Chapter 20

The Residue theorem

20.1 Calculating integrals via residues

Recall last time when we showed the residue of a function $g(z)$ is the coefficient of the $\frac{1}{z-z_0}$ term in the Laurent expansion for $g(z)$. Each point where $g(z)$ diverges as $\frac{b_{-1}}{z-z_0}$ is called a _pole_ of $g(z)$ with b_{-1} the _residue_ of the pole.

The idea of calculating real-valued integrals via residues is not at all obvious. When we reviewed the one-dimensional integral calculus earlier in the course, we covered how one can integrate a wide range of different functions. Special examples like Frullani's integral and Feynman integration produced more options. Here we expand the repertoire even more. Probably the single most useful integration technique is via residues. If you recall from our example given in Chapter 19, the only integral you need is the integral of powers of $z - z_0$ and only one of those integrals is nonzero. Couple this with a Laurent expansion about any singular point and you can almost see the methodology emerging. One issue often leads to confusion, though. Lot's of confusion. It is how to relate an integral from $-\infty$ to $+\infty$ along the real axis to a closed contour integral, where we can use the residue theorem. We employ examples below to describe how this works. Please look carefully at how this is done and try to master it. It is the key to becoming a practitioner on the calculus of residues.

Let's start with a careful description of the residue theorem. The residue theorem says if γ is a curve that encircles some poles of $g(z)$, each circled once in the counter-clockwise direction, then

$$\int_\gamma g(z)dz = 2\pi i \sum_{\text{poles inside } \gamma} (\text{residues of } g \text{ inside } \gamma) \tag{20.1}$$

The residue theorem is primarily used to integrate many different definite integrals (usually over the full real axis, but sometimes over smaller regions). In most cases, it does not help with indefinite integrals.

20.2 Examples of the residue theorem

We illustrate with a few examples here and then cover more in the laboratories. We start with an integral that works with simply identifying how the residue theorem works. This first example will not lead to an application to one-dimensional integrals.

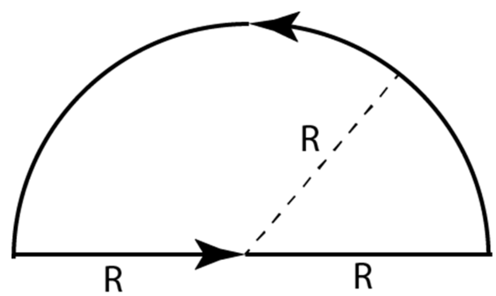

Figure 20.1: Path γ used for integrating the first residue theorem example. The path here is given for arbitrary R and our specific γ for the first example has $R = 2$.

Example: *Integrate* $\frac{1}{z^4+1}$ *over* γ.

Suppose we integrate $\frac{1}{z^4+1}$ over γ: $\int_\gamma \frac{dz}{z^4+1}$. The poles of $\frac{1}{z^4+1}$ occur when $z^4 + 1 = 0 \implies z^4 = -1$. A simple modification of the way we solved De Moivre's theorem allows us to find where the roots are. We have the four roots z_n ($n = 0$, 1, 2 and 3) are found via

$$z^4 = e^{i\pi+2in\pi} \implies z_n = e^{\frac{i\pi}{4}+\frac{in\pi}{2}}, n = 0, 1, 2, 3, \tag{20.2}$$

because $-1 = e^{i\pi}$. Note that all of these roots lie on the unit circle. You can immediately see that two poles lie inside γ, circled in blue in Fig. 20.2.

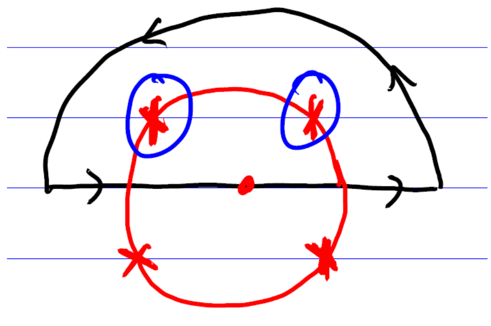

Figure 20.2: Schematic showing the two roots, circled in blue, that lie inside γ. These two points are where the two poles lie that are inside the contour.

