Chapter 35

The Frenet-Serret Apparatus

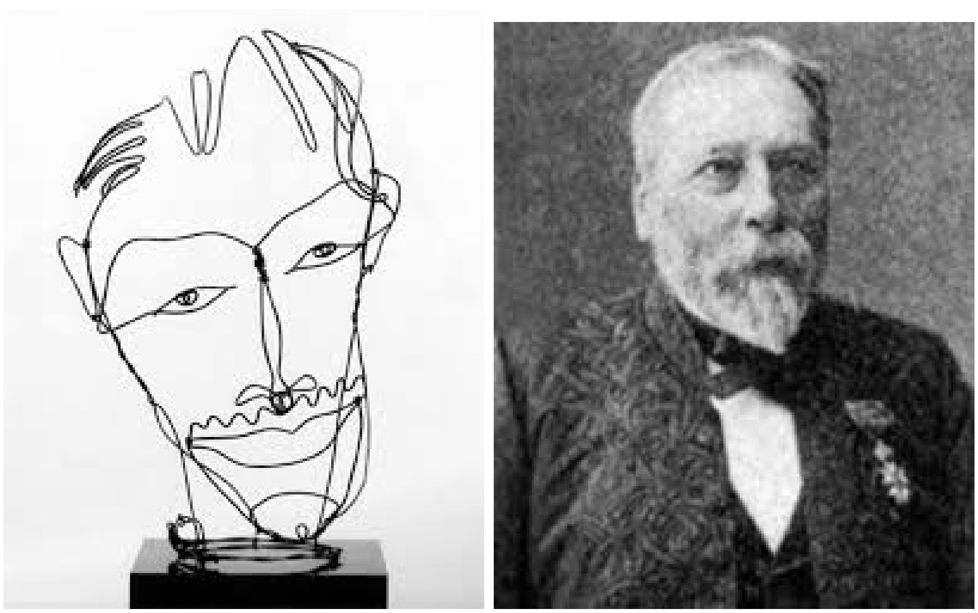

Figure 35.1: Unfortunately there is no easy to find picture of Jean Frédéric Frénet. So I substituted a wire sculpture by Alexander Calder that shows you the artwork in a three-dimensional curve. On the right is an image of Joseph Serret. These two mathematicians worked on the differential geometry of curves in the 19th century.

Differential geometry is a challenging subject. It incorporates the math behind Einstein's general relativity. You really need a full course to learn all of the details. And a lot of hard work! But there is one part of differential geometry that can be taught in just one lecture. It is the theory behind how we describe one-dimensional curves that lie in two or three dimensions. This includes the smooth curves you can draw on a piece of paper and those you can make by bending wire in three dimensions. Let's get started.

We discuss the differential geometry of 2D and 3D curves. Consider a curve in 3D (2D can be found by restricting to $\alpha_3(t) = 0$):

$$\vec{\alpha}(t) = \alpha_1(t)\hat{\imath} + \alpha_2(t)\hat{\jmath} + \alpha_3(t)\hat{k} = \Big(\alpha_1(t), \alpha_2(t), \alpha_3(t)\Big). \tag{35.1}$$

You should already know that $\frac{d\vec{\alpha}}{dt}$ is the velocity of the particle moving on the curve and $|\frac{d\vec{\alpha}}{dt}|$ is the speed. The velocity points in the direction of the unit tangent vector $\vec{T} = \frac{d\vec{\alpha}}{dt}/|\frac{d\vec{\alpha}}{dt}|$.

We start with a nontrivial curve to see how this approach works. Consider the helix $\alpha(t) = (r\cos t, r\sin t, ht)$. The velocity and tangent vector are

$$\frac{d\vec{\alpha}}{dt}(t_0) = (-r\sin t_0, r\cos t_0, h) \tag{35.2}$$

and

$$\vec{T} = (-r\sin t_0, r\cos t_0, h)\frac{1}{\sqrt{r^2+h^2}} \tag{35.3}$$

at time $t = t_0$.

