Chapter 13

The Line Integral and the Curl

13.1 Line Integrals

We have already investigated line integrals when we looked at calculating the arc length. We found

$$S = \int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx \tag{13.1}$$

And the picture in Fig. 13.1 should remind you why this was this way.

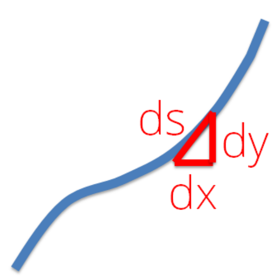

Figure 13.1: Schematic of the line integral. The blue curve is the continuous function $y(x)$ that we are seeking the arc length of. The red triangle shows how to relate the differential of the arc length to the differentials in the x and y directions.

Using the Pythagorean theorem, we immediately see that $ds^2 = dx^2 + dy^2$

or,

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{13.2}$$

when the curve is expressed in the form $y = f(x)$.

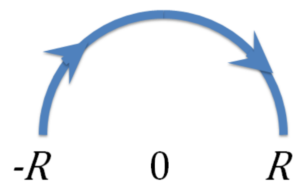

Figure 13.2: Line integral curve for the integration over a semicircle. One must separate a full circle into pieces like this, because a function is single-valued (one value $y = f(x)$ for each x); otherwise, it is not well-defined.

Let's be concrete and work out the example of the line integral over a semicircle of radius r in the clockwise direction (see Fig. 13.2). Here The y-component of the semicircle satisfies

$$y = f(x) = \sqrt{R^2 - x^2}. \tag{13.3}$$

Computing the derivative yields

$$\frac{df(x)}{dx} = -\frac{x}{\sqrt{R^2 - x^2}}. \tag{13.4}$$

Plugging into the formula for the arc length then becomes

$$S = \int_{-R}^{R} \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx = \int_{-R}^{R} \sqrt{1 + \frac{x^2}{R^2 - x^2}}\, dx \tag{13.5}$$

$$= \int_{-R}^{R} \frac{R}{\sqrt{R^2 - x^2}}\, dx. \tag{13.6}$$

This integral can be evaluated directly via a simple trigonometric substitution. Let $x = R\sin\theta$, so that $dx = R\cos\theta\, d\theta$. Then we have that the arc length becomes

$$S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \frac{R^2\cos\theta}{R\sqrt{1 - \sin^2\theta}} = R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = \pi R, \tag{13.7}$$

exactly as expected (recall that the circumference of a circle of radius R is $2\pi R$).

The wrinkle we have now for the line integrals we want to consider is that we are integrating functions *that have vector dot products* as the integrand. Consider the integral over the same curve, but now the integrand is not the scalar 1, but is the vector dot product given by $\vec{F}(x)\cdot\vec{t}$ where $\vec{F}(x) = -y\hat{\imath} + x\hat{\jmath} = -\sqrt{R^2-x^2}\,\hat{\imath} + x\hat{\jmath}$.

The unit vector $\hat{t}$ for each step dx is taken to be the tangent to the curve at the point $(x, f(x)) = (x, y)$. This tangent vector is proportional to $\hat{\imath} + \frac{df}{dx}\hat{\jmath} = \hat{\imath} - \frac{x}{\sqrt{R^2-x^2}}\hat{\jmath}$. Putting this all together gives us

$$\hat{t} = \frac{\sqrt{R^2-x^2}\,\hat{\imath} - x\hat{\jmath}}{R}. \tag{13.8}$$

So $\vec{F}(x)\cdot\hat{t} = -\sqrt{R^2-x^2}\frac{\sqrt{R^2-x^2}}{R} - \frac{x^2}{R} = \frac{x^2-R^2+x^2}{R} = -R$, which turns out to be independent of x. This won't always happen. So the inegral then becomes

$$\int \vec{F}\cdot\hat{t}\,ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx(-R)\frac{R}{R^2-x^2} = -R(\pi R) = -\pi R^2 \tag{13.9}$$

The key to calculating these integrals is to carefully work out each term and then you are left with an ordinary integral at the end (which you should be able to evaluate). Note that there are many steps. Just work through these problems slowly and remember all the different steps you need to assemble to finish the evaluation of the final result.

