Physics 155 HW # 13

1.) Solve

$$\left(D^8 + 4D^7 - 2D^6 - 20D^5 + D^4 + 40D^3 - 8D^2 - 32D + 16\right) y = 0 \quad (1)$$

where $D = \dfrac{d}{dt}$. Hint: Recall that the roots of the form $\dfrac{p}{q}$ must have p divide the "a_0 term" and q divide the "a_n term"

2.) Solve

$$\left(D^2 - 2D + 5\right) y = te^t \sin(2t) \quad (2)$$

with $y(0) = 0$ and $\dot{y}(0) = 1$.

3.) Abel's identity for the Wronskian - We will work out the explicit result for third-order differential equations, but it holds for all n.

Consider the equation

$$y^{(3)} + a_2(t)y^{(2)} + a_1(t)y^{(1)} + a_0(t)y = 0 \quad (3)$$

a.) The Wronskian $W(t)$ is

$$W(t) = \det \begin{pmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t) \\ y_1^{(2)}(t) & y_2^{(2)}(t) & y_3^{(2)}(t) \end{pmatrix} \quad (4)$$

where $y_i(t)$ satisfy the differential equation.

Show that

$$\frac{dW(t)}{dt} = \det\begin{pmatrix} y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t) \\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t) \\ y_1^{(2)}(t) & y_2^{(2)}(t) & y_3^{(2)}(t) \end{pmatrix} + \det\begin{pmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1^{(2)}(t) & y_2^{(2)}(t) & y_3^{(2)}(t) \\ y_1^{(2)}(t) & y_2^{(2)}(t) & y_3^{(2)}(t) \end{pmatrix}$$

$$+ \det\begin{pmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t) \\ y_1^{(3)}(t) & y_2^{(3)}(t) & y_3^{(3)}(t) \end{pmatrix}$$

That is, the sum of the determinants of three 3×3 matrices with the corresponding 1st, 2nd, and 3rd rows differentiated. Note that the first two determinants vanish.

b.) Substitute in for $y_1^{(3)}, y_2^{(3)}, y_3^{(3)}$ via the differential equation and add appropriate multiples of rows together to show that

$$\frac{dw}{dt} = -a_2(t)W \tag{5}$$

Hence $W(t) = C\exp\left[-\int^t a_2(t')dt'\right]$, which is Abel's identity. Note that this implies either $W(t) = 0$ on the entire interval $a \leq x \leq b$ (since $C = 0$) or $W(t) \neq 0$ anywhere. We proved this result using other techniques in class.

4.) Consider the 2nd-order equation with constant coefficients similar to lab:

$$L\ddot{Q} + R\dot{Q} + \frac{1}{c}Q = E_0 e^{i\omega_0 t} \tag{6}$$

where at the end of the calculations, the physical result will be $\mathrm{Re}\{Q(t)\}$

a.) Show the steady state result is

$$Q(t) = \frac{E_0}{i\omega_0 R - \omega_0^2 L + \frac{1}{c}} e^{i\omega_0 t} \tag{7}$$

by choosing $Q(t) = Ae^{i\omega_0 t}$ and solving for A. This is essentially the method of undetermined coefficients to find the particular solution.

b.) Show that the steady state current is

$$I(t) = \frac{E_0}{R + i\left(\omega_0 L - \frac{1}{\omega_0 c}\right)} e^{i\omega_0 t} \tag{8}$$

$$= \frac{E_0}{\sqrt{R^2 + \left(\omega_0 L - \frac{1}{\omega_0 c}\right)^2}} e^{i(\omega_0 t - \delta)} \tag{9}$$

where $\delta = \tan^{-1}$(something involving ω, R, and c).

5.) a.) Reparameterize the curve

$$\alpha(t) = (\cosh(t), \sinh(t), t) \tag{10}$$

by arc length. Then compute its Frenet-Serret apparatus.

b.) Show that the following curve is a unit speed curve

$$\alpha(s) = \frac{1}{2}\left(s + \sqrt{s^2+1}, \frac{1}{s+\sqrt{s^2+1}}, \sqrt{2}\ln(s+\sqrt{s^2+1})\right) \tag{11}$$

Compute its Frenet-Serret apparatus.

6.) a.) If $\alpha(s)$ is a unit speed curve, prove that

$$\frac{d\vec{\alpha}}{ds} \cdot \left(\frac{d^2\vec{\alpha}}{ds^2} \times \frac{d^3\vec{\alpha}}{ds^3}\right) = \kappa^2\tau \tag{12}$$

b.) Show that the osculating plane through $\alpha(0)$ is $\perp$ to

$$\frac{d\vec{\alpha}}{ds}(0) \times \frac{d^2\vec{\alpha}}{ds^2}(0) \tag{13}$$

if $\kappa \neq 0$.