Chapter 31

Applications of First-Order Differential Equations

In this chapter, we will apply first-order differential equations to a number of different problems motivated by physics. This includes a description of air resistance in moving objects, Newton's law of cooling, and a proper treatment of rocket motion when we take into account the fact that the mass of the rocket changes as the fuel is burned and expelled.

31.1 Models of Air Resistance

Consider first the case of no air resistance—a mass m is dropped from a height in the presence of gravity (the acceleration due to gravity is g). If there is no air resistance then Newton says that

$$m\frac{dv}{dt} = mg, \tag{31.1}$$

where we take the positive direction to be *downward*. Rearranging, we find that

$$\frac{dv}{dt} = g \implies \int_0^v d\bar{v} = \int_0^t g\,d\bar{t} \implies v = gt, \tag{31.2}$$

under the assumption that the particle is dropped with no initial velocity.

Now assume the object feels air resistance proportional to $|v|$, $F = -k|v|$. Note that here we have $|v| \geq 0$ so $|v| = v$. This implies that

$$m\frac{dv}{dt} = mg - kv, \tag{31.3}$$

or, after rearranging

$$\frac{dv}{dt} = -\frac{k}{m}v + g. \tag{31.4}$$

This is a standard first-order linear differential equation. Let's use our methodology to solve it. First, we identify that

$$q(t) = \frac{k}{m} \tag{31.5}$$

and

$$r(t) = g. \tag{31.6}$$

Next, we find the integral of $q(\bar{t})$ subject to the initial and final conditions on the time variable:

$$\int_0^t d\bar{t}q(\bar{t}) = \frac{k}{m}t \tag{31.7}$$

$$e^{-\int_0^t d\bar{t}q(\bar{t})} = e^{-\frac{k}{m}t}. \tag{31.8}$$

Now we are ready to substitute into our general formula, which yields

$$v(t) = e^{-\frac{k}{m}t}\int_0^t dt' e^{\frac{k}{m}t'}g \tag{31.9}$$

$$= \frac{m}{k}ge^{-\frac{k}{m}t}(e^{\frac{k}{m}t} - 1) \tag{31.10}$$

$$= \frac{mg}{k}(1 - e^{-\frac{k}{m}t}). \tag{31.11}$$

Now we check our result.First the initial condition. Letting $t = 0$, we immediately fin that $v(0) = 0$. For the differential equation, we must take a derivative. We find that

$$\frac{dv}{dt} = -\frac{mg}{k}\left(-\frac{k}{m}\right)e^{-\frac{k}{m}t} = ge^{-\frac{k}{m}t}. \tag{31.12}$$

We use this to check the original equation and find that

$$g - \frac{k}{m}v = g - g(1 - e^{-\frac{k}{m}t}) = ge^{-\frac{k}{m}t} = \frac{dv}{dt}. \tag{31.13}$$

Hence, the solution is verified.

For air resistance proportional to v^2, we have a *nonlinear* first-order differential equation, given by

$$m\frac{dv}{dt} = mg - k'v^2. \tag{31.14}$$

Rearranging to put it into standard form, we have

$$\frac{dv}{dt} + \frac{k'}{m}v^2 = g, \tag{31.15}$$

or

$$dv + \left(\frac{k'}{m}v^2 - g\right) dt = 0. \tag{31.16}$$

This is an exact differential equation, so we can immediately solve it by isolating all v terms on the left and all t terms on the right, as follows:

$$\frac{dv}{\frac{k'}{m}v^2 - g} = -dt. \tag{31.17}$$

Now, we integrate both sides, being sure to incorporate the initial conditions for v and t

$$\int_0^v \frac{-d\bar{v}}{\frac{k'}{m}\bar{v}^2 - g} = \int_0^t dt = t. \tag{31.18}$$

Sinmplifying, we have

$$\frac{m}{k'}\int_0^v \frac{d\bar{v}}{\frac{gm}{k'} - \bar{v}^2} = t. \tag{31.19}$$

