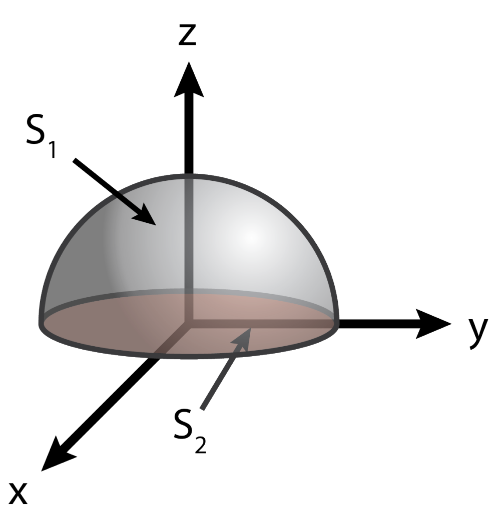
Physics 155 HW # 5

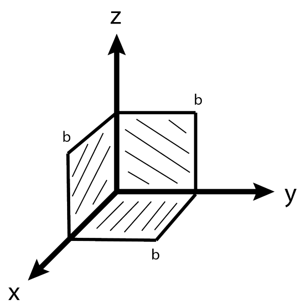
1.) An electrostatic field satisfies

$$\vec{E}(\vec{r}) = \lambda(\hat{\imath} yz + \hat{\jmath} xz + \hat{k} xy) \tag{1}$$

where λ = constant. Use Gauss' law to find the total charge enclosed by the surface shown in the figure S, the hemisphere S_1 given by $z = \sqrt{R^2 - x^2 - y^2}$ and the circle S_2 given by $x^2 + y^2 = R^2$ at $z = 0$.



2.) Some surface integrals can be evaluated much more easily by using symmetry or determining when integrands are constant. Try to use these techniques to evaluate the following surface integrals:



a.) $\vec{F} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$

$S =$ 3 squares each of side b that lie in the respective planes of the axes

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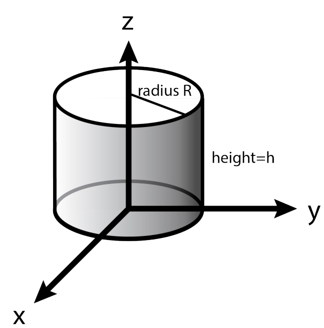
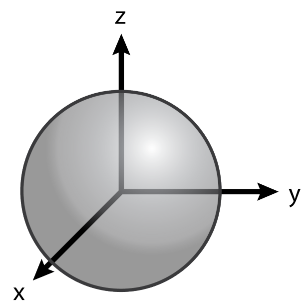


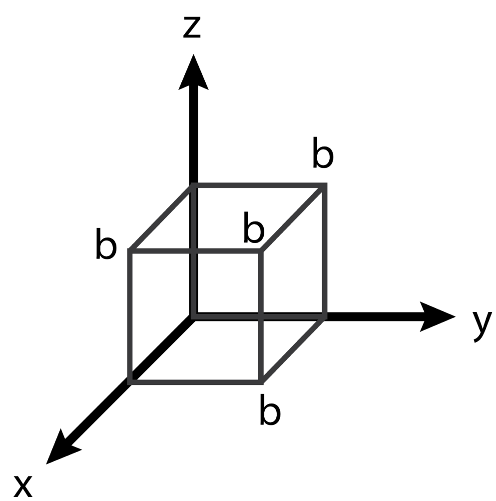
b.) $\vec{F} = (\hat{\imath}x + \hat{\jmath}y)\ln(x^2 + y^2)$

$S =$ cylinder including top and bottom of radius R and height h. The bottom sits on the xy-plane.



c.) $\vec{F} = (\hat{\imath}x + \hat{\jmath}y + \hat{k}z)e^{-(x^2+y^2+z^2)}$

$S =$ surface of a sphere of radius R centered at the origin



d.) $\vec{F} = \hat{\imath}\, E(x)$, where $E(x)$ is an arbitrary scalar function of x

