Physics 155 HW # 11

Each problem is worth 25 points!

1.) Consider the general 2x2 matrix:

$$M = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \tag{1}$$

Determine the eigenvalues for arbitrary a, b, d.

Determine the two <u>orthonormal</u> eigenvectors.

Hint: The second eigenvector can be found by orthogonality, or by keeping track of the $\pm$ (the latter is preferred).

2.) Ion dynamics

Given $V(\vec{x}) = V(x_0) + \frac{1}{2}\vec{\delta x} \cdot K(x_0) \cdot \vec{\delta x}$ as we derived in lecture, compute the force on each ion due to a displacement from equilibrium $x_i = x_i^0 + \delta x_i$:

$$\text{Force on } i\text{th ion} = -\left.\frac{\partial V(\vec{x})}{\partial x_i}\right|_{x_i = x_i^0 + \delta x_i} \tag{2}$$

Verify that

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = -K \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{pmatrix} \tag{3}$$

(We will verify for the case where we compute for the eigenvalue of the K matrix where we set $k = \dfrac{5}{4}$, so that $x_1^0 = -1$, $x_2^0 = 0$, $x_3^0 = 1$). Using this force, Newton says

$$m\frac{d^2}{dt^2}\begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{pmatrix} = -K \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{pmatrix} \tag{4}$$

We seek the so-called normal modes of the motion. Using $\delta x_j = \delta x_j^0 e^{-i\omega t}$ with the same ω for each ion, show that

$$\omega^2 \begin{pmatrix} \delta x_1^0 \\ \\ \delta x_2^0 \\ \\ \delta x_3^0 \end{pmatrix} = \frac{1}{m} K \begin{pmatrix} \delta x_1^0 \\ \\ \delta x_2^0 \\ \\ \delta x_3^0 \end{pmatrix} \tag{5}$$

$\Rightarrow \delta x_j^0$ is an eigenvector of K and $\omega = \sqrt{\frac{\text{eigenvalue}}{m}}$

Find the three frequencies if $m = 1$.

Find the three eigenvectors (Hint: $E = k = \frac{5}{4}$ is one eigenvalue, use this to get a quadratic equation).

Discuss why your results are what they are. Could you have guessed the eigenvectors?

This problem can be solved by hand.

Substitute in the numbers as soon as you can to make it easier, and use the hint that one eigenvalue (not frequency) is $\frac{5}{4}$.