Chapter 15

Line Integrals and the Gradient

15.1 Path Independence of the Line Integral

Previously we saw that if $\int_a^b \vec{F}\cdot\hat{t}\,ds$ is independent of the path, then $\nabla\times F = 0$. The converse, $\nabla\times F = 0$ implies $\int_a^b \vec{F}\cdot\hat{t}\,ds$ is independent of the path required the region we were in to be "simply connected" which essentially means "has no holes." But we haven't focused on that too much because most physics problems are in simply connected regions of space.

Now we look at a third possibility. Suppose $\vec{F}(\vec{r}) = \nabla\psi(\vec{r})$ with ψ a scalar function and $\nabla = \hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}$ the gradient operator. Note that the gradient operator converts a scalar function into a vector field, but the vector field is not arbitrary, it has constraints due to the fact that it is defined via the gradient operator. We let s be the arc length parametrization of the path $\vec{r}(s) = \hat{i}x(s) + \hat{j}y(s) + \hat{k}z(s)$. In this case, we then have that the tangent vector satisfies $\hat{t} = \hat{i}\frac{dx}{ds} + \hat{j}\frac{dy}{ds} + \hat{k}\frac{dz}{ds}$. (Note that arc length parameterization means the parameter s used to parametrically determine the curve via the three functions $(x(s), y(s), z(s))$ *is the arc length traveled along the path*; determining such a parameterization is not trivial to do, but we assume it has been done.) The scalar dot product of the vector field (defined via the gradient) with the tangent vector becomes

$$\vec{F}\cdot\hat{t} = \frac{d\psi}{dx}\frac{dx}{ds} + \frac{d\psi}{dy}\frac{dy}{ds} + \frac{d\psi}{dz}\frac{dz}{ds}. \tag{15.1}$$

It is clear, that if we think in terms of the chain rule, we have $\vec{F}\cdot\hat{t} = \frac{d\psi}{ds}$. This means we can integrate it immediately when we perform the line integral.

We find

$$\int_{(x,y,z)}^{(x',y',z')} \vec{F} \cdot \hat{t}\, ds = \int_{(x,y,z)}^{(x',y',z')} \frac{d\psi}{ds}\, ds = \psi(x',y',z') - \psi(x,y,z), \tag{15.2}$$

because it is a perfect differential. Since this depends only on the endpoints, it is *independent of the path*, as we claimed.

So if we integrate from some fixed point (x_0, y_0, z_0) to (x, y, z), we have

$$\tilde{\psi}(x,y,z) = \int_{(x_0,y_0,z_0)}^{(x,y,z)} \vec{F} \cdot \hat{t}\, ds \tag{15.3}$$

is a well defined function, because the integral is path independent; that is, for each point (x, y, z), the function is well defined and single valued. Now suppose we take the following specific paths: the first path c_1 is given by $(x_0, y_0, z_0) \to (x_0, y, z)$ and the secodn path c_2 is given by $(x_0, y, z) \to (x, y, z)$. The first integral path is independent of x (because $x = x_0$ throughout the path), so the derivative is nonzero only for the second path, or

$$\frac{d\tilde{\psi}}{dx} = \frac{d}{dx}\int_{c_2} \vec{F} \cdot \hat{t}\, ds = \frac{d}{dx}\int_{x_0}^{x} \vec{F}(\bar{x}, y, z) \cdot \hat{\imath}\, d\bar{x} = F_x(x,y,z) \tag{15.4}$$

using similar paths for y and z, we get $\frac{d}{dy}\tilde{\psi}(x,y,z) = F_y$ and $\frac{d}{dz}\tilde{\psi}(x,y,z) = F_z$.

So this is how one can construct a scalar function ψ, whose gradient is $\vec{F}$. If $\int \vec{F} \cdot \hat{t}\, ds$ is independent of path, then we can always find a function ψ such that $\vec{F} = \vec{\nabla}\psi$. Be sure you can understand the construction of such a function.

