How to Find an Example

Alexander Shen

LIRMM / CNRS, University of Montpellier

Outline

Be creative!

Albrecht Duerer,
Melancholia, 1514

Magic Square: Definition

Magic Square: Definition

- $1, 2, 3, \ldots, 15, 16$

Magic Square: Definition

- $1, 2, 3, \ldots, 15, 16$
- $1, 2, 3, \ldots, n^2$
  for $n \times n$

Magic Square: Definition

- $1, 2, 3, \ldots, 15, 16$
- $1, 2, 3, \ldots, n^2$ for $n \times n$
- the same sum in columns, rows, diagonals

How to Find Magic Squares

How to Find Magic Squares

- Duerer: a proof that magic square of size 4 (made of $1, 2, \ldots, 16$) exists

How to Find Magic Squares

- Duerer: a proof that magic square of size 4 (made of $1, 2, \ldots, 16$) exists
- no magic square of size 2 made of $1, 2, 3, 4$ – why?

How to Find Magic Squares

- Duerer: a proof that magic square of size 4 (made of $1, 2, \ldots, 16$) exists
- no magic square of size 2 made of $1, 2, 3, 4$ – why?
- spoiler

How to Find Magic Squares

- Duerer: a proof that magic square of size 4 (made of $1, 2, \ldots, 16$) exists
- no magic square of size 2 made of $1, 2, 3, 4$ – why?
- spoiler

a	b
c	d

How to Find Magic Squares

- Duerer: a proof that magic square of size 4 (made of $1, 2, \ldots, 16$) exists
- no magic square of size 2 made of $1, 2, 3, 4$ – why?
- spoiler

$$\begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array}$$

$a + b = a + c$

How to Find Magic Squares

- Duerer: a proof that magic square of size 4 (made of $1, 2, \ldots, 16$) exists
- no magic square of size 2 made of $1, 2, 3, 4$ – why?
- spoiler

$$\begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array}$$

$$a + b = a + c \quad \Rightarrow \quad b = c$$

How to Find Magic Squares

- Duerer: a proof that magic square of size 4 (made of $1, 2, \ldots, 16$) exists
- no magic square of size 2 made of $1, 2, 3, 4$ – why?
- spoiler

a	b
c	d

$$a + b = a + c \quad \Rightarrow \quad b = c$$

- what about size 3 – made of $1, 2, 3, \ldots, 9$?

Outline

Looking for 3×3 Magic Square

Looking for 3×3 Magic Square

- magic square exist for all $n > 2$

Looking for 3×3 Magic Square

- magic square exist for all $n > 2$
- 3×3: brute force search feasible?

Looking for 3×3 Magic Square

- magic square exist for all $n > 2$
- 3×3: brute force search feasible?
- all permutations of 9 digits:
  $9 \times 8 \times \ldots \times 1 = 362880$

Looking for 3×3 Magic Square

- magic square exist for all $n > 2$
- 3×3: brute force search feasible?
- all permutations of 9 digits:
  $9 \times 8 \times \ldots \times 1 = 362880$
- OK for computers, but…

Looking for 3×3 Magic Square

- magic square exist for all $n > 2$
- 3×3: brute force search feasible?
- all permutations of 9 digits:
  $9 \times 8 \times \ldots \times 1 = 362880$
- OK for computers, but…
- what is the row/column sum?

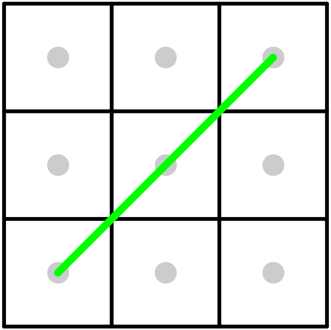
Looking for 3×3 Magic Square

- magic square exist for all $n > 2$
- 3×3: brute force search feasible?
- all permutations of 9 digits:
  $9 \times 8 \times \ldots \times 1 = 362880$
- OK for computers, but…
- what is the row/column sum?
- $1 + 2 + 3 + \ldots + 8 + 9 = 45$

Looking for 3×3 Magic Square

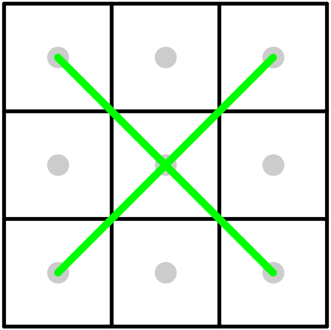
- magic square exist for all $n > 2$
- 3×3: brute force search feasible?
- all permutations of 9 digits:
  $9 \times 8 \times \ldots \times 1 = 362880$
- OK for computers, but…
- what is the row/column sum?
- $1 + 2 + 3 + \ldots + 8 + 9 = 45$
- $45/3 = 15$

