Physics 155 HW # 4

1.) We will complete the derivation of V_d for even d. Start from

$$\begin{aligned} V_{2n} = \int_0^R dr\, r^{2n-1} \int_0^\pi (\sin\theta_1)^{2n-2} d\theta_1 \int_0^\pi (\sin\theta_2)^{2n-3} d\theta_2 ... \\ ... \int_0^\pi (\sin\theta_{2n-3})^2 d\theta_{2n-3} \int_0^\pi (\sin\theta_{2n-2}) d\theta_{2n-2} \int_0^{2\pi} d\theta_{2n-1} \end{aligned} \tag{1}$$

$$\text{Let } I_n = \int_0^\pi (\sin\theta)^n d\theta = \begin{cases} \dfrac{(n-1)!!}{2^{\frac{n}{2}}(\frac{n}{2})!}\pi & n \text{ even} \\ \\ \dfrac{2^{\frac{n+1}{2}}(\frac{n-1}{2})!}{n!!} & n \text{ odd} \end{cases} \tag{2}$$

Recall that the double factorial sign is just like a factorial, but we only include *every other* integer. For example, $9!! = 1 \times 3 \times 5 \times 7 \times 9$.

$$\text{Using } \int_0^{2\pi} d\theta_{2n-1} = 2\pi \qquad \int_0^R dr\, r^{2n-1} = \frac{R^{2n}}{2n} \tag{3}$$

$$\text{gives } \boxed{V_{2n} = \frac{R^{2n} \times 2\pi}{2n} \times I_{2n-2} I_{2n-3} ... I_1} \tag{4}$$

Evaluate for $n = 1$, 2, 3, and 4, and then see the pattern to guess or deliberately derive the result for arbitrary n.

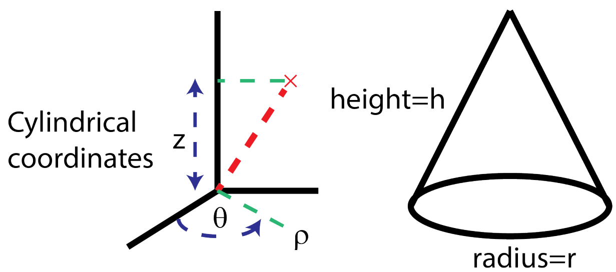
2.) The porcupine: consider a unit cube in $d = 10$ dimensions. Fill in the following table for lengths of edges and "diagonals":

type of edge or diagonal	length of diagonal	number of equivalent diagonals
(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)	1	10
(1, 1, 0, 0, 0, 0, 0, 0, 0, 0)	$\sqrt{2}$	90
(1, 1, 1, 0, 0, 0, 0, 0, 0, 0)		
(1, 1, 1, 1, 0, 0, 0, 0, 0, 0)		
(1, 1, 1, 1, 1, 0, 0, 0, 0, 0)		
(1, 1, 1, 1, 1, 1, 0, 0, 0, 0)		
(1, 1, 1, 1, 1, 1, 1, 0, 0, 0)		
(1, 1, 1, 1, 1, 1, 1, 1, 0, 0)		
(1, 1, 1, 1, 1, 1, 1, 1, 1, 0)		
(1, 1, 1, 1, 1, 1, 1, 1, 1, 1)		

Be careful with your counting. Recall the cube has eight corners and four diagonals. Think of how to determine the different possibilities for the different diagonals, but do not double count them.

3.) Use cylindrical coordinates to find the moment of inertia of a right cone with uniform mass density ρ_0, height h, and maximal radius r. Express your result as a number times mr^2. Choose the rotation axis to be the z-axis.

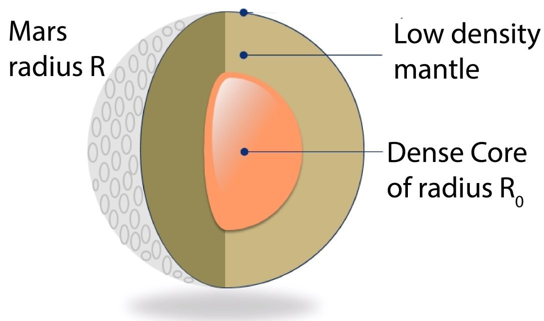
Use coordinates as follows:



$$\begin{aligned} 0 \leq z \leq h \\ 0 \leq \rho \leq r \end{aligned} \tag{5}$$

4.) Mars has a radius of 3,390 km and an average density of 3,934 kg/m^3. Its polar moment of inertia is $0.365MR^2$.

We will build a copy of Mars using a high-density core planet-building material of density 8,000 kg/m^3 and a low-density outer crust/mantle made of planet-building material with density 1,000 kg/m^3.



Adjust R_0 so that the mass M = the mass of Mars, keeping the outer radius R fixed. Report your value for R_0 and then compute the moment of inertia over the polar axis. Express in units of MR^2 and compare to the exact answer and explain how one can fix this discrepancy by keeping the mass and radius fixed and using a planet-building material of density 5,000 kg/m^3 too.

Note you are asked to *explain how you would do it* with the additional intermediate density material, not to actually calculate it.

5.) A charge of +1 is placed at (1,0,0), and a charge of -1 is placed at (-1,0,0). Find a formula for the electric field vector:

$$\begin{aligned} \text{at } (0,y,0) &\text{ for } -\infty < y < \infty \\ \text{repeat at } (2,y,0) &\text{ for } -\infty < y < \infty \\ \text{and } (10,y,0) &\text{ for } -\infty < y < \infty \end{aligned} \qquad (6)$$