Chapter 2

Everything you want to know about series but were afraid to ask

2.1 The geometric series

The book by Toeplitz discusses Zeno's paradox — that one can never go from point A to point B because it takes an infinite number of steps, if one takes half as big a step each time. Since one needs an infinite number of steps, one can never make it. But the flaw in Zeno's argument is that the sum of all of the steps is *finite*, which is why we can do it. We will prove

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2 \tag{2.1}$$

next.

We perform the proof in full generality, using an abstract x in the *geometric series*. Our goal is to show that

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \tag{2.2}$$

for $|x| < 1$. The strategy is to use the multiply by one trick. We have

$$\begin{aligned}&\left(1 + x + x^2 + x^3 + x^4 + \cdots + x^N\right)\frac{(1-x)}{(1-x)}\\ &\quad= \left(1 - x + x - x^2 + x^2 - \cdots - x^N + x^N - x^{N+1}\right)\frac{1}{(1-x)}.\end{aligned} \tag{2.3}$$

Note how all the terms except the first and the last have pairs appearing—one with a plus sign (black) and one with a minus sign (red)—so they will cancel. This shows that

$$1 + x + x^2 + x^3 + x^4 + \cdots + x^N = \frac{1 - x^{N+1}}{1 - x}. \tag{2.4}$$

If $|x| < 1$, then as $N \to \infty$, one will have $|x|^{N+1} \to 0$ (be sure you understand why).

So we take the limit $N \to \infty$ to obtain

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x} \tag{2.5}$$

for $|x| < 1$. This sum is called the geometric series. It is an important result to remember (both the final answer and the methodology used to derive it). It will come up again many times.

2.2 The (dreaded) Taylor series

Finally, we talk about a MacLaurin and Taylor series. These are essentially the same thing. But to be completely precise, we note that the MacLaurin series is a Taylor series expanded about the point $x_0 = 0$. The Taylor series can be expanded about any point x_0. It is important to remember this, because most examples are MacLaurin series, but there are times when a Taylor series about a different point is needed.

Suppose we want a polynomial that has the same function value and same first n derivatives of a function at the origin. How do we make such a thing? Obviously the first term is the value of the function at the origin, or f(0). The second term must have the correct slope, so the coefficient of x is $\frac{df(0)}{dx}$, which we denote as $f^{(1)}(0)$, with the superscript indicating the number of derivatives. The third term is proportional to x^2. Since the second derivative of x^2 is 2, we must have its term be $f^{(2)}(0)$ multiplied by $\frac{x^2}{2}$. Hence, we have already found that the quadratic polynomial that satisfies this criterion is

$$f(0) + f^{(1)}(0)x + \frac{1}{2} f^{(2)}(0)x^2. \tag{2.6}$$

To get the full series, we just continue in the same fashion, recalling that

$$\frac{d^n}{dx^n} x^n = n! \tag{2.7}$$

yields

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!}, \tag{2.8}$$

which, one can see, includes the quadratic polynomial we derived above.

As a check in your understanding, you should be able to take the nth derivative of the series, and evaluate it at $x = 0$, and show it is equal to $f^{(n)}(0)$. You must assume that you an interchange the order of taking a derivative and performing the sum when you carry out this calculation. Note that the result in Eq. (2.7) holds at every point x_0! This is why the form for the Taylor series will be exactly the same as the MacLaurin series (with the only change being $x \to x - x_0$).

Now we are ready for a worked-out example. Evaluate the MacLaurin series of $\sqrt{1+x}$. We need to calculate a number of different derivatives and evaluate them at $x = 0$.

$$\frac{d}{dx}\sqrt{1+x} = \frac{1}{2}\frac{1}{\sqrt{1+x}}\bigg|_{x=0} = \frac{1}{2} \tag{2.9}$$

$$\frac{d^2}{dx^2}\sqrt{1+x} = \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{(1+x)^{\frac{3}{2}}}\bigg|_{x=0} = -\frac{1}{4} \tag{2.10}$$

$$\frac{d^3}{dx^3}\sqrt{1+x} = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{(1+x)^{\frac{5}{2}}}\bigg|_{x=0} = \frac{3}{8} \tag{2.11}$$

At this stage, it is common for students, who are "technicians," to try to recognize the pattern and simply "hope" it continues that way forever. But it is better to carefully verify (via an induction argument), as a "practitioner" would, that it really holds. We will not go through these details here. Instead, we just show what we explicitly derived:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots \tag{2.12}$$

which follows from

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{4}\left(\frac{x^2}{2}\right) + \frac{3}{8}\left(\frac{x^3}{6}\right) + \dots \tag{2.13}$$

You should try to do the full inductive argument to determine the general series on your own if you have never done this before.

2.3 Hyperbolic functions

This material does not exactly fit here, but it is useful for you to have a quick review of hyperbolic functions, which I have seen cause more than their fair share of consternation.

We begin with some basic definitions. The hyperbolic cosine is defined as

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \tag{2.14}$$

and it is an *even* function of x. The hyperbolic sine is defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \tag{2.15}$$

and it is an *odd* function of x. The derivatives follow immediately from the definitions:

$$\frac{d\cosh(x)}{dx} = \frac{d}{dx}\frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh(x) \tag{2.16}$$

and

$$\frac{d\sinh(x)}{dx} = \frac{d}{dx}\frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh(x). \tag{2.17}$$

Recall how similar this is to the conventional trig functions except for some signs. Indeed, this is common with hyperbolics. Another example is the following identity

$$\cosh^2(x) - \sinh^2(x) = \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x}) = 1. \tag{2.18}$$

In summary, we have shown that

$$\frac{d\cosh(x)}{dx} = \sinh(x), \tag{2.19}$$

$$\frac{d\sinh(x)}{dx} = \cosh(x), \tag{2.20}$$

and

$$\cosh^2(x) = 1 + \sinh^2(x). \tag{2.21}$$

Plots of the hyperbolic functions are shown in Fig. 2.1.

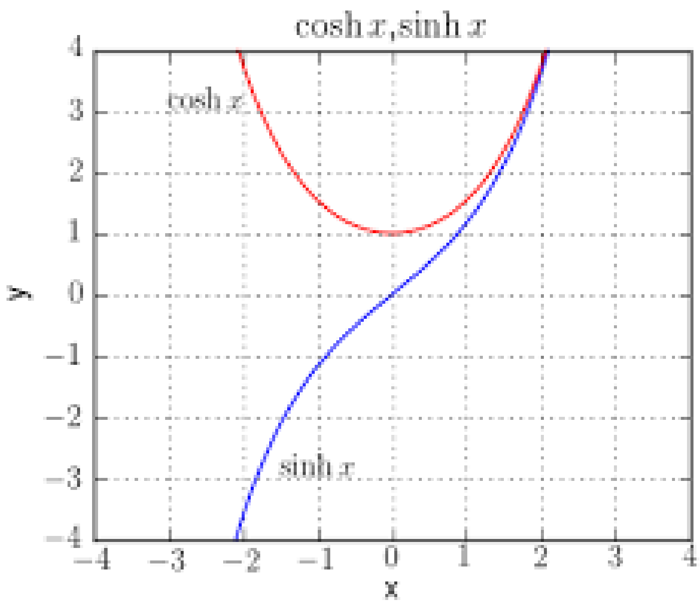

Figure 2.1: Plot of hyperbolic cosine (red) and sine (blue). One can immediately see their even and odd characters. The hyperbolic cosine is always larger than 1 and always large than the hyperbolic sine.