Hint: the Center



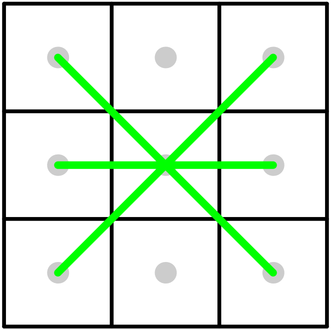
summing up four lines…

Hint: the Center



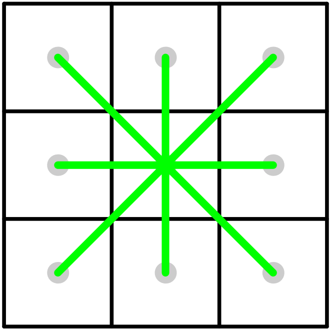
summing up four lines…

Hint: the Center



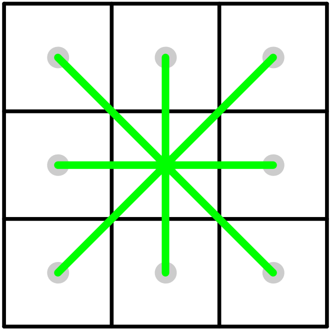
summing up four lines…

Hint: the Center



summing up four lines…

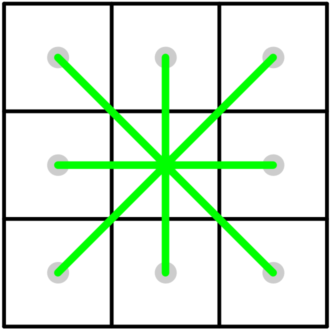
Hint: the Center



summing up four lines…

$4S = \text{total sum} + 3 \cdot \text{center}$

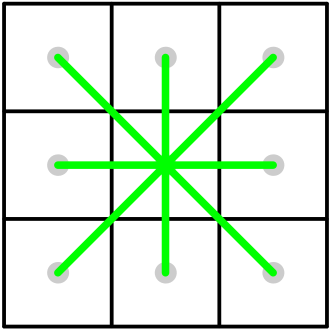
Hint: the Center



summing up four lines...

$$4S = 3S + 3 \cdot \text{center}$$

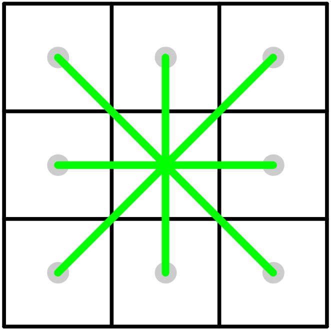
Hint: the Center



summing up four lines...

$S/3 =$ center

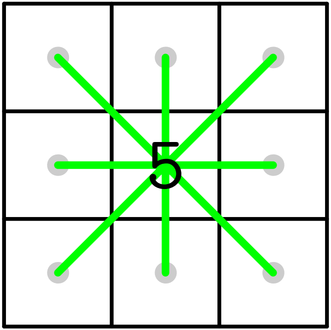
Hint: the Center



summing up four lines...

$S/3 = \text{center} = 5$

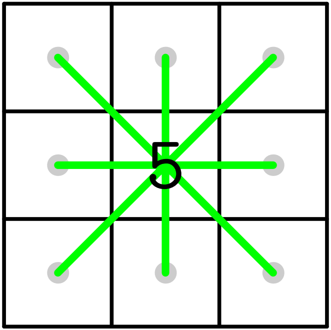
Hint: the Center



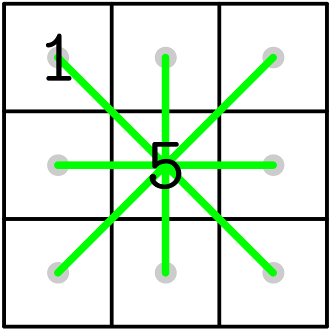
summing up four lines...

$S/3 = \text{center} = 5$

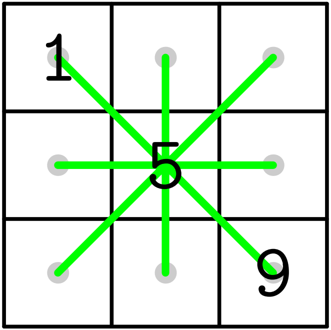
1: Corner or Middle?



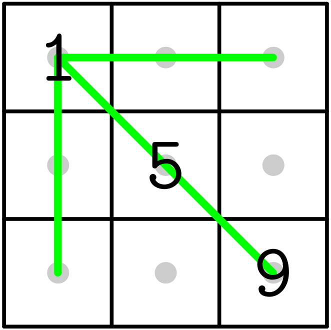
1: Corner or Middle?



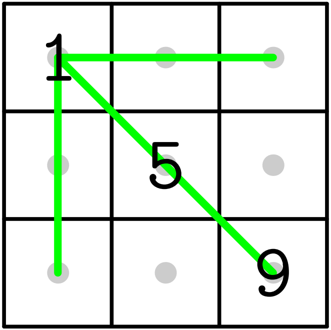
1: Corner or Middle?



1: Corner or Middle?

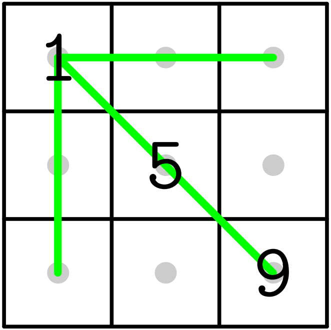


1: Corner or Middle?



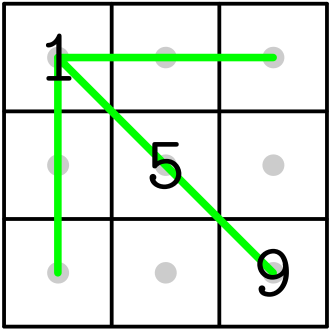
$14 = 5 + 9$

1: Corner or Middle?



