Chapter 34

Method of Undetermined Coefficients

34.1 Introduction

As we saw last time, the method of variation of parameters is an exceedingly painful way to find the particular solution of a linear inhomogeneous differential equation, which we call y_p. While it will always work, there is an alternative, called the method of undetermined coefficients. It is essentially a procedure that allows you to guess and verify your guess using some standard procedures. You can think of it as providing a sufficiently general guess (with some adjustable parameters) and a procedure that shows you how to adjust the parameters to make it work.

The two realistic guesses that we can make are polynomials if $f(t)$ is a polynomial in t and exponentials if $f(t)$ has exponentials (or sines and cosines) in it. The only subtlety is if the guess is already one of the terms in the homogeneous solutions, then we need to multiply by powers of t that are higher than what was in the homogeneous solution set. Some examples will make these restrictions clear.

The method of undetermined coefficients is a method that is best taught by looking through a few examples to see the different types of issues that arise with the method. Since it involves guessing, it is only as good as the initial guess that is made. We illustrate a number of standard guesses that will work for certain inhomogeneous functions $f(t)$.

34.2 Example 1

Our first example is the same problem we tackled last time. The procedure was long and complicated before. Here we will find it will go a bit faster. We start from our old friend, the differential equation

$$(D^4 - 16)y = \cos(t) \tag{34.1}$$

We make the guess for $y_p(t)$ to be given by the form $y_p(t) = a\cos(t)$, with a a yet to be determined constant. Why does this seem reasonable? Because the fourth derivative of a cosine is proportional to a cosine. Hence, we have

$$D^4(a\cos(t)) = a\cos(t), \tag{34.2}$$

so, by substituting into the equation, we find that

$$a(1-16)\cos(t) = \cos(t) \implies a = -\frac{1}{15}, \quad y_p = -\frac{1}{15}\cos(t). \tag{34.3}$$

You can see that we simply force the given form to work and it yields the answer with very little work. This is much simpler than the solution we found before using variation of parameters.

34.3 Example 2

Here, we examine a differential equation, where the function f includes a term that is part of the solution of the homogeneous equation. This will require our guess to be a bit more sophisticated. The differential equation is

$$(D^3 - D)y = 1 + t. \tag{34.4}$$

We must first find the homogeneous solution. The polynomial in D is given by

$$P(D) = (D^3 - D) = D(D-1)(D+1), \tag{34.5}$$

and is easily factorized. The roots of the polynomial are $0, 1, -1$. Using the roots of the polynomial, we immediately find that the homogeneous solution is a linear combination of 1, e^t, and e^{-t}.

Now our first guess for y_p might have been $at + b$ but when we plug in b, we get $P(D)b = 0$. This occurs, because b solves the homogeneous equation.

So we need to multiply by an additional factor of t to try at^2+bt as our guess for y_p. We now plug into the differential equation to see if we can make it work. We operate the differential polynomial on y_p to find $D^3(at^2+bt)=0$ and $-D(at^2+bt)=-2at-b$. Hence to make $P(D)y_p=1+t$, we must pick $b=-1$ and $a=-\frac{1}{2}$. This gives us $y_p=-\frac{1}{2}t^2-t$, which we can immediately check. And it works!

34.4 Example 3

Here, we look at a more complicated example where the homogeneous solution represented in y_p is a nontrivial function of t. The differential equation we wish to solve is

$$(D^2-5D+6)y=e^{2t}+\cos(t). \tag{34.6}$$

We find the homogeneous solutions using our standard methods. The roots of $P(D)$ are

$$r_\pm=\frac{5}{2}\pm\frac{1}{3}\sqrt{25-24}=\frac{5}{2}\pm\frac{1}{2}=2,\ 3. \tag{34.7}$$

This means $y_{\text{homog}}(t)$ is a linear combination of e^{2t} and e^{3t}. Our initial guess for y_p would have been $ae^{2t}+be^{it}+ce^{-it}$, but e^{2t} is in the homogeneous solution, so we revise by multiplying the homogeneous solution by an extra power of t to give us our ansatz $y_p(t)=ate^{2t}+be^{it}+ce^{-it}$. You may wonder why we did not use just a number times $\cos(t)$. We are forced into this situation because there are both even order and odd order derivatives in the differential equation (recall, the derivative of a sin is a cos, and so on).

