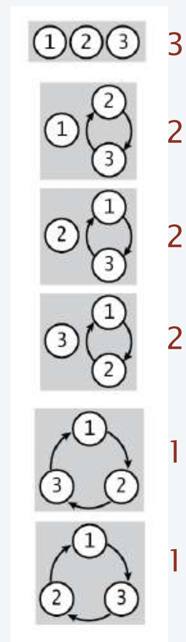
are known as *Stirling numbers of the first kind.*

Notation: ${N \brack k}$

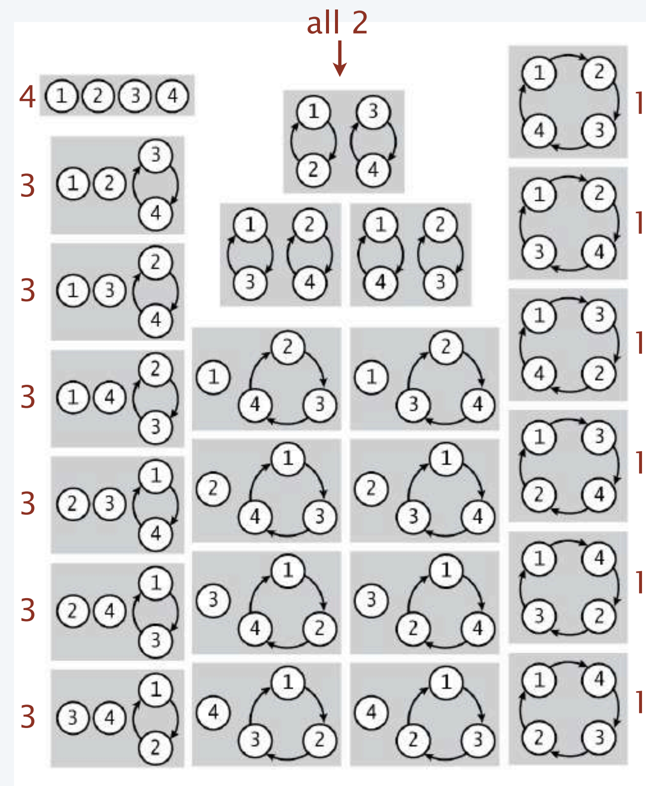
${1 \brack 1} = 1$



${2 \brack 1} = 1 \quad {2 \brack 2} = 1$



${3 \brack 1} = 2 \quad {3 \brack 2} = 3 \quad {3 \brack 3} = 1$



${4 \brack 1} = 6 \quad {4 \brack 2} = 11 \quad {4 \brack 3} = 6 \quad {4 \brack 4} = 1$

Stirling numbers of the first kind (cycle numbers)

Fundamental identity

$$P(z,u) = \sum_{p \in p} \frac{z^{|p|}}{|p|!} u^{cycles(p)} = \sum_{N \geq 0} \sum_{k \geq 0} \begin{bmatrix} N \\ k \end{bmatrix} \frac{z^N}{N!} u^k = \frac{1}{(1-z)^u}$$

