Chapter 30

Nonlinear first-order differential equations

30.1 Introduction to nonlinear differential equations

You might have thought that we are done with our simple world of first-order differential equations because we derived a formula to solve them all. But no!. This is one of the few cases where we actually can solve a number of nonlinear differential equations (in general, nonlinear differential equations are so hard people spend their entire scientific careers studying them). Nonlinear equations are very rich. All of the complexities of the weather, nonlinear effects like the so-called "butterfly effect" all derive from nonlinear differential equations. So, let's jump in and get started.

A nonlinear first order differential equation takes the form

$$\frac{dy}{dt} = f(t, y) = -\frac{M(t, y)}{N(t, y)} \tag{30.1}$$

so that

$$M(t, y)dt + N(t, y)dy = 0. \tag{30.2}$$

Note that the reduction into the M and N form is *not unique*. This is sometimes a cause for confusion, but it really is an opportunity that we will exploit later.

There are five known general methods to solve these nonlinear differential equations. If none of these work, you need to solve them numerically. But even that is not so simple ···

30.2 Type 1: Reducible to Linear

This is one of the easiest. Find a way to convert from a nonlinear to a linear and then solve it.

Example: The Bernouli equation

$$\frac{dy}{dt} + q(t)y = r(t)y^n. \tag{30.3}$$

Note how the nonlinearity arises from the power n on the last term. If it is zero, or one, we have a linear equation. Everything else is nonlinear.

Let $w = y^{1-n}$, Then $dw = (1-n)y^{-n}dy$, which implies that $\frac{dy}{dt} = \frac{1}{1-n}y^n\frac{dw}{dt}$ or, substituting into the original equation

$$\frac{1}{1-n}y^n\frac{dw}{dt} + q(t)y = r(t)y^n. \tag{30.4}$$

We then divide by y^n and multiply by $1-n$ to obtain

$$\frac{dw}{dt} + (1-n)q(t)y^{1-n} = (1-n)r(t). \tag{30.5}$$

This still looks nonlinear, but express in terms of w to find

$$\frac{dw}{dt} + (1-n)q(t)w = (1-n)r(t) \tag{30.6}$$

which is a first-order *linear* differential equation!

We will have the chance to practice some problems on this type of differential equation in the homework.

30.3 Type 2: Separable

Separable equations occur when M *does not* depend explicitly on y and N *does not* depend explicitly on t. That is, $M(t,y) = M(t)$ and $N(t,y) = N(y)$. In this case, we have that the differential equation can be rewritten as

$$M(t)dt + N(y)dy = 0. \tag{30.7}$$

Move one term to the right and integrate both sides

$$\int_{t_0}^{t} M(\bar{t})d\bar{t} = -\int_{y_0}^{y} N(\bar{y})d\bar{y}, \quad \text{with} \quad y(t_0) = y_0. \tag{30.8}$$

If the integrals can be done, we now are left with an ordinary equation to solve.

Example: $\frac{dy}{dt} = ty^2$, which can be rewritten as $\frac{1}{y^2}dy - tdt = 0$, and we set $y(1) = a$, as the initial condition. Performing the integrals yields

$$\int_{a}^{y} \frac{1}{\bar{y}^2}d\bar{y} = \int_{1}^{t} \bar{t}d\bar{t} \implies -\frac{1}{\bar{y}}\bigg|_a^y = \frac{t^2-1}{2} \tag{30.9}$$

$$\frac{1}{a} - \frac{1}{y} = -\frac{1}{2}(1-t^2) \implies y(t) = \frac{1}{\frac{1}{a}+\frac{1}{2}(1-t^2)} = \frac{2a}{2+a-at^2}, \tag{30.10}$$

so we get

$$y(t) = \frac{2a}{2+a-at^2}. \tag{30.11}$$

Now comes the important point many students forget to do. We need to check to ensure we actually solved the original problem. $y(1) = a$ checks out, as well as

$$\frac{dy}{dt} = \frac{-2a \times 2at}{-(2+a-at^2)^2} = ty^2, \tag{30.12}$$

so everything works as it should.

