Physics 155 HW # 10

1.) Using the ideas developed in the in-class problems, determine the largest factor that divides the determinants below, and then verify that these results are correct by computing the determinant via row reduction. Hint: use prime factorization of each row when considered as a 5-digit number.

a.) $\begin{pmatrix} 1 & 5 & 8 & 1 & 0 \\ 1 & 4 & 5 & 0 & 8 \\ 6 & 8 & 8 & 2 & 0 \\ 3 & 9 & 9 & 2 & 8 \\ 7 & 9 & 9 & 8 & 0 \end{pmatrix}$ b.) $\begin{pmatrix} 1 & 0 & 0 & 6 & 2 \\ 7 & 5 & 3 & 3 & 5 \\ 3 & 1 & 4 & 3 & 4 \\ 8 & 0 & 0 & 0 & 2 \\ 5 & 8 & 3 & 5 & 7 \end{pmatrix}$

2.) Recalling the second part of the problems, calculate the determinants of the following two matrices using the formulas developed in the in-class problems (think Van der Monde).

a.) $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & 3 & 5 & -6 \\ 16 & 4 & 9 & 25 & 36 \\ 64 & -8 & 27 & 125 & -216 \\ 256 & 16 & 81 & 625 & 1,296 \end{pmatrix}$ b.) $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & \sqrt{2} & 1 & 2 & 7 \\ 1 & 2 & 1 & 4 & 49 \\ -1 & 2\sqrt{2} & 1 & 8 & 343 \\ 1 & 4 & 1 & 16 & 2,401 \end{pmatrix}$

3.) Consider the set of functions on the interval $[-1,1]$ with a scalar product $(f \cdot g) = \int_{-1}^{1} f(x)g(x)\,dx$. Starting from the spanning set $(1, ax+b, \alpha x^2+\beta x+\gamma)$, determine the first three orthonormal polynomials that determine the basis vectors for the vector space (because it is infinite-dimensional, there is an infinite number of them).

4.) Consider a vector space given by the set of functions $f(x)$ on the interval $[a,b]$ that are differentiable k times. Consider the set of k functions $\{f_1(x), f_2(x), ..., f_k(x)\}$. We want to show they are independent, which means

$$c_1 f_1(x) + c_2 f_2(x) + ... + c_k f_x(x) = 0 \tag{1}$$

has a solution only for $c_1 = c_2 = ... = c_k = 0$.

Pick a point x_0, then we have

$$c_1 f_1(x_0) + c_2 f_2(x_0) + ... + c_k f_k(x_0) = 0 \tag{2}$$

Differentiate the above formula up to k times and evaluate at x_0. Then

$$\det \begin{pmatrix} f_1(x_0) & f_2(x_0) & \cdots & f_k(x_0) \\ f_1'(x_0) & f_2'(x_0) & \cdots & f_k'(x_0) \\ \vdots & & & \vdots \\ f_1^{(k)}(x_0) & f_2^{(k)}(x_0) & \cdots & f_k^{(k)}(x_0) \end{pmatrix} \neq 0 \tag{3}$$

Shows independence of the functions (the converse is not true, the Wronskian at a point x_0 can be zero for independent functions). This determinant is called the Wronskian.

Evaluate the Wronskian for

$$\{1, x, x^2, ..., x^k\} \tag{4}$$

and show it is nonzero. Since this holds for arbitrary k, it shows the vector space is infinite-dimensional and that the set off all polynomials spans it.

5.) Consider the set of vectors

$$\{(1\,1\,1\,0), (1\,1\,2\,2), (0\,0\,2\,3), (-1\ \ -1\,0\,1), (-2\ \ -2\,1\,3)\} \qquad (5)$$

Find the dimension of the subset spanned by these vectors by finding the largest set of spanning vectors.