Chapter 33

Linear Differential Equations with Constant Coefficients

33.1 Description of the basic method

Suppose we have an nth-order linear differential equation with constant coefficients

$$y^{(n)} + p_1 y^{(n-1)} + p_2 y^{(n-2)} + \cdots + p_{n-1} y^{(1)} + p_n y = f(t) \tag{33.1}$$

and its associated homogeneous equation

$$y^{(n)} + p_1 y^{(n-1)} + p_2 y^{(n-2)} + \cdots + p_{n-1} y^{(1)} + p_n y = 0. \tag{33.2}$$

Here $p_1, p_2, \cdots, p_n$ are constants. Recall we use the superscript (n) to denote an n-fold derivative. We will use D to denote $\frac{d}{dt}$ and then the differential equation becomes

$$(D^n + p_1 D^{n-1} + p_2 D^{n-2} + \cdots + p_{n-1} D + p_n) y = f, \tag{33.3}$$

which is a polynomial in D with real coefficients "acting" on the function y. The polynomial in D is called an *operator* and it *acts* on a function. Let

$$P(D) = D^n + p_1 D^{n-1} + p_2 D^{n-2} + ... + +p_{n-1} D + p_n. \tag{33.4}$$

We factorize it into products of monomials as follows:

$$P(D) = (D - r_1)^{k_1} (D - r_2)^{k_2} ... (D - r_m)^{k_m}, \tag{33.5}$$

where we have m distinct roots $\{r_i,\ i = 1, \cdots m\}$ and $k_1 + k_2 + ... + k_m = n$. The roots r_i are in general complex, but because the coefficients p_i are real, $P(D) = P^*(D)$, so the roots must come in *complex conjugate pairs* with $k_i = k_j$ if $r_i = r_j^*$. The corresponding homogeneous equation is

$$P(D)y = 0, \tag{33.6}$$

which can be solved by forming any linear combination of all n of the linearly independent solutions to the homogeneous equation (the inhomogeneous equation can be solved by the method of variation of parameters from Chapter 32). So if we find a y that satisfies

$$(D - r_i)^{k_i} y = 0, \tag{33.7}$$

then we must also have $P(D)y = 0$, because all derivative terms in the product of $P(D)$ commute with each other. This is due to the fact that the p_i (and r_i) are numbers and the derivative of a number is always zero: $Dp_i = Dr_i = 0$. (Be sure you understand this point, which is both subtle and important.)

Starting with the case where $k_i = 1$, we claim

$$(D - r_i)y = e^{r_i t} D(e^{-r_i t} y). \tag{33.8}$$

We check via the chain rule:

$$D(e^{-r_i t} y) = -r_i e^{-r_i t} y + e^{-r_i t} Dy \tag{33.9}$$

so

$$e^{r_i t} D(e^{-r_i t} y) = (D - r_i)y \tag{33.10}$$

which checks. Hence, we have

$$\begin{aligned}
(D - r_i)^{k_i} y &= (D - r_i)^{k_i - 1} \left[e^{r_i t} D(e^{-r_i t} y)\right] && (33.11)\\
&= (D - r_i)^{k_i - 2} [e^{r_i t} D \left(e^{-r_i t} e^{r_i t} D(e^{-r_i t} y)\right] && (33.12)\\
&= (D - r_i)^{k_i - 2} e^{r_i t} D^2 (e^{-r_i t} y). && (33.13)
\end{aligned}$$

So we find the general identity

$$(D - r_i)^{k_i} y = e^{r_i t} D^{k_i} (e^{-r_i t} y). \tag{33.14}$$

This can be immediately integrated to find that

$$(D - r_i)^{k_i} y = 0 \implies e^{-r_i t} y = c_1 + c_2 t + c_3 t^2 + \cdots + c_{k_i} t^{k_i - 1} \tag{33.15}$$

or that

$$y(t) = c_1 e^{r_i t} + c_2 t e^{r_i t} + \cdots + c_{k_i} t^{k_i - 1} e^{r_i t} \tag{33.16}$$

is the general solution of $(D - r_i)^{k_i} y = 0$. Putting this together, the solution to $P(D)y = 0$ is a linear combination of the following functions:

$$\begin{aligned}
&r_1 : e^{r_1 t},\, t e^{r_1 t},\, t^2 e^{r_1 t}, \cdots, t^{k_1 - 1} e^{r_1 t} \\
&r_2 : e^{r_2 t},\, t e^{r_2 t},\, t^2 e^{r_2 t}, \cdots, t^{k_2 - 1} e^{r_2 t} \\
&\cdots \\
&r_m : e^{r_m t},\, t e^{r_m t},\, t^2 e^{r_m t}, \cdots, t^{k_m - 1} e^{r_m t}
\end{aligned}$$

where one can show, by calculating the Wronskian, that these functions are all independent. The full solution then becomes a linear combination of these functions plus any particular solution.