30.4 Type 3: Reducible to separable

For this case, the restrictions on M and N are a bit looser. We require only that $M(t,y)$ and $N(t,y)$ are *homogeneous of degree k*. You have probably not encountered homogeneous functions yet. This condition requires

$$M(\lambda t, \lambda y) = \lambda^k M(t,y), \quad N(\lambda t, \lambda y) = \lambda^k N(t,y), \tag{30.13}$$

where λ is a number. To solve this type of problem, we let $y(t) = tu(t)$. The original equation

$$M(t,y) + N(t,y)\frac{dy}{dt} = 0, \tag{30.14}$$

will be simplified by this transformation. First compute the derivative of y with respect to t using this substitution:

$$\frac{dy}{dt} = u + t\frac{du}{dt}. \tag{30.15}$$

Next, substitute into the differential equation, to give

$$M(t, y) + N(t, y)\frac{dy}{dt} = M(t, tu) + N(t, tu)(u + t\frac{du}{dt}) = 0. \tag{30.16}$$

Now, we are ready to use the homogeneity. Note we think of $t = \lambda$ to remove the factor of t from both arguments of M and N. This yields

$$t^k M(1, u) + t^k N(1, u)(u + t\frac{du}{dt}) = 0. \tag{30.17}$$

We divide out the common factor of t^k

$$M(1, u) + uN(1, u) + tN(1, u)\frac{du}{dt} = 0. \tag{30.18}$$

Now, rearrange the equation with all u-dependent terms on the right and all t-dependent terms on the left:

$$\frac{dt}{t} = -\frac{N(1, u)}{M(1, u) + uN(1, u)}du. \tag{30.19}$$

Here $\frac{dt}{t}$ only has t dependence and $-\frac{N(1,u)}{M(1,u)+uN(1,u)}$ only has u dependence. Then we just need to integrate both sides

$$\ln\left(\frac{t}{t_0}\right) = -\int_{u_0}^{u} d\bar{u}\frac{N(1, \bar{u})}{M(1, \bar{u}) + \bar{u}N(1, \bar{u})}, \quad \text{with} \quad u_0 = \frac{y(t_0)}{t_0} \tag{30.20}$$

This will solve the problem if we can do the integral and if we can solve the resulting algebraic problem.

Example: $(t^2 + y^2)\,dt + 2ty\,dy = 0$, with $y(t_0) = y_0$ when $t_0 > 0$. First note that $M = t^2 + y^2$ is homogeneous of degree 2 and $N = 2ty$ is also homogeneous of degree 2. Using our "cookbook" solution routine, we have

$$\ln\left(\frac{t}{t_0}\right) = -\int_{u_0}^{u} d\bar{u}\frac{2\bar{u}}{1 + \bar{u}^2 + 2\bar{u}^2} = -\int_{u_0}^{u} d\bar{u}\frac{2\bar{u}}{1 + 3\bar{u}^2}. \tag{30.21}$$

The integral can be done as follows:

$$= -\int_{u_0}^{u} \frac{d\bar{u}^2}{1+3\bar{u}^2} = -\frac{1}{3}\ln(1+3\bar{u}^2)\Big|_{u_0}^{u}. \tag{30.22}$$

Hence, we obtain

$$\left(\frac{t}{t_0}\right)^3 = \frac{1+3(u_0)^2}{1+3u^2}. \tag{30.23}$$

Now, we have to solve the algebraic equation for u (and eventually y):

$$t^3(1+3u^2) = (t_0)^3\left(1+3(u_0)^3\right). \tag{30.24}$$

Substituting in $y = tu$ gives

$$t^3 + 3ty^2 = (t_0)^3 + 3t_0(y_0)^2, \tag{30.25}$$

which is solved by

$$y = \sqrt{\frac{(t_0)^3 - t^3 + 3t_0(y_0)^2}{3t}}. \tag{30.26}$$

Now we check the answer: $y(t_0) = \sqrt{(y_0)^2} = y_0$, which checks out. Also,

$$\frac{dy}{dt} = \frac{1}{2}\frac{\left(-\frac{1}{3t^2}((t_0)^3 + 3t_0(y_0)^2) - \frac{2}{3}t\right)}{\sqrt{\frac{(t_0)^3 - t^3 + 3t_0(y_0)^2}{3t}}}\frac{t}{t} \tag{30.27}$$

$$\implies 2ty\frac{dy}{dt} = -\frac{1}{3t}((t_0)^3 - t^3 + 3t_0(y_0)^2) - t^2 = -(y^2+t^2) \tag{30.28}$$

which also checks out!

30.5 Type 4: Exact differential

This case corresponds to the case when the functions M and N appear to come from derivatives of the same function F. We start with our nonlinear equation

$$M(t,y)dt + N(t,y)dy = 0. \tag{30.29}$$

If we have that M and N are so-called exact differentials, given by

$$M(t,y) = \frac{\partial F(t,y)}{\partial t} \quad \text{and} \quad N(t,y) = \frac{\partial F(t,y)}{\partial y}, \tag{30.30}$$

then, we have

$$M(t,y)dt + N(t,y)dy = \frac{\partial F(t,y)}{\partial t}dt + \frac{\partial F(t,y)}{\partial y}dy \tag{30.31}$$

$$= dF(t,y) = 0 \implies F(t,y) = \text{ constant } = F(t_0, y_0) \tag{30.32}$$

which solves the problem.

