Existence Proofs

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Outline

Know What Are You Looking For

Proofs For Existential Statements

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- what does the proof look like?

Proofs For Existential Statements

- what does the proof look like?
- it depends

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Proofs For Existential Statements

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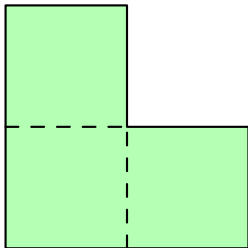- one example is enough

Cutting Figures

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congruent pieces: of the same shape and size

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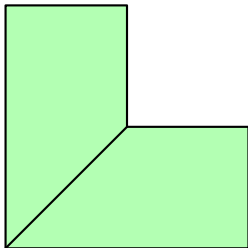
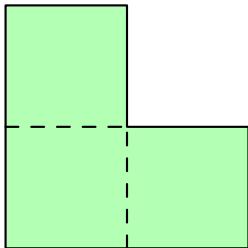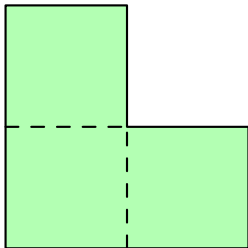*congruent* pieces: of the same shape and size

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Prove that this figure can be cut into 2 congruent pieces

what about 4 pieces?

Spoiler

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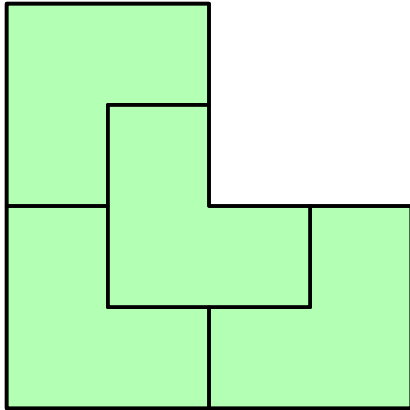
Outline

The Octagon
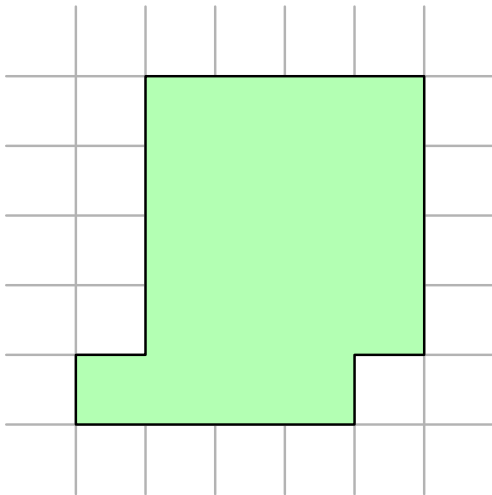
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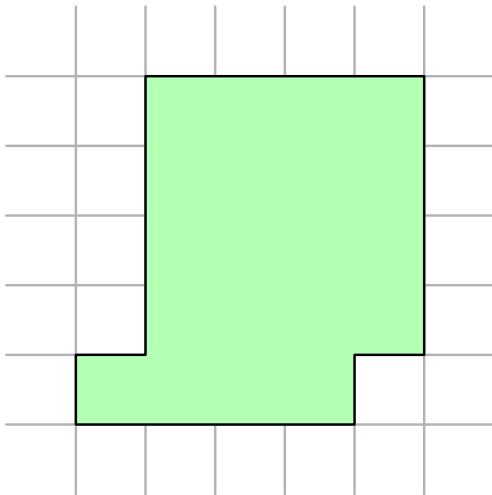
split into two congruent pieces

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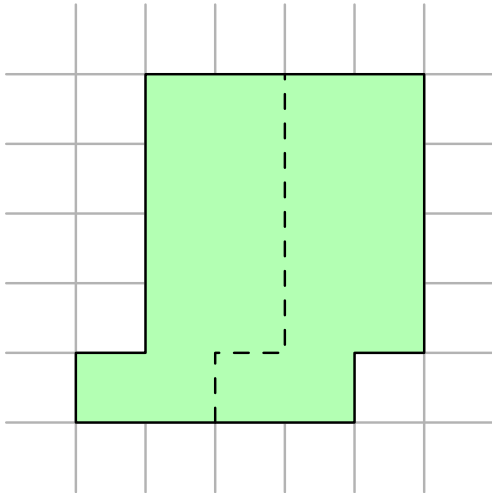
Spoiler
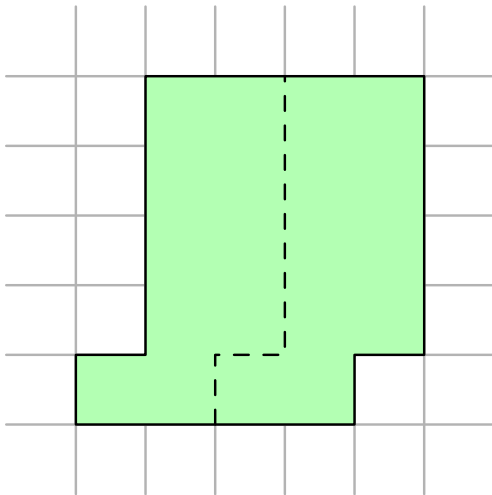
what about three congruent pieces?

Outline

Tensegrities

Tensegrities

- drinking straws and thread

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- there exists a "tensegrity": a solid construction; straws do not touch each other; connected by threads

Not Allowed

Tensegrity Finished
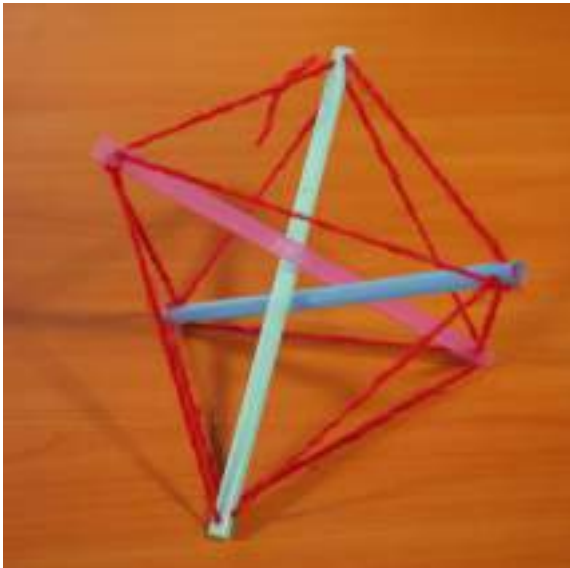
A Tensegrity: Animation

Tensegrities in the Real Life

made by Kenneth Snelson, a student of Buckminster Fuller (who invented the word and made many of them)

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- you say: $7125 = 57 \cdot 125$

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- divisible by 7: 14, 21, 28, 35, 42, 49, 56, 63
- but what if we asked for a number that becomes 57 times smaller?
- you say: $7125 = 57 \cdot 125$
- no need to explain how you found it

No Rights for Teachers

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- a divisible by 7: only $a = 7$ works
- $10^k = 8 \times X$; 10 and 100 not multiples of 8
- 1000 works, $X = 125$
- also $71250 = 57 \times 1250$, etc.

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- easy: $4 + 7$ (also $1 + 2 + 3 + 5 = 11$)

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- if the weights are $1, 2, 3, 4, 5, 6$?
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- what about weights $2, 4, 6, 8, 10, 12$?
- hint: just changing the units

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- NP-complete $\approx$ infeasible

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- claim: *an object with some property exists*
- proof: *an example*
- one example is enough
- no need to disclose the sources
- beware: claim may be false!