If all r_i are real, the above functions are also real. If any r_i are complex, they occur in complex conjugate pairs and when $r_i = r_j^*$ we also have $k_i = k_j$. In this case, changing the functions to

$$\frac{1}{2} t^\alpha (e^{r_i t} + e^{r_j t}) \tag{33.17}$$

and

$$\frac{1}{2i} t^\alpha (e^{r_i t} - e^{r_j t}) \tag{33.18}$$

and using $r_i = a_i + i b_i$ and $r_j = a_i - i b_i$ with a_i, b_i real, then gives the functions

$$t^\alpha e^{a_i t} \cos b_i t \tag{33.19}$$

and

$$t^\alpha e^{a_i t} \sin b_i t \tag{33.20}$$

as the *real function basis* to use for solving these problems. Note that here, we have $0 \le \alpha \le k_i - 1$.

33.2 Examples

One really needs to work through some examples before you can feel comfortable with being able to solve problems using this methodology. Here is our first example. Consider the quartic differential operator

$$D^4 - 4D^3 + 4D^2 = P(D). \tag{33.21}$$

Solve $P(D)y = 0$. We start by factorizing $P(D)$:

$$P(D) = D^2(D^2 - 4D + 4) = D^2(D-2)^2 \tag{33.22}$$

so functions that solve the homogeneous equation are $1, t, e^{2t}$,and te^{2t}. First check that these functions are indeed independent. The Wronskian becomes

$$\det W = \det \begin{pmatrix} 1 & t & e^{2t} & te^{2t} \\ 0 & 1 & 2e^{2t} & (1+2t)e^{2t} \\ 0 & 0 & 4e^{2t} & (4+4t)e^{2t} \\ 0 & 0 & 8e^{2t} & (12+8t)e^{2t} \end{pmatrix}. \tag{33.23}$$

We find the determinant by row reduction. Subtract twice the third row from the fourth and then take the product of the diagonal elements:

$$\det W = \det \begin{pmatrix} 1 & t & e^{2t} & te^{2t} \\ 0 & 1 & 2e^{2t} & (1+2t)e^{2t} \\ 0 & 0 & 4e^{2t} & (4+4t)e^{2t} \\ 0 & 0 & 0 & 4e^{2t} \end{pmatrix} = 16e^{4t} \neq 0. \tag{33.24}$$

So the functions are linearly independent.

Second example: Start from the differential operator

$$P(D) = D^5 + D^4 - D^3 - 3D^2 + 2. \tag{33.25}$$

We know we need to factorize this. Let us use a theorem from algebra—a polynomial with integer coefficients can only have rational roots of the form $r = \frac{p}{q}$, where p and q are integers. We must have p divides the constant term in the polynomial and q divides the coefficient of the highest power. So in this case $p = \pm 1, \ \pm 2$ (constant coefficient is 2), $q = \pm 1$ (highest-power coefficient is 1). Then our possible real roots of the polynomial are $\frac{p}{q} = 1, -1, 2$, and -2. The only way to see if these are roots is to substitute in for D and see if the polynomial vanishes.

Substituting into the polynomial shows that the only roots are $r = 1$ and $r = -1$ (which means there are multiple roots or complex roots). In any case, we know that $(D+1)(D-1) = D^2 - 1$ divides $P(D)$. Performing the division yields

$$P(D) = (D^2 - 1)(D^3 + D^2 - 2) \tag{33.26}$$

Repeating the p and q method on the cubic factor of $P(D)$, we find

$$P(D) = (D^2 - 1)^2(D + 1)(D^2 + 2D + 2). \tag{33.27}$$

Then using the quadratic formula, we find the remaining roots are $-1 \pm i$. So the solution y is a linear combination of $e^t, te^t, e^{-t}, e^{-t}\cos t$, and $e^{-t}\sin t$.

