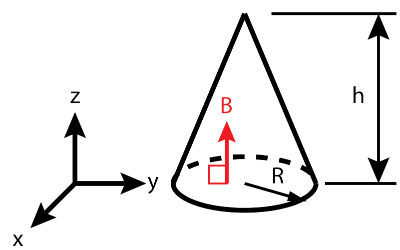
Physics 155 HW # 6

1.) a.) One of Maxwell's equations is $\vec{\nabla} \cdot \vec{B} = 0$ with $\vec{B}$ the magnetic field. Use the divergence theorem to prove that

$$\iint_S \vec{B} \cdot \hat{n}\, dS = 0$$

for any closed surface S.

b.) Determine the flux of a uniform magnetic field $\vec{B}$ through the curved surface of a right circular cone (radius R, height h) oriented so $\vec{B}$ is <u>normal</u> to the base of the cone as shown here.



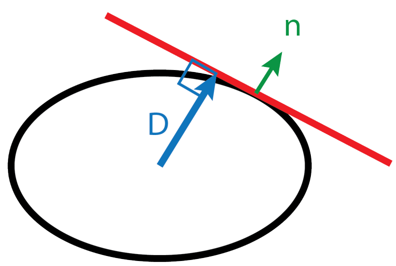
Another way of saying this is $\vec{B} = B\hat{k}$ with B a constant.

2.) Let S be the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ and let $D(x, y, z) =$ the distance from the origin to the plane tangent to S at point (x, y, z) (the tangent plane is $\perp$ to the normal $\hat{n}$).

a.) Show that if $\vec{F}(\vec{r}) = \hat{\imath}\,\dfrac{x}{a^2} + \hat{\jmath}\,\dfrac{y}{b^2} + \hat{k}\,\dfrac{z}{c^2}$, then $\vec{F} \cdot \hat{n} = D^{-1}$ with $\hat{n}$ the outward normal at (x, y, z).

b.) Show $\iint_S D^{-1}(x,y,z)\,dS = \frac{4\pi}{3}\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right)$

Hint: $\hat{n}$ Lies in the direction $\frac{x}{a^2}\,\hat{\imath}+\frac{y}{b^2}\,\hat{\jmath}+\frac{z}{c^2}\,\hat{k}$



$D(x,y,z)$ = length of vector $\vec{D}$ which is parallel to $\hat{n}$ but runs through the origin. $\vec{D}$ intersects the red tangent plane.

For part (b), use the divergence theorem and the fact that $V_{ellipsoid} = \frac{4\pi}{3}abc$

3.) We will show that if $\vec{\nabla}\cdot\vec{G}=0$, then $\vec{G}=\vec{\nabla}\times\vec{H}$ and that if $\vec{G}=\vec{\nabla}\times\vec{H}$, then $\vec{\nabla}\cdot\vec{G}=0$.

a.) Prove that if $\vec{G}=\vec{\nabla}\times\vec{H}$, then $\vec{\nabla}\cdot\vec{G}=0$ by directly evaluating the derivatives.

b.) Suppose $\vec{\nabla} \cdot \vec{G} = 0$, show that

$$\begin{aligned} H_x &= 0, \quad H_y = \int_{x_0}^{x} G_z(x', y, z)dx' \\ H_z &= -\int_{x_0}^{x} G_y(x', y, z)dx' + \int_{y_0}^{y} G_x(\underset{\substack{\uparrow \\ \text{note!}}}{x_0}, y', z)dy' \end{aligned} \tag{1}$$

satisfies $\vec{G} = \vec{\nabla} \times \vec{H}$. Note that the first argument of G_x is x_0!

4.) Determine the cases where one can write $\vec{G} = \vec{\nabla} \times \vec{H}$. If it is possible, find $\vec{H}$ (hint: use problem 3).

a.) $\vec{G} = \hat{\imath}y + \hat{\jmath}z + \hat{k}x$

b.) $\vec{G} = B_0\hat{k} \quad B_0 = \text{constant}$

c.) $\vec{G} = \hat{\imath}x^2 - \hat{\jmath}y^2$

d.) $\vec{G} = 2\hat{\imath}x - \hat{\jmath}y - \hat{k}z$

e.) $\vec{G} = 2\hat{\imath}x - \hat{\jmath}y + \hat{k}z$

5.) Show that the following vector fields have both $\vec{\nabla} \times \vec{F} = \vec{0}$ and $\vec{\nabla} \cdot \vec{F} = 0$.

a.) $(x^2 - y^2)\hat{\imath} - 2xy\hat{\jmath}$

b.) $e^x(\cos(y)\hat{\imath} - \sin(y)\hat{\jmath})$

c.) $\dfrac{1}{2}\ln(x^2 + y^2)\hat{\imath} - \tan^{-1}\left(\dfrac{y}{x}\right)\hat{\jmath} \quad$ for $x > 0$