Chapter 12

The Divergence Theorem

12.1 Defining divergence

Recall from last lecture, where we showed if we consider a small rectangular prism about the point (x_0, y_0, z_0) with widths dx, dy, dz, then:

$$\frac{1}{\delta V}\int_{\delta S}\vec{F}\cdot\hat{n}\,ds = \frac{\rho(x,y,z)}{\epsilon_0} \tag{12.1}$$

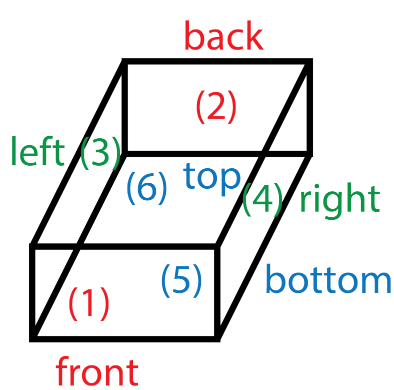

Figure 12.1: Rectangular prism employed for defining the divergence.The sides are labeled as follows: (1) front; (2) back; (3) left; (4) right; (5) bottom and (6) top.

There are 6 faces to the prism. Let's do the surface integral::

$$\int_{\delta S} \vec{F} \cdot \hat{n}\, ds = \int_{S_1} \vec{F} \cdot \hat{n}\, ds + \int_{S_2} \vec{F} \cdot \hat{n}\, ds + \int_{S_3} \vec{F} \cdot \hat{n}\, ds + \int_{S_4} \vec{F} \cdot \hat{n}\, ds + \int_{S_5} \vec{F} \cdot \hat{n}\, ds + \int_{S_6} \vec{F} \cdot \hat{n}\, ds. \tag{12.2}$$

We denoted the corresponding contributions according to the colors in Fig. 12.1.

side	normal	coordinate	area
1	$-\hat{j}$	$x\hat{i} + \left(y - \frac{dy}{2}\right)\hat{j} + z\hat{k}$	$dx\,dz$
2	$\hat{j}$	$x\hat{i} + \left(y + \frac{dy}{2}\right)\hat{j} + z\hat{k}$	$dx\,dz$
3	$-\hat{i}$	$\left(x - \frac{dx}{2}\right)\hat{i} + y\hat{j} + z\hat{k}$	$dy\,dz$
4	$\hat{i}$	$\left(x + \frac{dx}{2}\right)\hat{i} + y\hat{j} + z\hat{k}$	$dy\,dz$
5	$-\hat{k}$	$x\hat{i} + y\hat{j} + \left(z - \frac{dz}{2}\right)\hat{k}$	$dx\,dy$
6	$\hat{k}$	$x\hat{i} + y\hat{j} + \left(z + \frac{dz}{2}\right)\hat{k}$	$dx\,dy$

Using the results of the table above, we can directly compute the total surface integral for the small volume element of the rectangular prism:

$$\begin{aligned}\int_S \vec{F} \cdot \hat{n}\, ds &= \left[F_y\left(x, y + \frac{dy}{2}, z\right) - F_y\left(x, y - \frac{dy}{2}, z\right)\right] dx\,dz \\ &+ \left[F_x\left(x + \frac{dx}{2}, y, z\right) - F_x\left(x - \frac{dx}{2}, y, z\right)\right] dy\,dz \\ &+ \left[F_z\left(x, y, z + \frac{dz}{2}\right) - F_z\left(x, y, z - \frac{dz}{2}\right)\right] dx\,dy.\end{aligned} \tag{12.3}$$

Using $\delta V = dx\,dy\,dx$, we define the divergence to be

$$\left(\frac{1}{\delta V}\int_{\delta S} \vec{F} \cdot \hat{n} dS\right) = \text{ divergence}, \tag{12.4}$$

which then becomes

$$\text{divergence} = \left(\frac{\partial F_y}{\partial y} + \frac{\partial F_x}{\partial x} + \frac{\partial F_z}{\partial z}\right)\Bigg|_{(x,y,z)}. \tag{12.5}$$

This can be rewritten as

$$\text{divergence} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \vec{F}(x, y, z) = \vec{\nabla} \cdot \vec{F}(x, y, z). \tag{12.6}$$

This then leads to our final result for the divergence

$$\vec{\nabla} \cdot \vec{F}(x, y, z) = \frac{\partial}{\partial x} F_x(x, y, z) + \frac{\partial}{\partial y} F_y(x, y, z) + \frac{\partial}{\partial z} F_z(x, y, z), \tag{12.7}$$

with $\vec{\nabla}$ called the divergence operator.

