Data Structures and Algorithms

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Session: Properties of Line Segment

Introduction [1]

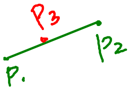
- branch of computer science that studies algorithms for solving geometric problems
- has applications in computer graphics, robotics, VLSI design, computer-aided design, molecular modeling, etc.
- input is a set of geometric objects, such as a set of points, a set of line segments, vertices of a polygon, etc
- output is response to a query about the objects, or a new geometric object

Eg: Do two line segments intersect

Convex hull

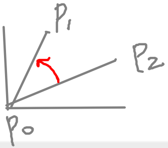
[1] Chapter 33, CLRS, Third Edition

Properties of Line Segment



- A convex combination of two distinct points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ is any point $p_3 = (x_3, y_3)$ such that for some α in the range $0 \leq \alpha \leq 1$, we have $x_3 = \alpha x_1 + (1 - \alpha)x_2$ and $y_3 = \alpha y_1 + (1 - \alpha)y_2$ i.e. $\boxed{p_3 = \alpha p_1 + (1 - \alpha)p_2}$
- Consider 3 points p_0 at $(0, 0)$, p_1 at (x_1, y_1), and p_2 at (x_2, y_2)
  - Determine whether a segment $\overline{p_0 p_1}$ is clockwise or counter-clockwise from the other segment $\overline{p_0 p_2}$ with respect to the point p_0
  - Do the 2 line segments intersect?

Do $(P_1 P_2)$ & $(P_3 P_4)$ intersect?

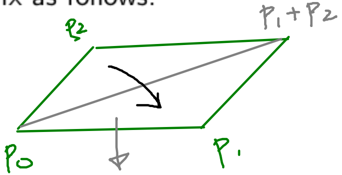


Direction of p_1 from p_2 with respect to p_0

- Consider 3 points p_0 at $(0,0)$, p_1 at (x_1, y_1), and p_2 at (x_2, y_2)
- To determine whether p_1 is clockwise or counter-clockwise from p_2 with respect to the origin $(0,0)$, compute the determinant of a matrix as follows:

$$x_2 y_2 - x_1 y_1 = \begin{vmatrix} x_2 & x_1 \\ y_2 & y_1 \end{vmatrix} = p_2 \times p_1 \neq p_1 \times p_2 = \begin{array}{c} \begin{array}{cc} p_1 & p_2 \end{array} \\ \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \end{array}$$

$$= x_1 * y_2 - x_2 * y_1$$



Signed Area = $p_1 \times p_2$

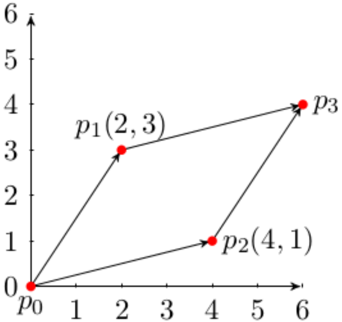
- If this result is positive, then p_1 is clockwise from p_2, else it is counter-clockwise

$P_2 \times P_1 = -(P_1 \times P_2)$

P_2 is counter clockwise from P_1

Direction of p_1 from p_2 with respect to $p_0 = (0,0)$

Cross Product of vectors p_1 and p_2



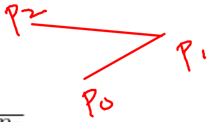
Computing Cross Product

$$
\begin{aligned}
p_3 &= p_1 + p_2 \\
&= (x_1 + x_2,\ y_1 + y_2) \\
&= (2 + 4,\ 3 + 1) \\
&= (6, 4)
\end{aligned}
$$

$$
\begin{aligned}
p_1 \times p_2 &= x_1 * y_2 - x_2 * y_1 \\
&= (2 * 1 - 4 * 3) \\
&= -10
\end{aligned}
$$

Since, the result of $p_1 \times p_2$ is negative, p_1 is counter-clockwise from p_2, with respect to the origin p_0

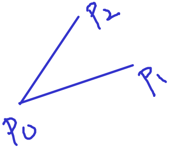
Determine whether consecutive segments turn left or right

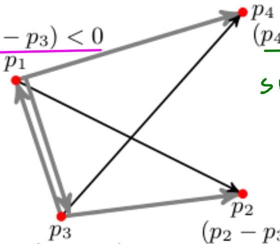


- Consider two line segments $\overline{p_0p_1}$ and $\overline{p_1p_2}$
- Do they turn left or right at point p_1 ≡ Does P_2 turn counter clockwise or clockwise from P_0 wrt P_1
- We simply check whether directed segment $\overrightarrow{p_0p_2}$ is clockwise or counterclockwise with respect to directed segment $\overrightarrow{p_0p_1}$
- compute the cross product $(p_2 - p_0) \times (p_1 - p_0)$

Determine whether two line segments intersect

$(p_1 - p_0) \times (p_2 - p_0) \geq 0 \Rightarrow p_1$ turns clockwise from p_2 w.r.t. p_0