We consider another example. If $\vec{F}(\vec{x})$ is proportional to $\hat{r} = \frac{\hat{\imath}x+\hat{\jmath}y+\hat{k}z}{\sqrt{x^2+y^2+z^2}} = \hat{\imath}\frac{x}{r} + \hat{\jmath}\frac{y}{r} + \hat{k}\frac{z}{r}$ and is a function of r only, then $\int \vec{F}\cdot\vec{t}\,ds$ depends only on the initial and final points.

Proof:

$$\int_1^2 \vec{F}(r)\cdot\hat{t}\,ds = \int_1^2 \left[F_x(r)dx + F_y(r)dy + F_z(r)\right]dz. \tag{13.10}$$

This form follows because $ds = \sqrt{dx^2+dy^2+dz^2}$ and $\hat{t} = \hat{\imath}\frac{dx}{ds} + \hat{\jmath}\frac{dy}{ds} + \hat{k}\frac{dz}{ds}$. Thus, we find the unit vector times the differential of the arc length becomes

$$\hat{t}\,ds = \left(\hat{\imath}\frac{dx}{ds} + \hat{\jmath}\frac{dy}{ds} + \hat{k}\frac{dz}{ds}\right)ds = \hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz. \tag{13.11}$$

Taking the dot product with $\vec{F}(r)$ then gives the integral in Eq. (13.10). But, we also have that $dr = \frac{dr}{dx}dx + \frac{dr}{dy}dy + \frac{dr}{dz}dz = \frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz$, since $r = \sqrt{x^2 + y^2 + z^2}$, so our specific vector field can also be expressed as

$$\vec{F}(r) = \left(\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right) f(r) \tag{13.12}$$

according to the hypothesis of the problem. Using the result in Eq. (13.11) yields the following for the dot product

$$\vec{F}(r) \cdot \hat{t}\, ds = f(r) \left(\frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz\right) = f(r)\, dr. \tag{13.13}$$

This then means that we can re-express the integral as

$$\int_1^2 \vec{F}(\vec{r}) \cdot \hat{t}\, ds = \int_1^2 f(r)\, dr, \tag{13.14}$$

which is a function of r_1 and r_2 only (by the fundamental theorem of calculus). So the integral cares only about endpoints not the paths between them. If the endpoints are the same, we must have that the integral vanishes

$$\oint \vec{F}(\vec{r}) \cdot \hat{t}\, ds = 0. \tag{13.15}$$

So a function $\vec{F}(\vec{r})$ that can be expressed in th form $f(r)\hat{r}$ has no circulation, since $\oint \vec{F}(\vec{r}) \cdot \hat{t}\, ds = 0$. The electrostatic field is an example of a vector field with this property. It has no circulation.

13.2 Curl

The curl will be defined analogous to the way we defined the divergence. Examine the circulation over a small path and divide the circulation by the area enclosed by the path. You have a problem like this that you will have to do on the homework. Here we examine a simplification of the general definition and examine the curl in the $x - y$ plane using a rectangle for the path (see Fig. 13.3).

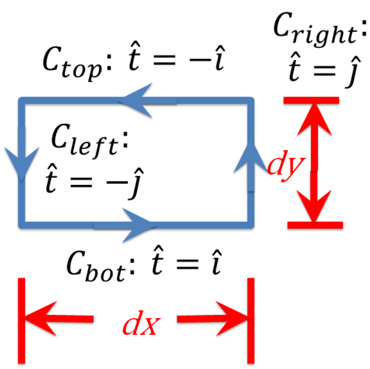

Figure 13.3: Example of how one can define the curl.

Since dx and dy are small, we immediately see that we can approximate $\oint \vec{F}(x,y,z) \cdot \hat{t}\, ds$ by

$$\oint \vec{F}(x,y,z) \cdot \hat{t}\, ds = \int_{c_{\text{top}}} \vec{F} \cdot \hat{t}\, ds + \int_{c_{\text{left}}} \vec{F} \cdot \hat{t}\, ds + \int_{c_{\text{bottom}}} \vec{F} \cdot \hat{t}\, ds + \int_{c_{\text{right}}} \vec{F} \cdot \hat{t}\, ds$$

$$= -F_x\left(x, y + \frac{\Delta y}{2}, z\right)\Delta x - F_y\left(x - \frac{\Delta x}{2}, y, z\right)\Delta y$$

