Chapter 8

The vanishing sphere and other multidimensional integrals

8.1 Three-dimensional integrals

Recall we discussed how to integrate volumes and areas in cubic, cylindrical, and spherical coordinates in Chapter 7. Now we will discuss some other examples:

Example 1: What is the mass of a sphere of radius R with constant mass density ρ_0? To find the total mass, we add each volume element, weighted by the mass density. Since the mass density is a constant, it does not change the volume integral. Rather, it just multiplies the result, as you can see below.

$$M = \int_0^R dr \int_{-1}^1 r\, d\cos\theta \int_0^{2\pi} r\rho_0\, d\phi \tag{8.1}$$

$$= \int_0^R r^2\rho_0 dr \times 2 \times 2\pi = \frac{4\pi}{3} R^3 \rho_0 \tag{8.2}$$

as you might have guessed.

Example 2: Now we try this again, but with a mass density that *varies with the radius*, such as $\rho(r) = \rho_0(\frac{r}{R})^\alpha$. The integral now becomes

$$M = \int_0^R dr r^2 \rho_0 (\frac{r}{R})^\alpha \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi. \tag{8.3}$$

The integral over the angles is trivial and yields a factor of 4π. One can then complete the integral as follows:

$$= \frac{4\pi\rho_0}{R^\alpha}\int_0^R dr r^{\alpha+2} = \frac{4\pi\rho_0}{R^\alpha}\frac{r^{\alpha+3}}{\alpha+3}\bigg|_0^R = \frac{4\pi\rho_0}{R^\alpha}\frac{R^{\alpha+3}}{\alpha+3} = \frac{4\pi R^3}{\alpha+3}\rho_0. \tag{8.4}$$

Note that e have a nice check of this formula. If we set $\alpha = 0$, then the same mass will be found as we calculated before in Example 1. Note further that as α increases, the mass goes down. Do you understand why this is so?

8.2 The vanishing sphere

Let's examine the phenomenon of the "vanishing sphere"—as the number of dimensions d increases, the volume of the sphere in d dimensions approaches *zero* for any finite radius.

We start by defining a sphere in d-dimensions. The position vector $\vec{x}$ is an d-tuple, given by $\vec{x} = (x_1, x_2, x_3, x_4, ...x_d)$. A sphere is defined by the volume inside the set

$$\vec{x}\cdot\vec{x} \leq R^2. \tag{8.5}$$

This implies that

$$(x_1)^2 + (x_2)^2 + (x_3)^2 + ...(x_d)^2 \leq R^2. \tag{8.6}$$

Now we need to determine spherical coordinates in d dimensions. We work by generalizing the spherical coordinates from $3d$, where we have r and two angles θ and ϕ, to d "hyper-spherical" coordinates given by r, θ_1, ..., θ_{d-1}. The procedure is a s follows. We start with the first Cartesian coordinate x_1, which we define in terms of the r-coordinate and the first angle θ_1

$$x_1 = r\cos(\theta_1). \tag{8.7}$$

One should think of this as the projection of the vector $\vec{x}$ on the first Cartesian axis, which defines the angle θ_1. The perpendicular component of $\vec{x}$ now has a length given by $r\sin\theta_1$. So, its projection onto the second axis is

$$x_2 = \underbrace{r\sin(\theta_1)}_{length\ of\ \perp\ component}\cos(\theta_2). \tag{8.8}$$

The procedure is then iterated. Each subsequent perpendicular component is of length of the previous perpendicular component multiplied by the sine of the latest angle. Hence, we have

$$x_3 = \underbrace{r\sin(\theta_1)\sin(\theta_2)}_{length\ of\ \perp\ component}\cos(\theta_3). \tag{8.9}$$

This continues until we get to the last angle. It is different. The second-to-last angle is defined via

$$x_{d-1} = \underbrace{r\sin(\theta_1)\sin(\theta_2)\sin(\theta_3)...\sin(\theta_{d-2})}_{length\ of\ \perp\ component}\cos(\theta_{d-1}). \tag{8.10}$$

And then the last one is

$$x_d = \underbrace{r\sin(\theta_1)\sin(\theta_2)\sin(\theta_3)...\sin(\theta_{d-2})\sin(\theta_{d-1})}_{length\ of\ final\ \perp\ component} \tag{8.11}$$

where the final cosine is replaced by a sine representing the last perpendicular component.