The next step is to find the residues. The residue of $g(z) = \frac{1}{z^4+1}$ at z_n is

$$\text{Res at } z_n = \lim_{z \to z_n} \frac{z - z_n}{z^4 + 1}. \tag{20.3}$$

We know this because the pole is an isolated pole (implying all terms in the Laurent expansion with b_{-n} vanish for $n \geq 2$). This then allows us to compute the residue by multiplying by $(z - z_n)$ and taking the limit as $z \to z_n$. Because $g(z)$ has a singularity, we find this limit becomes $\frac{0}{0}$. To evaluate such an indeterminate form, we use l'Hôpital's rule and differentiate the numerator and denominator separately:

$$\text{Res at } z_n = \lim_{z \to z_n} \frac{\frac{d}{dz}(z - z_n)}{\frac{d}{dz}(z^4 + 1)} = \frac{1}{4z_n^3}. \tag{20.4}$$

So the residue for $n = 0$ becomes

$$\frac{1}{4e^{\frac{3i\pi}{4}}} = \frac{1}{4}e^{-\frac{3i\pi}{4}} = \frac{-1-i}{4\sqrt{2}} \tag{20.5}$$

and at $n = 1$, the residue becomes

$$\frac{1}{4e^{\frac{9i\pi}{4}}} = \frac{1}{4}e^{-\frac{9i\pi}{4}} = \frac{1-i}{4\sqrt{2}} \tag{20.6}$$

and the integral becomes $2\pi i$ times the sum of the two residues

$$2\pi i \left(\frac{1}{4\sqrt{2}}\right)(-1 - i + 1 - i) = \frac{\pi}{\sqrt{2}}. \tag{20.7}$$

We finally end up with the result of

$$\int_\gamma \frac{1}{z^4+1} dz = \frac{\pi}{\sqrt{2}}. \tag{20.8}$$

Our next example will help us evaluate an indefinite integral over the real axis. The key strategy we need to use here is the "add zero" trick. We need to find a contour in the complex plane, which is constructed out of two pieces: (i) one which will become the integral over the real axis and (ii) the other which will equal zero and hence add nothing to the final result, Once we have obtained such a closed contour, we can then use the residue theorem to evaluate the integral. Because all we did was add zero, the integral remains equal to the original value of the integral over the real axis. Now that is cool!

But how do we find a piece of the contour integral that will be zero? It turns out to be rather simple to do. Think of the integral over a semicircle of radius R given in Fig. 20.1. The horizontal part of the semicircle becomes the entire real axis as $R \to \infty$. What about the other piece, given by the circular part of the semicircle? The perimeter is πR. If the integrand decays faster than $1/R$ as $R \to \infty$, then the total integral over that arc is equal to zero as $R \to \infty$! And that is how we do it. Note that it is critically important to verify this for any integral you want to evaluate. Sometimes doing this shows how you "close the contour" one way or another.

Example: *Show how to use the residue theorem to evaluate an indefinite integral over the real axis.* We examine the following integral

$$\int_{-\infty}^{\infty} dx \frac{1}{1+x^{2n}}, n = 1, 2, ... \tag{20.9}$$

Note that if we close the integral on the upper half plane, as discussed above and shown in Fig. 20.1 for $R \to \infty$, then we add a term of size $\approx \frac{\pi R}{1+R^{2n}} \to 0$ as $R \to \infty$ (because $n \geq 1$), so

$$\int_{-\infty}^{\infty} dx \, \frac{1}{1+x^{2n}} = \int_\gamma dz \, \frac{1}{1+z^{2n}} = 2\pi i \sum \text{residues inside } \gamma. \tag{20.10}$$