Our third example starts with the differential operator $P(D) = D^4 - 16$. We show how to solve the inhomogeneous equation $P(D)y = \cos t$.

First, we must solve the homogeneous equation, which can be factorized "by inspection":

$$D^4 - 16 = (D^2 - 4)(D^2 + 4) = (D - 2)(D + 2)(D - 2i)(D + 2i). \tag{33.28}$$

The general solution is then a linear combination of $e^{2t}, e^{-2t}, \cos 2t$, and $\sin 2t$ plus a particular solution.

To find the particular solution, we use variation of parameters. Recall this method is rather painful with all the algebraic manipulations we need to do. To begin, we write the particular solution as a linear combination of different functions multiplied by the homogeneous solutions, as follows:

$$y_p(t) = A_1(t)e^{2t} + A_2(t)e^{-2t} + A_3(t)\cos 2t + A_4(t)\sin 2t. \tag{33.29}$$

We find the four equations that we need to solve are (recall they involve higher order derivatives for each subsequent equation)

$$\begin{aligned}
\dot{A}_1 e^{2t} + \dot{A}_2 e^{-2t} + \dot{A}_3 \cos 2t + \dot{A}_4 \sin 2t &= 0 && (33.30)\\
\dot{A}_1 2e^{2t} + \dot{A}_2(-2)e^{-2t} + \dot{A}_3(-2)\sin 2t + \dot{A}_4(2)\cos 2t &= 0 && (33.31)\\
\dot{A}_1 4e^{2t} + \dot{A}_2 4e^{-2t} - \dot{A}_3 4\cos 2t - \dot{A}_4 4\sin 2t &= 0 && (33.32)\\
\dot{A}_1 8e^{2t} + \dot{A}_2(-8)e^{-2t} + \dot{A}_3 8\sin 2t - 8\dot{A}_4(2)\cos 2t &= \cos t. && (33.33)
\end{aligned}$$

We reorganize the equations into a matrix form given by

$$\begin{pmatrix} e^{2t} & e^{-2t} & \cos 2t & \sin 2t \\ 2e^{2t} & -2e^{-2t} & -2\sin 2t & 2\cos 2t \\ 4e^{2t} & 4e^{-2t} & -4\cos 2t & -4\sin 2t \\ 8e^{2t} & -8e^{-2t} & 8\sin 2t & -8\cos 2t \end{pmatrix} \begin{pmatrix} \dot{A}_1 \\ \dot{A}_2 \\ \dot{A}_3 \\ \dot{A}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos t \end{pmatrix} \tag{33.34}$$

Now we use row reduction to solve it. This requires a lot of algebra. We zero out the first column

$$\begin{pmatrix} e^{2t} & e^{-2t} & \cos 2t & \sin 2t \\ 0 & -4e^{-2t} & -2(\cos 2t + \sin 2t) & 2(\cos 2t - \sin 2t) \\ 0 & 0 & -8\cos 2t & -8\sin 2t \\ 0 & -16e^{-2t} & 8(-\cos 2t + \sin 2t) & -8(\cos 2t + \sin 2t) \end{pmatrix} \begin{pmatrix} \dot{A}_1 \\ \dot{A}_2 \\ \dot{A}_3 \\ \dot{A}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos t \end{pmatrix}. \tag{33.35}$$

Then the second column

$$\begin{pmatrix} e^{2t} & e^{-2t} & \cos 2t & \sin 2t \\ 0 & -4e^{-2t} & -2(\cos 2t + \sin 2t) & 2(\cos 2t - \sin 2t) \\ 0 & 0 & -8\cos 2t & -8\sin 2t \\ 0 & 0 & 16\sin 2t & -16\cos 2t \end{pmatrix} \begin{pmatrix} \dot{A}_1 \\ \dot{A}_2 \\ \dot{A}_3 \\ \dot{A}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos t \end{pmatrix}. \tag{33.36}$$

And finally the third

$$\begin{pmatrix} e^{2t} & e^{-2t} & \cos 2t & \sin 2t \\ 0 & -4e^{-2t} & -2(\cos 2t + \sin 2t) & 2(\cos 2t - \sin 2t) \\ 0 & 0 & -8\cos 2t & -8\sin 2t \\ 0 & 0 & 0 & -16\sec 2t \end{pmatrix} \begin{pmatrix} \dot{A}_1 \\ \dot{A}_2 \\ \dot{A}_3 \\ \dot{A}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos t \end{pmatrix}. \tag{33.37}$$