Having to divide by the speed to determine the tangent vector can be cumbersome. Hence, the easiest curves to deal with are unit speed curves parametrized by the arc length s, which is given by the familiar formula from calculus

$$s(t) = \int_0^t dt\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}. \tag{35.4}$$

Note that we can think of this as integrating the *speed* of the curve as a function of time, which obviously leads to *how far* we moved along the curve.

Let's see an example for how to compute the arc length as a function of time. We have that the speed on our helix example is $\sqrt{r^2+h^2}$, so the arc length $s(t)$ satisfies $\int_0^t dt\sqrt{r^2+h^2} = t\sqrt{r^2+h^2}$. We next invert this relationship to find the *time* as a function of *arc length*. This is simple here and yields $t(s) = \frac{s}{\sqrt{r^2+h^2}}$. This means we can write

$$\vec{\alpha}(s) = \left(r\cos\frac{s}{\sqrt{r^2+h^2}}, r\sin\frac{s}{\sqrt{r^2+h^2}}, h\frac{s}{\sqrt{r^2+h^2}}\right), \tag{35.5}$$

by replacing t with s. When a curve is parametrized by its arc length, instead of some arbitrary time, it becomes a unit-speed curve and the velocity is automatically a unit vector, which is equal to the tangent vector and satisfies

$$\vec{T}(s) = \frac{d\vec{\alpha}}{ds} = \left(-r\sin\frac{s}{\sqrt{r^2+h^2}}, r\cos\frac{s}{\sqrt{r^2+h^2}}, h\right)\frac{1}{\sqrt{r^2+h^2}}. \qquad (35.6)$$

You should check to confirm that this tangent vector is indeed a unit vector.

In general, the tangent vector tells us precisely where the particle is going in the next instant. To determine the curve for longer times, we need to know how the tangent vector changes with time. For a straight line $\vec{T}$ does not change but for something like a circle it does.

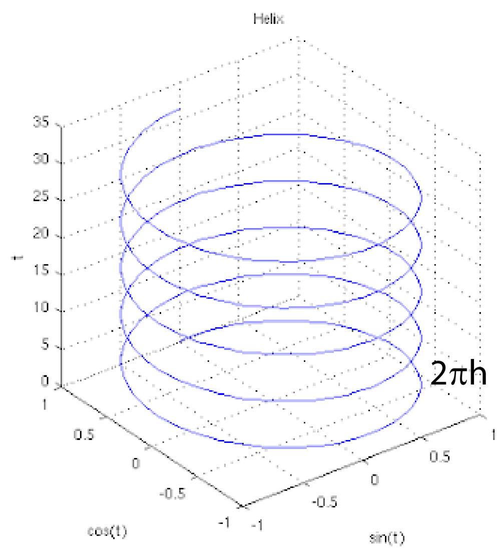

Figure 35.2: Schematic of a helix with radius r. As the curve winds around by an angle of 2π, the curve rises by a distance of $2\pi h$.

In general, when the path curves $\vec{T}$ changes with time. Since $\vec{T}$ is a unit vector, $\frac{d\vec{T}}{ds}$ is perpendicular to $\vec{T}$ (be sure you know why). We define

$$\frac{d\vec{T}}{ds} = \kappa(s)\vec{N} \qquad (35.7)$$

where $\kappa(s)$ is the curvature and $\vec{N}$ is the principal normal vector for the curve. Note that the derivative of a unit vector must be perpendicular to the unit vector, but it can have an arbitrary length. This is why we need to introduce $\kappa(s)$, which is given by the length of the derivative of the tangent vector.

