Chapter 25

Scalar Products and Orthonormal Bases

Last chapter, where we defined vector spaces, showed us how to define a scalar product and a norm and we even started looking at the dimensionality of a vector space. Today, we continue this theme and develop the so-called Gram-Schmidt orthogonalization procedure.

Supposed I have a set of four 4-tuples. How do I tell whether they span $\mathbb{R}^4$? (Spanning means any vector in $\mathbb{R}^4$ can be expressed as a linear combination of the given 4 vectors.)

Hence, if they span R^4, then any vector (x_1, x_2, x_3, x_4) can be written in terms of the 4 vectors. Let's say they are: $u_1 = (1,1,1,1), u_2 = (1,-1,1,-1), u_3 = (1,2,3,4), u_4 = (1,0,2,0)$ Then we want to solve $c_1\vec{u_1} + c_2\vec{u_2} + c_3\vec{u_3} + c_4\vec{u_4} = \vec{x}$ for nonzero c's. This can be written in matrix form as $Mc = x$, with M being

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 3 & 2 \\ 1 & -1 & 4 & 0 \end{pmatrix} \tag{25.1}$$

If $\det M \neq 0$, then these equations can be solved. So we compute the determinant. Note that one column has two zeros in it. So we expand by minors on the fourth column.

$$\det M = (-1) \cdot 1 \cdot \det \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \\ 1 & -1 & 4 \end{pmatrix} - (2) \cdot \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & 4 \end{pmatrix} \tag{25.2}$$

$$= -(4-3-2-2+3+4) - 2(-4+2-1+1+2-4) = -4+8 = 4 \quad (25.3)$$

Since this is nonzero, a solution exists. Hence, the vectors span R^4. But since the dimension of R^4 is 4, we need at least 4 vectors to span, so these vectors do form a basis for R^4. One can see that finding the coordinates in this basis is painful because the basis vectors are not orthogonal.

In general, having orthonormal basis vectors is convenient for doing calculations. One set for R^n is just $(1,0,0,...),(0,1,0,....),...,(0,0,1,0,...)$, the so-called Cartesian basis. But there is another basis that is easy to understand when one basis vector must be $\vec{e_1} = (1,1,1,...,1)\frac{1}{\sqrt{N}}$ then the other orthonormal vectors are $\vec{e_2} = (1,-1,0,0,...,0)\frac{1}{\sqrt{2}}$, $\vec{e_3} = (1,1,-2,0,...,0)\frac{1}{\sqrt{6}}$, ..., $\vec{e_n} = (1,1,1,...,1,-n)\frac{1}{\sqrt{n^2+n-1}}$. One can check $\vec{e_i}\cdot\vec{e_j} = 0$ directly for $i \neq j$ and $\vec{e_i}\cdot\vec{e_i} = 1$ by construction.

So is there a direct way to construct an orthonormal basis from a set of independent spanning vectors? The answer is yes and the procedure is called *Gram-Schmidt orthogonalization.*

Let $\vec{u_1}, \vec{u_2}, ..., \vec{u_n}$ = spanning, independent set. Then let $\vec{e_1} = \frac{\vec{u_1}}{\sqrt{(\vec{u_1}\cdot\vec{u_1})}}$ so $\vec{e_1}$ has a unit norm. Then let $\vec{e_2} = \frac{\vec{u_2}-\vec{e_1}(\vec{e_1}\cdot\vec{u_2})}{\text{normalization}}$ and then normalize. This projects $\vec{u_2}$ onto $\vec{e_1}$ and removes that projection from $\vec{e_2}$ so that $\vec{e_2}\cdot\vec{e_1} = 0$. Now proceed the same way for the other cases -

$$\vec{e_3} = \frac{\vec{u_3} - \vec{e_1}(\vec{e_1}\cdot\vec{u_3}) - \vec{e_2}(\vec{e_2}\cdot\vec{u_3})}{\text{normalization}} \tag{25.4}$$

