Chapter 9

Feynman or Parametric Integration

9.1 Powers and Gaussians

This technique is usually called differentiating under the integral sign. Take a function that depends on α and x. Under proper converging conditions

$$\frac{d}{dx}\int f(x,\alpha)dx = \int \frac{\delta}{\delta x} f(x,\alpha)\,dx \tag{9.1}$$

The most common example of this comes with a variant of the Gaussian integral. Suppose we want to find

$$\int_0^\infty x^n e^{-x^2}\,dx. \tag{9.2}$$

Recall, that we already worked out the integral of the Gaussian without any power. It was

$$\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}. \tag{9.3}$$

Since the integrand is even, we also know the integral from 0 to infinity is given by

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}. \tag{9.4}$$

Now, we want to investigate powers of x times Gaussians. For $n = 2m+1$ odd, we let $x^2 = u$ and $2x\,dx = du$. The integral transforms to

$$\int_0^\infty x^{2m+1} e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty u^m e^{-u}\,du. \tag{9.5}$$

The hardest part of this technique is figuring out where to introduce a parameter that we differentiate with respect to. Recall that a derivative of an exponential returns the exponential times a derivative of the exponent. This motivates us to introduce α in the following fashion:

$$\int_0^\infty x^{2m+1} e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty u^m e^{-u}\,du = \frac{1}{2}(-1)^m \int_0^\infty \frac{d^m}{d\alpha^m} e^{-\alpha u}\,du\bigg|_{\alpha=1}. \tag{9.6}$$

Convince yourself that the derivative, evaluated when $\alpha = 1$ does indeed produce the required power. Now, we complete the calculation by performing the integral first. We have

$$\begin{aligned}
\int_0^\infty x^{2m+1} e^{-x^2}\,dx &= \frac{1}{2}(-1)^n \frac{d^m}{d\alpha^m}\int_0^\infty e^{-\alpha u}\,du\bigg|_{\alpha=1} && (9.7)\\
&= \frac{1}{2}(-1)^m \frac{d^m}{d\alpha^m}\left(-\frac{1}{\alpha}e^{-\alpha u}\Big|_0^\infty\right)\bigg|_{\alpha=1} && (9.8)\\
&= \frac{1}{2}(-1)^m \frac{d^m}{d\alpha^m}\frac{1}{\alpha}\bigg|_{\alpha=1} && (9.9)\\
&= \frac{1}{2}(-1)^m(-1)^m m!\,\frac{1}{\alpha^{n+1}}\bigg|_{\alpha=1} && (9.10)\\
&= \frac{1}{2}m!. && (9.11)
\end{aligned}$$

Be sure you can evaluate the m-fold derivative of $1/\alpha$ to be able to find this final result. To be rigorous, prove it via induction.

Now for $n = 2m$ even, we begin in the same way

$$\int_0^\infty x^{2m} e^{-x^2}\,dx = (-1)^n \int_0^\infty \frac{d^m}{d\alpha^m} e^{-\alpha x^2}\bigg|_{\alpha=1} dx \tag{9.12}$$

Next, we set $y = \sqrt{\alpha}\,x$, $dy = \sqrt{\alpha}\,dx$ to remove the α from the exponent and we do the integral and the subsequent derivatives:

$$\int_0^\infty x^{2m} e^{-x^2}\,dx = (-1)^n \frac{d^m}{d\alpha^m} \int_0^\infty e^{-y^2} \frac{1}{\sqrt{\alpha}}\,dy\bigg|_{\alpha=1} \tag{9.13}$$

$$= (-1)^m \frac{d^m}{d\alpha^m} \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{\alpha}}\bigg|_{\alpha=1} \tag{9.14}$$

$$= (-1)^m (-1)^m \frac{\sqrt{\pi}}{2} \frac{1}{2} \frac{3}{2} \frac{5}{2} \cdots \frac{2m-1}{2} \frac{1}{a^{m+\frac{1}{2}}}\bigg|_{\alpha=1} \tag{9.15}$$

$$= \frac{\sqrt{\pi}}{2} \frac{(2m-1)!!}{2^m}. \tag{9.16}$$

So we have

$$\int_0^\infty x^n e^{-x^2}\,dx = \begin{cases} \frac{\sqrt{\pi}}{2} \frac{(n-1)!!}{2^{\frac{n}{2}}}, & \text{if } n \text{ even} \\ \\ \frac{1}{2}\left(\frac{n-1}{2}\right)!, & \text{if } n \text{ odd} \end{cases}. \tag{9.17}$$

Clearly, this method is powerful. But it often is quite confusing. The biggest issue is how do we put in the parameter we differentiate with respect to? One way is as we saw before—we add the parameter to an argument of a function. Another way is to add an entire function into the integrand. We show how this works next. Consider the following integral.

$$\int_0^\infty \frac{\sin x}{x}\,dx \tag{9.18}$$

Adding an α to the argument of the sin function does not help us. Instead, we introduce $e^{-\alpha x}$ with $\alpha > 0$, and choose the limit where $\alpha \to 0$. This will reproduce the original integral.

