Chapter 17

The Laplacian in Cylindrical and Spherical Coordinates

17.1 Cylindrical Coordinates, the hard way

Cylindrical coordinates and their unit vectors are given in Fig. 17.1. Note that in curvilinear coordinate systems, it is common for the unit vectors to depend on position. For example, the radial unit vector is oriented along a radius at an angle θ in the $x-y$ plane. *This direction will change as the position changes*; then the unit vector changes direction as well. When we compute derivatives, *we need to take into account the change of direction of these unit vectors.* It is very common for students to have trouble understanding this aspect of unit vectors in curvilinear coordinates. Be sure that you can clearly see how this works; think about what happens to these vectors as the coordinates change.

We begin with the definitions of the different unit vectors in cylindrical coordinates. If you remember how this is done in polar coordinates, you will find it is essentially the same here. The first one is the radial unit vector, pointing in the direction of the position vector projected to the plane:

$$\hat{e}_r = \frac{\hat{i}x + \hat{j}y}{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta. \tag{17.1}$$

The final result uses the facts that $\frac{x}{r} = \cos\theta$ and $\frac{y}{r} = \sin\theta$. Next is $\hat{e}_\theta$, pointing in the tangential direction in the plane. It is easy to construct because there is only one vector perpendicular to $\hat{e}_r$ (recall the “easy” way to construct such a vector—we interchange the horizontal and vertical components,

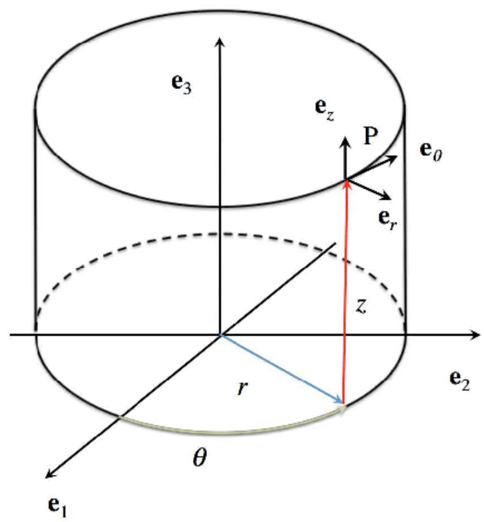

Figure 17.1: Cylindrical coordinates and unit vectors (note that sometimes ρ is used for r and ϕ for θ, but you should always be able to figure it out by the context). The cylindrical coordinates are polar coordinates for the $x-y$ plane and the Cartesian z-coordinate. The radial unit vector is in a radial direction in the $x-y$ plane, the θ unit vector is in the tangential direction to the circle in the $x-y$ plane and the z-component unit vector is along the z-axis. Note that we use the symbol $\hat{e}_\alpha$ to denote these different unit vectors (including $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ for the standard $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ Cartesian unit vectors.)

changing the sign of one of them—one only has to decide on the overall sign to use)

$$\hat{e}_\theta = \frac{-\hat{\imath}y + \hat{\jmath}x}{r} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta. \tag{17.2}$$

These two relations can be inverted as well, so we have

$$\hat{\imath} = \cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta \tag{17.3}$$

and

$$\hat{\jmath} = \sin\theta\hat{e}_r + \cos\theta\hat{e}_\theta. \tag{17.4}$$

As r changes $\hat{e}_r$ and $\hat{e}_\theta$ don't change. As z changes $\hat{e}_r$ and $\hat{e}_\theta$ don't change. But as θ changes, we have

$$\frac{\partial\hat{e}_r}{\partial\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta = \hat{e}_\theta \tag{17.5}$$

and

$$\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{\imath}\cos\theta - \hat{\jmath}\sin\theta = -\hat{e}_r. \tag{17.6}$$

Note that Cartesian unit vectors always have vanishing derivatives with respect to any parameter.

The derivative of a unit vector is always perpendicular to the unit vector. Do you know why? By definition, a unit vector satisfies $\hat{e}_\alpha \cdot \hat{e}_\alpha = 1$. So the derivative with respect to some parameter γ gives $(\partial_\gamma \hat{e}_\alpha) \cdot \hat{e}_\alpha + \hat{e}_\alpha \cdot (\partial_\gamma \hat{e}_\alpha) = 0$, because the length of the unit vector is a constant, independent of any parameter. Since the two terms are the same, we see that $(\partial_\gamma \hat{e}_\alpha) \cdot \hat{e}_\alpha = 0$, or the derivative of the unit vector is *perpendicular* to the unit vector. Note that this does not say that the derivative of the unit vector must also be a unit vector! It turned out this way here, but does not need to be true in general.

