Chapter 6

How to integrate

6.1 Elementary examples of integration

The Toeplitz book starts by showing how to determine the derivatives of trigonometric functions, which I will assume you all know: $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$, $\frac{d}{dx}\tan(x) = \frac{1}{[\cos(x)]^2}$ and so on.

It also describes the inverse trigonometric functions, which are defined, typically, on the range from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, although this can vary. The restricted range is to ensure that the inverse trigonometric functions are single-valued, which we discussed in the last lecture (pictures of the inverse functions can be found in the Toeplitz book).

The critically important idea that we will use here and in the lab is the question of the derivatives of the inverse functions. If $y = \sin(x)$ so $\arcsin(y) = x$, then $\frac{dx}{dy} = \frac{1}{cos(x)} = \frac{1}{\sqrt{1-y^2}}$ so $\frac{d}{dy}\arcsin(y) = \frac{1}{\sqrt{1-y^2}}$. Note that we can replace $\cos(x)$ by $\sqrt{1-y^2}$, because $\sin(x) = y$. One might be worried about whether we choose the plus root or the minus root. This is determined by looking at the angles being considered for the given problem. Similarly, $\frac{d}{dy}\arccos(y) = \frac{-1}{\sqrt{1-y^2}}$ and $\frac{d}{dy}\arctan(y) = \frac{1}{1+y^2}$. These identities become important in evaluating integrals as we now show.

So, we want to generate a general set of methods for how to integrate different functions. We already know how to integrate polynomials, since

$$\int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} & n \neq -1 \\ \ln(x) & n = -1 \end{cases} \tag{6.1}$$

$$\implies \int (ax+b)^n dx = \begin{cases} \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} & n \neq -1 \\ \frac{1}{a}\ln(ax+b) & n = -1 \end{cases} \tag{6.2}$$

If we let $P_n(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ denote an n^{th} degree polynomial, then we know immediately how to integrate the polynomial. What about a more complicated integral, like a polynomial divided by a linear function? We can use the above formulas to integrate $\int \frac{P_n(x)}{x-a} dx$ via

$$\int \frac{P_n(x)}{x-a} dx = \int \frac{P_n(x) - P_n(a)}{x-a} dx + \int \frac{P_n(a)}{x-a} dx \tag{6.3}$$

$$= \int \frac{a_1(x-a) + a_2(x^2-a^2) + a_3(x^3-a^3) + ... + a_n x^n}{x-a} dx + P_n(a)\ln(x-a) \tag{6.4}$$

$$\int dx[a_1 + a_2(x+a) + a_3(x^2+ax+a^2) + ... + a_n(x^{n-1} + ax^{n-2} + ... + a^{n-1})] + P_n(a)\ln(x-a) \tag{6.5}$$

which can be integrated. Note how we used the relation in deriving the geometric series to go from the second to the third line—in other words, all of these integrals can be evaluated analytically, because we can factor the polynomial appropriately after using first the add zero trick and then replacing the remaining polynomial by finite-sum expansions.

What about even more complicated integrals, like $\int \frac{P_n(x)}{(x-a)^m} dx$? This can be written (after using the add-zero trick) as

$$\int \frac{P_n(x) - P_n(a)}{(x-a)} \frac{dx}{(x-a)^{m-1}} + \int \frac{P_n(a)}{(x-a)^m} dx \tag{6.6}$$

with the second term being able to be integrated, and $[P_n(x) - P_n(a)]/(x-a)$ being a polynomial of degree $n-1$. So we can reduce $\int \frac{P_n(x)}{(x-a)^m} dx$ to known integrals plus an integral of the form $\int \frac{P_{n-1}(x)}{(x-a)^{m-1}} dx$. Continue iterating this way until the remaining polynomial becomes a constant, or the exponent on the denominator becomes 1. At this point, all integrals are known how to do by our previous work. Hence, this implies that all integrals of the form

$$\int \frac{P_n(x)}{(x-a)^m} dx \tag{6.7}$$

can also be integrated. Note, that it does not say working out the results for individual cases will be easy. In general, it will not be simple to do so. But the path to doing so is completely known.

Note: old-schoolers, like myself, use integral tables to find these integrals, because the procedure is very tedious to do by hand. If you know mathematica, or wolfram alpha, then this is a very useful exercise that these symbolic manipulation packages can do for you.

What about more complex denominators? We know $\int \frac{dx}{x^2+1} = \arctan(x)$ and

$$\int \frac{dx}{(x-a)(x-b)} = \int dx \left(\frac{1}{x-a} - \frac{1}{x-b} \right) \frac{1}{(a-b)} = \frac{1}{a-b} \ln \left(\frac{x-a}{x-b} \right). \tag{6.8}$$

The Toeplitz book shows how you can generalize these results to $\int \frac{dx}{Ax^2+Bx+C}$. In fact, similar to rational functions, one can integrate any integral of the form $\int \frac{P_n(x)}{(Ax^2+Bx+C)^m} dx$, as well.

If we have an integral of a rational function of $\cos(x)$ and $\sin(x)$, that can be integrated to let $t = \tan(\frac{x}{2})$, so that $\frac{dx}{dt} = \frac{2}{1+t^2}$ but one can show that $\frac{2t}{1+t^2} = \sin(x)$, and $\frac{1-t^2}{1+t^2} = \cos(x)$. So any integral of a rational function of $\sin(x)$ and $\cos(x)$ can be written as an integral of a rational function of t.