We operate the differential equation on our ansatz. This requires us to calculate a few different terms. We find that

$$\begin{aligned}(D^2-5D+6)&(ate^{2t}+be^{it}+ce^{-it})=\\ &=a(4+4t)e^{2t}-5a(1+2t)e^{2t}+6ate^{2t}-be^{it}-5ibe^{it}+6be^{it}\\ &-ce^{-it}+5ice^{-it}+6ce^{-it}\\ &=e^{2t}+\frac{1}{2}e^{it}+\frac{1}{2}e^{-it},\end{aligned} \tag{34.8}$$

after using the definition of the cosine in terms of exponentials. Hence we have

$$a(4+4t)-a(5+10t)+a6t=1, \tag{34.9}$$

which implies that $a = -1$. We also have

$$-b - 5ib + 6b = \frac{1}{2} \tag{34.10}$$

or

$$b = \left(\frac{1}{10}\right)\frac{1}{1-i}. \tag{34.11}$$

Finally, we have

$$-c + 5ic + 6c = \frac{1}{2}, \tag{34.12}$$

which implies that

$$c = \left(\frac{1}{10}\right)\frac{1}{1+i}. \tag{34.13}$$

Putting this all together yields

$$y_p(t) = -te^{2t} + \frac{1}{5}\mathrm{Re}\left\{\frac{e^{it}}{1-i}\right\} = -te^{2t} + \left(\frac{1}{5}\right)\mathrm{Re}\left\{\frac{1+i}{2}e^{it}\right\}. \tag{34.14}$$

Calculating the real parts finally gives us our answer

$$y_p(t) = -te^{2t} + \frac{1}{10}[\cos(t) - \sin(t)]. \tag{34.15}$$

The full solution of the differential equation then becomes

$$y(t) = c_1 e^{2t} + c_2 e^{3t} - te^{2t} + \frac{1}{10}[\cos(t) - \sin(t)]. \tag{34.16}$$

I hope you can see that this method is far superior to variation of parameters when it works (that is, when you can make the right guess).

34.5 Example 4

Moving on, the next equation to solve is

$$(D^2 + 2D + 2)y = t\cos(2t) + \sin(2t) \tag{34.17}$$

The roots of $P(D)$ are

$$r_{\pm} = -1 \pm \frac{1}{2}\sqrt{4-8} = -1 \pm i \tag{34.18}$$

so $y_{\text{homog}}(t) =$ linear combination of $e^{-t}\cos(t)$ and $e^{-t}\sin(t)$. Our guess for $y_p(t)$ is

$$y_p(t) = a\cos(2t) + bt\cos(2t) + c\sin(2t) + dt\sin(2t). \tag{34.19}$$

Again, it might seem odd to have both constants and linear terms multiplying the trig functions, but this is needed when we take the derivatives, as you will soon see. In other words, we add $a\cos(2t)$ because the derivatives will remove the power of t, and we add $dt\sin(2t)$, because we are likely to need it.

We now plug the ansatz into the equation to find

$$\begin{aligned}(D^2+2D+2)y_p &= -4a\cos(2t) - 4a\sin(2t) + 2a\cos(2t) - 4tb\cos(2t)\\ &\quad - 4b\sin(2t) - 4tb\sin(2t) + 2b\cos(2t) + 2tb\cos(2t) - 4c\sin(2t)\\ &\quad + 4c\cos(2t) + 2c\sin(2t) - 4td\sin(2t) + 4d\cos(2t) + 4td\cos(2t)\\ &\quad + 2d\sin(2t) + 2td\sin(2t)\\ &\quad = t\cos(2t) + \sin(2t). \end{aligned} \tag{34.20}$$

We now have to combine terms and solve for the coefficients. The coefficients of the following functions must satisfy these coupled linear equations:

$$\begin{aligned} \cos(2t):&\qquad -2a + 2b + 4c + 4d = 0 &(34.21)\\ t\cos(2t):&\qquad 0a - 2b + 0c + 4d = 1 &(34.22)\\ \sin(2t):&\qquad -4a - 4b - 2c + 2d = 1 &(34.23)\\ t\sin(2t):&\qquad 0a - 4b + 0c - 2d = 0. &(34.24)\end{aligned}$$