We have to perform the integral on the left hand side. Let $\frac{gm}{k'} = a^2$ and expand in partial fractions. This allows the integrals to be performed as logarithms:

$$\frac{m}{k'}\int_0^v \frac{d\bar{v}}{a^2 - \bar{v}^2} = \frac{m}{k'}\int_0^v \frac{d\bar{v}}{2a}\left(\frac{1}{a-\bar{v}} + \frac{1}{a+\bar{v}}\right) \tag{31.20}$$

$$= \frac{1}{2}\sqrt{\frac{m}{gk'}}\left(-\ln\frac{a-v}{a} + \ln\frac{a+v}{a}\right) \tag{31.21}$$

$$= \frac{1}{2}\sqrt{\frac{m}{gk'}}\ln\left(\frac{\sqrt{\frac{gm}{k'}} + v}{\sqrt{\frac{gm}{k'}} - v}\right). \tag{31.22}$$

Our next step is to exponentiate both sides

$$e^{2t\sqrt{\frac{gk'}{m}}} = \frac{\sqrt{\frac{gm}{k'}} + v}{\sqrt{\frac{gm}{k'}} - v} \tag{31.23}$$

and solve for $v(t)$

$$\left(1 + e^{2t\sqrt{\frac{gk'}{m}}}\right)v = \sqrt{\frac{gm}{k'}}\left(-1 + e^{2t\sqrt{\frac{gk'}{m}}}\right). \tag{31.24}$$

We obtain

$$v(t) = \sqrt{\frac{gm}{k'}} \left(\frac{1 - e^{-2t\sqrt{\frac{gk'}{m}}}}{1 + e^{-2t\sqrt{\frac{gk'}{m}}}} \right) \tag{31.25}$$

$$= \sqrt{\frac{gm}{k'}} \left(\frac{e^{t\sqrt{\frac{gk'}{m}}} - e^{-t\sqrt{\frac{gk'}{m}}}}{e^{t\sqrt{\frac{gk'}{m}}} + e^{-t\sqrt{\frac{gk'}{m}}}} \right) \tag{31.26}$$

$$= \sqrt{\frac{gm}{k'}} \tanh \sqrt{\frac{gk'}{m}} t, \tag{31.27}$$

where the last line follows by recalling that $\cosh(x) = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$.

Now we need to check to verify the solution is correct. First check that $v(0) = 0$, which obviously holds since $\tanh(0) = 0$. Next, verify the differential equation. Th derivative of v with respect to t is

$$\frac{dv}{dt} = \sqrt{\frac{gm}{k'}} \sqrt{\frac{gk'}{m}} \operatorname{sech}^2 \sqrt{\frac{gk'}{m}} t = g \operatorname{sech}^2 \sqrt{\frac{gk'}{m}} t. \tag{31.28}$$

But we have a hyperbolic trig identity given by $1 - \tanh^2 x = \operatorname{sech}^2 x$, so we find that

$$\frac{dv}{dt} = g - \frac{k'}{m} v^2. \tag{31.29}$$

Hence, we have verified that the solution is indeed correct.

Note that this nonlinear equation is also a Bernoulli equation (and also a Ricatti equation). For the Bernoulli equation, we could use the standard method for those equations to solve this. As an exercise, you might want to verify that you obtain the same answer working with our general procedure for Bernoulli equations.

31.2 Newton's Law of Cooling

Let $T(t)$ be the temperature of an object at time t and T_0 be the ambient temperature of the surroundings. Then Newton's Law of Cooling is

$$\frac{dT}{dt} = -k(T - T_0) \tag{31.30}$$

where k is a constant. It says the rate of change of he temperature is proportional to the difference in temperature of the object and the surrounding "heat bath."