Using this new divergence notation allows us to re-express Gauss' law as

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}. \tag{12.8}$$

12.2 The divergence theorem

Consider a closed surface surrounding a volume V:

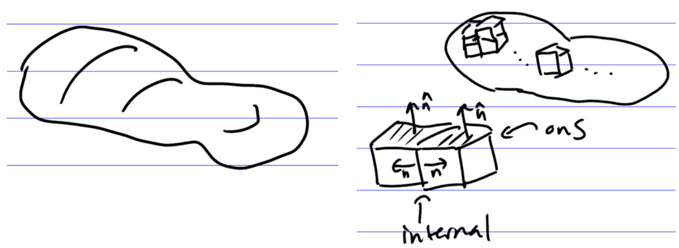

Figure 12.2: On the left, we show a general volume that will be integrated. On the right, it is broken into small volumes for carrying out the integration. Note how the outward pointing normals cancel for all internal boundaries, but are unpaired for the outer boundaries.

Any face on S has a unit normal which points out and accumulates $\vec{F} \cdot \hat{n}$ terms in the integral, but all internal faces have two unit normals $\hat{n}$, one in each direction (+ and -) on the boundary surfaces of two adjacent prisms, which *cancel* when integrated over their respective surfaces. So we learn that

$$\int_S \vec{F} \cdot \hat{n}\, dS = \sum_i \int_{S_i} \vec{F} \cdot \hat{n}\, dS, \tag{12.9}$$

where S_i is the surface of one of the cubes in the volume. But we know for each small cube:

$$\int_{S_i} \vec{F} \cdot \hat{n} dS = \int_{V_i} \vec{\nabla} \cdot \vec{F}\, dV. \tag{12.10}$$

Since $\frac{1}{V_i}\int_{S_i} \vec{F}\cdot\hat{n}\,dS = \vec{\nabla}\cdot\vec{F}$, hence

$$\int_S \vec{F}\cdot\hat{n}\,dS = \sum_i \int_{S_i} \vec{F}\cdot\hat{n}\,dS = \sum_i \int_{V_i} \vec{\nabla}\cdot\vec{F}\,dV = \int_V \vec{\nabla}\cdot\vec{F}\,dV \qquad (12.11)$$

or,

$$\int_S \vec{F}\cdot\hat{n}\,dS = \int_V \vec{\nabla}\cdot\vec{F}\,dV, \qquad (12.12)$$

which is called the divergence theorem.

Example: This is a cute example that allows you to compute a volume by integrating a surface area.

$$\vec{F} = \vec{r} = xi + yj + zk \qquad (12.13)$$

$$\vec{\nabla}\cdot\vec{F} = 1 + 1 + 1 = 3 \qquad (12.14)$$

$$\frac{1}{3}\int_S \vec{r}\cdot\hat{n}dS = \frac{1}{3}\int_V \vec{\nabla}\cdot\vec{r}dV = \frac{1}{3}\int_V 3dV = V \qquad (12.15)$$

so,

$$\frac{1}{3}\int_S \vec{r}\cdot\hat{n}dS = V \qquad (12.16)$$

Check: Volume of a sphere

$$V = \frac{1}{3}\int_S \vec{r}\cdot\hat{n}dS \qquad (12.17)$$

with $\hat{n} = \hat{r} = \vec{r}/r =$ unit vector in radial direction, so $\vec{r}\cdot\hat{n} = \vec{r}\cdot\vec{r}/r = r$ and we have

$$V = \frac{1}{3}\int_S rdS. \qquad (12.18)$$

The r comes out of the integral because we are integrating over the surface, which has a fixed radius. We obtain

$$V = \frac{r}{3}\int_S dS = \frac{r}{3}\times 4\pi r^2 = \frac{4}{3}\pi r^3. \qquad (12.19)$$

This checks out!