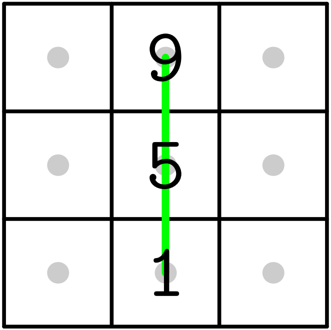
$14 = 5 + 9 = 6 + 8$

1: Corner or Middle?

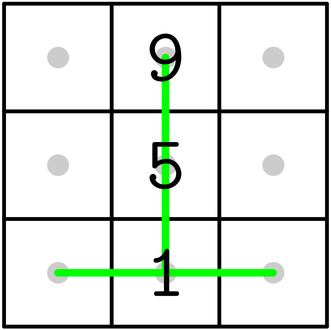


$$14 = 5 + 9 = 6 + 8 = 7 + 7$$

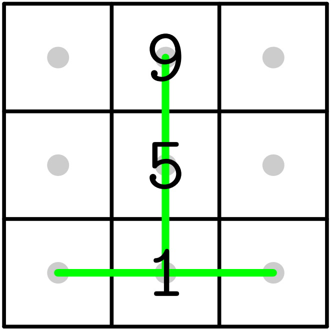
1: Middle!



1: Middle!

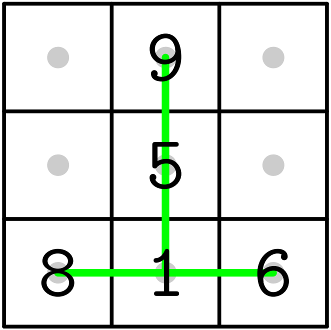


1: Middle!



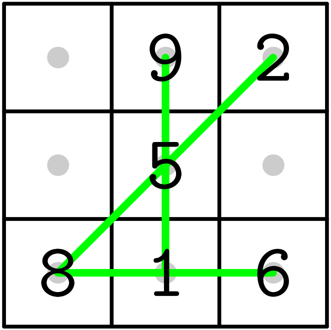
$$14 = 5 + 9 = 8 + 6$$

1: Middle!



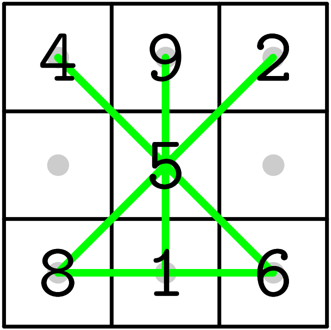
$14 = 5 + 9 = 8 + 6$

1: Middle!



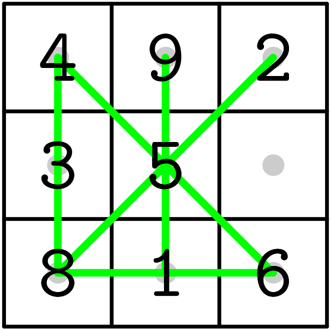
$$14 = 5 + 9 = 8 + 6$$

1: Middle!



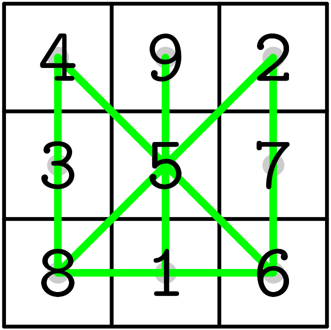
$$14 = 5 + 9 = 8 + 6$$

1: Middle!



$$14 = 5 + 9 = 8 + 6$$

1: Middle!



$$14 = 5 + 9 = 8 + 6$$

Outline

Magic Squares

Narrowing the Search

Multiplicative Magic Squares

More Puzzles

Integer linear combinations

Paths in a Graph

Products, Not Sums

Products, Not Sums

- magic square: *sums* for rows, columns, diagonals are the same

Products, Not Sums

- magic square: *sums* for rows, columns, diagonals are the same
- what about *products*?

Products, Not Sums

- magic square: *sums* for rows, columns, diagonals are the same
- what about *products*?
- we cannot hope to use $1, 2, 3, 4, 5, 6, 7, 8, 9$

Products, Not Sums

- magic square: *sums* for rows, columns, diagonals are the same
- what about *products*?
- we cannot hope to use $1, 2, 3, 4, 5, 6, 7, 8, 9$
- 7 appears in some products but not in the others

Products, Not Sums

- magic square: *sums* for rows, columns, diagonals are the same
- what about *products*?
- we cannot hope to use $1, 2, 3, 4, 5, 6, 7, 8, 9$
- 7 appears in some products but not in the others
- arbitrary different positive integers allowed

Products, Not Sums

- magic square: *sums* for rows, columns, diagonals are the same
- what about *products*?
- we cannot hope to use $1, 2, 3, 4, 5, 6, 7, 8, 9$
- 7 appears in some products but not in the others
- arbitrary different positive integers allowed
- is it possible?