An example is:

$$\int \frac{dx}{[\sin(x)]^m} = \int \left(\frac{1+t^2}{2t} \right)^m \frac{2}{1+t^2} \, dt = \int \frac{(1+t^2)^{(m-1)}}{2^{m-1} t^m} \, dt, \tag{6.9}$$

which can be integrated, but the result won't be written out explicitly here.

We next consider integrals that include terms that behave like square roots—particularly $\sqrt{a^2-x^2}$ for $-a \leq x \leq a$. Any rational function of x and $\sqrt{a^2-x^2}$ can be integrated as follows: Let $x = a\cos(\phi)$, so that $dx = -a\sin(\phi)d\phi$, and$\sqrt{a^2-x^2} = a\sin(\phi)$. A rational function of x and $\sqrt{a^2-x^2}$ (multiplied byb dx) becomes a rational function of $\cos(\phi)$ and $\sin(\phi)$ (multiplied by $\sin(\phi)$), which remains a rational function of $\cos(\phi)$ and $\sin(\phi)$. We just explained how to integrate such quantities. So this approach can be further extended to all rational functions of x and $\sqrt{Ax^2+Bx+C}$, but they require figuring out carefully all the different possible cases (and really requires using integral tables or wolfram alpha in order to ensure the final result is correct).

This approach goes no further. If we have a rational function of x and $\sqrt{Ax^4+Bx^3+Cx^2+Dx+E}$, one cannot integrate this without introducing what are called elliptic functions. Elliptic functions are similar to trigonometric functions but are defined on ellipses, not circles. This topic is beyond what we will cover here in this class, but don't be afraid if you encounter these functions in the future. They can be handled just like sines and cosines. They satisfy lots of similar identities and so forth.

6.2 Gaussian and Frullani integrals

I want to end with showing two more integrals. The first is the Gaussian integral:

$$\int_{-\infty}^{\infty} dx e^{-x^2} = \sqrt{\pi}. \tag{6.10}$$

You may wonder, where in the heck does $\sqrt{\pi}$ come from? It turns out to be very simple and follows from a really neat trick that everyone should know.

We begin by letting $I = \int_{-\infty}^{\infty} dx e^{-x^2}$. Then, we have $I^2 = \int_{-\infty}^{\infty} dx e^{-x^2} \times \int_{-\infty}^{\infty} dy e^{-y^2}$ This is an integral is over the entire $x-y$ plane. Now convert from Cartesian coordinates to polar coordinates:

$$I^2 = \int_0^{\infty} dr \int_0^{2\pi} r e^{-r^2}\, d\theta, \tag{6.11}$$

since $x^2 + y^2 = r^2$. The integral over θ gives 2π, resulting in

$$I^2 = 2\pi \int_0^{\infty} dr\, r e^{-r^2}. \tag{6.12}$$

Let $r^2 = u$, so that $2r\, dr = du$. This then gives

$$I^2 = \pi \int_0^{\infty} du\, e^{-u} = \pi \left(-e^{-u}\right)\Big|_0^{\infty} = \pi(-0 - (-1)) = \pi. \tag{6.13}$$

Taking the square root then yields $I = \sqrt{\pi}$!

The last thing I want to show is Frullani's theorem. This is a theorem that is little known, but extremely powerful if you encounter a function to integrate that satisfies its conditions. The Frullani integral is

$$\int_0^{\infty} \frac{f(ax) - f(bx)}{x} dx = [f(0) - f(\infty)] \ln\left(\frac{b}{a}\right). \tag{6.14}$$

The functions must decay fast enough, as discussed below, for the integral to make sense.

Proof: Examine the double integral $\int_a^b dx \int_0^\infty dy\,(-f'(xy))$, by integrating over y first:

$$\int_a^b dx \left(\frac{-f(xy)}{x}\right)\Big|_{y=0}^{y=\infty} = \int_a^b dx[f(0)-f(\infty)]\frac{1}{x} = \ln\left(\frac{b}{a}\right)[f(0)-f(\infty)]. \tag{6.15}$$

Now integrate instead over x first:

$$\int_0^\infty dy \left(\frac{-f(xy)}{y}\right)\Big|_{x=a}^{x=b} = \int_0^\infty dy \frac{f(ay)-f(by)}{y}. \tag{6.16}$$

So, we have established that

$$\int_0^\infty dx \frac{f(ax)-f(bx)}{x} = [f(0)-f(\infty)]\ln\left(\frac{b}{a}\right), \tag{6.17}$$

which is the Frullani result.

Example:

$$\int_0^\infty dx\,\frac{e^{-ax}-e^{-bx}}{x} = \text{a Frullani integral with } f(x)=e^{-x} \tag{6.18}$$

$$= \ln\left(\frac{b}{a}\right)$$

$$\int_0^\infty dx \frac{b\sin(ax)-a\sin(bx)}{abx^2} = \text{ Frullani with } f(x)=\frac{\sin(x)}{x} \tag{6.19}$$

$$= \ln\left(\frac{b}{a}\right)$$

and so on. This is an interesting integral to remember.