Chapter 29

First Order Linear Differential Equations

29.1 Definition and Method of Solution

The most general linear first order differential equation is

$$\frac{d}{dt}y(t) + q(t)y(t) = r(t) \tag{29.1}$$

where we want to find the function $y(t)$ given $q(t)$ and $r(t)$. For most differential equations, but particularly for linear ones, the way we solve them is to first find a solution to the *homogeneous problem* $[r(t) = 0]$

$$\frac{d}{dt}y(t) + q(t)y(t) = 0 \tag{29.2}$$

and add to that solution a *particular solution* to

$$\frac{d}{dt}y(t) + q(t)y(t) = r(t) \tag{29.3}$$

called the *inhomogeneous problem.*

Here is how we solve the homogeneous problem:

$$\frac{d}{dy}y(t) + q(t)y(t) = 0 \implies \frac{d}{dt}y(t) = -q(t)y(t). \tag{29.4}$$

Divide both sides by $y(t)$

$$\frac{1}{y(t)}\frac{d}{dt}y(t) = -q(t), \tag{29.5}$$

and rewrite in terms of the logarithmic derivative

$$\frac{d}{dt}\ln y(t) = -q(t). \tag{29.6}$$

Now we integrate

$$\int_{t_0}^{t} d\ln y(t) = -\int_{t_0}^{t} q(\bar{t})\, d\bar{t} \tag{29.7}$$

to find

$$\ln\frac{y(t)}{y(t_0)} = -\int_{t_0}^{t} q(\bar{t})d\bar{t} \tag{29.8}$$

or, after exponentiating

$$y(t) = y(t_0)\exp\left[-\int_{t_0}^{t} q(\bar{t})d\bar{t}\right] \tag{29.9}$$

where $y(t_0)$ is a constant determined by the value y has at $t = t_0$, since the exponential term when $t = t_0$ is 1.

How to solve the inhomogeneous problem:

$$\frac{dy}{dt} + q(t)y(t) = r(t) \tag{29.10}$$

We multiply by the integrating factor that we found above, $\exp\left[\int_{t_0}^{t} q(\bar{t})d\bar{t}\right]$

$$\exp\left[\int_{t_0}^{t} q(\bar{t})d\bar{t}\right]\frac{dy}{dt} + \exp\left[\int_{t_0}^{t} q(\bar{t})d\bar{t}\right]q(t)y(t) = \exp\left[\int_{t_0}^{t} q(\bar{t})d\bar{t}\right]r(t). \tag{29.11}$$

But, the left hand side is a perfect differential

$$\text{LHS} = \frac{d}{dt}\left(y(t)\exp\left[\int_{t_0}^{t} q(\bar{t})d\bar{t}\right)\right]. \tag{29.12}$$

So

$$\int_{t_0}^{t} d\left(y(t)\exp\left[\int_{t_0}^{t} q(\bar{t})d\bar{t}\right]\right) = \int_{t_0}^{t}\exp\left[\int_{t_0}^{t'} q(\bar{t})d\bar{t}\right]r(t')\,dt' \tag{29.13}$$

$$y(t)\exp\left[\int_{t_0}^{t} q(\bar{t})d\bar{t}\right] - y(t_0) = \int_{t_0}^{t}\exp\left[\int_{t_0}^{t'} q(\bar{t})d\bar{t}\right]r(t')\,dt'. \tag{29.14}$$

Note that we integrate the q function to t' first and then integrate the entire integrand over t' on the right-hand side. Hence, we have

$$y(t) = y(t_0)e^{-\int_{t_0}^{t} q(\bar{t})\,d\bar{t}} + e^{-\int_{t_0}^{t} q(\bar{t})\,d\bar{t}} \int_{t_0}^{t} e^{\int_{t_0}^{t'} q(\bar{t})d\bar{t}} r(t')\,dt' \tag{29.15}$$

where the first term is the homogeneous solution (multiplied by a constant term) and the second is the particular solution. This is a general solution, with one parameter to fit, $y(t_0)$. This solution is also unique. The proof is given in the book. It requires $q(t)$ to be continuous and bounded on $[t_0, t]$. You may be surprised by the fact that we can find a closed-form solution for all first-order linear differential equations. But we can!