Spoiler

Spoiler

- $2^{x+y} = 2^x \cdot 2^y$

Spoiler

- $2^{x+y} = 2^x \cdot 2^y$
- exponentiation: addition $\rightarrow$ multiplication

Spoiler

- $2^{x+y} = 2^x \cdot 2^y$
- exponentiation: addition $\rightarrow$ multiplication

$$\begin{array}{|c|c|c|}\hline 4 & 9 & 2 \\ \hline 3 & 5 & 7 \\ \hline 8 & 1 & 6 \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{|c|c|c|}\hline 2^4 & 2^9 & 2^2 \\ \hline 2^3 & 2^5 & 2^7 \\ \hline 2^8 & 2^1 & 2^6 \\ \hline \end{array}$$

Spoiler

- $2^{x+y} = 2^x \cdot 2^y$
- exponentiation: addition $\to$ multiplication

$$\begin{array}{|c|c|c|}\hline 4 & 9 & 2 \\ \hline 3 & 5 & 7 \\ \hline 8 & 1 & 6 \\ \hline \end{array} \quad \to \quad \begin{array}{|c|c|c|}\hline 2^4 & 2^9 & 2^2 \\ \hline 2^3 & 2^5 & 2^7 \\ \hline 2^8 & 2^1 & 2^6 \\ \hline \end{array}$$

- numbers rather big — find smaller ones?

Spoiler

- $2^{x+y} = 2^x \cdot 2^y$
- exponentiation: addition $\to$ multiplication

$$\begin{array}{|c|c|c|} \hline 4 & 9 & 2 \\ \hline 3 & 5 & 7 \\ \hline 8 & 1 & 6 \\ \hline \end{array} \quad \to \quad \begin{array}{|c|c|c|} \hline 2^4 & 2^9 & 2^2 \\ \hline 2^3 & 2^5 & 2^7 \\ \hline 2^8 & 2^1 & 2^6 \\ \hline \end{array}$$

- numbers rather big — find smaller ones?
- less than 300?

Spoiler

- $2^{x+y} = 2^x \cdot 2^y$
- exponentiation: addition $\to$ multiplication

$$\begin{array}{|c|c|c|}\hline 4 & 9 & 2 \\ \hline 3 & 5 & 7 \\ \hline 8 & 1 & 6 \\ \hline \end{array} \quad \to \quad \begin{array}{|c|c|c|}\hline 2^4 & 2^9 & 2^2 \\ \hline 2^3 & 2^5 & 2^7 \\ \hline 2^8 & 2^1 & 2^6 \\ \hline \end{array}$$

- numbers rather big — find smaller ones?
- less than 300?
- divide by 2

Spoiler

- $2^{x+y} = 2^x \cdot 2^y$
- exponentiation: addition $\to$ multiplication

$$\begin{array}{|c|c|c|}\hline 4 & 9 & 2 \\ \hline 3 & 5 & 7 \\ \hline 8 & 1 & 6 \\ \hline \end{array} \quad \to \quad \begin{array}{|c|c|c|}\hline 2^4 & 2^9 & 2^2 \\ \hline 2^3 & 2^5 & 2^7 \\ \hline 2^8 & 2^1 & 2^6 \\ \hline \end{array}$$

- numbers rather big — find smaller ones?
- less than 300?
- divide by 2
- less than 40?

Spoiler–2

Spoiler–2

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 0 \\ \hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline \end{array} \quad \to \quad \begin{array}{|c|c|c|}\hline 2 & 4 & 1 \\ \hline 1 & 2 & 4 \\ \hline 4 & 1 & 2 \\ \hline \end{array}$$

Spoiler–2

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 0 \\ \hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{|c|c|c|}\hline 2 & 4 & 1 \\ \hline 1 & 2 & 4 \\ \hline 4 & 1 & 2 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|}\hline 0 & 2 & 1 \\ \hline 2 & 1 & 0 \\ \hline 1 & 0 & 2 \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{|c|c|c|}\hline 1 & 9 & 3 \\ \hline 9 & 3 & 1 \\ \hline 3 & 1 & 9 \\ \hline \end{array}$$

Spoiler–2

1	2	0
0	1	2
2	0	1

$\rightarrow$

2	4	1
1	2	4
4	1	2

$\times$

0	2	1
2	1	0
1	0	2

$\rightarrow$

1	9	3
9	3	1
3	1	9

2	36	3
9	6	4
12	1	18

Outline

Magic Squares

Narrowing the Search

Multiplicative Magic Squares

More Puzzles

Integer linear combinations

Paths in a Graph

100??? **Divisible By** 9127

100??? **Divisible By** 9127

- there is a 6-digit number starting with 100 and divisible by 9127

100??? **Divisible By** 9127

- there is a 6-digit number starting with 100 and divisible by 9127
- not so many candidates

100??? **Divisible By** 9127

- there is a 6-digit number starting with 100 and divisible by 9127
- not so many candidates
- **for** $i = 100\,000$ upto $100\,999$:

      if i is a multiple of 9127:

          print (i)

100??? **Divisible By** 9127

- there is a 6-digit number starting with 100 and divisible by 9127
- not so many candidates
- **for** $i =$ 100 000 upto 100 999:

      if *i* is a multiple of 9127:

          print (*i*)
- pencil and paper solution?