Our next task is to calculate the derivatives of the Cartesian coordinates with respect to the cylindrical coordinates. Since we have $x = r\cos\theta$, $y = r\sin\theta$, and z, we find

$$\frac{\partial x}{\partial r} = \cos\theta \tag{17.7}$$

and

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \tag{17.8}$$

for x. For y we have

$$\frac{\partial y}{\partial r} = \sin\theta \tag{17.9}$$

and

$$\frac{\partial y}{\partial \theta} = r\cos\theta. \tag{17.10}$$

Next, we carefully work out the chain rule for derivatives with respect to the three cylindrical coordinates. We obtain

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial}{\partial z}\frac{\partial z}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \tag{17.11}$$

for the r-coordinate. We obtain

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial}{\partial z}\frac{\partial z}{\partial \theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y} \tag{17.12}$$

for the θ-coordinate. We now use these results to find the gradient operator in cylindrical coordinates. We find that

$$\begin{aligned}
\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} &= \hat{\imath}\cos^2\theta\frac{\partial}{\partial x} + \cancel{\hat{\jmath}\sin\theta\cos\theta\frac{\partial}{\partial x}} + \cancel{\hat{\imath}\cos\theta\sin\theta\frac{\partial}{\partial y}} + \hat{\jmath}\sin^2\theta\frac{\partial}{\partial y} \\
&\quad + \hat{\imath}\sin^2\theta\frac{\partial}{\partial x} - \cancel{\hat{\jmath}\sin\theta\cos\theta\frac{\partial}{\partial x}} - \cancel{\hat{\imath}\sin\theta\cos\theta\frac{\partial}{\partial y}} + \hat{\jmath}\cos^2\theta\frac{\partial}{\partial y} \\
&= \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y},
\end{aligned} \tag{17.13}$$

since the cross terms cancel. So in cylindrical coordinates, we have the gradient operator is

$$\nabla = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_z\frac{\partial}{\partial z}. \tag{17.14}$$

The Laplacian is found by properly computing the dot product of the gradient with itself. Using the derivatives we computed already above for the unit vectors, we immediately find that

$$\begin{aligned}
\nabla\cdot\nabla &= \left[\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_z\frac{\partial}{\partial z}\right]\cdot\left[\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_z\frac{\partial}{\partial z}\right] \\
&= \hat{e}_r\cdot\hat{e}_r\frac{\partial^2}{\partial r^2} + \hat{e}_\theta\frac{1}{r}\cdot\left(\frac{\partial}{\partial\theta}\hat{e}_r\right)\frac{\partial}{\partial r} + \hat{e}_\theta\cdot\hat{e}_\theta\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \hat{e}_z\cdot\hat{e}_z\frac{\partial^2}{\partial x^2} \\
&= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}.
\end{aligned} \tag{17.15}$$

You should check that we only included the nonzero terms in the second line. All others vanish either due to derivatives or due to unit vectors being perpendicular. One needs to do this analysis carefully. In the last line, we compressed the derivatives in the radial direction to one term instead of two.

17.2 Cylindrical Coordinates, the easy way

The easy way involves looking at how functions of the cylindrical coordinates behave. Consider a function $f(r,\theta,z)$. We start by looking at the lowest-order change in the function due to small changes in the coordinates

$$\Delta f = \frac{\partial f}{\partial r}\Delta r + \frac{\partial f}{\partial\theta}\Delta\theta + \frac{\partial f}{\partial z}\Delta z + ... \tag{17.16}$$

Next, we define a vector corresponding to the shifts (measured in a length) along each unit vector direction corresponding to the changes in the coordinates

$$\Delta\vec{s} = \hat{e}_r\Delta r + \hat{e}_\theta r\Delta\theta + \hat{e}_z\Delta z \tag{17.17}$$

and recognize that the change in f is actually a directional derivative

$$\Delta f = \left(\hat{e}_r\frac{\partial f}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial f}{\partial\theta} + \hat{e}_z\frac{\partial f}{\partial z}\right)\cdot\Delta\vec{s} = \vec{\nabla}f\cdot\Delta\vec{s}. \tag{17.18}$$

From this, we can immediately find the gradient

$$\nabla f = \left(\hat{e}_r\frac{\partial f}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial f}{\partial\theta} + \hat{e}_z\frac{\partial f}{\partial z}\right). \tag{17.19}$$

To find the divergence in cylindrical coordinates, we either take derivatives like above, or proceed similar to pages 42-43 and problem II-20 of *Div, Grad, and Curl* to find the divergence. In either case, one has

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}, \tag{17.20}$$

where the subscripts on F denote the components in terms of the unit vectors in cylindrical coordinates. Using $\vec{F} = \vec{\nabla}f$ gives

$$\begin{aligned}\nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right) + \frac{\partial^2 f}{\partial z^2}\\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}f\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}.\end{aligned} \tag{17.21}$$

That is it for cylindrical coordinates. We now move on to spherical coordinates, which is a lot harder.