Now, assume that the vector field is the gradient of a scalar function. We have the curl satisfies

$$\begin{aligned}
\nabla \times F &= \hat{\imath}\left(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right) + \hat{\jmath}\left(\frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z\right) + \hat{k}\left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right) \\
&= \hat{\imath}\left(\frac{\partial^2\psi}{\partial y\partial z} - \frac{\partial^2\psi}{\partial z\partial y}\right) + \hat{\jmath}\left(\frac{\partial^2\psi}{\partial z\partial x} - \frac{\partial^2\psi}{\partial x\partial z}\right) + \hat{k}\left(\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial y\partial x}\right) \\
&= 0
\end{aligned} \tag{15.5}$$

because mixed second partial derivatives are independent of the order of the derivatives.

So we see that the three different results: (i) $\int_c \vec{F} \cdot \hat{t}\, ds$ is independent of path; (ii) $\nabla \times F = 0$ and (iii) $\vec{F} = \vec{\nabla}\psi$ are all equivalent. The issue of simple-connectedness is subtle and discussed further in *Div, Grad, and Curl*. One needs to understand it to properly apply the equivalence of these three results.

15.2 Laplace's and Poisson's Equations

We will apply this general result to Maxwell's equations and electric fields. This will also introduce us to Laplace's and Poisson's equations.

Consider a static electric field $\vec{E}$. Maxwell's equations tell us that $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ and $\nabla \times \vec{E} = 0$. The fact that the curl vanishes implies that we can find a scalar potential such that $\vec{E} = -\nabla\phi$, which automatically gives $\nabla \times E = 0$. (Note that the scalar function here is defined with a minus sign. Obviously, it is a small generalization of the discussion given above.) Using the potential in the other Maxwell equation (Gauus' law) then gives $-\nabla \cdot \nabla\phi = \frac{\rho}{\epsilon_0}$ or $\nabla^2\phi = -\frac{\rho}{\epsilon_0}$. The symbol ∇^2 is the Laplacian (often called "del squared"). We compute it just like you might think—it is the divergence of the gradient, or

$$\begin{aligned}
\nabla^2\phi &= \nabla \cdot \nabla\phi && (15.6)\\
&= \nabla \cdot \left(\hat{\imath}\partial_x \phi + \hat{\jmath}\partial_y \phi + \hat{k}\partial_z \phi\right) && (15.7)\\
&= \partial_x^2 \phi + \partial_y^2 \phi + \partial_z^2 \phi, && (15.8)
\end{aligned}$$

where we used symbols $\partial_x = \frac{\partial}{\partial x}$ and the like as a shorthand notation for partial derivatives. Hence, we say that the Laplacian satisfies

$$\text{Laplacian} = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{15.9}$$

Poisson's equation is

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}, \tag{15.10}$$

and it is used in problems where one has free charges and we are trying to determine the scalar potential. But many electrostatic problems are to find the fields in a space where there are no free charges. Then we have Laplace's equation

$$\nabla^2\phi = 0. \tag{15.11}$$

In both of these cases, the idea is that we solve Poisson's or Laplace's equation for the scalar potential ϕ first, and then use the gradient to determine the electric field.

Both of these equations are not enough to determine the solution. They are subject to different boundary conditions. These boundary conditions are often given in terms of the potentials or fields at the boundaries surfaces of the problem (usually metals). It is easiest to see how this works by going through some examples.

Example 1

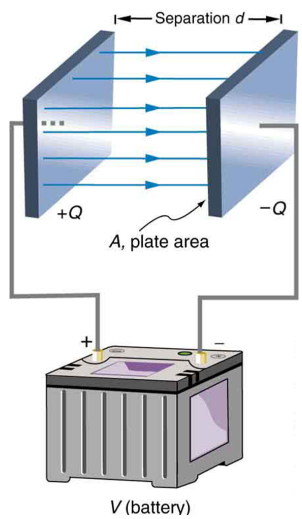

Figure 15.1: Parallel-plate capacitor set up. In a real capacitor, the plates are finite in size and of area A, as depicted in the above figure, but we will work with *infinite* plates, which are infinite parallel planes, separated by a distance d.

What is the potential and electric field of an infinite parallel plate capacitor with $\phi = 0$ on the left plane, $\phi = V_0$ on the right plane and the separation between the planes being d.