100??? **Divisible By** 9127

- there is a 6-digit number starting with 100 and divisible by 9127
- not so many candidates
- **for** $i = 100\,000$ upto 100 999:

      if i is a multiple of 9127:

          print (i)
- pencil and paper solution?
- $100\,000/9127 = 10.9565\ldots$

100??? **Divisible By** 9127

- there is a 6-digit number starting with 100 and divisible by 9127
- not so many candidates
- **for** $i = 100\,000$ upto $100\,999$:
      if i is a multiple of 9127:
          print (i)
- pencil and paper solution?
- $100\,000/9127 = 10.9565\ldots$
- $11 \times 9127 = 100\,397$

100??? **Divisible By** 9127

- there is a 6-digit number starting with 100 and divisible by 9127
- not so many candidates
- **for** $i = 100\,000$ upto $100\,999$:

      if i is a multiple of 9127:

          print (i)
- pencil and paper solution?
- $100\,000/9127 = 10.9565\ldots$
- $11 \times 9127 = 100\,397$
- next one is too big

Always Remainder One

Always Remainder One

- There exists a three-digit number N that produces remainder 1 when divided by $2, 3, 4, 5, 6, 7$

Always Remainder One

- There exists a three-digit number N that produces remainder 1 when divided by $2, 3, 4, 5, 6, 7$
- 1 is OK (but only one digit!)

Always Remainder One

- There exists a three-digit number N that produces remainder 1 when divided by $2, 3, 4, 5, 6, 7$
- 1 is OK (but only one digit!)
- what about $N - 1$?

Always Remainder One

- There exists a three-digit number N that produces remainder 1 when divided by $2, 3, 4, 5, 6, 7$
- 1 is OK (but only one digit!)
- what about $N - 1$?
- a multiple of $2, 3, 4, 5, 6, 7$

Always Remainder One

- There exists a three-digit number N that produces remainder 1 when divided by $2, 3, 4, 5, 6, 7$
- 1 is OK (but only one digit!)
- what about $N - 1$?
- a multiple of $2, 3, 4, 5, 6, 7$
- $2 \times 2 \times 3 \times 5 \times 7$

Always Remainder One

- There exists a three-digit number N that produces remainder 1 when divided by $2, 3, 4, 5, 6, 7$
- 1 is OK (but only one digit!)
- what about $N - 1$?
- a multiple of $2, 3, 4, 5, 6, 7$
- $2 \times 2 \times 3 \times 5 \times 7 = 420$

Always Remainder One

- There exists a three-digit number N that produces remainder 1 when divided by $2, 3, 4, 5, 6, 7$
- 1 is OK (but only one digit!)
- what about $N - 1$?
- a multiple of $2, 3, 4, 5, 6, 7$
- $2 \times 2 \times 3 \times 5 \times 7 = 420$
- $N - 1 = 420$

Always Remainder One

- There exists a three-digit number N that produces remainder 1 when divided by $2, 3, 4, 5, 6, 7$
- 1 is OK (but only one digit!)
- what about $N - 1$?
- a multiple of $2, 3, 4, 5, 6, 7$
- $2 \times 2 \times 3 \times 5 \times 7 = 420$
- $N - 1 = 420$; $N = 421$

Always Remainder One

- There exists a three-digit number N that produces remainder 1 when divided by $2, 3, 4, 5, 6, 7$
- 1 is OK (but only one digit!)
- what about $N - 1$?
- a multiple of $2, 3, 4, 5, 6, 7$
- $2 \times 2 \times 3 \times 5 \times 7 = 420$
- $N - 1 = 420$; $N = 421$
- others?

Always Remainder One

- There exists a three-digit number N that produces remainder 1 when divided by $2, 3, 4, 5, 6, 7$
- 1 is OK (but only one digit!)
- what about $N - 1$?
- a multiple of $2, 3, 4, 5, 6, 7$
- $2 \times 2 \times 3 \times 5 \times 7 = 420$
- $N - 1 = 420$; $N = 421$
- others? $420 \cdot 2 + 1 = 841$

Always Remainder One

- There exists a three-digit number N that produces remainder 1 when divided by $2, 3, 4, 5, 6, 7$
- 1 is OK (but only one digit!)
- what about $N - 1$?
- a multiple of $2, 3, 4, 5, 6, 7$
- $2 \times 2 \times 3 \times 5 \times 7 = 420$
- $N - 1 = 420$; $N = 421$
- others? $420 \cdot 2 + 1 = 841$
- 420×3 is too big

Perfect Square Starts with 31415

Perfect Square Starts with 31415

- Find an integer n such that $n^2 = 31415\ldots$

Perfect Square Starts with 31415

- Find an integer n such that $n^2 = 31415\ldots$
- finite decimal fraction is enough:
  $(10x)^2 = 100x^2$

Perfect Square Starts with 31415

- Find an integer n such that $n^2 = 31415\ldots$
- finite decimal fraction is enough:
  $(10x)^2 = 100x^2$
- $\sqrt{31415} = 177.242771\ldots$

Perfect Square Starts with 31415

- Find an integer *n* such that $n^2 = 31415\ldots$
- finite decimal fraction is enough:
  $(10x)^2 = 100x^2$
- $\sqrt{31415} = 177.242771\ldots$
- $177.242771^2 = 31414.9998718\ldots$

Perfect Square Starts with 31415

- Find an integer n such that $n^2 = 31415\dots$
- finite decimal fraction is enough:
  $(10x)^2 = 100x^2$
- $\sqrt{31415} = 177.242771\dots$
- $177.242771^2 = 31414.9998718\dots$
- $177.243^2 = 31415.081049$

