Chapter 32

Linear differential equations

32.1 Introduction

In this chapter, we discuss the general solution of linear differential equations. These equations can be of arbitrary order (involving arbitrary order derivatives), but the function and all derivatives appear *linearly*, so they can only be multiplied by a function of t not a function of y or any of its derivatives).

Hence, the inhomogeneous nth-order linear differential equation is given by

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y(t) = f(t). \qquad (32.1)$$

There is an existence and uniqueness theorem about the solutions of these types of equations. If $a_n(t) \neq 0$ anywhere and all a's are continuous, then the solution, given a set of initial conditions, is *unique*. This is a very powerful theorem about these differential equation—it says that if we find a solution, it is *the* solution.

In general, we find that the solutions are given by linear combinations of the n linearly independent solutions of the homogeneous equation ($f = 0$) plus a particular solution of the inhomogeneous equation. This is just like what we did before for first-order linear differential equations.

32.2 Basis Set of Solutions

Define a set of n linearly independent solutions to the homogeneous equations with $\{y_i(t)|i = 1, \cdots, n\}$ such that for some t_0 we have

$$\left(y_1(t_0), y_1^{(1)}(t_0), y_1^{(2)}(t_0), \cdots, y_1^{(n-1)}(t_0)\right) = (1, 0, 0, \cdots, 0) \tag{32.2}$$

$$\left(y_2(t_0), y_2^{(1)}(t_0), y_2^{(2)}(t_0), \cdots, y_2^{(n-1)}(t_0)\right) = (0, 1, 0, \cdots, 0)$$

$$\vdots$$

$$\left(y_k(t_0), y_k^{(1)}(t_0), \cdots, y_k^{(k-1)}(t_0), \cdots, y_k^{(n-1)}(t_0)\right) = (0, 0, \cdots, \underset{k^{\text{th}}}{1}, \cdots, 0)$$

$$\vdots$$

$$\left(y_n(t_0), y_n^{(1)}(t_0), \cdots, y_n^{(n-1)}(t_0)\right) = (0, 0, 0, \cdots, 1) \tag{32.3}$$

Then, the Wronskian of these functions at t_0 is defined to be

$$W = \det \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = n, \tag{32.4}$$

so $W \neq 0$. This basis is useful for solving the homogeneous problem with $y(t_0) = c_1$, $y^{(1)}(t_0) = c_2$, ..., $y^{(n-1)}(t_0) = c_n$ because the solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + ... + c_n y_n(t), \tag{32.5}$$

as you can easily verify.

32.3 The Wronskian

Previously, when we discussed the Wronskian, we noted that if $W \neq 0$, the system of functions was linearly independent, but if $W = 0$, they might still be independent. We now show that if the functions in the set $\{y_1, \cdots, y_n\}$ solve a homogeneous differential equation, then the Wronskian cannot vanish. Here is why. Suppose $\{y_1, \cdots, y_n\}$ are linearly independent. Form $y(t) =$

$c_1y_1(t) + c_2y_2(t) + ... + c_ny_n(t)$ for some set of constants $\{c_i|i = 1, \cdots n\}$. We pick the constants such that

$$\begin{aligned} y(t_0) = c_1y_1(t_0) + c_2y_2(t_0) + \cdots + c_ny_n(t_0) &= 0 \\ y^{(1)}(t_0) = c_1y_1^{(1)}(t_0) + c_2y_2^{(1)}(t_0) + \cdots + c_ny_n^{(1)}(t_0) &= 0 \\ &\vdots \\ y^{(n-1)}(t_0) = c_1y_1^{(n-1)}(t_0) + c_2y_2^{(n-1)}(t_0) + \cdots + c_ny_n^{(n-1)}(t_0) &= 0, \end{aligned} \tag{32.6}$$

which can be done since we assumed that

$$\det \begin{pmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1^{(1)}(t_0) & y_2^{(1)}(t_0) & \cdots & y_n^{(1)}(t_0) \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{pmatrix} = 0 \tag{32.7}$$

(recall that we are trying to show that we *cannot* have the Wronskian vanish if the functions are linearly independent). This means that $y(t) = 0$ for all t, because $y(t) = 0$ solves the differential equation *and* the initial conditions. Since the solution of the differential equations is unique, this implies that $y(t) = c_1y_1(t) + c_2y_2(t) + ... + c_ny_n(t) = 0$ for all t with coefficients c_i not all zero. But that implies $\{y_1, ..., y_n\}$ are not linearly independent, which is a contradiction. So we cannot have $\{y_1, ..., y_n\}$ linearly independent *and* the Wronskian vanishing at $t = t_0$ This implies that the Wronskian won't ever vanish for a set of linearly independent functions!