29.2 Examples

We never really understand how this works until we try ourselves. Let's start with some worked examples. Consider the following equation:

$$\frac{dy}{dt} + \frac{1}{1-t}y = 1 - t \tag{29.16}$$

for $0 \le t < 1$; we want the solution that also satisfies $y(0) = 1$.

Solving is essentially a "cookbook" given the form we have determined above. First find the integral of q:

$$\int_{t_0}^{t} q(\bar{t})\,d\bar{t} = \int_0^t \frac{1}{1-\bar{t}}\,d\bar{t} = -\ln(1-\bar{t})\Big|_0^t = -\ln(1-t). \tag{29.17}$$

Next, plug into the full solution

$$y(t) = y(t_0)\exp\left[\ln(1-\bar{t})\right] + \exp\left[\ln(1-t)\right] \int_0^t \exp\left[-\ln(1-t')\right](1-t')\,dt' \tag{29.18}$$

$$= 1 \times (1-t) + (1-t)\int_0^t dt' \frac{1-t'}{1-t'} \tag{29.19}$$

$$= 1 - t + (1+t)t \tag{29.20}$$

$$= (1-t)(1+t) \tag{29.21}$$

$$= 1 - t^2. \tag{29.22}$$

OK, we worked through the solution, but it is relatively easy to make an error, so it is very wise to verify the solution actually solves the equation we set out to solve. So, we check the solution:

$$\frac{d}{dt}(1-t^2) = -2t \tag{29.23}$$

$$-2t + \frac{1-t^2}{1-t} = -2t + (1+t) = 1-t \tag{29.24}$$

which checks out.

Let's try another example.

$$\frac{d}{dt}y + 2y = e^{-t} \tag{29.25}$$

with $y(0) = 3$. First, we have

$$\int_0^t q(\bar{t})\, d\bar{t} = \int_0^t 2\, d\bar{t} = 2t. \tag{29.26}$$

Plugging into the solution yields

$$y(t) = y(0)e^{-2t} + e^{-2t}\int_0^t e^{2t'}e^{-t'}\, dt' \tag{29.27}$$

$$= 3e^{-2t} + e^{-2t}(e^t - 1) \tag{29.28}$$

$$= 2e^{-2t} + e^{-t}. \tag{29.29}$$

Check:

$$y(0) = 2 + 1 = 3, \tag{29.30}$$

which checks. In addition, we have

$$\frac{d}{dt}y(t) = -4e^{-2t} - e^{-t} \tag{29.31}$$

$$\frac{d}{dt}y(t) + 2y(t) = -4e^{-2t} - e^{-t} + 4e^{-2t} + 2e^{-t} = e^{-t}, \tag{29.32}$$

which also checks.

Here is our final example:

$$\frac{d}{dt}y - 2ty = t, \tag{29.33}$$

with $y(0) = 1$. We first integrate q

$$\int_0^t q(\bar{t})d\bar{t} = \int_0^t -2\bar{t}d\bar{t} = -t^2, \tag{29.34}$$

and then form the solution as before

$$\begin{aligned} y(t) &= e^{t^2} + e^{t^2} \int_0^t e^{-t'^2} t'\, dt' && (29.35) \\ &= e^{t^2} - \frac{1}{2} e^{t^2} \int_0^t (-2t) e^{-t^2}\, dt' && (29.36) \\ &= e^{t^2} - \frac{1}{2} e^{t^2} \left(e^{-t^2} - 1 \right) && (29.37) \\ &= \frac{3}{2} e^{t^2} - \frac{1}{2}, && (29.38) \end{aligned}$$

since $y(0) = 1$. Now check

$$\frac{dy}{dt} - 2ty = 3te^{t^2} - 3te^{t^2} + t = t, \tag{29.39}$$

which also checks.

Essentially all linear first order differential equations can be solved by straightforward integrals. Nonlinear first order differential equations are another story. They will be covered next,