Now to find the volume element, it is just like before in $3d$. Each dimension uses the length of the corresponding perpendicular component. One needs to be careful about the last term:

$$dr\times r d\theta_1\times r\sin(\theta_1)d\theta_2\times r\sin(\theta_1)\sin(\theta_2)d\theta_3...r\sin(\theta_1)\sin(\theta_2)...\sin(\theta_{d-2})d\theta_{d-1}. \tag{8.12}$$

Here r goes from $0\to\infty$, $\theta_1...\theta_{d-2}$ from $0\to\pi$, and θ_{d-1} from $0\to 2\pi$. So an integral for the volume in d-dimensions is:

$$V_d = \int_0^R dr r^{d-1}\int_0^\pi \sin(\theta_1)^{d-2}d\theta_1\int_0^\pi \sin(\theta_2)^{d-3}d\theta_2\cdots\int_0^\pi \sin(\theta_{d-2})d\theta_{d-2}\int_0^{2\pi} d\theta_{d-1} \tag{8.13}$$

Hence, to find the volume, we need to integrate $\int_0^\pi(\sin\theta)^\alpha d\theta = I_\alpha$, with $u = (\sin\theta)^{\alpha-1}$ and $v' = \sin(\theta)$, using integration by parts with $v = -\cos(\theta)$. This yields

$$I_\alpha = -(\sin\theta)^{\alpha-1}\cos\theta\Big|_0^\pi + \int_0^\pi(\alpha-1)(\sin\theta)^{\alpha-2}\cos\theta\cos\theta\,d\theta. \tag{8.14}$$

Simplifying, we find

$$I_\alpha = 2\delta_{\alpha=1} + (\alpha-1)\int_0^\pi(\sin\theta)^{\alpha-2}(1-\sin^2\theta)\,d\theta. \tag{8.15}$$

Using the definition of I_α, we find a recurrence relation between the integrals, given by

$$I_\alpha = 2\delta_{\alpha=1} + (\alpha - 1)\left(I_{\alpha-2} - I_\alpha\right). \tag{8.16}$$

We re-arrange the recurrence relation to its final form, which is

$$\alpha I_\alpha = (\alpha - 1) I_{\alpha-2} + 2\delta_{\alpha=1}. \tag{8.17}$$

The recurrence relation then tells us that $I_1 = 2$ (which can also be verified from integrating the definition of I_1). Hence, we have the odd integrals satisfy $I_3 = \frac{2}{3} I_1 = \frac{4}{3}$, $I_5 = \frac{4}{5} I_3 = \frac{4}{5} \times \frac{2}{3} I_1 = \frac{16}{15}$, $I_7 = \frac{6}{7} I_5 = \frac{6\times4\times2}{7\times5\times3} \times 2 = \frac{32}{35}$, and so on. One can recognize the clear pattern. We therefore have

$$I_{2n+1} = \frac{2^n n!}{(2n+1)!!} \times 2 = \frac{2^{n+1} n!}{(2n+1)!!}. \tag{8.18}$$

To be completely rigorous, one should verify this result by induction. The proof should take two to three lines and it is useful for you to try it yourself.

Now, onto the even integrals. We have

$$I_2 = \int_0^\pi (\sin\theta)^2 \, d\theta = \frac{1}{2} \int_0^\pi \left(\sin^2\theta + \cos^2\theta\right) \, d\theta = \frac{\pi}{2}. \tag{8.19}$$

Using this initial result, $I_2 = \frac{\pi}{2}$, we can then generate all other even integrals from the recurrence relation: $I_4 = \frac{3}{4} I_2 = \frac{3}{8}\pi$, $I_6 = \frac{5}{6} I_4 = \frac{5*3}{6*4}\frac{\pi}{2}$, and so on. Again, the pattern is clear, so we know that

$$I_{2n} = \frac{(2n-1)!!}{2^n n!}\pi. \tag{8.20}$$

Again, you should verify this is correct by induction.

In order to compute the volumes of the spheres, we first consider the case where d is an odd number. Evaluating the initial volume integral is now straightforward, because all angular integrals are known. We have

$$V_d = \int_0^R r^{d-1} \, dr \times 2\pi \times I_{d-2} \times I_{d-3} \times I_{d-4} \cdots I_2 \times I_1. \tag{8.21}$$

The radial integral can be done and we group the angular integrals to give

$$V_d = \frac{1}{d} R^d \times 2\pi \times (I_1 I_3 I_5 \cdots I_{d-2}) \times (I_2 I_4 I_6 \cdots I_{d-3}). \tag{8.22}$$

Now, we substitute in the results for each of the angular integrals:

$$V_d = \frac{1}{d}R^d \times 2\pi \times 2 \times \left(2 \times \frac{2}{3}\right) \times \left(2 \times \frac{2}{3} \times \frac{4}{5}\right) \times \left(2 \times \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7}\right) \cdots \left(\frac{2^{\frac{d-1}{2}}(\frac{d-3}{2})!}{(d-2)!!}\right)$$
$$\times \frac{\pi}{2} \times \left(\frac{\pi}{2} \times \frac{3}{4}\right) \times \left(\frac{\pi}{2} \times \frac{3}{4} \times \frac{5}{6}\right) \cdots \left(\frac{\pi(d-4)!!}{2^{\frac{d-3}{2}}(\frac{d-3}{2})!}\right). \tag{8.23}$$

So,

$$V_d = \frac{1}{d}R^d \times 2\pi \times \left(2 \times \frac{\pi}{2}\right) \times \left(2 \times \frac{2}{3}\right) \times \left(\frac{\pi}{2} \times \frac{3}{4}\right) \times \left(2 \times \frac{2}{3} \times \frac{4}{5}\right)$$
$$\times \left(\frac{\pi}{2} \times \frac{3}{4} \times \frac{5}{6}\right) \cdots \times \left(\frac{2\pi}{d-3}\right)\left(\frac{2 \times 2 \times 4 \cdots (d-3)}{3 \times 5 \cdots (d-2)}\right) \tag{8.24}$$