32.4 Method of variation of parameters

There is no simple way to say this, so I will be blunt. The method of variation of parameters is just a tortuous method to use to find a particular solution to a differential equation. It does always work, but one should use other techniques (which are essentially good guessing) if at all possible instead of this method. But, this method will *always* work. So if you have no other choice, it is a method of last resort.

Variation of parameters is used to find particular solutions to inhomogeneous differential equations. Suppose the differential equation that we want to solve is

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y^{(1)} + a_0(t)y = f(t). \tag{32.8}$$

We seek to find a solution of the form

$$y_p(t) = A_1(t)y_1(t) + A_2(t)y_2(t) + \cdots + A_n(t)y_n(t). \tag{32.9}$$

Then, we have

$$\begin{aligned} \frac{dy_p}{dt} &= A_1(t)\dot{y}_1(t) + A_2(t)\dot{y}_2(t) + \cdots + A_n(t)\dot{y}_n(t) \\ &+ \dot{A}_1(t)y_1(t) + \dot{A}_2(t)y_2(t) + \cdots + \dot{A}_n(t)y_n(t). \end{aligned} \tag{32.10}$$

We choose to set $\dot{A}_1(t)y_1(t) + \dot{A}_2(t)y_2(t) + \cdots + \dot{A}_n(t)y_n(t) = 0$. This is a condition on the functions $A_1, \cdots, A_n$. Calculating the second derivative, we find

$$\begin{aligned} \frac{d^2y_p}{dt^2} &= A_1(t)\ddot{y}_1(t) + A_2(t)\ddot{y}_2(t) + \cdots + A_n(t)\ddot{y}_n(t) \\ &+ \dot{A}_1(t)\dot{y}_1(t) + \dot{A}_2(t)\dot{y}_2(t) + \cdots + \dot{A}_n(t)\dot{y}_n(t). \end{aligned} \tag{32.11}$$

Again, we choose to set $\dot{A}_1(t)\dot{y}_1(t) + \dot{A}_2(t)\dot{y}_2(t) + \cdots + \dot{A}_n(t)\dot{y}_n(t) = 0$.

We continue this way for the higher derivatives up to

$$\begin{aligned} \frac{d^ny_p}{dt^n} &= A_1y_1^{(n)} + A_2y_2^{(n)} + \cdots + A_ny_n^{(n)} \\ &+ \dot{A}_1y_1^{(n-1)} + \dot{A}_2y_2^{(n-1)} + \cdots + \dot{A}_ny_n^{(n-1)}. \end{aligned} \tag{32.12}$$

We also choose to set $\dot{A}_1y_1^{(n-1)} + \dot{A}_2y_2^{(n-1)} + ... + \dot{A}_ny_n^{(n-1)} = 0$.

Now, because all terms without derivatives with respect to t of A's vanish due to the fact that the y_i's solve the homogeneous equation, we find that the differential equation becomes the following set of equations (which includes the original equation and all of the "constraints" we chose earlier):

$$\dot{A}_1(t)y_1(t) + \dot{A}_2(t)y_2(t) + \cdots + \dot{A}_n(t)y_n(t) = 0 \tag{32.13}$$

$$\dot{A}_1(t)\dot{y}_1(t) + \dot{A}_2(t)\dot{y}_2(t) + \cdots + \dot{A}_n(t)\dot{y}_n(t) = 0 \tag{32.14}$$

$$\vdots$$

$$\dot{A}_1y_1^{(n-2)} + \dot{A}_2y_2^{(n-2)} + \cdots + \dot{A}_ny_n^{(n-2)} = 0 \tag{32.15}$$

$$\dot{A}_1y_1^{(n-1)} + \dot{A}_2y_2^{(n-1)} + \cdots + \dot{A}_ny_n^{(n-1)} = \frac{f(t)}{a_n(t)}. \tag{32.16}$$