Distribution

$$P(z,u) = \sum_{N \geq 0} u(u+1)\ldots(u+N-1)\frac{z^N}{N!}$$

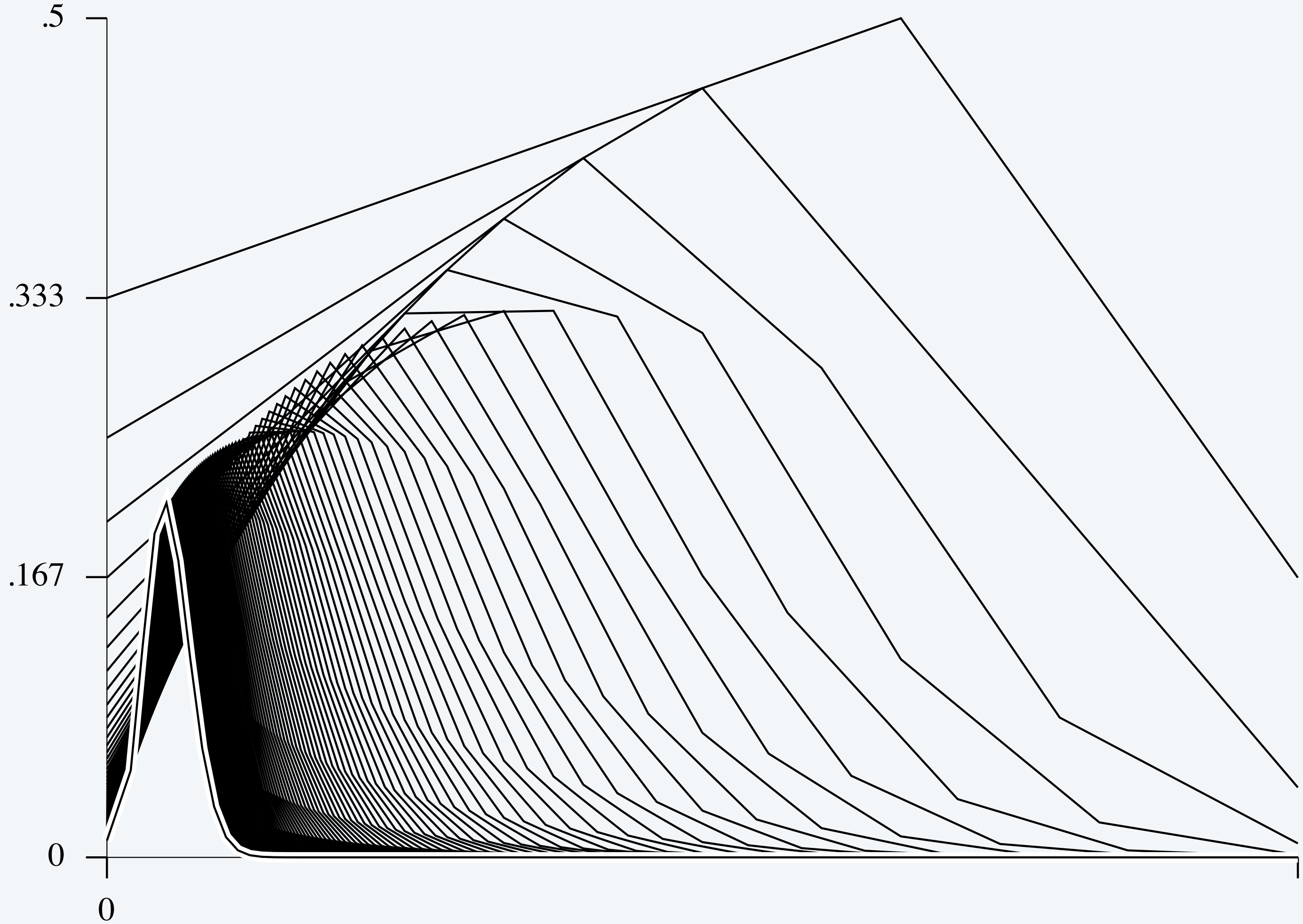
(Taylor's theorem)

$\begin{bmatrix} N \\ k \end{bmatrix}$

N \ k	1	2	3	4	5	6	7
1	1						
2	1	1					
3	2	3	1				
4	6	11	6	1			
5	24	50	35	10	1		
6	120	274	225	85	15	1	

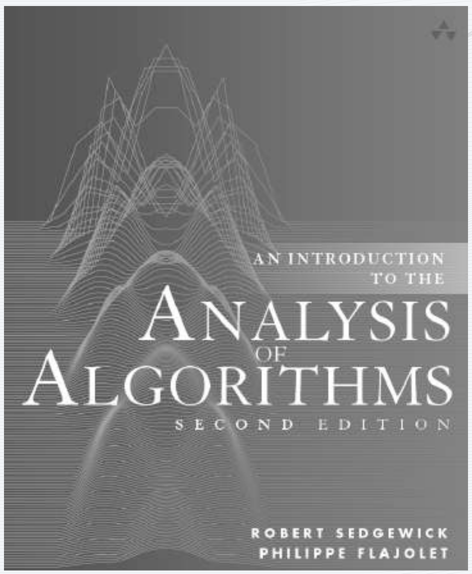
$[u^k]u(u+1)(u+2)(u+3) \longrightarrow$ (row 4)