We finally simplify in a couple of steps:

$$V_d = \frac{1}{d}R^d \times (2\pi)^{\frac{d-1}{2}} \times \frac{1}{2^{\frac{d-3}{2}}\left(\frac{d-3}{2}\right)!} \times \frac{2^{\frac{d-1}{2}}\left(\frac{d-3}{2}\right)!}{(d-2)!!}, \tag{8.25}$$

$$V_d = \frac{1}{d}R^d(2\pi)^{\frac{d-1}{2}}\frac{2}{(d-2)!!}, \tag{8.26}$$

$$V_d = R^d(2\pi)^{\frac{d-1}{2}} \times \frac{2}{d!!} = (\sqrt{2\pi}R)^d\sqrt{\frac{2}{\pi}}\frac{1}{d!!}. \tag{8.27}$$

Note how, for fixed R and as d increases, we eventually have $d > 2\pi R^2$, so that $\frac{(\sqrt{2\pi}R)^d}{d!!} \to 0$ as $d \to \infty$. This implies that the Volume of a sphere vanishes as $d \to \infty$!

Why?

Recall that volumes are measured *relative to the unit cube* (which always has volume equal to 1). Note that the diagonal of a unit cube has length $\sqrt{d} \to \infty$ as $d \to \infty$, so the unit cube looks like a "porcupine" as $d \to \infty$. This implies that any finite radius sphere lies in a small volume inside the unit cube, so $V_d \to 0$ as $d \to \infty$.

In general, high dimensional integrals are hard to evaluate except by a Monte Carlo method—the Monte Carlo method picks a volume that encloses the object that is being integrated. Next, one chooses a random point uniformly in the volume. If the randomly chosen point lies inside the sphere, then we add 1 to the sum. If not, then we don't add 1. Hence, the volume is estimated to be the total number of points in sphere divided by the total number in the volume and multiplied by the volume.

Lets try with a python code for odd dimensions with a unit radius sphere. We will use $100,000,000$ randomly chosen points, which gives an accuracy of better than 3 digits for low d. Recall that

$$V_d = \left(\sqrt{2\pi}R\right)^d \sqrt{\frac{2}{\pi}}\,\frac{1}{d!!}, \tag{8.28}$$

so we have $V_3 = \frac{4}{3}\pi$, $V_5 = \frac{8}{15}\pi^2$, $V_7 = \frac{16}{105}\pi^3$, $\cdots$, $V_{19} = \frac{1024}{19!!}\pi^4$. We compare the two results in Table 8.1.

	Monte Carlo	exact
1:	2.0	2.0
3:	4.18854	4.18879
5:	5.26270	5.26379
7:	4.72316	4.72477
9:	3.29940	3.29851
11:	1.88584	1.88410
13:	0.91628	0.91063
15:	0.36438	0.38144
17:	0.14811	0.14098
19:	0.03670	0.04662

Table 8.1: Comparison of a Monte Carlo integration to the exact results for unit spheres of different dimensions.

Note how, from $d = 13$ and beyond, it can be seen from the table the the numbers become less and less accurate.

8.3 Moments of Inertia

Over the years, I have noticed that students really struggle with knowing how to set up the integral for a moment of inertia and evaluate it. We will spend an entire laboratory working on this problem, after which you should become a "practitioner.". We introduce some of the issues here.

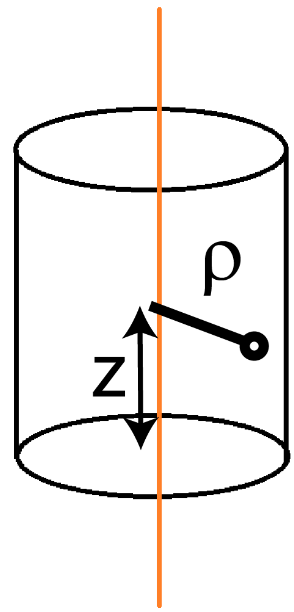

Figure 8.1: Schematic of the cylinder and the axis of rotation (orange) about which the cylinder rotates. Note how the *distance* to the axis is given by ρ and is independent of the angle and the height.

Note that a moment of inertia simply involves an integral similar to the ones used to find the mass given a particular mass density (constant or not); we looked at these in Examples 1 and 2 above. The difference is that we must add an additional factor of l^2, where l is the perpendicular distance of the particular volume element from the axis of rotation. It is this last step that often stymies students. Look carefully at the geometry and see whether you can see how this is done.

As an example, we show a cylinder in black and the axis of rotation in orange in Fig. 8.1. If you recall cylindrical coordinates, you should be able to immediately see that the perpendicular distance of some point within the cylinder to the rotation axis is just given by the radial coordinate ρ (it is independent of z or θ). Hence, we have $l = \rho$, and the additional factor needed in the integral will be a factor of $l^2 = \rho^2$. We examine this in complete detail in the lab.