17.3 Spherical Coordinates

We begin the discussion of spherical coordinates by doing it the "fast way". Recall that we compute the change in a function f

$$\Delta f = \frac{\partial f}{\partial r}\Delta r + \frac{\partial f}{\partial\theta}\Delta\theta + \frac{\partial f}{\partial\phi}\Delta\phi \tag{17.22}$$

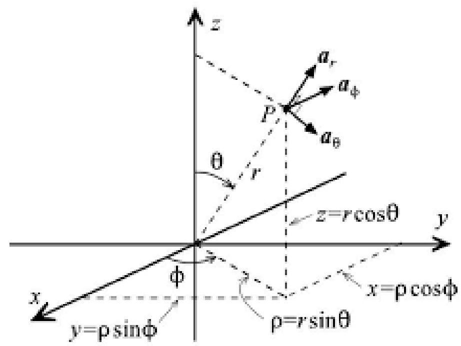

Figure 17.2: Unit vectors for spherical coordinates. Note that the physicist and mathematics conventions for θ and ϕ are reversed. It is confusing. I am sorry. Here we see the unit vectors. Just replace the label a with $\hat{e}$ for our convention.

and define a vector $\Delta\vec{s}$ that corresponds to the lengths displaced along each unit vector due to the changes in the coordinates

$$\Delta\vec{s} = \hat{e}_r \Delta r + \hat{e}_\theta r\Delta\theta + \hat{e}_\phi r\sin\theta\Delta\phi \tag{17.23}$$

and recognize that we have a directional derivative

$$\Delta f = \vec{\nabla} f\cdot\Delta\vec{s} = \left(\hat{e}_r\frac{\partial f}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial f}{\partial\theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\right)\cdot\Delta\vec{s}. \tag{17.24}$$

The gradient can then be extracted "by inspection"

$$\vec{\nabla} f = \hat{e}_r\frac{\partial f}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial f}{\partial\theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}. \tag{17.25}$$

In other words, we have

$$\vec{\nabla} = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{17.26}$$

in spherical coordinates.

We can next work out the Laplacian $\vec{\nabla}\cdot\vec{\nabla} f$, which requires us to work out derivatives of the unit vectors $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$ with respect to changes in r, θ, ϕ or we can use the formula for the divergence in spherical coordinates which can be derived as discussed in the book. We show the former here.

To begin, let's find $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$ as before. The radial unit vector is in the direction $\vec{r}/r$

$$\hat{e}_r = \frac{\hat{i}x + \hat{j}y + \hat{k}z}{r}, \tag{17.27}$$

the $\hat{e}_\theta$ vector lies in the plane formed by the z axis and $\hat{e}_r$

$$\hat{e}_\theta = \text{unit vector in } z,\ \rho \text{ plane perpendicular to } \hat{e}_r. \tag{17.28}$$

We first find the unit vector in the $x - y$ plane along the projection of $\vec{r}$, denoted $\hat{e}_\rho$

$$\hat{e}_\rho = \frac{\hat{i}x + \hat{j}y}{r \sin\theta} \tag{17.29}$$

so

$$\hat{e}_\theta = -\cos\theta \hat{e}_\rho + \sin\theta \hat{k}. \tag{17.30}$$

Simplifying these two results yields

$$\hat{e}_r = \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta \tag{17.31}$$

and

$$\hat{e}_\theta = \hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta. \tag{17.32}$$

. Our last unit vector is in the ϕ direction. We find it by simple cross product

$$\hat{e}_\phi = \hat{e}_r \times \hat{e}_\theta = -\hat{i}\sin\phi + \hat{j}\cos\phi. \tag{17.33}$$

Now, on to the derivatives. No unit vector depends on r. For the θ and ϕ, note that

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta \tag{17.34}$$

and

$$\frac{\partial \hat{e}_r}{\partial \phi} = \sin\theta \hat{e}_\phi \tag{17.35}$$

for the radial unit vector. We have

$$\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r \tag{17.36}$$

and

$$\frac{\partial \hat{e}_\theta}{\partial \phi} = \cos\theta \hat{e}_\phi \tag{17.37}$$

for the θ direction unit vector. Finally, we have

$$\frac{\partial \hat{e}_\phi}{\partial \theta} = 0 \tag{17.38}$$

and

$$\frac{\partial \hat{e}_\phi}{\partial \phi} = -\sin\theta \hat{e}_r - \cos\theta \hat{e}_\theta \tag{17.39}$$

for the ϕ direction unit vector. We now employ these results in calculating the Laplacian

$$\begin{aligned}
\nabla^2 &= \left(\hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} \right) \cdot \left(\hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} \right) \\
&= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\cos\theta}{\sin\theta}\frac{1}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2} \\
&= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial \theta} \right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}.
\end{aligned} \tag{17.40}$$

In the last line, we have compressed the notation for the derivatives.

If we use

$$\vec{\nabla} \cdot F = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi} \tag{17.41}$$

on

$$F = \vec{\nabla} f = \hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}, \tag{17.42}$$

we find

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}, \tag{17.43}$$

which also agrees.

The bottom line from all of this work is that calculating the divergence, gradient, curl and Laplacian in other coordinate systems requires patience to carry out. It is our classic "French cooking" exercise!