Residues occur when $z^{2n} = -1$ (so we solve by generalizing De Moivre's theorem again), $z_k = e^{\frac{i\pi}{2n}+\frac{ik\pi}{n}}$ for $0 \leq k \leq 2n$. The residues are equally spaced on the unit circle and those with $k = 0, 1, ..., n-1$ lie inside γ (those with $n \leq k < 2n$ lie outside γ), because they are inside the upper half plane (above the real axis).

$$\text{Residue at } z_n = \lim_{z \to z_n} \frac{(z - z_n)}{1 + z^{2n}} \overset{\text{l'}Hopital}{=} \frac{1}{2n z_k^{2n-1}} = -\frac{z_k}{2n}, \tag{20.11}$$

since $z_k^{2n} = -1$. Hence, we find that

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^{2n}} = \sum_{k=0}^{n-1} 2\pi i \frac{1}{2n} (-e^{\frac{i\pi}{2n} + \frac{ik\pi}{n}}) \tag{20.12}$$

The sum looks like it is hard to evaluate, but if you think of our devious math tricks, maybe you will think that perhaps there might be some way to simplify it. First, we pull out the factors independent of k:

$$= -\frac{\pi i}{n} e^{\frac{i\pi}{2n}} \sum_{k=0}^{n-1} e^{i\frac{k\pi}{n}}. \tag{20.13}$$

Now we see that we have something that looks like a finite piece of the geometric series. We always treat this by multiplying by $(1 - x)/(1 - x)$ for $x = e^{i\pi/n}$ here:

$$= -\frac{\pi i}{n} e^{\frac{i\pi}{2n}} \frac{1 - e^{i\pi}}{1 - e^{\frac{i\pi}{n}}}. \tag{20.14}$$

Rationalizing the denominator then gives us

$$= -\frac{2\pi i}{n} e^{\frac{i\pi}{2n}} \frac{1 - e^{-\cdot\frac{i\pi}{n}}}{(1 - e^{\frac{i\pi}{n}})(1 - e^{-\frac{i\pi}{n}})}. \tag{20.15}$$

Simplifying, we have

$$= -\frac{2\pi i}{n} \frac{e^{\frac{i\pi}{2n}} - e^{-\frac{i\pi}{2n}}}{1 - e^{\frac{i\pi}{n}} - e^{-\frac{i\pi}{n}} + 1}. \tag{20.16}$$

Now, we recall the definition of sin in terms of exponentials:

$$= -\frac{2\pi i}{n} \frac{2i \sin \frac{\pi}{2n}}{2 - 2\cos \frac{\pi}{n}}, \tag{20.17}$$

and we simplify the final result y collecting all the factors

$$= \frac{2\pi}{n} \frac{\sin \frac{\pi}{2n}}{1 - \cos \frac{\pi}{n}}. \tag{20.18}$$

But, we are not done yet. The following manipulations are things that you might not easily recognize. But with practice, eventually you should be able to do them too. We have

$$1 - \cos\frac{\pi}{n} = 1 - \left(\cos^2\left(\frac{\pi}{2n}\right) - \sin^2\left(\frac{\pi}{2n}\right)\right) \tag{20.19}$$

after using the double angle formula for the cosine. Then we replace $\sin^2$ by $1 - \cos^2$ and vice versa

$$= 1 - 2\cos^2\left(\frac{\pi}{2n}\right) + 1 = 2\left(1 - \cos^2\left(\frac{\pi}{2n}\right)\right) = 2\sin^2\left(\frac{\pi}{2n}\right). \tag{20.20}$$