$$+ F_x\left(x, y - \frac{\Delta y}{2}, z\right)\Delta x + F_y\left(x + \frac{\Delta x}{2}, y, z\right) \tag{13.16}$$

$$= \Delta x \Delta y \left[\frac{F_x\left(x, y - \frac{\Delta y}{2}, z\right) - F_x\left(x, y + \frac{\Delta y}{2}, z\right)}{\Delta y} + \right.$$

$$\left.\frac{F_y\left(x + \frac{\Delta x}{2}, y, z\right) - F_y\left(x - \frac{\Delta x}{2}, y, z\right)}{\Delta x}\right] \tag{13.17}$$

$$= \Delta x \Delta y \left(\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}\right)(x,y,z). \tag{13.18}$$

Similarly, if we did this about rectangles in the $y - z$ or $z - x$ planes, we would get

$$\Delta y \Delta z \left(\frac{\delta F_z}{\delta y} - \frac{\delta F_y}{\delta z}\right)(x,y,z) \tag{13.19}$$

and

$$\Delta z \Delta x \left(\frac{\delta F_x}{\delta z} - \frac{\delta F_z}{\delta x}\right)(x,y,z) \tag{13.20}$$

We describe the rectangles by their unit normals that we get by curling the fingers of our right hand in the direction of the path of the integral, so the thumb points in the direction of the normal to the enclosed area. So the xy rectangle is associated with $\hat{k}$, the yz rectangle with $\hat{i}$, and the xz rectangle with $\hat{j}$. These results are summarized in Fig. 13.4.

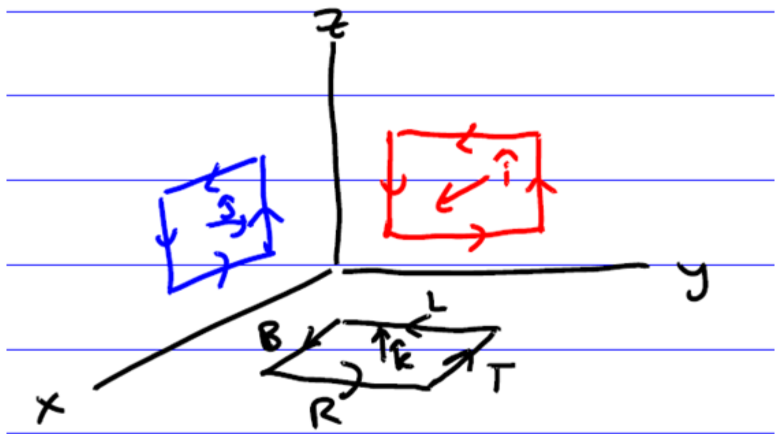

Figure 13.4: Schematic of how to determine the curl using rectangular paths in the three different Cartesian planes.

The curl of $\vec{F}$ is a vector, so we multiply our calculations by the respective normals and add. Schematically (see Fig. 13.4), we have

$$\oint \frac{\vec{F}\cdot\hat{t}\,ds}{\text{area enclosed}} \implies \hat{i}\left(\frac{\delta F_z}{\delta y}-\frac{\delta F_y}{\delta z}\right)+\hat{j}\left(\frac{\delta F_x}{\delta z}-\frac{\delta F_z}{\delta x}\right)+\hat{k}\left(\frac{\delta F_y}{\delta x}-\frac{\delta F_x}{\delta y}\right). \tag{13.21}$$

Because the area and the path both $\to 0$, we call

$$\lim_{\to 0}\oint \frac{\vec{F}\cdot\hat{t}],ds}{\text{area enclosed}} = \text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F}. \tag{13.22}$$

Since

$$\det\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z}\\ F_x & F_y & F_z\end{vmatrix} = \hat{i}\left(\frac{\delta F_z}{\delta y}-\frac{\delta F_y}{\delta z}\right)+\hat{j}\left(\frac{\delta F_x}{\delta z}-\frac{\delta F_z}{\delta x}\right)+\hat{k}\left(\frac{\delta F_y}{\delta x}-\frac{\delta F_x}{\delta y}\right), \tag{13.23}$$

which is the same as the expression in Eq. (13.21). Hence, the familiar formula for the curl, given by $\nabla \times \vec{F} = \det |...|$.