Chapter 38

Applications of Poisson's Theorem

Poisson's Theorem showed that a regularized Fourier series (solution of the heat equation at a radius $r \to 1$) could be brought as close as possible to any function $f(\theta)$ defined on the perimeter. If $f(\theta)$ was continuously differentiable, we could even prove *pointwise* convergence, which says at each θ, the Fourier series converges to $f(\theta)$.

Before going on, let me share a little story with you. When I was in graduate school, I took a graduate course in real analysis. The professor was a nice guy, but definitely did not feel like grading our work, so at the end of the term he simply posted a note, everyone in the class receives an "A." In this class, we spent much of the semester discussing different forms of continuity. I believe there were at least a dozen different forms of continuity—pointwise continuous, uniformly continuous, equicontinuous, piecewise continuous, and many, many more. It turns out that most of the ideas for different ways to define continuous and differentiable functions stem from extensions of Fourier's original work. So, we delve into this topic a bit further next.

We will discuss only two other forms of convergence (aside from pointwise convergence). The first is called *uniform* convergence, which says for some finite N, we can find a truncated Fourier series

$$S_N(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta} \tag{38.1}$$

with the standardly defined Fourier coefficients

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta f(\theta)e^{-in\theta}, \tag{38.2}$$

such that for large enough N,

$$|S_N(\theta) - f(\theta)| < \epsilon, \tag{38.3}$$

for *all* $0 \leq \theta \leq 2\pi$. Unlike pointwise continuity, where for each θ we can take the limit as N approaches infinity, here we want to keep N fixed and finite, but show the convergence nevertheless occurs for all θ. This is called the *Weierstrauss Theorem.*

The second kind of continuity is called *least squares* continuity. Here, we want to show that

$$\int_{-\pi}^{\pi} d\theta |S_N(\theta) - f(\theta)|^2 < \epsilon \tag{38.4}$$

or the average mean square deviation is small. Note that the results can err by a large amount at isolated points and not affect the integral's value. The least-squares continuity naturally leads to Parseval's equality, given by

$$2\pi \sum_{n=-\infty}^{\infty} |a_n|^2 = \int_{-\pi}^{\pi} d\theta\, |f(\theta)|^2. \tag{38.5}$$

Parseval's identity can be very useful in applications. It is worthwhile to remember it. The detailed proof is given in the book. I encourage you to read it.

The main focus of this lecture is to tidy up some discussions we had earlier of thinking of the Fourier series as a change of basis in the infinite dimensional space of functions.

Here, we think of the orthonormal unit vectors as $u_n = e^{in\theta}$. This is a complex vector space so

$$u_n \cdot u_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, \left(e^{-in\theta}\right)^* e^{-im\theta} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, e^{in\theta} e^{-im\theta} = \delta_{nm}. \tag{38.6}$$

The exponential functions are unit vectors that are orthogonal and there are an infinite number of them. So we can imagine expanding a function $f(\theta)$ in the basis of the u_n's. This means

$$f(\theta) = \sum_{n=-\infty}^{\infty} a_n u_n. \tag{38.7}$$

To find the expansion coefficients a_n, we take the inner product of $f(\theta)$ with u_m on the right:

$$f(\theta) \cdot u_m = \sum_{n=-\infty}^{\infty} a_n u_n \cdot u_m = \sum_{n=-\infty}^{\infty} a_n \delta_{nm} = a_m. \tag{38.8}$$

Hence, $a_m = f \cdot u_m = \int d\theta f(\theta) \exp(-im\theta)/2\pi$. This is the same function as we had before!

Parseval's equality becomes just an expansion for the norm of the vector f calculated two different ways:

$$\|f\|^2 = \frac{1}{2\pi} \int d\theta \, |f(\theta)|^2 = \sum_{n=-\infty}^{\infty} |a_n|^2. \tag{38.9}$$

As we discussed earlier in the course, we can have dot products between two functions. Computing them is easiest done with the Fourier coefficients, as shown below:

$$\begin{aligned} f \cdot g &= \frac{1}{2\pi} \int d\theta f(\theta) g^*(\theta) \\ &= \frac{1}{2\pi} \int d\theta \sum_n a_n e^{in\theta} \sum_m b_m^* e^{-im\theta} \\ &= \sum_{nm} a_n b_m^* \delta_{nm} \\ &= \sum_n a_n b_n^*. \end{aligned} \tag{38.10}$$

Now, because $\|f\|^2 = \sum_n |a_n|^2$, $\|g\|^2 = \sum_n |b_n|^2$, we can define the angle between f and g via

$$\cos\alpha = \frac{f \cdot g}{\|f\| \, \|g\|} = \frac{\sum_n a_n b_n^*}{\sqrt{\left(\sum_n |a_n|^2\right)\left(\sum_m |b_m|^2\right)}} \tag{38.11}$$

and the cosine is well defined because the Schwartz inequality says

$$\left| \sum_n a_n b_n^* \right| \leq \sqrt{\left(\sum_n |a_n|^2 \right) \left(\sum_m |b_m|^2 \right)}. \tag{38.12}$$

The Fourier series also satisfies the triangle inequality, given by

$$\|f+g\| \leq \|f\| + \|g\| \tag{38.13}$$

or

$$\sqrt{\sum_n |a_n + b_m|^2} \leq \sqrt{\sum_n |a_n|^2} + \sqrt{\sum_n |b_n|^2}. \tag{38.14}$$

Proof:

$$\|f+g\|^2 = \|f\|^2 + 2f \cdot g + \|g\|^2 \leq \|f\|^2 + 2|f \cdot g| + \|g\|^2. \tag{38.15}$$

But the Schwartz inequality says $2|f \cdot g| \leq 2\|f\|\,\|g\|$, so that we have

$$\begin{aligned} \|f+g\|^2 &\leq \|f\|^2 + 2\|f\|\,\|g\| + \|g\|^2 \\ &\leq (\|f\| + \|g\|)^2. \end{aligned} \tag{38.16}$$

This implies that $\|f+g\| \leq \|f\| + \|g\|$.

And that is it. We are done! I hope you enjoyed learning this material as much as I did in assembling it. Please go forward and pursue physics with your newfound knowledge. Be a practitioner, not a technician. Always think. Always ask why.