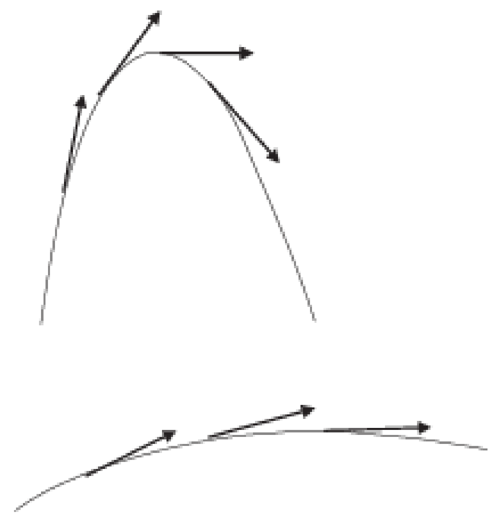

Figure 35.3: Illustration of different curvatures. In the top curve, the tangent vector rotates rapidly as we move along the curve. This is a curve with a large curvature. In the bottom curve, the tangent vector changes its direction more slowly. This has a small curvature and is closer to a straight line (which has no curvature).

Lets explore why we call κ a curvature. Consider the path of radius r that traces out a circle on the $x - y$ plane. It's unit speed parametrization ia given by

$$\vec{\alpha}(s) = \left(r \cos \frac{s}{r}, r \sin \frac{s}{r}, 0\right), \tag{35.8}$$

with

$$\vec{T}(s) = \frac{d\vec{\alpha}(s)}{ds} = \left(-\sin \frac{s}{r}, \cos \frac{s}{r}, 0\right); \tag{35.9}$$

one can immediately verify that this is a unit speed curve because $|\vec{T}| = 1$. Now, we find the principal normal vector and the curvature via a second derivative:

$$\frac{d\vec{T}}{ds} = \frac{1}{r}\left(-\cos \frac{s}{r}, -\sin \frac{s}{r}, 0\right). \tag{35.10}$$

By computing the length of the derivative, we find that $\kappa(s) = \frac{1}{r}$, which is the conventional curvature for a circle; we also find that $\vec{N}(s) = (-\cos\frac{s}{r}, \sin\frac{s}{r}, 0)$. This all makes sense because a circle with a small radius curves much more than a circle with a large radius.

For our helix, we find

$$\frac{d\vec{T}}{ds} = \left(-r\cos\frac{s}{\sqrt{r^2+h^2}}, -r\sin\frac{s}{\sqrt{r^2+h^2}}, 0\right)\frac{1}{r^2+h^2} = \kappa(s)\vec{N}. \quad (35.11)$$

Hence, we learn that

$$\vec{N}(s) = \left(-\cos\frac{s}{\sqrt{r^2+h^2}}, -\sin\frac{s}{\sqrt{r^2+h^2}}, 0\right) \quad (35.12)$$

and

$$\kappa(s) = \frac{r}{r^2+h^2}. \quad (35.13)$$

Note that the curvature of the helix is a constant, independent of s, but the helix is not a circle. In other words, not all curves with constant curvature are circles.

We have found two perpendicular vectors, $\vec{T}$ and $\vec{N}$, related to the motion of the particle on the curve. One more and we will have an orthonormal basis for the three-dimensional space. Obviously, we obtain the third vector via a cross product. So we define the binormal vector $\vec{B}(s)$ to satisfy $\vec{B} = \vec{T}\times\vec{N}$. Now for every s, we have $\vec{T}(s)$, $\vec{N}(s)$, and $\vec{B}(s)$ forming an orthonormal basis for the three-dimensional space. As s changes, so do $\vec{T}(s)$, $\vec{N}(s)$, and $\vec{B}(s)$; but the three vectors always remain orthonormal and point in the directions of an *instantaneous* coordinate system determined by the curve. We already found that $\frac{d\vec{T}}{ds} = \kappa(s)\vec{N}$. When we take the derivative of the principal normal vector, all we know is that the derivative is perpendicular to $\vec{N}$. This means it can have components along $\vec{T}$ and $\vec{B}$. But the component along $\vec{T}$ is fixed already by the curvature. This is because $\vec{T}(s)\cdot\vec{N}(s) = 0$. If we differentiate that expression, we find

$$\frac{d\vec{T}}{ds}\cdot\vec{N} + \vec{T}\cdot\frac{d\vec{N}}{ds} = 0. \quad (35.14)$$

