Chapter 4

Tangents and Logarithms

4.1 Optimizations with constraints

The book by Toeplitz describes how a significant effort was expended by mathematicians on finding tangents to curves and how calculus makes it much easier to do. We will examine, as an example, one such derivative problem—for the set of rectangles of fixed perimeter p, what shape has the largest area?

Recall that area = length×width, and perimeter = 2×(length + width).

So if we let w denote the width and p denote the perimeter, then we have

$$l = \text{length} = \frac{p}{2} - w. \tag{4.1}$$

The area satisfies

$$A = l \times w = \left(\frac{p}{2} - w\right) \times w = \left(\frac{p}{2}\right) w - w^2. \tag{4.2}$$

Differentiating to determine the maximum gives

$$\frac{dA}{dw} = \frac{p}{2} - 2w = 0 \to w = \frac{p}{4}. \tag{4.3}$$

Solving for the length, then yields

$$l = \frac{p}{2} - \frac{p}{4} = \frac{p}{4} = w \to l = w, \tag{4.4}$$

which means the shape is a square (because the length is equal to the width)!

Some people would like to solve this problem in a much simpler way by saying such a rectangle must be a special rectangle. But the only special rectangle we know is a square. So it must be square. Sometimes such arguments are deep and meaningful because the arguments are based on symmetry principles. Other times, it is just good luck that it gives the right answer. It is safer, at this stage of your career, to err on the side of producing a rigorous argument to support such statements, rather than falling back on a "symmetry" or "unique" argument.

4.2 Trigonometric tables

I want to take the remainder of our time discussing sine and logarithm tables. Back in the period from 200 BC to the early to mid 1900's, nearly all calculations were done with tables or instruments that acted as tables, such as slide rules.

The Greeks constructed sine tables, while logarithms didn't come about until the early 1600's. The accuracy needed was one part in ten million, or 7 digits, in order to perform astronomical calculations accurately enough. In other words, this is what Kepler needed to establish his laws of planetary motion. Trying to develop such tables was very demanding but it was a key to technological advance at the time.

The generation of a sine table was known to the Greeks. A 30–60–90 triangle gives $\sin(30°)$. The construction of a regular pentagon gives $\sin(36°)$. Archimedes formulas, which can be employed to produce $\sin\left(\frac{x}{2}\right)$ from $\sin(x)$, give us $\sin(15°)$ and $\sin(18°)$. The Greeks also knew sine addition formulas such as

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta). \tag{4.5}$$

So using this, one obtains $\sin(18° - 15°) = \sin(3°)$. Archimedes again gives $\sin(1\frac{1}{2}°)$ and $\sin(\frac{3}{4}°)$. Then, they use the identity

$$\frac{\sin(x)}{\sin(y)} < \frac{x}{y} \text{ for } 0 < y < x < 90°, \tag{4.6}$$

but recall that the angles must be expressed in radians for all of these calculations.

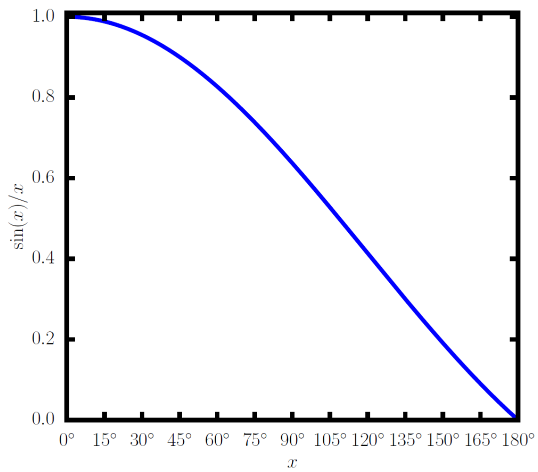

Figure 4.1: The function $\sin(x)/x$ for the range from 0 to π. Note how it is always a decreasing function of x. Such a function is called monotonic.

The identity follows from $\frac{\sin(x)}{x} < 1$ being a monotonic decreasing function of x (see Fig. 4.1) so

$$\frac{\sin(x)}{x} < \frac{\sin(y)}{y} \text{ for } y < x. \tag{4.7}$$

Cross multiplying gives us

$$\frac{\sin(x)}{\sin(y)} < \frac{x}{y} \text{ for } y < x. \tag{4.8}$$

This general result implies that

$$\frac{\sin(1\frac{1}{2}^\circ)}{\sin(1^\circ)} < \frac{3}{2} \text{ or } \frac{2}{3}\sin\left(1\frac{1}{2}^\circ\right) < \sin(1^\circ) \tag{4.9}$$

and

$$\frac{\sin(1^\circ)}{\sin(\frac{3}{4}^\circ)} < \frac{4}{3} \text{ or } \sin(1^\circ) < \frac{4}{3}\sin\left(\frac{3}{4}^\circ\right). \tag{4.10}$$

Using the values for $\sin(1\frac{1}{2}^\circ)$ and $\sin(\frac{3}{4}^\circ)$ pins (or better squeezes) the value of $\sin(1^\circ)$ to 4-5 digits of accuracy. They then computed $\sin(\frac{1}{2}^\circ)$ from Archimedes and finally used the addition formula to generate sine tables for every $\frac{1}{2}^\circ$.

4.3 Tables of logarithms

Log tables were even more valuable. Multiplying two 7-digit numbers to keep 7 digits of accuracy was tedious. But using a logarithm table reduced multiplication to addition because $\ln(xy) = \ln(x)+\ln(y)$, or calculating roots to division because $\ln(\sqrt[n]{x}) = \frac{1}{n}\ln(x)$.

