Chapter 11

Surface Integrals

11.1 How to construct the integral of a vector on a surface

We have already learned about vector-valued functions. Suppose $\vec{F}(x, y, z)$ is such a vector-valued function. While one could imagine integrating the individual components of $\vec{F}$ over the surface, in many applications, it is more appropriate to integrate a scalar over the surface. This requires us to construct a scalar at each point of the vector field. We do this by taking the dot product with the unit outward pointing normal on the surface. The surface integral over a surface S is then defined to be

$$\int_S \vec{F}(x, y, z) \cdot \hat{n}(x, y, z)\, dS \tag{11.1}$$

where $\hat{n}$ is the unit normal vector at the point (x, y, z) on the surface (pointing in an *outward* direction). This is illustrated in Fig. 11.1. We take the dot product (which yields a scalar number at each point) and sum over the surface. Note that this procedure, in principle, is quite straightforward to implement, but one has to be careful to ensure that it is done correctly.

The book by Schey describes in detail how one constructs the unit vector $\hat{n}(x, y, z)$ for a surface defined by $z = f(x, y)$. The result is

$$\hat{n}(x, y, z) = \frac{-\hat{\imath}\frac{df}{dx} - \hat{\jmath}\frac{df}{dy} + \hat{k}}{\sqrt{1 + \left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2}}. \tag{11.2}$$

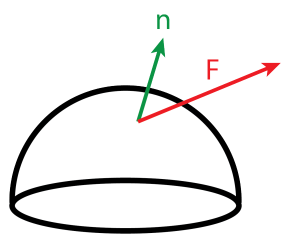

Figure 11.1: Normal vector (green) and vector field (red) on the surface of a hemisphere.

The unit vector is found by determining the vector perpendicular to the two linearly independent tangent vectors to the surface, as described in *Div, Grad, and Curl*. To perform the integration, we break the surface up into tiny surface elements, each of area ΔS_i, and compute the vector function at (x_i, y_i, z_i) in the center of each surface patch ΔS. Then we find the unit vector $\hat{n}$, take the dot product with the vector field F, multiply by ΔS_i and add up the results of all of the patches

$$\int_S \vec{F}(x,y,z)\cdot\hat{n}\,dS \sim \sum_i \Delta S_i \vec{F}(x_i,y_i,z_i)\cdot\hat{n}(x_i,y_i,z_i). \tag{11.3}$$

This is illustrated in Fig. 11.2.

Let us next evaluate the surface area of a hemisphere of radius R. The image of the hemisphere can be found in Fig. 11.1. Since the integral involves the patch area multiplied by $\hat{n}\cdot F$. If we choose $\vec{F}=\hat{n}$, then the scalar number we multiply the area patch magnitude by will be 1 and we will be calculating the surface area of the hemisphere. We will work in spherical coordinates for the hemispherical cap and polar coordinates for the circular bottom. The integral separates into two pieces as follows:

$$\int_S \vec{F}\cdot\hat{n}\,dS = \int_0^{\frac{\pi}{2}} R^2\sin\theta\,d\theta \int_0^{2\pi} d\phi\,\hat{n}\cdot\hat{n} + \int_0^R dr\int_0^{2\pi} r\,d\theta\,\hat{n}\cdot\hat{n}. \tag{11.4}$$

The area element is $R\,d\theta\,R\sin\theta\,d\phi$ for the spherical cap as worked out before, and $dr\,r\,d\theta$ for the circle. We also have $\hat{n}\cdot\hat{n}=1$ since $\hat{n}$ is a unit vector; note that in the clever way we have worked out this integral, we never need

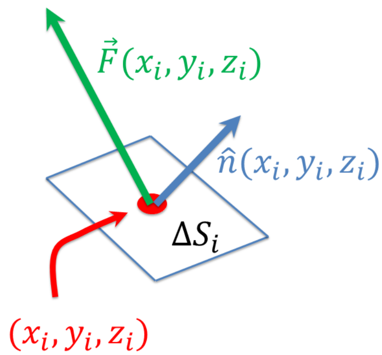

Figure 11.2: Strategy for calculating the surface integral. We first create the area patches ΔS_i. Then, using the normal vector to the patch $\hat{n}$, we take the dot product of the normal with the vector field. This is a number that we multiply by the patch area and add up to determine the total integral.