This means that

$$\vec{T}\cdot\frac{d\vec{N}}{ds} = -\frac{d\vec{T}}{ds}\cdot\vec{N} = -\kappa(s)\vec{N}\cdot\vec{N} = -\kappa(s), \quad (35.15)$$

because the principal normal vector is a unit vector. Hence, the component along $\vec{T}$ is $-\kappa(s)$. We define the component of the derivative of the principal normal along $\vec{B}$ to be the torsion. It satisfies

$$\frac{d\vec{N}}{ds} \cdot \vec{B} = \tau(s). \tag{35.16}$$

Then, noting that $\vec{N} \cdot \vec{B} = 0$, we find that

$$\frac{d\vec{B}}{ds} = -\tau(s)\vec{N}. \tag{35.17}$$

Be sure you understand why the derivative of the binormal vector has no component along $\vec{T}$, which comes from the definition of the principal normal.

For the helix, we have

$$\begin{aligned}
\vec{B}(s) &= \vec{T}(s) \times \vec{N}(s) \\
&= \left(-r\sin\frac{s}{\sqrt{r^2+h^2}}, r\cos\frac{s}{\sqrt{r^2+h^2}}, h\right)\frac{1}{\sqrt{r^2+h^2}} \\
&\qquad \times \left(-\cos\frac{s}{\sqrt{r^2+h^2}}, -\sin\frac{s}{\sqrt{r^2+h^2}}, 0\right) \\
&= \det\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -r\frac{1}{\sqrt{r^2+h^2}}\sin\frac{s}{\sqrt{r^2+h^2}} & r\frac{1}{\sqrt{r^2+h^2}}\cos\frac{s}{\sqrt{r^2+h^2}} & h\frac{1}{\sqrt{r^2+h^2}} \\ -\cos\frac{s}{\sqrt{r^2+h^2}} & -\sin\frac{s}{\sqrt{r^2+h^2}} & 0 \end{vmatrix} \\
&= \hat{\imath}\frac{h}{\sqrt{r^2+h^2}}\sin\frac{s}{\sqrt{r^2+h^2}} + \hat{\jmath}\left(-\frac{h}{\sqrt{r^2+h^2}}\cos\frac{s}{\sqrt{r^2+h^2}}\right) \\
&\qquad + \hat{k}\left(\frac{r}{\sqrt{r^2+h^2}}\sin^2\frac{s}{\sqrt{r^2+h^2}} + \frac{r}{\sqrt{r^2+h^2}}\cos^2\frac{s}{\sqrt{r^2+h^2}}\right).
\end{aligned} \tag{35.18}$$

So, we find that

$$\vec{B}(s) = \frac{1}{\sqrt{r^2+h^2}}\left(h\sin\frac{s}{\sqrt{r^2+h^2}}, -h\cos\frac{s}{\sqrt{r^2+h^2}}, r\right). \tag{35.19}$$

To compute the torsion $\tau(s)$, we now compute the derivative of the binormal vector. Note that this is an easier way to obtain the torsion, because the derivative of the binormal lies along the principal normal vector and the length of the derivative immediately yields the torsion. Computing, we find that

$$\frac{d\vec{B}}{ds} = \frac{1}{\sqrt{r^2+h^2}}\left(\frac{h}{\sqrt{r^2+h^2}}\cos\frac{s}{\sqrt{r^2+h^2}}, \frac{h}{\sqrt{r^2+h^2}}\sin\frac{s}{\sqrt{r^2+h^2}}, 0\right). \tag{35.20}$$

We can either compute the length of this vector, or, in this case, we find it is easier to compute via the dot product with $\vec{N}$:

$$-\frac{d\vec{B}}{ds}\cdot\vec{N} = \tau(s) = -\frac{1}{r^2+h^2}\left(-h\cos^2\frac{s}{\sqrt{r^2+h^2}} - h\sin\frac{s}{\sqrt{r^2+h^2}}\right). \tag{35.21}$$

