Physics 155 HW # 12

1.) Solve the following linear first-order differential equations.

a.) $\dfrac{dy}{dt}+y = \dfrac{1}{1+t^2}$ $\quad y(0) = 0$ $\quad$ *This answer can be left as an integral*

b.) $\dfrac{dy}{dt} + \dfrac{2}{t}y = \dfrac{\cos(t)}{t^2}$ $\quad y(\pi) = 0$ $\quad t > 0$

c.) $t\dfrac{dy}{dt} + 2y = \sin(t)$ $\quad y(\frac{\pi}{2}) = 1$

2.) Consider the Bernoulli equation

$$\frac{dy}{dt} + P(t)y = q(t)y^n \tag{1}$$

a.) Solve the equation directly when $n = 0$ and $n = 1$.

b.) Use the substitution $w = y^{1-n}$ to reduce to a linear equation.

c.) Solve $t^2\dfrac{dy}{dt} + 2ty - y^3 = 0$

d.) Solve $\dfrac{dy}{dt} = \epsilon y - \sigma y^2$ for $\epsilon > 0$ and $\sigma > 0$

3.) In general, integrating factors are hard to find. But if we have an equation

$$M(t, y)dt + N(t, y)dy = 0 \tag{2}$$

such that $\dfrac{dM}{dy} \neq \dfrac{dN}{dt}$, then we can introduce a Q such that

$$\frac{d(QM)}{dy} = \frac{d(QN)}{dt} \text{ or } Q\frac{\partial M}{\partial y} - Q\frac{\partial N}{\partial t} = N\frac{\partial Q}{\partial t} - M\frac{\partial Q}{\partial y} \tag{3}$$

If $Q = Q(t)$ only (no y-dependence), then

$$\frac{1}{Q}\frac{dQ}{dt} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) \tag{4}$$

or

$$Q = \exp\left[\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) dt\right] \tag{5}$$

For this to work, $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial t}\right)$ must be independent of y.

Similarly, if Q has no t-dependence, then $Q = \exp\left[\int \dfrac{1}{M}\left(\dfrac{\partial N}{\partial t} - \dfrac{\partial M}{\partial y}\right) dy\right]$ is the integrating factor.

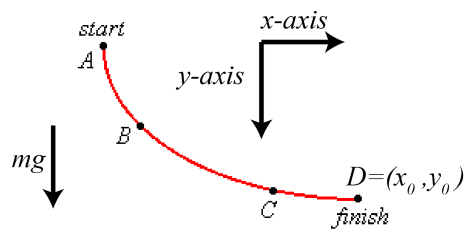
Use this information to solve the following (by finding the right integrating factor):

a.) $ty\,dt + (t^2 + te^y)dy = 0$

b.) $3t^2y\,dt + (y^4 - t^3)dy = 0$

c.) $(y^2 - 2t^2y)dt + (2t^3 - ty)dy = 0$

4.) Brachistochrone problem - Find the curve for which a particle slides without friction from start (A) to finish (D) in the minimum time (under the influence of gravity). Choose the y-axis to point *downward.*



It can be shown (you don't need to) that the minimum curve satisfies $\left[1+\left(\frac{dy}{dx}\right)^2\right] y = k^2$ with $k^2 > 0$ a positive constant.

a.) Solve this equation for $\frac{dy}{dx}$

b.) Introduce a new variable $y = k^2 \sin^2(t)$ and show that the equation becomes $2k^2 \sin^2(t)\, dt = dx$

c.) Let $\theta = 2t$ and show the solution that has $x = 0$ when $y = 0$ satisfies

$$x = \frac{k^2}{2}(\theta - \sin\theta) \tag{6}$$

$$y = \frac{k^2}{2}(1 - \cos\theta) \tag{7}$$

These equations are the parametric equations of a cycloid. We need to adjust k to go through (x_0, y_0), but you are not asked to do that.

5.) Show that $y(t) = t$ and $y(t) = \frac{1}{t}$ both solve the equation $t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = 0$. Use these results plus variation of parameters to solve $t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = t$ for $0 < a \leq t \leq b$.