Stirling numbers of the first kind (cycle numbers) distribution



ANALYTIC COMBINATORICS

PART ONE



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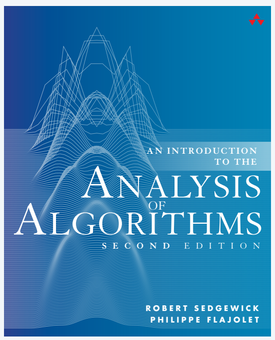
- **Exercises**

7e.Perms.Exs

Exercise 7.29

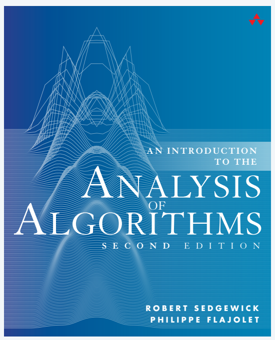
Arrangements.



Exercise 7.29 An *arrangement* of N elements is a sequence formed from a subset of the elements. Prove that the EGF for arrangements is $e^z/(1-z)$. Express the coefficients as a simple sum and give a combinatorial interpretation of that sum.

Exercise 7.45

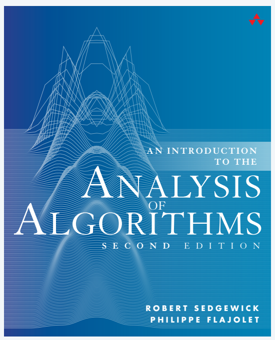
Inversions in involutions.



Exercise 7.45 Find the CGF for the total number of inversions in all involutions of length N. Use this to find the average number of inversions in an involution.

Exercise 7.61

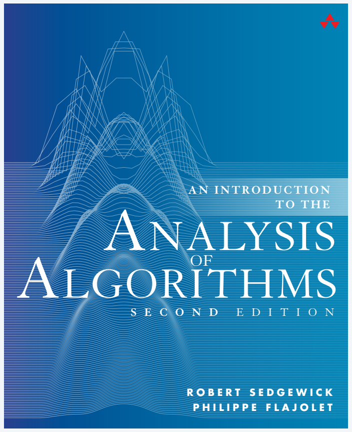
Cycle length distribution.



Exercise 7.61 Use asymptotics from generating functions (see §5.5) or a direct argument to show that the probability for a random permutation to have j cycles of length k is asymptotic to the Poisson distribution $e^{-\lambda}\lambda^j/j!$ with $\lambda = 1/k$.

Assignments for next lecture

1. Read pages 345-413 in text.

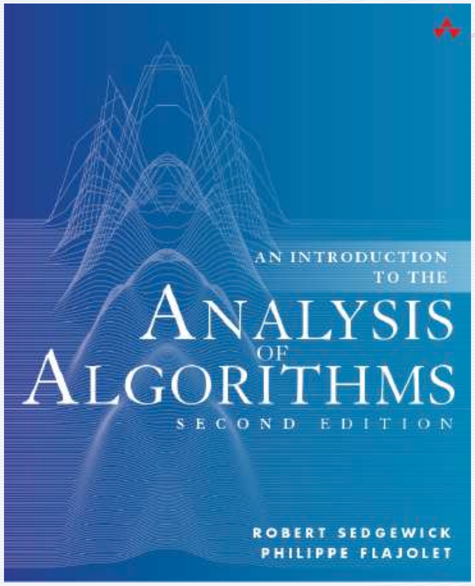


2. Run experiments to validate mathematical results.

Experiment 1. Generate 1000 random permutations for $N = 100$, 1000, and 10,000 and compare the average number of cycles and 1-cycles with the values predicted by analysis.

Experiment 2. *Extra credit.* Validate the results of Exercise 7.61 for N = 1000 and k = 10 by generating 10,000 random permutations and plotting the histogram of occurences of cycles of length 10.

3. Write up solutions to Exercises 7.29, 7.45, and 7.61.



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