Perfect Square Starts with 31415

- Find an integer n such that $n^2 = 31415\ldots$
- finite decimal fraction is enough: $(10x)^2 = 100x^2$
- $\sqrt{31415} = 177.242771\ldots$
- $177.242771^2 = 31414.9998718\ldots$
- $177.243^2 = 31415.081049$
- $177\,243^2 = 31\,415\,081\,049$

Perfect Square Starts with 31415

- Find an integer n such that $n^2 = 31415\ldots$
- finite decimal fraction is enough:
  $(10x)^2 = 100x^2$
- $\sqrt{31415} = 177.242771\ldots$
- $177.242771^2 = 31414.9998718\ldots$
- $177.243^2 = 31415.081049$
- $177\,243^2 = 31\,415\,081\,049$
- $177\,244^2 = 31\,415\,435\,536$

Two Perfect Squares Starting with 31415

Two Perfect Squares Starting with 31415

- Prove that there is another *n* with a different first digit whose square also starts with 31415

Two Perfect Squares Starting with 31415

- Prove that there is another n with a different first digit whose square also starts with 31415
- can we use the same trick?

Two Perfect Squares Starting with 31415

- Prove that there is another *n* with a different first digit whose square also starts with 31415
- can we use the same trick?
- $\sqrt{3141.5} = 56.0490856\ldots$

Two Perfect Squares Starting with 31415

- Prove that there is another *n* with a different first digit whose square also starts with 31415
- can we use the same trick?
- $\sqrt{3141.5} = 56.0490856\ldots$
- $5605^2 = 31\,416\,025$: too big

Two Perfect Squares Starting with 31415

- Prove that there is another *n* with a different first digit whose square also starts with 31415
- can we use the same trick?
- $\sqrt{3141.5} = 56.0490856\ldots$
- $5605^2 = 31\,416\,025$: too big
- $560491^2 = 314\,150\,161\,081$: OK

Two Perfect Squares Starting with 31415

- Prove that there is another *n* with a different first digit whose square also starts with 31415
- can we use the same trick?
- $\sqrt{3141.5} = 56.0490856\ldots$
- $5605^2 = 31\,416\,025$: too big
- $560491^2 = 314\,150\,161\,081$: OK
- answer: 177 243 and 560 491.

Outline

Magic Squares

Narrowing the Search

Multiplicative Magic Squares

More Puzzles

Integer linear combinations

Paths in a Graph

7 **and** 13

Imagine a country with 7/13 florins coins.
Two people, each has many coins of each type.
Can one pay 6 florins to the other?

7 **and** 13

Imagine a country with 7/13 florins coins.
Two people, each has many coins of each type.
Can one pay 6 florins to the other?

- $6 = 13 - 7$

7 **and** 13

Imagine a country with 7/13 florins coins.
Two people, each has many coins of each type.
Can one pay 6 florins to the other?

- $6 = 13 - 7$
- what about paying 1 florin?

7 **and** 13

Imagine a country with 7/13 florins coins.
Two people, each has many coins of each type.
Can one pay 6 florins to the other?

- $6 = 13 - 7$
- what about paying 1 florin?
- yes: $2 \times 7 - 13 = 1$

7 **and** 13

Imagine a country with 7/13 florins coins.
Two people, each has many coins of each type.
Can one pay 6 florins to the other?

- $6 = 13 - 7$
- what about paying 1 florin?
- yes: $2 \times 7 - 13 = 1$ or $7 - 6 = 1$

7 **and** 13

Imagine a country with 7/13 florins coins.
Two people, each has many coins of each type.
Can one pay 6 florins to the other?

- $6 = 13 - 7$
- what about paying 1 florin?
- yes: $2 \times 7 - 13 = 1$ or $7 - 6 = 1$
- what about 2 florins?

7 **and** 13

Imagine a country with 7/13 florins coins.
Two people, each has many coins of each type.
Can one pay 6 florins to the other?

- $6 = 13 - 7$
- what about paying 1 florin?
- yes: $2 \times 7 - 13 = 1$ or $7 - 6 = 1$
- what about 2 florins?
- $4 \times 7 - 2 \times 13 = 28 - 26 = 2$

7 **and** 13

Imagine a country with 7/13 florins coins.
Two people, each has many coins of each type.
Can one pay 6 florins to the other?

- $6 = 13 - 7$
- what about paying 1 florin?
- yes: $2 \times 7 - 13 = 1$ or $7 - 6 = 1$
- what about 2 florins?
- $4 \times 7 - 2 \times 13 = 28 - 26 = 2$ or $2 \times 1 = 2$

7 **and** 13

Imagine a country with 7/13 florins coins.
Two people, each has many coins of each type.
Can one pay 6 florins to the other?

- $6 = 13 - 7$
- what about paying 1 florin?
- yes: $2 \times 7 - 13 = 1$ or $7 - 6 = 1$
- what about 2 florins?
- $4 \times 7 - 2 \times 13 = 28 - 26 = 2$ or $2 \times 1 = 2$
- any (integer) amount is payable

7 **and** 13

Imagine a country with 7/13 florins coins.
Two people, each has many coins of each type.
Can one pay 6 florins to the other?