Example:

$$(y^2 - t^2)\, dt + 2ty\, dy = 0, \text{ with } y(t_0) = y_0. \tag{30.33}$$

By playing around a bit, we can see immediately that

$$F(t,y) = ty^2 - \frac{1}{3}t^3, \quad \frac{\partial F}{\partial t} = y^2 - t^2, \quad \frac{\partial F}{\partial y} = 2ty. \tag{30.34}$$

So,

$$ty^2 - \frac{1}{3}t^3 = t_0(y_0)^2 - \frac{1}{3}(t_0)^3, \tag{30.35}$$

because $F(t,y)$ is a constant. Rearranging to solve for $y(t)$ gives us

$$y^2 = \frac{t_0(y_0)^2 - \frac{1}{3}((t_0)^3 - t^3)}{t}, \tag{30.36}$$

and

$$y(t) = \sqrt{\frac{3t_0(y_0)^2 - (t_0)^3 + t^3}{3t}}. \tag{30.37}$$

Check: $y(t_0) = \sqrt{(y_0)^2} = y_0$, which checks out. Next, compute the derivative:

$$\frac{dy}{dt} = \frac{1}{2}\frac{-\frac{1}{3t^2}(3t_0(y_0)^2 - (t_0)^3 + t^3) + (\frac{1}{3t}) \times 3t^2}{\sqrt{\frac{3t_0(y_0)^2) - (t_0)^3 + t^3}{3t}}}, \tag{30.38}$$

Multiplying by $2ty$, we find

$$2ty\frac{dy}{dt} = -\frac{1}{3t}\left(3t_0y_0^2 - t_0^3 + t^3\right) + t^2 = -(y^2 - t^2) \tag{30.39}$$

which works!

30.6 Type 5: Reducible to exact

This is both the hardest type to recognize and the hardest to solve.

The technique is motivated by the so-called integrating factor that we used to integrate the linear one-dimensional problems and uses the fact that M and N are not uniquely determined, just their ratio is.

If we find an F and a Q such that

$$\frac{dF(t,y)}{dt} = Q(t,y)M(t,y), \quad \frac{dF(t,y)}{dy} = Q(t,y)N(t,y). \tag{30.40}$$

Then, we have

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial y}dy = 0 \implies \frac{dy}{dt} = -\frac{\frac{\partial F}{\partial t}}{\frac{\partial F}{\partial y}}, \tag{30.41}$$

or

$$\frac{dy}{dt} = -\frac{Q(t,y)M(t,y)}{Q(t,y)N(t,y)} = -\frac{M(t,y)}{N(t,y)}. \tag{30.42}$$

Then $F(t,y) = F(t_0,y_0)$ is a constant yields the solution again. To check if an equation is exact, check the mixed derivatives, which must be equal

$$\frac{\partial^2 F}{\partial t \partial y} = \frac{\partial^2 F}{\partial y \partial t} \implies \frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \tag{30.43}$$

Example:

$$(3t^4 - y)dt + tdy = 0. \tag{30.44}$$

We see that $\tilde{F} = \frac{3}{5}t^5 - yt$ does not work. But,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) = \frac{1}{t}(-1-1) = -\frac{2}{t} = \text{ independent of } y. \tag{30.45}$$

So, let

$$Q(t) = e^{-\int^t \frac{2}{t'}dt'} = e^{-2\ln t} = \frac{1}{t^2} \tag{30.46}$$

be the integrating factor. Multiply by Q to get

$$\left(3t^2 - \frac{y}{t^2}\right)dt + \frac{1}{t}dy = 0. \tag{30.47}$$

Now,

$$\frac{\partial \tilde{M}}{\partial y} = -\frac{1}{t^2} = \frac{\partial \tilde{N}}{\partial t} \tag{30.48}$$

so

$$F = t^3 + \frac{y}{t}. \tag{30.49}$$

And, we have

$$t^3 + \frac{y}{t} = (t_0)^3 + \frac{y_0}{t_0} \implies y(t) = t\left((t_0)^3 - t^3 + \frac{y_0}{t_0}\right). \tag{30.50}$$

Check: $y(t_0) = y_0$ - this checks! Also,

$$\frac{dy}{dt} = \left((t_0)^3 - t^3 + \frac{y_0}{t_0}\right) + t(-3t^2) = -\left(-\frac{y}{t} + 3t^3\right) \implies \frac{1}{t}\frac{dy}{dt} + \left(3t^2 - \frac{y}{t^2}\right) = 0, \tag{30.51}$$

so it checks!