Hence, we find that

$$\tau(s) = \frac{h}{r^2+h^2}. \tag{35.22}$$

These five results, $\kappa(s)$, $\tau(s)$, $\vec{T}(s)$, $\vec{N}(s)$, and $\vec{B}(s)$ are called the Frenet-Serret apparatus. They can be painful to compute (lots of derivatives and lots of dot or cross products). But they provide the full description of a curve moving in a three dimensional space (in other words, no higher order derivatives are needed).

Furthermore, all the derivatives of these vectors are determined by κ and τ. We have already established this, but we show how these results follow directly with a compact notation given by

$$\frac{d}{ds}\begin{pmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{pmatrix}. \tag{35.23}$$

Proof: We know by the definition of κ and $\vec{N}$, that $\frac{d\vec{T}}{ds} = \kappa\vec{N}$, so the first row is true. In addition, $\frac{d\vec{N}}{ds}$ is found via

$$0 = \vec{T}\cdot\vec{N} \implies \frac{d\vec{T}}{ds}\cdot\vec{N} + \vec{T}\cdot\frac{d\vec{N}}{ds} = 0. \tag{35.24}$$

Then we have that

$$\kappa\vec{N}\cdot\vec{N} + \vec{T}\cdot\frac{d\vec{N}}{ds} = 0, \tag{35.25}$$

from the definition of the principal normal vector and the curvature, Hence,

$$\vec{T}\cdot\frac{d\vec{N}}{ds} = -\kappa \tag{35.26}$$

since $\vec{N} \cdot \vec{N} = 1$. Similarly,

$$\vec{N} \cdot \vec{B} = 0 \implies \frac{d\vec{N}}{ds} \cdot \vec{B} + \vec{N} \cdot \frac{d\vec{B}}{ds} = 0 \implies \frac{d\vec{N}}{ds} \cdot \vec{B} = -\vec{N} \cdot \frac{d\vec{B}}{ds} = \tau \quad (35.27)$$

and $\vec{N} \cdot \vec{N} = 1 \implies \frac{d\vec{N}}{ds} \cdot \vec{N} = 0$, so the second row is true. For the third row, we know that $\frac{d\vec{B}}{ds} \cdot \vec{N} = -\tau$. Furthermore,

$$\vec{B} \cdot \vec{T} = 0 \implies \frac{d\vec{B}}{ds} \cdot \vec{T} + \vec{B} \cdot \frac{d\vec{T}}{ds} = 0 \implies \frac{d\vec{B}}{ds} \cdot \vec{T} = -\kappa \vec{B} \cdot \vec{N} = 0. \quad (35.28)$$

Finally, since a unit vector satisfies $\frac{d\vec{B}}{ds} \cdot \vec{B} = 0$, we have that the third row is true.

The equations above are the Frenet-Serret equations. We have some final definitions for jargon lovers: The *osculating plane* to α at s is the plane through $\alpha(s)$ that is perpendicular to $\vec{B}(s)$. The *normal plane* is the plane perpendicular to $\vec{T}$. The *rectifying plane* is the plane perpendicular to $\vec{N}$.

If $\alpha(s)$ is a planar curve, then $\vec{B}$ = constant and $\tau = 0$. The tangent vector shows the line the curve instantaneously moves on. The osculating plane is the plane the curve instantaneously moves on. The torsion describes how that plane rotates.

A helix can be defined to be a curve for which a unit vector $\vec{u}$ exists that satisfies $\vec{T} \cdot \vec{u}$ = constant . For our helix, $\vec{u} = (0, 0, 1)$. The book proves that if α is a helix, then $\tau = c\kappa$ where c is some constant. For our example above, we have that $c = h/r$. This result is called Lancret's Theorem.

I hope you enjoyed this short primer on differential geometry.