to determine $\hat{n}(x, y, z)$, because its square is always 1. So, after performing the integrals, we have

$$\int_S \vec{F} \cdot \hat{n}\, dS = R^2 \left(-\cos\theta \Big|_0^{\frac{\pi}{2}} \right) 2\pi + \frac{r^2}{2}\Big|_0^R 2\pi \tag{11.5}$$

$$= 2\pi R^2 + \pi R^2 \tag{11.6}$$

$$= 3\pi R^2. \tag{11.7}$$

Let's redo the same problem, but now with $\vec{F} = x\hat{\imath} = R\sin\theta\cos\phi$. In this case, we do need to compute the normal vector. On the spherical hemisphere part of the integral, we have

$$\hat{n} = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} \tag{11.8}$$

$$= \frac{R\sin\theta\cos\phi\hat{\imath} + R\sin\theta\sin\phi\hat{\jmath} + R\cos\theta\hat{k}}{R} \tag{11.9}$$

$$= \sin\theta\cos\phi\hat{\imath} + \sin\theta\sin\phi\hat{\jmath} + \cos\theta\hat{k} \tag{11.10}$$

and on the circular bottom, we have $\hat{n} = -\hat{k}$ (recall that the normal points outwards for these integrals). So on the hemisphere, the integral becomes

$$\vec{F} \cdot \hat{n} = R\sin^2\theta\cos^2\phi \tag{11.11}$$

and on the circle $\vec{F} \cdot \hat{n} = 0$. This yields

$$\int_S \vec{F} \cdot \hat{n} \, dS = R^2 \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} d\phi \, \sin\theta \sin^2\theta \cos^2\phi + \int_0^R dr \int_0^{2\pi} r \, d\phi \times 0 \tag{11.12}$$

$$= R^2 \int_0^1 \left(1 - \cos^2\theta\right) \, d\cos\theta \int_0^{2\pi} \cos^2\phi \, d\phi \tag{11.13}$$

$$= R^2 \left(1 - \frac{1}{3}\right) \left(2\pi \frac{1}{2}\right) \tag{11.14}$$

$$= \frac{2}{3} \pi R^2. \tag{11.15}$$

Note how we converted the integral over θ into an integral over $\cos\theta$. It usually is easier to do it this way.

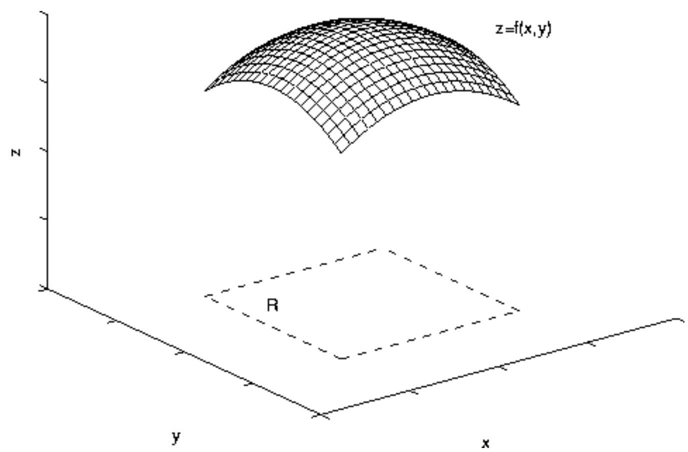

Figure 11.3: Surface integral projection onto a plane. We divide the surface into little patches on the surface and then map those little surfaces onto the plane below in order to integrate. The little patches need to have their area corrected if they lie at an angle to the plane that they are projected to below (see the next figure).