In each step, we project out the parts that are on the vectors we already have chosen. Consider our earlier example:

$$\{\vec{u_i}\} = \{(1,1,1,1),(1,-1,1,-1),(1,2,3,4),(1,0,2,0)\} \tag{25.5}$$

So $\vec{e_1} = \frac{1}{2}(1,1,1,1)$, $\vec{e_2} = \frac{(1,-1,1,-1)-0\cdot\vec{e_1}}{\text{norm}} = \frac{1}{2}(1,-1,1,-1)$

$$\vec{e_3} = \frac{(1,2,3,4) - \frac{1}{2}\cdot(1,1,1,1)\cdot 5 - \frac{1}{2}(1,-1,1,-1)\cdot(-1)}{\text{norm}} = \frac{1}{2}(-1,-1,1,1) \tag{25.6}$$

$$\vec{e_4} = \frac{(1,0,2,0) - \frac{1}{2}\cdot(1,1,1,1)\cdot\frac{3}{2} - \frac{1}{2}(1,-1,1,-1)\cdot\frac{3}{2} - \frac{1}{2}(-1,-1,1,1)\cdot\frac{1}{2}}{\text{norm}} \tag{25.7}$$

$$= (-\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4}) \implies \frac{1}{2}(-1,1,1,-1) \tag{25.8}$$

So the basis is $\frac{1}{2}(1,1,1,1), \frac{1}{2}(-1,-1,1,1)\frac{1}{2}(1,-1,1,-1)$ and $\frac{1}{2}(-1,1,1,-1)$, which is an orthonormal basis.

Suppose we now construct an orthonormal basis for the set of regular functions on $[0,1]$ with scalar product $\int_0^1 f(x)g(x)dx$. Start with the first vector being a constant: $\vec{e_0} = 1$. Then the second is a linear function: $\vec{e_1} = ax+b$; the third is a quadratic: $\vec{e_2} = \alpha x^2 + \beta x + \gamma + ...$ and so on. The first vector is already normalized since $\int_0^1 |\vec{e_0}|^2 dx = 1$. Now we apply Gram-Schmidt.

$$\int \vec{e_0}\vec{e_1} dx = 0 \implies \int_0^1 (ax+b)dx = 0 \implies \left(\frac{ax^2}{2} + bx\right)\Big|_0^1 = 0 \implies \frac{a}{2}+b = 0 \tag{25.9}$$

$$\int |\vec{e_1}|^2 dx = 1 \implies \int_0^1 (ax+b)^2 dx = \int_0^1 (a^2x^2+2abx+b^2)dx = a^2\frac{1}{3}+ab+b^2 = 1 \tag{25.10}$$

So $b = \frac{-a}{2}$, and $(\frac{a^2}{3} - \frac{a^2}{2} + \frac{a^2}{4}) = 1$ so $a^2 = 12 \implies a = 2\sqrt{3}$ and $b = -\sqrt{3}$. Therefore, $\vec{e_1} = 2\sqrt{3}x - \sqrt{3}$.

For e_2, we have three integrals to compute:

$$\int \vec{e_0}\vec{e_2} dx = 0 \implies \int_0^1 (\alpha x^2 + \beta x + \gamma) = 0 \implies \tfrac{\alpha}{3} + \tfrac{\beta}{2} + \gamma = 0; \tag{25.11}$$

$$\int \vec{e_1}\vec{e_2} dx = 0 \implies \int_0^1 (2\sqrt{3}x - \sqrt{3})(\alpha x^2 + \beta x + \gamma) = 0 \implies \tfrac{\sqrt{3}}{6}\alpha + \tfrac{\sqrt{3}}{6}\beta = 0; \tag{25.12}$$

and

$$\int |\vec{e_2}|^2 dx = 1 \implies \int_0^1 (\alpha x^2+\beta x+\gamma)^2 dx = 1 \implies \tfrac{\alpha^2}{5} + \tfrac{\alpha\beta}{2} + \tfrac{2\alpha\gamma+\beta^2}{4} + \beta\gamma + \gamma^2 = 1. \tag{25.13}$$

Solving these equations yields

$$\alpha = 6\sqrt{5}, \quad \beta = -6\sqrt{5} \quad \text{and} \quad \gamma = \sqrt{5}. \tag{25.14}$$

Hence, we have $\vec{e_2} = 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}$.

One can continue this procedure indefinitely, but it is complicated and tedious; it can be completed on a computer. Such collections of polynomials are called orthogonal polynomials.