We can rewrite this as the matrix equation

$$\begin{pmatrix} -2 & 2 & 4 & 4\\ 0 & -2 & 0 & 4\\ -4 & -4 & -2 & 2\\ 0 & -4 & 0 & -2 \end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix} = \begin{pmatrix} 0\\ 1\\ 1\\ 0\end{pmatrix} \tag{34.25}$$

and use row reduction to solve it. This involves the following steps:

$$\left(\begin{array}{cccc|c} -1 & 1 & 2 & 2 & 0\\ 0 & -1 & 0 & 2 & \frac{1}{2}\\ -2 & -2 & -1 & 1 & \frac{1}{2}\\ 0 & -2 & 0 & -1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cccc|c} -1 & 1 & 2 & 2 & 0\\ 0 & -1 & 0 & 2 & \frac{1}{2}\\ 0 & -4 & -5 & -3 & \frac{1}{2}\\ 0 & -2 & 0 & -1 & 0 \end{array}\right) \tag{34.26}$$

$$\Longrightarrow \left(\begin{array}{rrrr|r} -1 & 1 & 2 & 2 & 0 \\ 0 & -1 & 0 & 2 & \frac{1}{2} \\ 0 & 0 & -5 & -11 & -\frac{3}{2} \\ 0 & 0 & 0 & -5 & -1 \end{array}\right) \tag{34.27}$$

We can now find the values of a, b, c, and d:

$$d = \frac{1}{5} \tag{34.28}$$

$$-5c - \frac{11}{5} = -\frac{3}{2} \Longrightarrow -5c = \frac{7}{10} \Longrightarrow c = -\frac{7}{50} \tag{34.29}$$

$$b = 2d - \frac{1}{2} = \frac{2}{5} - \frac{1}{5} \Longrightarrow b = -\frac{1}{10} \tag{34.30}$$

$$-a - \frac{1}{10} - \frac{7}{25} + \frac{2}{5} = 0 \Longrightarrow a = \frac{-5 - 14 + 20}{50} \Longrightarrow a = \frac{1}{50}. \tag{34.31}$$

Hence, we have the following for the general solution $y(t)$:

$$\begin{aligned} y(t) &= c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t) \\ &+ \frac{1}{50}\cos(2t) - \frac{1}{10}t\cos(2t) - \frac{7}{50}\sin(2t) + \frac{1}{5}t\sin(2t). \end{aligned} \tag{34.32}$$

34.6 Example 5

Our final example is the equation

$$P(D)y = e^{\alpha t}, \tag{34.33}$$

where α is not equal to any of the roots $r_1, \cdots, r_n$ of $P(D)$. Then for the general solution $y(t) = y_{\text{homog}}(t) + y_p(t)$, we can make the guess $y_p(t) = ae^{\alpha t}$. The particular solution satisfies

$$P(D)ae^{\alpha t} = aP(\alpha)e^{\alpha t}, \tag{34.34}$$

implying the differential equation gets replaced by an *algebraic* equation. The solution follows immediately as

$$a = \frac{1}{P(\alpha)}, \tag{34.35}$$

since $P(\alpha) \neq 0$ due to the fact that α is not a root of $P(D)$. This shows us that when $f(t)$ = exponential whose exponent is not a root of $P(D)$, then finding $y_p(t)$ is **very easy** to do!

To summarize the method, if $f(t)$ is a polynomial, we guess a polynomial of the same degree unless there is a polynomial as a homogeneous solution, in which case extra factors of t will be needed. If $f(t)$ is an exponential, the above approach shows how to do it when none of the exponents are roots of $P(D)$. If any of them are, we need to once again include higher powers of t to be able to make it work. Finally, in many cases when $f(t)$ is a sine or cosine, we need to include both terms in the ansatz if the differential equation has any odd powers of D in it (and extra powers of t if they appear in the homogeneous solution $\cdots$).