- $6 = 13 - 7$
- what about paying 1 florin?
- yes: $2 \times 7 - 13 = 1$ or $7 - 6 = 1$
- what about 2 florins?
- $4 \times 7 - 2 \times 13 = 28 - 26 = 2$ or $2 \times 1 = 2$
- any (integer) amount is payable
- for every integer c, the equation $7x + 13y = c$ has integer solutions

15 **and** 21

15/21 coins; how to pay 6?

15 **and** 21

15/21 coins; how to pay 6?

- $6 = 21 - 15$

15 **and** 21

15/21 coins; how to pay 6?

- $6 = 21 - 15$
- 8?

15 **and** 21

15/21 coins; how to pay 6?

- $6 = 21 - 15$
- 8? obstacle: all payable amounts are multiples of 3;

15 **and** 21

15/21 coins; how to pay 6?

- $6 = 21 - 15$
- 8? obstacle: all payable amounts are multiples of 3; no chances for 8 or 1

15 **and** 21

15/21 coins; how to pay 6?

- $6 = 21 - 15$
- 8? obstacle: all payable amounts are multiples of 3; no chances for 8 or 1
- 3?

15 **and** 21

15/21 coins; how to pay 6?

- $6 = 21 - 15$
- 8? obstacle: all payable amounts are multiples of 3; no chances for 8 or 1
- 3? OK: $6 = 21{-}15;\ 9 = 15{-}6,\ 3 = 9{-}6;$

15 **and** 21

15/21 coins; how to pay 6?

- $6 = 21 - 15$
- 8? obstacle: all payable amounts are multiples of 3; no chances for 8 or 1
- 3? OK: $6 = 21{-}15;\ 9 = 15{-}6,\ 3 = 9{-}6;$ unfolding: $9 = 2{\times}15 - 21,$
  $3 = 3{\times}15 - 2{\times}21$

15 **and** 21

15/21 coins; how to pay 6?

- $6 = 21 - 15$
- 8? obstacle: all payable amounts are multiples of 3; no chances for 8 or 1
- 3? OK: $6 = 21{-}15;\ 9 = 15{-}6,\ 3 = 9{-}6;$ unfolding: $9 = 2{\times}15 - 21,$ $3 = 3{\times}15 - 2{\times}21$
- any multiple of 3 is payable

15 **and** 21

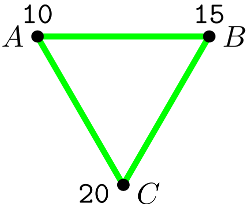
15/21 coins; how to pay 6?

- $6 = 21 - 15$
- 8? obstacle: all payable amounts are multiples of 3; no chances for 8 or 1
- 3? OK: $6 = 21-15;\ 9 = 15-6,\ 3 = 9-6;$ unfolding: $9 = 2\times 15 - 21$, $3 = 3\times 15 - 2\times 21$
- any multiple of 3 is payable
- the equation $15x + 21y = c$ has integer solutions $\Leftrightarrow$ c is a multiple of 3

Outline

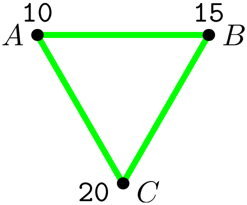
Hotels and Paths

- Hotels $A(10)$, $B(15)$, $C(20)$; change every night for $10 + 15 + 20 = 45$ nights



Hotels and Paths

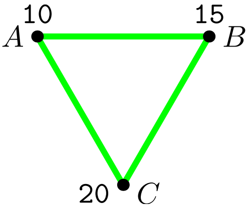
- Hotels $A(10)$, $B(15)$, $C(20)$; change every night for $10 + 15 + 20 = 45$ nights



- repeat 5 times a path of length 9

Hotels and Paths

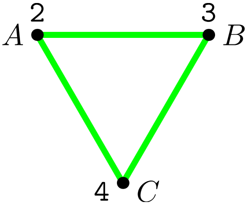
- Hotels $A(10)$, $B(15)$, $C(20)$; change every night for $10 + 15 + 20 = 45$ nights



- repeat 5 times a path of length 9
- different endpoints

Hotels and Paths

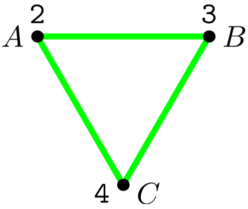
- Hotels $A(10)$, $B(15)$, $C(20)$; change every night for $10 + 15 + 20 = 45$ nights



- repeat 5 times a path of length 9
- different endpoints

Hotels and Paths

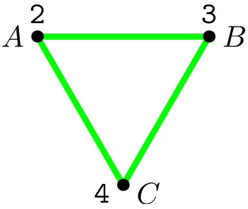
- Hotels $A(10)$, $B(15)$, $C(20)$; change every night for $10 + 15 + 20 = 45$ nights



- repeat 5 times a path of length 9
- different endpoints
- every second point is C

Hotels and Paths

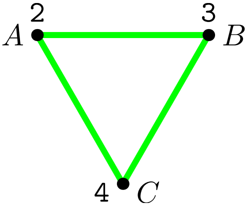
- Hotels $A(10)$, $B(15)$, $C(20)$; change every night for $10 + 15 + 20 = 45$ nights