$$I(\alpha) = \int_0^\infty \frac{\sin x}{x} e^{-\alpha x}\,dx \tag{9.19}$$

But now the derivative satisfies

$$\frac{d}{d\alpha} I(\alpha) = \frac{d}{d\alpha} \int_0^\infty \frac{\sin x}{x} e^{-\alpha x}\,dx \tag{9.20}$$

$$= \int_0^\infty \frac{\sin x}{x} \frac{d}{d\alpha} e^{-\alpha x}\,dx \tag{9.21}$$

$$= \int_0^\infty \frac{\sin x}{x} (-x)\, e^{-\alpha x}\,dx \tag{9.22}$$

$$= -\int_0^\infty \sin x\, e^{-\alpha x}. \tag{9.23}$$

But $\sin x = \text{Im}\,[e^{ix}]$ (recall that $e^{ix} = \cos x + i\sin x$). So, we have

$$\frac{d}{d\alpha}I(\alpha) = -\text{Im}\int_0^\infty e^{-\alpha x + ix}\,dx. \tag{9.24}$$

Now we can evaluate the integral to find

$$\frac{d}{d\alpha}I(\alpha) = -\text{Im}\frac{1}{-\alpha + i}e^{-\alpha x + ix}\bigg|_0^\infty \tag{9.25}$$

$$= \text{Im}\frac{1}{-\alpha + i}. \tag{9.26}$$

We next evaluate the imaginary part. We use the multiply by one trick, so that the imaginary part becomes simple:

$$\frac{d}{d\alpha}I(\alpha) = \text{Im}\frac{-\alpha - i}{(-\alpha - i)(-\alpha + i)} \tag{9.27}$$

$$= \text{Im}\frac{-\alpha - i}{\alpha^2 + 1} \tag{9.28}$$

$$= -\frac{1}{1 + \alpha^2}. \tag{9.29}$$

This is a differential equation. But do not panic. We actually know how to solve it. Recall that $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$ so

$$I(\alpha) = -\tan^{-1}x + C. \tag{9.30}$$

How do we find the constant? It is not quite so obvious at first, but with a little thought, we try to examine the limit where $\alpha \to \infty$. Then the integral $\to 0$. But $\tan^{-1}(\infty) = \frac{\pi}{2}$. So we can find the constant C via

$$-\frac{\pi}{2} + C = 0 \implies C = \frac{\pi}{2}. \tag{9.31}$$

Putting this all together tells us that

$$I(\alpha) = \frac{\pi}{2} - \tan^{-1}\alpha \tag{9.32}$$

and for $\alpha = 0$,

$$I(0) = \frac{\pi}{2}. \tag{9.33}$$

Hence

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}. \tag{9.34}$$

Clearly this approach has a lot of surprises. Sometimes we convert evaluating an integral into solving a differential equation (we will cover how to solve such equations in great generality later in these notes). The key point is it takes practice to learn how to make the approach work. There are only a handful of things we can try to make it work. One simply tries them to see if Feynman integration will work for a given problem.

Overall, this is an impressive method to remember. It requires some creativity to employ the technique.

We will do one more example:

$$\int_0^1 \frac{x^2-1}{\ln x}\,dx \tag{9.35}$$

It is not obvious how to enter the parameter here. It turns out the trick is to replace the exponent 2 with α. Then $\frac{d}{d\alpha}x^\alpha = \frac{d}{d\alpha}e^{\alpha\ln x} = \ln x\, e^{\alpha \ln x} = \ln x\, x^\alpha$ (if you have not seen this before, be sure to remember it—you can differentiate an expression with respect to its exponent by converting it to an exponential). Using this, we rewrite the integral as

$$I(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\ln x}\,dx \tag{9.36}$$

The simplification occurs when we differentiate with respect to α (which is the common theme in this approach):

$$\begin{aligned}
\frac{d}{d\alpha}I(\alpha) &= \int_0^1 \frac{d}{d\alpha}\frac{x^\alpha-1}{\ln x}\,dx && (9.37)\\
&= \int_0^1 \frac{\ln x\, x^\alpha}{\ln x}\,dx && (9.38)\\
&= \int_0^1 x^\alpha\,dx && (9.39)\\
&= \left.\frac{x^{\alpha+1}}{\alpha+1}\right|_0^1 && (9.40)\\
&= \frac{1}{\alpha+1} && (9.41)
\end{aligned}$$

Now we need to integrate with respect to α to get the solution:

$$I(\alpha) = \int d\alpha \frac{1}{\alpha+1} + C = \ln(\alpha+1) + C. \tag{9.42}$$

If $\alpha = 0$, then $x^\alpha = 1$ and the integral vanishes $I(\alpha) = 0$. This implies that $\ln(1) + C = 0$, so $C = 0$. Finally, we have

$$I(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\ln x}\, dx = \ln(\alpha+1) \tag{9.43}$$

For $\alpha = 2$,

$$I(2) = \int_0^1 \frac{x^2 - 1}{\ln x}\, dx = \ln 3. \tag{9.44}$$

You might recall that we already solved the integral for $\alpha = 1$, which gives $\ln 2$.