Chapter 14

Stokes Theorem

14.1 Defining Stokes Theorem

Stokes theorem extends the result that holds for small surfaces given by

$$\oint \vec{F} \cdot \hat{t}\, ds = \int\int \vec{\nabla} \times \vec{F} \cdot \hat{n}\, dS \tag{14.1}$$

to large surfaces. The first integral is a line integral of the vector field dotted into the tangent vector of the curve. The second integral is an area integral of the curl of the vector field dotted into the normal unit vector perpendicular to the area.

Focus on Fig. 14.1. It shows a surface, denoted by the symbol Σ, a normal vector at one point, denoted by $\hat{n}$, and the boundary of the area, denoted $\partial\Sigma$. We consider a line integral over the closed path $\partial\Sigma$.. A capping area is defined to be any area that has the above curve as its boundary. The area given by Σ is an example of a capping area.

If you are having any trouble with this concept, think of putting a rubber sheet or a soap film with its boundary constrained to lie of $\partial\Sigma$. Any shape you can deform to is a capping area.

Stokes theorem says

$$\oint \vec{F} \cdot \hat{t}\, ds = \int\int_{\text{capping area}} \vec{\nabla} \times \vec{F} \cdot \hat{n}\, dS \tag{14.2}$$

To prove Stokes theorem, break the capping area into small pieces. For each

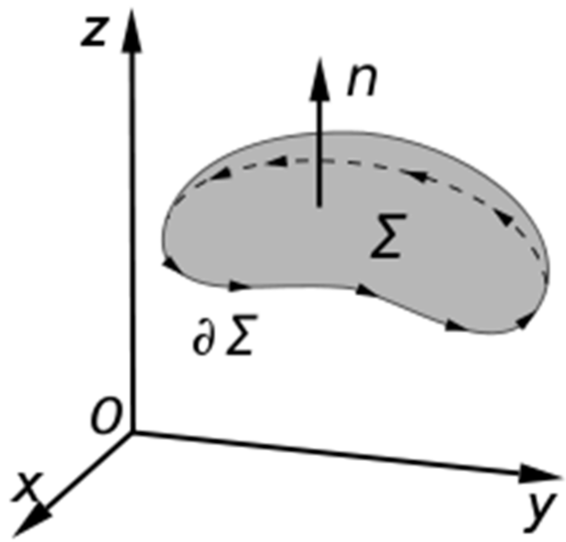

Figure 14.1: Schematic of Stokes theorem. The curve, denoted $\partial\Sigma$, is the line integral over the boundary of the area. The area, denoted Σ, can be thought of as any area that has the underlying boundary curve as its edge.

small piece, or area $\triangle S_i$, we already showed

$$\oint_{\text{piece i}} \vec{F} \cdot \hat{t}\, ds = \int_{\triangle S_i} \vec{\nabla} \times \vec{F} \cdot \hat{n}\, dS. \tag{14.3}$$

Fig. 14.2 shows that when you hook line integrals over different areas together, the integrals over the interior paths cancel (red lines in Fig. 14.2).

So, when we add all up all of the paths, we get a curve that goes around the boundary. We add all areas up and we obtain the surface integral over the full surface. This is Stokes theorem.

One of the hardest concepts to become comfortable with is given by the question What is the meaning of curl? In essence, curl means a vector field has *some kind of rotational character*, but this is subtle. We show four examples of the curl in the next section. You will clearly see that the notion of a curl is not always so obvious.

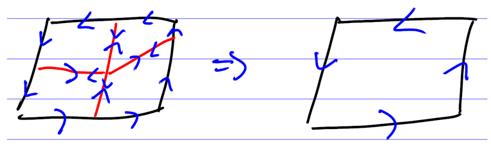

Figure 14.2: Figure illustrating the crux of Stokes theorem. As we integrate over the small line integrals, we find the internal lines (given in red) will have their integrals all cancel, because they are integrated over in both directions. This means the sum over all of those integrals is given by the line integral over the boundary of the curve only,

14.2 Examples of Stokes Theorem

We describe four examples of the curl of vector fields.

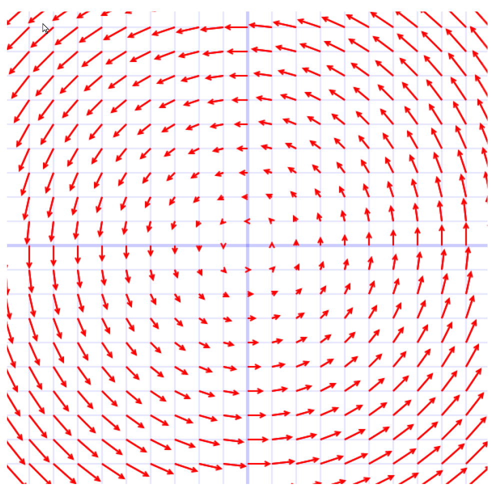

Figure 14.3: The vector field given by $\vec{F} = -y\hat{i} + x\hat{j}$.