- repeat 5 times a path of length 9
- different endpoints
- every second point is C
- A

Hotels and Paths

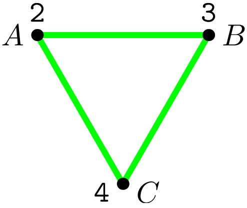
- Hotels $A(10)$, $B(15)$, $C(20)$; change every night for $10 + 15 + 20 = 45$ nights



- repeat 5 times a path of length 9
- different endpoints
- every second point is C
- AC

Hotels and Paths

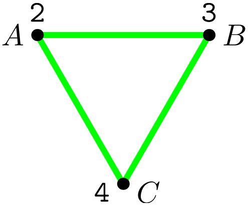
- Hotels $A(10)$, $B(15)$, $C(20)$; change every night for $10 + 15 + 20 = 45$ nights



- repeat 5 times a path of length 9
- different endpoints
- every second point is C
- ACA

Hotels and Paths

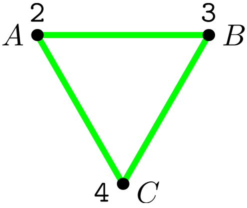
- Hotels $A(10)$, $B(15)$, $C(20)$; change every night for $10 + 15 + 20 = 45$ nights



- repeat 5 times a path of length 9
- different endpoints
- every second point is C
- $ACAC$

Hotels and Paths

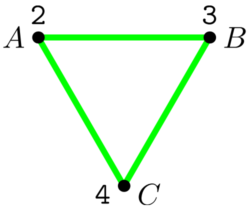
- Hotels $A(10)$, $B(15)$, $C(20)$; change every night for $10 + 15 + 20 = 45$ nights



- repeat 5 times a path of length 9
- different endpoints
- every second point is C
- $ACACB$

Hotels and Paths

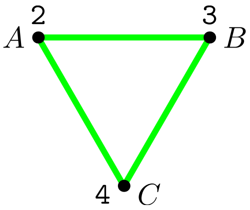
- Hotels $A(10)$, $B(15)$, $C(20)$; change every night for $10 + 15 + 20 = 45$ nights



- repeat 5 times a path of length 9
- different endpoints
- every second point is C
- $ACACBC$

Hotels and Paths

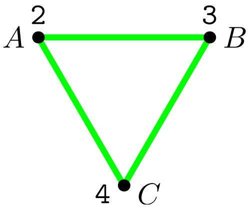
- Hotels $A(10)$, $B(15)$, $C(20)$; change every night for $10 + 15 + 20 = 45$ nights



- repeat 5 times a path of length 9
- different endpoints
- every second point is C
- $ACACBCB$

Hotels and Paths

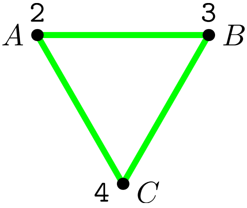
- Hotels $A(10)$, $B(15)$, $C(20)$; change every night for $10 + 15 + 20 = 45$ nights



- repeat 5 times a path of length 9
- different endpoints
- every second point is C
- $ACACBCBC$

Hotels and Paths

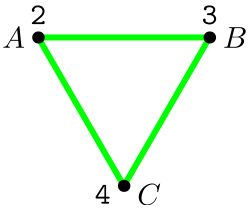
- Hotels $A(10)$, $B(15)$, $C(20)$; change every night for $10 + 15 + 20 = 45$ nights



- repeat 5 times a path of length 9
- different endpoints
- every second point is C
- $ACACBCBCB$

Hotels and Paths

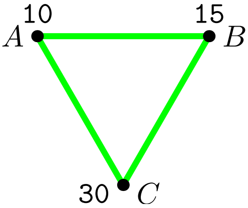
- Hotels $A(10)$, $B(15)$, $C(20)$; change every night for $10 + 15 + 20 = 45$ nights



- repeat 5 times a path of length 9
- different endpoints
- every second point is C
- $ACACBCBCB$ (5 times)

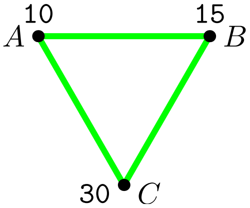
When a Path Does Not Exist

- Hotels $A(10)$, $B(15)$, $C(30)$; change every night for $10 + 15 + 30 = 55$ nights



When a Path Does Not Exist

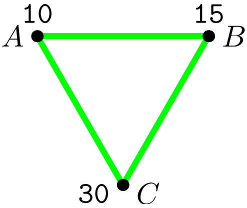
- Hotels $A(10)$, $B(15)$, $C(30)$; change every night for $10 + 15 + 30 = 55$ nights



- possible or not?

When a Path Does Not Exist

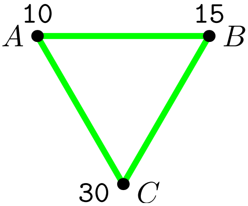
- Hotels $A(10)$, $B(15)$, $C(30)$; change every night for $10 + 15 + 30 = 55$ nights



- possible or not?
- obstacle: too many vouchers for C

When a Path Does Not Exist

- Hotels $A(10)$, $B(15)$, $C(30)$; change every night for $10 + 15 + 30 = 55$ nights



- possible or not?
- obstacle: too many vouchers for C
- to use them all we need at least 29 others