Check: Rectangular prism, given in Fig. 12.1.

Start by identifying the six different normal vectors and the areas of each of the faces. Then we assemble the results via

$$V = \frac{1}{3}\int_S \vec{r}\cdot\hat{n}\,dS = \frac{1}{3}\left[-y_1ab + y_2ab - x_3ac + x_4ac - z_5bc + z_6bc\right], \quad (12.20)$$

since the normals point as described in the table after Eq. (12.2). But, $y_2 - y_1 = c$, $x_4 - x_3 = b$, $z_6 - z_5 = a$ and

$$V = \frac{1}{3}\int_S \vec{r}\cdot\hat{n}dS = \frac{1}{3}abc3 = abc. \quad (12.21)$$

We end by discussing divergence in cylindrical and spherical coordinates. Cylindrical:

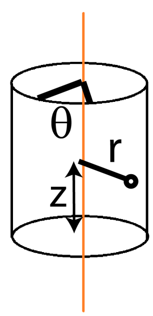

Figure 12.3: Schematic of cylindrical coordinates ρ, θ, and z.

The unit vectors are as follows:

$$\hat{e_z} = \hat{k} = \text{ unit vector in the } z \text{ direction;} \quad (12.22)$$

$$\hat{e_r} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}} = \frac{x\hat{i} + y\hat{j}}{r} = \text{ unit vector in the } r \text{ direction;} \quad (12.23)$$

and

$$\hat{e_\theta} = \frac{y\hat{i} - x\hat{j}}{r} = \text{ unit vector perpendicular to } z \text{ and } r. \quad (12.24)$$

The divergence in cylindrical coordinates is then

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}. \quad (12.25)$$

For spherical coordinates, the book *Div, Grad, Curl, and all that* interchanges θ and ϕ—this is common, math uses the notation in the book and physics uses the interchanged notation. We use our notation, not the book's, because ours is more standard for physics. Yes, it is a pain, yes it will cause confusion. But you need to learn the language of physicists as you become one.

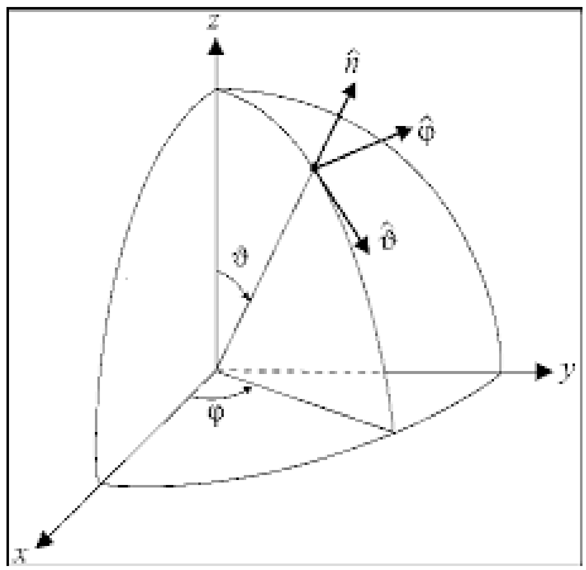

Figure 12.4: Physicist's notation for spherical coordinates (r, θ, ϕ) and the corresponding unit vectors.

The unit vectors satisfy

$$\hat{e_\phi} = \hat{e_r} \times \hat{e_\theta}. \tag{12.26}$$

The radial direction is

$$\hat{e_r} = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{r} \tag{12.27}$$

and the *theta* direction is

$$\hat{e_\theta} = z\frac{(x\hat{\imath} + y\hat{\jmath})}{r\sqrt{x^2 + y^2}} - \hat{k}\frac{\sqrt{x^2 + y^2}}{r}. \tag{12.28}$$

We need to work out the above cross product to get the ϕ direction (it lies in the x-y plane).

In spherical coordinates, the divergence then becomes

$$\vec{\nabla} \cdot \vec{F} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \theta}(\sin(\theta) F_\theta) + \frac{1}{r \sin(\theta)} \frac{\partial F_\phi}{\partial \phi} \tag{12.29}$$

which is $\theta \to \phi$, $\phi \to \theta$ from the book result.