Suppose $T(0) = T_1$. Find the solution for all t. Starting from the differential equation $\frac{dT}{dt} + kT = kT_0$, we can identify the two functions $q(t) = +k$ and $r(t) = +kT_0$. Then we employ our standard procedure:

$$\int_0^t d\bar{t}\, q(\bar{t}) = kt \tag{31.31}$$

So we find that

$$T(t) = T_1 e^{-kt} + e^{-kt} \int_0^t dt' e^{kt'} kT_0. \tag{31.32}$$

Performing the integral yields

$$T(t) = T_1 e^{-kt} + e^{-kt} kT_0 \frac{1}{k}(e^{kt} - 1), \tag{31.33}$$

and after simplifying, we have

$$T(t) = T_1 e^{-kt} + T_0(1 - e^{-kt}). \tag{31.34}$$

As always, we need to check our results. First the initial condition is given by $T(0) = T_1$ and then the differential quation:

$$\begin{aligned} \frac{dT}{dt} &= -kT_1 e^{-kt} + kT_0 e^{-kt} && (31.35) \\ &= -k(T_1 e^{-kt} + T_0(1 - e^{-kt}) - T_0) && (31.36) \\ &= -k(T(t) - T_0). && (31.37) \end{aligned}$$

Hence, our solution checks.

Suppose the object cooling is a hot cup of coffee. $T_1 = 200°\mathrm{F}$, $T_0 = 70°\mathrm{F}$ and it cools to 190°F in one minute. How long before it reaches 150°?

Solution: Use our general solution after plugging in T_1 and T_0.

$$T(t) = 200e^{-kt} + 70(1 - e^{-kt}) \tag{31.38}$$

Use the criterion in he problem to determine k:

$$190 = 200e^{-k} + 70(1 - e^{-k}) \implies 120 = 130e^{-k}, \tag{31.39}$$

or, after solving for k

$$k = -\ln\frac{120}{130} = 0.08. \tag{31.40}$$

Now we have all the information needed to answer the posed question. We find

$$150 = 200e^{-0.08t} + 70(1 - e^{-0.08t}), \tag{31.41}$$

which simplifies to

$$80 = 130e^{-0.08t}. \tag{31.42}$$

Take the logarithm to find

$$0.08t = -\ln\frac{8}{13}. \tag{31.43}$$

Plugging in he numbers yields

$$t = 6.075 \text{ minutes}. \tag{31.44}$$

So drink your Starbucks quickly!

31.3 Variable Mass (Rocket) Problem

Newton says the time rate of change of the momentum is equal to he applied force, or

$$F = \frac{dp}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}. \tag{31.45}$$

Note how the rate of change of momentum has two contributions. One from the acceleration and one from the changing mass of the rocket.

We consider a rocket with initial mass m_0 expelling fuel at a rate β. Then $m(t) = m_0 - \beta t$. Note that this is an *input* into our problem. We work with a fixed rate of fuel being expelled. Assume the gravitational constant is a constant g such that $F = -m(t)g$; this means the rocket is not going too far from the radius of the earth—of course more complicated treatments are possible. With all of these assumptions, Newton tells us that

$$(m_0 - \beta t)\frac{dv}{dt} - \beta v_{\text{exhaust}} = -(m_0 - \beta t)g \tag{31.46}$$

with an initial velocity $v(0) = v_0$. Rearranging the equation for the time derivative of the velocity gives us

$$\frac{dv}{dt} = -g + \frac{\beta v_{\text{ex}}}{m_0 - \beta t}, \tag{31.47}$$

which is a separable differential equation. So we integrate both sides

$$\int_{v_0}^{v} d\bar{v} = \int_0^t \left(-g + \frac{\beta v_{\text{ex}}}{m_0 - \beta \bar{t}} \right) d\bar{t}, \tag{31.48}$$

and find that

$$v - v_0 = -gt - \beta v_{\text{ex}} \left(\frac{1}{\beta} \right) \ln \left(\frac{m_0 - \beta t}{m_0} \right) \tag{31.49}$$

or

$$v(t) = v_0 - gt + v_{\text{ex}} \ln \left(\frac{m_0}{m_0 - \beta t} \right). \tag{31.50}$$

Increasing the exhaust velocity v_{exhaust} or increasing the rate that mass is expelled from the rocket β will increase the maximal speed at the time the fuel runs out.

One can extend this analysis to that of a *relativistic rocket*, but we will not do that here.