Chapter 22

Determinants

22.1 Definition of a determinant

The determinant of a matrix A with matrix elements a_{ij} is

$$\det A = \sum_{\substack{\text{permutations} \\ P \text{ of } n \text{ objects}}} (-1)^p a_{p_1 1} a_{p_2 2} a_{p_3 3} ... a_{p_n n}, \tag{22.1}$$

where $(p_1, p_2, ..., p_n)$ is a permutation of $(1, 2, ..., n)$ and

$$(-1)^p = \begin{cases} 1 & \text{if } p \text{ is an even permutation} \\ -1 & \text{if } p \text{ is an odd permutation.} \end{cases} \tag{22.2}$$

We illustrate this with some examples. The permutations are all possible reorderings of the set of numbers. For example, the six permutations of 3 numbers are 123, 231, 312, 132, 213, and 321. How do we determine whether a permutation is even or odd? Consider 312. To get back to 123, we need to

$$\begin{aligned} &\text{interchange 31 so then } 312 \to 132 \\ &\text{interchange 32 so then } 132 \to 123. \end{aligned} \tag{22.3}$$

In this case, these are two pair permutations. The number of pair permutations determines whether a permutation is classified as even or odd. An even permutation involves an even number of pair permutations. This implies that the even permutations make up one half of the $n!$ permutations of n numbers and odd permutations are the other half.

While this form for the definition of the determinant is simple to use for small matrices, it rapidly becomes painful to work with. Start with a 2×2 matrix

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \underbrace{a_{11}a_{22}}_{12\to 12} - \underbrace{a_{21}a_{12}}_{12\to 21}, \tag{22.4}$$

where the 12 permutation is even (+1) and the 21 permutation is odd (−1). The 3×3 matrix satisfies

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \underbrace{a_{11}a_{22}a_{23}}_{123\to 123} + \underbrace{a_{31}a_{12}a_{23}}_{123\to 312} + \underbrace{a_{21}a_{32}a_{13}}_{123\to 231} - \underbrace{a_{11}a_{32}a_{23}}_{123\to 132} - \underbrace{a_{31}a_{22}a_{13}}_{123\to 321} - \underbrace{a_{21}a_{12}a_{33}}_{123\to 213} \tag{22.5}$$

Where the permutations $123, 312, 231$ are even and the permutations $132, 321, 213$ are odd.

22.2 Properties of determinants

We now examine properties of the determinant. It turns out that the definition of the determinant implies a number of different powerful properties. These properties allow us to calculate determinants much more easily than having to sum $n!$ products of n terms. We start with some notation by introducing the completely antisymmetric tensor ϵ. We write

$$\det A = \sum_p \epsilon_{p_1p_2\ldots p_n} a_{p_1 1} a_{p_2 2} \ldots a_{p_n n} \tag{22.6}$$

with the completely antisymmetric tensor satisfying

$$\epsilon_{p_1p_2\ldots p_n} = \begin{cases} 0 & \text{any two } p_i\text{'s are the same} \\ 1 & p_1\ldots p_n \text{ is an even permutation} \\ -1 & p_1\ldots p_n \text{ is an odd permutation.} \end{cases} \tag{22.7}$$

Properties of the determinant:

1) If a row is all zeros, then $\det A = 0$ (because an element from each row appears in the product of each term, so each product has a zero in it). As a

result, the determinant ends up as the sum of $n!$ terms that are all zero and the determinant is 0.

2) If a column has all zeros, then $\det A = 0$ (the argument for this is similar to the one above).

3) If a row is multiplied by a constant λ, the determinant is multiplied by λ (since each row has an element that appears in one factor of each term in the sum for the determinant). The same thing happens for an entire column multiplied by a constant.

4) If two rows of a matrix are interchanged, the sign of the determinant changes ($\det A \to -\det A$). The interchange of a row is like adding a permutation to each term in the determinant. This changes the even permutations to odd permutations, and *vice versa*—hence, the sign of the determinant changes.