So, the solution is

$$\dot{A}_4 = -\frac{1}{16}\cos t \cos 2t \tag{33.38}$$

$$\dot{A}_3 = \frac{1}{16}\cos t \sin 2t \tag{33.39}$$

$$\begin{aligned} \dot{A}_2 &= \frac{1}{4}e^{2t}\bigg(-\frac{1}{8}\cos t \cos 2t \sin 2t \\ &- \frac{1}{8}\cos t \sin^2 2t - \frac{1}{8}\cos t \cos^2 2t + \frac{1}{8}\cos t \cos 2t \sin 2t\bigg), \end{aligned} \tag{33.40}$$

which becomes

$$\dot{A}_2 = -\frac{1}{32}e^{2t}\cos t. \tag{33.41}$$

Finally, we have

$$\dot{A}_1 = \frac{1}{32}e^{-2t}\cos t. \tag{33.42}$$

Each of these differential equations needs to be solved. We do this in turn next. For the first, we write the trig functions as sums of exponentials

$$\dot{A}_1 = \frac{1}{32}e^{-2t}\cos t = \frac{1}{64}\left(e^{(-2+i)t} + e^{(-2-i)t}\right), \tag{33.43}$$

which can be easily integrated to yield

$$A_1 = \frac{1}{64}\left(-\frac{1}{2-i}e^{(-2+i)t} - \frac{1}{2+i}e^{(-2-i)t}\right). \tag{33.44}$$

We do a similar thing to the second equation

$$\dot{A}_2 = -\frac{1}{32}e^{2t}\cos t = -\frac{1}{64}\left(e^{(2+i)t} + e^{(2-i)t}\right), \tag{33.45}$$

which yields

$$A_2 = -\frac{1}{64}\left(\frac{1}{2+i}e^{(2+i)t} + \frac{1}{2-i}e^{(2-i)t}\right). \tag{33.46}$$

Repeating for the third gives us

$$\dot{A}_3 = \frac{1}{16}\cos t \sin 2t = \frac{1}{64i}\left(e^{3it} + e^{it} - e^{-it} - e^{-3it}\right) \tag{33.47}$$

and

$$A_3 = -\frac{1}{64}\left(\frac{1}{3}e^{3it} + e^{it} + e^{-it} + \frac{1}{3}e^{-3it}\right). \tag{33.48}$$

And last, but not least, we have

$$\dot{A}_4 = -\frac{1}{16}\cos t \cos 2t = -\frac{1}{64}\left(e^{3it} + e^{it} + e^{-it} + e^{-3it}\right), \tag{33.49}$$

which is solved by

$$A_4 = -\frac{1}{64i}\left(\frac{1}{3}e^{3it} + e^{it} - e^{-it} - \frac{1}{3}e^{-3it}\right). \tag{33.50}$$

So we have our final results are

$$A_1 = -\frac{1}{32}\mathrm{Re}\left(\frac{e^{(-2+i)t}}{2-i}\right) \tag{33.51}$$

$$A_2 = -\frac{1}{32}\mathrm{Re}\left(\frac{e^{(2+i)t}}{2+i}\right) \tag{33.52}$$

$$A_3 = -\frac{1}{32}\left(\frac{1}{3}\cos 3t + \cos t\right) \tag{33.53}$$

$$A_4 = -\frac{1}{32}\left(\frac{1}{3}\sin 3t + \sin t\right) \tag{33.54}$$

and the particular solution becomes

$$A_1y_1 + A_2y_2 + A_3y_3 + A_4y_4 = y_p(t). \tag{33.55}$$

We substitute in the results from the homogeneous solutions and simplify to

$$y_p(t) = -\frac{1}{40}\cos t - \frac{1}{96}\cos t - \frac{1}{32}\cos t = -\frac{1}{15}\cos t. \tag{33.56}$$

You need to use the cosine difference formula to find this.

Of course, now that we see what the final answer is (which is easy to check), we could have simply guessed it by trying a number times $\cos t$. But it is comforting to know that we could work through the full variation of parameters to reach the final solution. The procedure is never fun though.