We put this together with our final result to obtain

$$\int_{-\infty}^{\infty} dx \frac{1}{1 + x^{2n}} = \frac{2\pi}{n} \frac{\sin\frac{\pi}{2n}}{2\sin^2\left(\frac{\pi}{2n}\right)} = \frac{\pi}{n}\csc\frac{\pi}{2n} \tag{20.21}$$

and, because the integrand is even, we get:

$$\int_0^{\infty} \frac{1}{1 + x^{2n}} = \frac{\pi}{2n}\csc\frac{\pi}{2n}, \tag{20.22}$$

which is a result that is hard to imagine one could evaluate analytically (or that the result is so simple). So, this is pretty remarkable stuff!
We have one final example:

$$I(a, b) = \int_0^{\pi} \frac{d\theta}{a + b\cos\theta}, \quad 0 < b < a. \tag{20.23}$$

This integral is only over a finite subset of the real axis, so our previous ideas do not look like they can be applied here. But the integrand is a function of sines and cosines. In this case, there is a new trick we can invoke. We want to change the integral from 0 to π to an integral over the unit circle. This works better if the integral goes from 0 to 2π, so we first extend the integral to twice the interval (now going from $-\pi$ to π) by using the fact that cosine is an even function and then we note that integrating over a whole period is independent of where one starts, so we can shift the integral by π without changing the final answer

$$I(a, b) = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}, \tag{20.24}$$

where the factor of 1/2 comes from extending the integral to $[-\pi, 0]$. We now change the variables to $z = e^{i\theta}, dz = ie^{i\theta}d\theta$, and $d\theta = \frac{dz}{iz}$. Then we also have $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right)$ and

$$I(a,b) = \frac{1}{2}\oint \frac{dz}{iz}\frac{1}{a + \frac{b}{2}(z + \frac{1}{z})} = \frac{1}{2i}\oint \frac{1}{\frac{b}{2}z^2 + az + \frac{b}{2}} \tag{20.25}$$

$$= \frac{1}{ib}\oint dz \frac{1}{z^2 + 2z\frac{a}{b} + 1} \tag{20.26}$$

The roots of the denominator are easily found because it is a quadratic equation in z. We have the two roots given by

$$z_\pm = -\frac{2a}{2b} \pm \frac{1}{2}\sqrt{\frac{4a^2}{b^2} - 4} = -\frac{a}{b} \pm \frac{\sqrt{a^2 - b^2}}{b}. \tag{20.27}$$

Since $z_+z_- = 1$ [check it directly from the values for $z_\pm$ or compare $(z - z_+)(z - z_-) = z^2 - (z_+ + z_-)z + z_+z_-$ to the polynomial in the denominator], there must be one root *inside* and one root *outside* the circle. But since $0 < b < a$, this implies that the $-$ root is outside the circle and the $+$ root is inside the circle. The denominator can be written as $= (z - z_+)(z - z_-)$, so the residue is

$$\frac{1}{z_+ - z_-} = \frac{1}{\frac{2}{b}\sqrt{a^2 - b^2}} = \frac{b}{2\sqrt{a^2 - b^2}}. \tag{20.28}$$

The then integral becomes

$$I(a,b) = \frac{1}{ib}2\pi i\frac{b}{2\sqrt{a^2 - b^2}} = \pi\frac{1}{\sqrt{a^2 - b^2}} \quad \text{for } 0 < b < a. \tag{20.29}$$

Hence, we have

$$\int_0^\pi \frac{d\theta}{a + b\cos\theta} = \pi\frac{1}{\sqrt{a^2 - b^2}}. \tag{20.30}$$

We can check this with a case that is trivial to evaluate, namely the case with $b = 0$. In this case, we are integrating a constant, and the result is $I(a, 0) = \pi/a$. The other case that can be exactly determined (although this is harder to immediately see) is the case with $a = b$. In this case, we find that $I(a, a) \to \infty$, and that is what our answer gives too.

Note that this last integral might seem to be a challenging one for you. It is. But as you become more sophisticated in your math abilities, you need to be able to push yourself to the limit (pun intended).

I hope you will agree this method of doing integrals is <u>*very*</u> powerful.