5) If two rows are proportional to each other, then $\det A = 0$. Take one row and multiply by a constant so it is equal to the second row. This changes the determinant by that factor according to property 4 above. Then, interchange the two rows—the matrix doesn't change, but $\det A = -\det A$, which implies that $\det A = 0$.

6) If row i can be written as $a_{ij} = b_{ij} + c_{ij}$ for $i \leq j \leq n$, then $\det A = \det B + \det C$ where B is the matrix A with the i^{th} row replaced by b_i and C is the matrix A with the i^{th} row replaced by c_i.

Proof: every place a_{ix} appears in a term in the determinant (one factor for each term), we replace the a_{ix} factor with $b_{ix} + c_{ix}$. Then we group all the b terms together and all the c terms together in the summation and we immediately see that $\det A = \det B + \det C$.

7) If we take α times row i and add to row j, it does not change the determinant. This is because we think of $b_{jx} = a_{jx}$ and $c_{jx} = a_{jx}\alpha$ but C has two rows proportional to one another, so $\det C = 0$ and $\det A = \det B + \det C$ implies that the determinant is unchanged.

8) The transpose of a matrix A^T interchanges the columns of the matrix with its rows—so $(A^T)_{ij} = a_{ji} = (A)_{ji}$. The determinant of $A^T = \det A$. This follows, since we can permute a product $a_{p_1 1} a_{p_2 2} a_{p_3 3} ... a_{p_n n}$ into the product $a_{1 p_1} a_{2 p_2} a_{3 p_3} ... a_{n p_n}$ by reordering the terms in the sum. The permutations remain even or odd, so overall, the determinant stays the same.

9) The ij cofactor of a matrix is defined to be $(-1)^{i+j}$ times the det of the $(n-1) \times (n-1)$ matrix whose elements are found by removing the i^{th}

row and the j^{th} column of A. The determinant satisfies

$$\det A = \sum_{j=1}^{n} a_{ij} c_{ij} \tag{22.8}$$

and similarly, we have

$$\det A = \sum_{i=1}^{n} a_{ij} c_{ij}. \tag{22.9}$$

This is often a useful way to find determinants if a row or column has many zeros.

10) If $C = AB$, such that $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$, then $\det C = \det A \det B$.

Proof:

$$\det C = \sum_{p} \epsilon_{p_1 \cdots p_n} c_{p_1 1} c_{p_2 2} \cdots c_{p_n n} \tag{22.10}$$

$$= \sum_{p} \epsilon_{p_1 \ldots p_n} \sum_{k_1=1}^{n} a_{p_1 k_1} b_{k_1 1} \sum_{k_2=1}^{n} a_{p_2 k_2} b_{k_2 2} \cdots \sum_{k_n=1}^{n} a_{p_n k_n} b_{k_n n} \tag{22.11}$$

$$= \sum_{p_1 \cdots p_n} \sum_{k_1 \cdots k_n} \epsilon_{p_1 \cdots p_n} a_{p_1 k_1} a_{p_2 k_2} \cdots a_{p_n k_n} b_{k_1 1} b_{k_2 2} \cdots b_{k_n n}. \tag{22.12}$$

So, we end up with

$$\sum_{k_1 \cdots k_n} \det A \cdot \epsilon_{k_1 \cdots k_n} b_{k_1 1} b_{k_2 2} \cdots b_{k_n n} \tag{22.13}$$

or

$$\det A \cdot \det B \tag{22.14}$$

because

$$\sum_{p_1 \cdots p_n} \epsilon_{p_1 \cdots p_n} a_{p_1 k_1} a_{p_2 k_2} \cdots a_{p_n k_n} = \det A \cdot \epsilon_{k_1 \cdots k_n}. \tag{22.15}$$

This last identity summarizes how the permutation $k_1 k_2 \cdots k_n$ of $12 \cdots n$ is even or odd according to whether the completely antisymmetric tensor is 1 or -1.