The point of these examples is that surface integrals are not frightening if you just work out all of the terms carefully and reduce them to integrals we already know how to do.

There is one other technical point we need to sort out. If we want to project the surface onto the plane, we map the dS elements onto the plane by projection.

$$dS = \frac{dx\,dy}{\cos\theta} = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} \tag{11.16}$$

See Figs. 11.3 and 11.4 to see how this works.

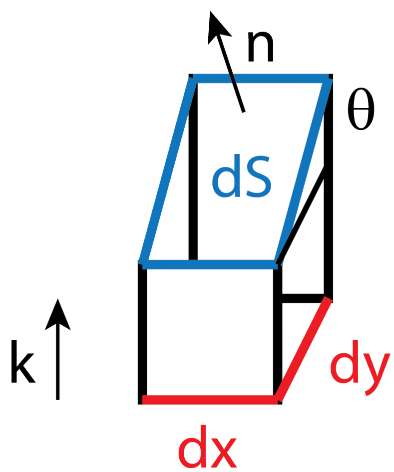

Figure 11.4: Schematic showing how the projection of the surface element dS at an angle of θ to the horizontal maps onto the element $dxdy$ in the plane. We see that $dS = dxdy/\cos\theta = dxdy/(\hat{n}\cdot\hat{k})$.

Recall for a surface $z = f(x, y)$, we had

$$\hat{n} = \frac{-\hat{i}\frac{df}{dx} - \hat{j}\frac{df}{dy} + \hat{k}}{\sqrt{1 + \left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2}}. \tag{11.17}$$

So we also have

$$\hat{n}\cdot\hat{k} = \frac{1}{\sqrt{1 + \left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2}}. \tag{11.18}$$

This then implies that (see Fig. 11.4)

$$dS = \sqrt{1 + \left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2}\,dx\,dy, \tag{11.19}$$

which is very similar to how we calculated arc lengths.

In summary, the general form of the integral is

$$\int_S \vec{F} \cdot \hat{n}\, dS = \int dx\, dy\, \vec{F}\,(x, y, f(x,y)) \cdot \hat{n}\,(x, y, f(x,y)) \sqrt{1 + \left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2} \tag{11.20}$$

The final thing we look at in this chapter is Gauss' Law, which is

$$\int_S \vec{E} \cdot \hat{n}\, dS = \frac{q_{\text{enclosed}}}{\epsilon_0} = \int dV \frac{\rho}{\epsilon_0}, \tag{11.21}$$

where ρ is the charge density. We want to examine what happens "locally." We examine a small volume and a small surface area.

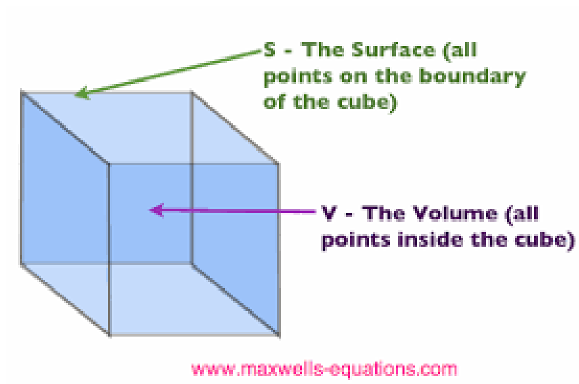

Figure 11.5: Volume and surface elements used in Gauss' law. The point (x, y, z) lies at the center of the cube.

Here, dS is the area of the prism and dV is the volume. This tells us that

$$\int_S \vec{E} \cdot \hat{n}\, dS = \int dV \frac{\rho}{\epsilon_0} = \int \frac{\rho(x,y,z)}{\epsilon_0}\, dV \tag{11.22}$$

We define

$$\frac{1}{\delta V} \int_{\delta S} \vec{E} \cdot \hat{n}\, dS = \text{"divergence"}. \tag{11.23}$$

We will see in the next chapter an alternative way to compute divergence, since the expression above is painful to try to evaluate.