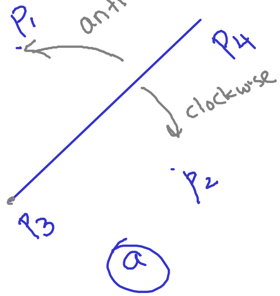
$(p_1 - p_3) \times (p_4 - p_3) < 0$

$(p_4 - p_1) \times (p_2 - p_1) < 0$

$sgn((p_4 - p_1) \times (p_2 - p_1)) = -sgn((p_3 - p_1) \times (p_2 - p_1))$

$(p_3 - p_1) \times (p_2 - p_1) > 0$

$(p_2 - p_3) \times (p_4 - p_3) > 0$



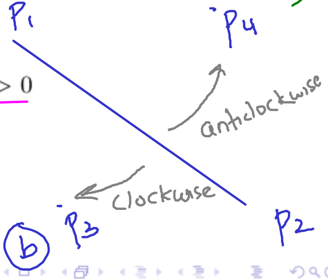


Figure: Case 1

$sgn((p_1 - p_3) \times (p_4 - p_3))$
$= -sgn((p_2 - p_3) \times (p_4 - p_3))$

Determine whether two line segments intersect

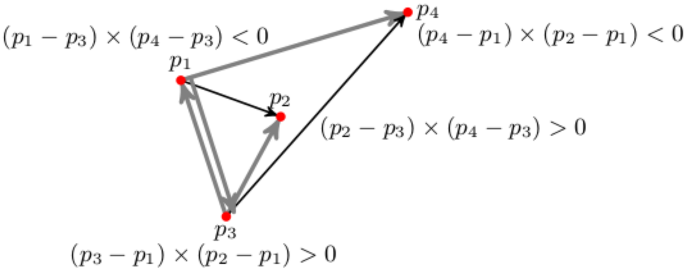


Figure: Case 2

Note: The lines do intersect but the line segments do not!

Determine whether two line segments intersect

$p_1\ p_2\ \&\ p_3$ are collinear $\iff (p_3 - p_1) \times (p_2 - p_1) = 0$

To check if p_3 lies on $p_1 p_2$,

check if

$(x_3 \in [x_1, x_2])$

&

$(y_3 \in [y_2, y_1])$

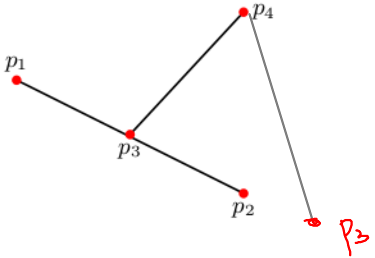


Figure: Case 3

$x_3 \in [\min(x_1, x_2), \max(x_1, x_2)]$

$y_3 \in [\min(y_1, y_2), \max(y_1, y_2)]$

Determine whether two line segments intersect

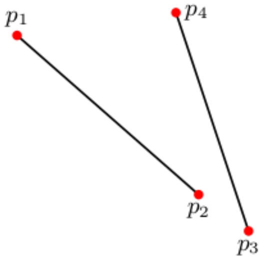
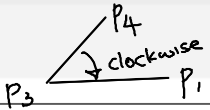


Figure: Case 4

Algorithm for detecting segments intersect



$\Rightarrow (p_1 - p_3) \times (p_4 - p_3) > 0$

Algorithm SegmentsIntersect(p_1,p_2,p_3,p_4)

$d_1 = Direction(p_3, p_4, p_1)$ → direction of p_1 from p_4 w.r.t p_3

$d_2 = Direction(p_3, p_4, p_2)$ (" " p_1 " p_3 " p_4)

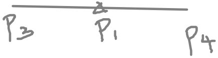
$d_3 = Direction(p_1, p_2, p_3)$

$d_4 = Direction(p_1, p_2, p_4)$

if $(\underbrace{(d_1 > 0 \text{ and } d_2 < 0) \text{ or } (d_1 < 0 \text{ and } d_2 > 0)}_{sign(d_1) = -sign(d_2)})$ **and** $(\underbrace{(d_3 > 0 \text{ and } d_4 < 0) \text{ or } (d_3 < 0 \text{ and } d_4 > 0)}_{sign(d_3) = -sign(d_4)})$ **then**

  return $TRUE$

else if $d_1 == 0$ **and** $ONSegment(p_3, p_4, p_1)$ **then** → check if



  return $TRUE$

else if $d_2 == 0$ **and** $ONSegment(p_3, p_4, p_2)$ **then**

  return $TRUE$

else if $d_3 == 0$ **and** $ONSegment(p_1, p_2, p_3)$ **then**

  return $TRUE$

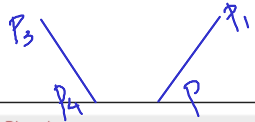
else if $d_4 == 0$ **and** $ONSegment(p_1, p_2, p_4)$ **then**

  return $TRUE$

else

  return $FALSE$ →

end if



Algorithm for detecting segments intersect

Algorithm Direction(p_i,p_j,p_k)
return $(p_k - p_i) * (p_j - p_i)$

Figure: Direction

Algorithm ONSegment(p_i,p_j,p_k)
if $min(x_i, x_j) \leq x_k \leq max(x_i, x_j)$ **and** $min(y_i, y_j) \leq y_k \leq max(y_i, y_j)$ **then**
  return $TRUE$
else
  return $FALSE$
end if

Figure: On-Segment

Thank you