These properties greatly simplify how one can compute a determinant. We show next how to calculate a determinant via row reduction with a concrete example. The strategy is to change the matrix by adding multiples of one row to another in order to make the entries below the diagonal all zeros.

The determinant is unchanged by all of these transformations. Here we go. The first equation took 3× the first row and subtracted it from the second. Then we subtract the first row from the fourth row. This makes zeros for all of the entries below the diagonal in the first column. Then we add 2× the second row to the third and subtract the second from the fourth. This zeroes out the second column below the diagonal. Finally, we take 3/10 of the third row and add to the fourth, to zero out the final subdiagonal element.

$$\begin{aligned} \det\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 5 & 1 & 0 \\ 0 & 2 & 6 & 0 \\ 1 & 1 & -2 & 3 \end{pmatrix} &= \det\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -8 & -12 \\ 0 & 2 & 6 & 0 \\ 0 & -1 & -5 & -1 \end{pmatrix} \\ &= \det\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -8 & -12 \\ 0 & 0 & -10 & -24 \\ 0 & 0 & 3 & 11 \end{pmatrix} \\ &= \det\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -8 & -12 \\ 0 & 0 & -10 & -24 \\ 0 & 0 & 0 & 3.8 \end{pmatrix}. \end{aligned} \tag{22.16}$$

Now, the matrix has zeroes all below the diagonal. If you think of it, only one term in the expansion of the determinant now contributes. It is the product of all of the diagonal elements. so $\det = 1 \times -1 \times -10 \times 3.8 = 38$. This is how we calculate determinants via row reduction.

The other way we can do it is via cofactors. Here we expand the cofactors running down the fourth column (in red)

$$\det\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 5 & 1 & 0 \\ 0 & 2 & 6 & 0 \\ 1 & 1 & -2 & 3 \end{pmatrix} = -4 \times \det\begin{pmatrix} 3 & 5 & 1 \\ 0 & 2 & 6 \\ 1 & 1 & -2 \end{pmatrix} + 3 \times \det\begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 1 \\ 0 & 2 & 6 \end{pmatrix}. \tag{22.17}$$

Now, we evaluate the determinant of the remaining 3×3 matrices (from the cofactors) using the definition of the determinant

$$\begin{aligned} \det &= -4(-12 + 30 + 0 - 2 - 0 - 18) + 3(30 + 0 + 18 - 0 - 36 - 2) \\ &= -4(-2) + 3(10) = 38. \end{aligned} \tag{22.18}$$

Lastly, we discuss the Hilbert matrix: $H_{ij} = \frac{1}{i+j-1}$ as an interesting matrix to determine determinants for. This set of matrices is constructed in such a way that the determinants become very mall very rapidly as the size of the matrix increases.

$$H_1 = 1, \tag{22.19}$$

$$H_2 = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}, \tag{22.20}$$

$$H_3 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \tag{22.21}$$

$$H_4 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} ... \tag{22.22}$$

Direct computation yields

$$\det H_1 = 1, \tag{22.23}$$

$$\det H_2 = 1 \cdot \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{22.24}$$

$$\begin{aligned} \det H_3 &= 1 \cdot \frac{1}{3} \cdot \frac{1}{5} \cdot + \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{4} - \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} - 1 \cdot \frac{1}{4} \cdot \frac{1}{4} - \frac{1}{5} \cdot \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{15} + \frac{1}{24} + \frac{1}{24} - \frac{1}{27} - \frac{1}{16} - \frac{1}{20}, \qquad \text{use lcd} = 16 \times 27 \times 5 = 2160 \\ &= \frac{144 + 90 + 90 - 80 - 135 - 108}{2160} = \frac{1}{2160}. \end{aligned} \tag{22.25}$$

As we can see, the determinants get small very fast. In fact, $\det H_4 = \frac{1}{6{,}048{,}000}$. We will work with these matrices in the lab.