Example 1: $\vec{F} = -y\hat{i} + x\hat{j}$ (see Fig. 14.3).

This vector field obviously rotates. Computing the curl (by using the definition in terms of derivatives) yields

$$\vec{\nabla} \times \vec{F} = \hat{k}(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}) = 2\hat{k}. \tag{14.4}$$

Indeed, this vector field, which clearly rotates, has a nonzero curl.

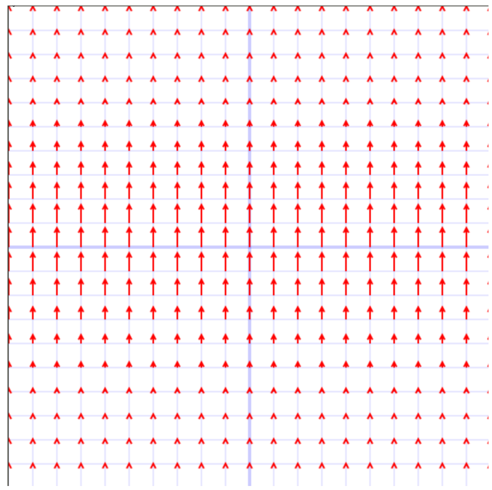

Figure 14.4: The vector field given by $\vec{F} = \hat{\jmath} e^{-0.05y^2}$.

Example 2: $\vec{F} = \hat{\jmath} e^{-0.05y^2}$ (see Fig. 14.4).
Our second example has its vector field satisfy

$$\vec{\nabla} \times \vec{F} = 0, \tag{14.5}$$

because F_y has no x or z dependence. Clearly for this vector field, we can see that there is no rotation, because the direction of the vectors never changes. So this concept of associating curl with rotation seems to work well.

Example 3: $\vec{F} = \hat{\jmath} e^{-0.05x^2}$ (see Fig. 14.5).
The curl for this vector field is given by

$$\vec{\nabla} \times \vec{F} = \hat{k}(-0.1xe^{-x^2}). \tag{14.6}$$

This curl is clearly $\neq 0$. But this "rotation" is subtle, since the direction of the vector field never changes either.

Example 4: $\vec{F} = \frac{-y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}$, with $(x,y) \neq (0,0)$ (see Fig. 14.6).
This vector field clearly looks like it has rotation, but,

$$\vec{\nabla} \times \vec{F} = \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right). \tag{14.7}$$

Evaluating the derivatives gives us

$$\vec{\nabla} \times \vec{F} = \hat{k}\left[\frac{1}{x^2+y^2} - \frac{x \times 2x}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{y \times 2y}{(x^2+y^2)^2}\right]. \tag{14.8}$$

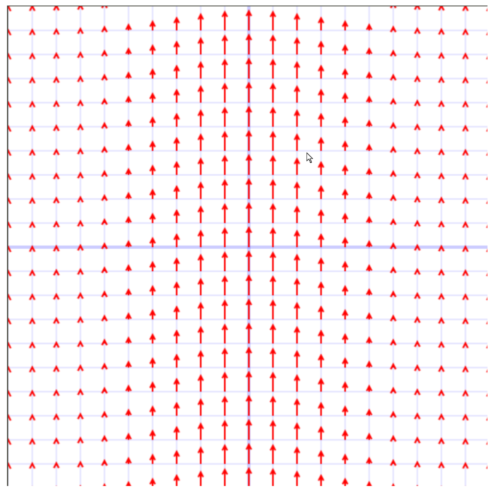

Figure 14.5: The vector field given by $\vec{F} = \hat{j}e^{-0.05x^2}$.

Simplifying, we find

$$\vec{\nabla} \times \vec{F} = \hat{k}\frac{x^2 + y^2 - 2x^2 + x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} = 0. \tag{14.9}$$

This shows us that curl is different from just thinking of it as the rotation of a vector field.

The concept and comfort level of the curl is one of the harder things ti develop as you work with vector fields. Take your time, think carefully, and you will be able to master it.

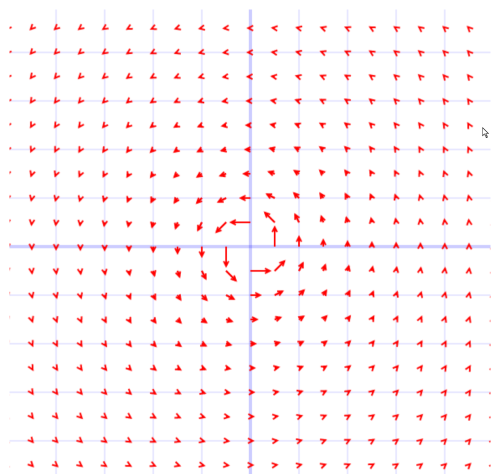

Figure 14.6: The vector field given by $\vec{F} = \frac{-y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}$, with $(x, y) \neq (0, 0)$.