This is a system of n equations in n unknowns with the determinant of the coefficient matrix being the matrix from which we compute the Wronskian of $\{y_1, \cdots, y_n\}$

$$W(t) = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ \dot{y}_1 & \dot{y}_2 & \cdots & \dot{y}_n \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}. \tag{32.17}$$

But the Wronskian is never zero, so we can always solve for $\dot{A}_1, \dot{A}_2, \cdots, \dot{A}_n$ for each t value. This implies that these n these equations can each be solved. The solution y_p is the particular solution to the differential equation.

32.5 Example

As an example, consider solving the following third-order inhomogeneous linear differential equation:

$$\dddot{y} + 3\ddot{y} + 2\dot{y} = -e^{-t} \tag{32.18}$$

The homogeneous equation is

$$\dddot{y} + 3\ddot{y} + 2\dot{y} = 0. \tag{32.19}$$

We let $y = e^{\alpha t}$ and require $(\alpha^3 + 3\alpha^2 + 2\alpha)e^{\alpha t} = 0$ in order to solve the homogeneous equation (we try this ansatz because the equation has *constant* coefficients).

Our next step is to factor the polynomial

$$\alpha(\alpha^2 + 3\alpha + 2) = \alpha(\alpha + 2)(\alpha + 1) = 0. \tag{32.20}$$

The solution are $\alpha = 0, -1, -2$, so the homogeneous equation is solved by

$$y_1 = 1, \quad y_2 = e^{-t}, \quad \text{and} \quad y_3 = e^{-2t}. \tag{32.21}$$

The Wronskian becomes

$$W(t) = \det \begin{pmatrix} 1 & e^{-t} & e^{-2t} \\ 0 & -e^{-t} & -2e^{-2t} \\ 0 & e^{-t} & 4e^{-2t} \end{pmatrix} = -4e^{-3t} + 2e^{-3t} = -2e^{-3t} \neq 0. \tag{32.22}$$

This implies that the three solutions are independent.

To solve the variation of parameters method, we must find the derivatives of the A_i functions via

$$\begin{pmatrix} 1 & e^{-t} & e^{-2t} \\ 0 & -e^{-t} & -2e^{-2t} \\ 0 & e^{-t} & 4e^{-2t} \end{pmatrix} \begin{pmatrix} \dot{A}_1 \\ \dot{A}_2 \\ \dot{A}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -e^{-t} \end{pmatrix}. \tag{32.23}$$

Solving this equation for $\dot{A}_1, \dot{A}_2$, and $\dot{A}_3$, we obtain

$$\dot{A}_3 = -\frac{1}{2}e^{t} \implies A_3 = -\frac{1}{2}e^{t}, \tag{32.24}$$

$$-e^{-t}\dot{A}_2 = -e^{t} \implies \dot{A}_2 = 1 \implies A_2 = t \tag{32.25}$$

and

$$\dot{A}_1 + e^{-t} - \frac{1}{2}e^{-t} = 0 \implies \dot{A}_1 = -\frac{1}{2}e^{-t} \implies \frac{1}{2}e^{-t}. \tag{32.26}$$

So,

$$y_p(t) = \frac{1}{2}e^{-t} \times 1 + t \times e^{-t} - \frac{1}{2}e^{t} \times e^{-2t} = te^{-t}. \tag{32.27}$$

To check this, we need the higher derivatives as well

$$\dot{y}_p = e^{-t} - te^{-t} = (1-t)e^{-t}, \tag{32.28}$$

$$\ddot{y}_p = -e^{-t} - (1-t)e^{-t} = (-2+t)e^{-t}, \tag{32.29}$$

and

$$\dddot{y}_p = e^{-t} + (2-t)e^{-t} = (3-t)e^{-t}. \tag{32.30}$$

Then, we find

$$\dddot{y}_p + 3\ddot{y}_p + 2\dot{y}_p = (3 - t - 6 + 3t + 2 - 2t)e^{-t} = -e^{-t} \tag{32.31}$$

which checks!