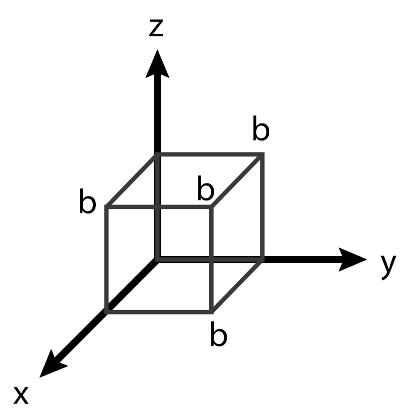
$S =$ surface of cube in the positive octant with one corner at the origin

3.) Verify the divergence theorem:

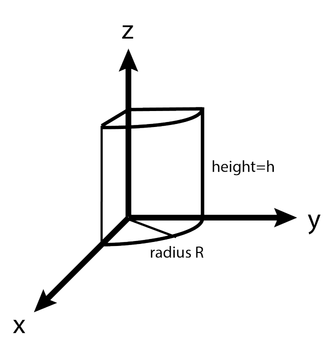
$$\int_S \vec{F} \cdot \hat{n}\, dS = \int_V \vec{\nabla} \cdot \vec{F}\, dV \tag{2}$$

For each of the following cases:

a.) $\vec{F} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$
$S =$ surface of a cube of edge b

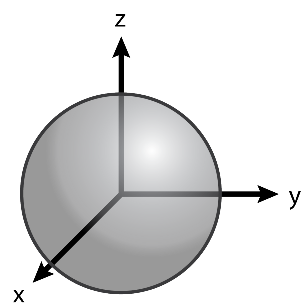


b.) $\vec{F} = \hat{e}_r r + \hat{e}_z z$ (cylindrical coordinates), $\hat{e}_r = \frac{x\hat{\imath}+y\hat{\jmath}}{\sqrt{x^2+y^2}} = \frac{x\hat{\imath}+y\hat{\jmath}}{r}$
$S =$ surface of a quarter-cylinder radius R, height h



hint: recall the divergence operator in cylindrical coordinates is different from Cartesian coordinates

c.) $\vec{F} = \hat{e}_r r^2$ (spherical coordinates), $\hat{e}_r = \frac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{r}$
$S =$ sphere of radius R centered at the origin



4.) Using the definition of the curl in Cartesian coordinates:

$$\vec{\nabla} \times \vec{F} = \hat{\imath}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \quad (3)$$

Find the curl of the following vector fields:

a.) $\hat{\imath}z^2 + \hat{\jmath}x^2 - \hat{k}y^2$

b.) $3\hat{\imath}xz - \hat{k}x^2$

c.) $\hat{\imath}e^{-y} + \hat{\jmath}e^{-z} + \hat{k}e^{-x}$

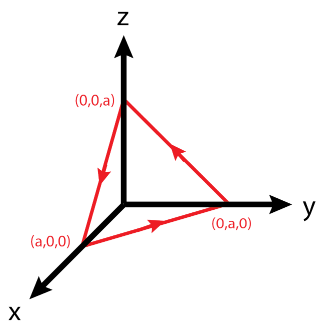
d.) $\hat{\imath}yz + \hat{\jmath}xz + \hat{k}xy$

e.) $-\hat{\imath}yz + \hat{\jmath}xz$

f.) $\hat{\imath}x + \hat{\jmath}y + \hat{k}(x^2 + y^2)$

g.) $\hat{\imath}xy + \hat{\jmath}y^2 + \hat{k}yz$

h.) $\dfrac{(\hat{\imath}x + \hat{\jmath}y + \hat{k}z)}{(x^2 + y^2 + z^2)^{3/2}} \qquad (x, y, z) \neq 0$



5.) a.)

$$\text{Calculate } \oint \vec{F} \cdot \hat{t}\, ds \text{ where } \vec{F} = \hat{k}(y + y^2) \tag{4}$$

over the perimeter of the triangle shown in the figure (integrate in the direction given by the arrows).

Hint: be sure to find the correct equation for the line on each plane to determine the tangent vector and integrate properly.

b.) Divide the result of part (a.) by the area of the triangle and take the limit $a \to 0$.

c.) Show that the result in part (b.) is

$$\hat{n} \cdot \vec{\nabla} \times \vec{F} \tag{5}$$

evaluated at (0,0,0), where $\hat{n}$ is the unit normal of the triangle which points away from the origin. Compute $\hat{n}$ carefully!