Since the planes are infinite, the problem is translationally invariant in y and z, so there is no dependence of the potential on y and z. (Be sure you understand this!) So we have

$$\frac{d^2\phi}{dx^2} = 0 \implies \phi = ax + b, \tag{15.12}$$

a linear function. Now we apply the boundary conditions: at $x = 0$, we have $\phi = 0$, which implies that $b = 0$ and at $x = d$, we have $\phi = V_0$, which implies that $a = \frac{V_0}{d}$. So

$$\phi = V_0 \frac{x}{d} = \text{a linear function.} \tag{15.13}$$

The electric field is now found by taking the gradient

$$\vec{E}(x) = -\nabla\phi(x) = \left(-\frac{V_0}{d}, 0, 0\right). \tag{15.14}$$

As expected, the electric field inside a parallel plate capacitor is a constant field, perpendicular to the plates.

Example 2: Spherical Capacitors

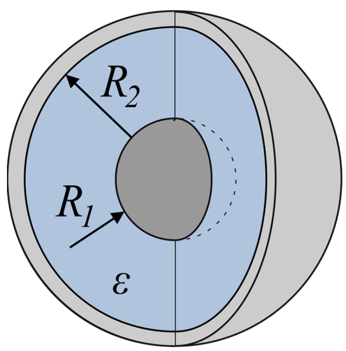

Figure 15.2: Schematic of a spherical capacitor, with inner radius R_1 and outer radius R_2. The region between the two spheres is filled with a dielectric with dielectric constant ϵ.

Because we are working in spherical coordinates, we will use the symbols Ψ instead of ϕ for the potential, in order to not confuse the potential with the angle ϕ used in spherical coordinates. The boundary conditions we have are $\Psi_{\text{inner}} = V_1$ and $\Psi_{\text{outer}} = 0$.

In spherical coordinates, The Laplacian becomes

$$\nabla^2\Psi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Psi}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{d^2\Psi}{d\phi^2}, \tag{15.15}$$

which you should have seen in one of your previous math or physics classes (the derivation is rather long, so if you have never seen it, go find an appropriate textbook or web resource to see the derivation). We do show another derivation in Chapter 17.

Because we have spherical symmetry, the potential is rotationally invariant, which means that there is no θ or ϕ dependence, so

$$\frac{d\Psi}{d\theta} = \frac{d\Psi}{d\phi} = 0. \tag{15.16}$$

Hence, we have

$$\nabla^2\Psi \to \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) = 0. \tag{15.17}$$

This means that

$$r^2\frac{d\Psi}{dr} = c_1 \implies \Psi = \int \frac{c_1}{\bar{r}^2}\,d\bar{r} + c_2 = -\frac{c_1}{r} + c_2. \tag{15.18}$$

Now we use the boundary conditions: at $r = R_1$, we have $\Psi = V_1$

$$\implies -\frac{c_1}{R_1} + c_2 = V_1. \tag{15.19}$$

Similarly, at $r = R_1$, we have $\phi = 0$ and

$$-\frac{c_1}{R_1} + c_2 = 0 \implies c_2 = \frac{c_1}{R_2}. \tag{15.20}$$

These two equations can be combined to yield

$$-\frac{c_1}{R_1} + \frac{c_1}{R_2} = V_1. \tag{15.21}$$

Solving, we find that

$$c_1\left(\frac{R_1 - R_2}{R_1R_2}\right) = V_1 \implies c_1 = \left(\frac{R_1R_2}{R_1 - R_2}\right)V_1. \tag{15.22}$$

Putting this all together, we find that the scalar potential becomes

$$\Psi(r) = \left(\frac{R_1R_2}{R_1 - R_2}\right)\frac{V_1}{r} - \frac{R_1}{R_2 - R_1}V_1 = \frac{R_1V_1}{R_2 - R_1}\left(\frac{R_2}{r} - 1\right). \tag{15.23}$$

We use the gradient to find the field. This becomes

$$\vec{E} = \frac{R_1R_2}{R_1 - R_2}\frac{V_1}{r^2}\hat{e}_r \tag{15.24}$$

for $R_1 \leq r \leq R_2$, and $E_\theta = E_\phi = 0$.