We start with an example. Compute $\sqrt[3]{36000}$. A log table with a specific step size tells us that

12809	35996.4763
12810	36000.0759

where the first entry is the step (or exponent) and the second is $(1 + x)^{step}$, where x is the value (step size) used in generating the table.

To solve this problem, we first interpolate to find $\log(3600) = 12809.98$. We then divide by 3 to get 4299.99. We next go to the table and find the step associated with this number (not shown here) to find the cube root (ans: 33.019272). So, one could compute complex things with these tables.

Generating these tables was mind numbing because it had to be done by hand. A four digit accuracy table required 2,300 steps. So, here are the first few steps of an example table with $x = 0.0001$.

step	$10,000 \times (1 + \frac{1}{10,000})^{step}$
0	10,000.0000
1	10,001.0000
2	10,002.0001
3	10,003.0003

The key trick to making the table is that one never uses multiplication. They are formed instead *by addition* using the following result:

$$a \times \left(1 + \frac{1}{10,000}\right) = a + \frac{a}{10,000}. \tag{4.11}$$

The addition is done in a simple fashion. Write down the previous number a. Take the same number and shift it four digits to the right. Then add them together, truncating terms that are beyond the desired accuracy. Then one repeats. Again. And again. And again.

Generating these tables in base 10 seemed to be most convenient for calculations, but to get 7 digits of accuracy requires 23,000,000 steps and no

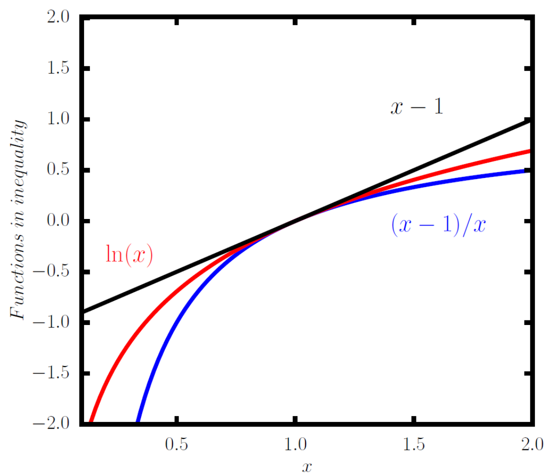

Figure 4.2: Functions used in the inequality. Blue is $\frac{x-1}{x}$, red is $\ln(x)$, and black is $x-1$. Note how they remain ordered even though they become equal at $x = 1$.

one could take up such work. This was further confounded by the fact that any error made at one step, invalidates all further entries in the table.

John Napier figured out some simplifications, that allowed one to actually construct such a table.

1. Compute a table with entries $(1 + \frac{1}{100})^{step}$ for each integer step.

2. Since steps in different tables are related by fixed ratios, compute every 100^{th} entry of a $(1 + \frac{1}{10000})^{step}$ table by multiplying the entries in the lower-precision table by that specific factor. Then fill in the first 100 entries of the higher precision table, and by simple addition, one finds all subsequent entries.

3. The remaining problem was the ratio between the low-precision and the high-precision table, which was not clear how to compute. Napier sought for an *absolute* logarithm table to resolve this issue.

4. To do so, Napier examined how the geometric table entries (called $f(x) = (1 + x)^{step}$) varied with respect to the table parameter x and found that
$$\lim_{x_1 \to x} \frac{f(x) - f(x_1)}{x - x_1} = \frac{c}{x} = f'(x). \tag{4.12}$$

So the simplest table would have $c = 1$, which defines the "absolute" logarithm table.

5. Napier further notices that if we compare the curves

$$\frac{x-1}{x}, \qquad \ln(x), \qquad x-1 \tag{4.13}$$

all three vanish at $x = 1$ (and hence meet each other) and the slopes are

$$\frac{1}{x^2}, \qquad \frac{1}{x}, \qquad 1 \tag{4.14}$$

respectively. Since $\frac{1}{x^2} < \frac{1}{x} < 1$, for $x > 1$ (and the opposite for $x < 1$), we find the curves remain strictly ordered with respect each other, as shown in Fig. 4.2. Hence,

$$\frac{x-1}{x}, \qquad \ln(x), \qquad x-1 \tag{4.15}$$

becomes, after dividing by $x - 1$,

$$\frac{1}{x} < \frac{\ln(x)}{x-1} < 1, \tag{4.16}$$

for all x. Now, let $x = \frac{a}{b} \to \frac{1}{a} < \frac{\ln(a)-\ln(b)}{a-b} < \frac{1}{b}$ for $a > b$. This identity allows the factor between different tables to be found and further allows one to compute the table entries more easily for the higher-precision tables.

What base did Napier use? Let $x = \frac{n+1}{n}$ in the inequality in Eq. (4.16). Then we find that

$$\frac{n}{n+1} < \frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} < 1 \tag{4.17}$$

or

$$\frac{n}{n+1} < \ln\left(1+\frac{1}{n}\right)^n < 1. \tag{4.18}$$

now, as we take $n \to \infty$, we find that $(1+\frac{1}{n})^n \to e$, so $\ln(e) = 1 \to \text{base} = e$.

They then used very clever tricks, described at the end of section 22 of the Toeplitz book, to compute tables with seven digits of accuracy. These tables were extremely influential for hundreds of years and were needed for many different calculations. The first important problem was allowing Kepler to develop his rules of planetary motion.

Note that we hardly use such tables anymore. But it is important that how they were constructed not become a lost art. There is much insight to be found from understanding how one constructed these tables.