Chapter 36

Dirichlet's Problem and Poisson's Theorem


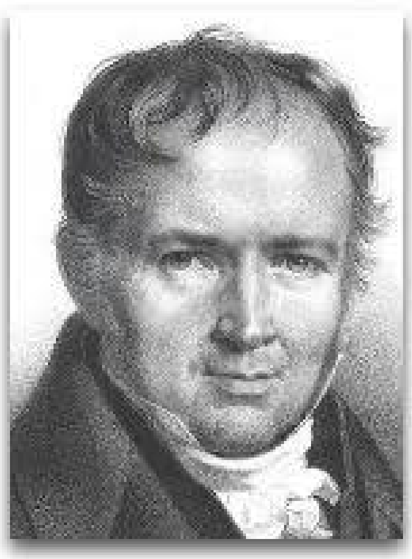

Figure 36.1: (Left) Peter Gustav Dirichlet and (right) Siméon-Denis Poisson. These are two famous mathematicians who worked on the Fourier series problem in the context of heat transport on a disk.

Consider a disc of radius $r = 1$ that has the temperature fixed with the boundary condition on the edge of the disc given by $U(r{=}1, \theta) = f(\theta)$. Here,

$U(r, \theta)$ is the temperature on the disc at the polar coordinates r and θ. Since we are in the steady state, the temperature distribution satisfies Laplace's equation:

$$\nabla^2 U(r, \theta) = 0. \tag{36.1}$$

In polar coordinates, we have that the Laplacian can be written in terms of derivatives with respect to r and θ as follows:

$$\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 U}{\partial \theta^2} = 0, \tag{36.2}$$

as we derived earlier. Note that $U(r, \theta)$ is continuous (and bounded) for $0 \leq r \leq 1$ and is periodic as θ increase by 2π, which implies that $U(r, \theta + 2\pi) = U(r, \theta)$.

We use the method of separation of variables to solve for $U(r, \theta)$. This technique is one to always consider when examining a partial differential equation that has a sum of derivatives with respect to independent variables. The approach makes an *ansatz* that the solution is a product of functions of each of the different independent variables. Here is how it works in this case. We let $U(r, \theta) = R(r)\Theta(\theta)$, then

$$\frac{\partial}{\partial r}\Big(rR'(r)\Theta(\theta)\Big) + \frac{1}{r}R(r)\Theta''(\theta) = 0. \tag{36.3}$$

Our next step is to divide both sides of the equation by $U/r = R\Theta/r$, which yields

$$\frac{r^2R'' + rR'}{R} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0. \tag{36.4}$$

Now we pause to look carefully at each term on the left hand side. The first term is a function of r only and the second term is a function of θ only. But the sum of both terms is equal to zero and this holds for *all* r and θ. This implies that we must have each function equal to opposite constants, so that

$$\frac{r^2R'' + rR'}{R} = n^2 = -\frac{\theta''}{\theta}. \tag{36.5}$$

Here, n^2 is a constant, but we allow it to be positive or negative. So

$$\Theta'' = -n^2\Theta. \tag{36.6}$$

This equation is easily solved; it is a differential equation you have seen many times. We have three cases:

$$\Theta(\theta) = \begin{cases} Ae^{in\theta} + Be^{-in\theta} & n^2 > 0 \\ A + B\theta & n^2 = 0 \\ Ae^{n\theta} + Be^{-n\theta} & n^2 < 0 \end{cases}. \tag{36.7}$$

The R equation becomes

$$r^2 R'' + rR' - n^2 R = 0. \tag{36.8}$$

This is a special type of equation called a Cauchy-Euler equation. You can see that each derivative term is multiplied by a corresponding power of r. Such an equation is solved by r raised to a power that is chosen carefully. The solutions are found as follows. We make the ansatz $R = r^\alpha$, which becomes an algebraic equation for α:

$$R(r) = r^\alpha \implies \alpha(\alpha-1)+\alpha-n^2 = 0 \implies \alpha^2-n^2 = 0 \implies \alpha = \pm n. \tag{36.9}$$

Hence, the solutions of this equation are

$$R(r) = \begin{cases} ar^n + br^{-n} & n^2 > 0 \\ a + b\ln(r) & n^2 = 0 \\ ar^{in} + br^{-in} & n^2 < 0 \end{cases}. \tag{36.10}$$

We now need to determine which of the different possible solutions are the correct ones to solve the problem. First, we need to require the periodicity requirement in θ, given by $\Theta(\theta + 2\pi) = \Theta(\theta)$. This implies that

$$n^2 > 0,\ \Theta(\theta) = Ae^{in\theta} + Be^{-in\theta} \text{ or } \Theta(\theta) = A,\ n^2 = 0. \tag{36.11}$$

The next requirement is that $R(r)$ remains bounded. We find that this implies that

$$R(r) = ar^n,\ n^2 > 0 \text{ or } a,\ n^2 = 0. \tag{36.12}$$

So the general result is

$$U(r,\theta) = \sum_{n=-\infty}^{+\infty} A_n r^{|n|} e^{in\theta}, \tag{36.13}$$

with A_n numbers that will be adjusted to solve the boundary condition at the edge of the disc. Note that the solutions for R with a given n^2 are correlated with the solutions for Θ with the same n^2. This is why the sum takes the given form above. Be sure you understand this clearly.

Then, for $r = 1$, we must require that

$$\sum_{n=-\infty}^{+\infty} A_n e^{in\theta} = f(\theta). \tag{36.14}$$

The question is, for arbitrary $f(\theta)$, can we find A_n's such that

$$f(\theta) = \sum_{n=-\infty}^{\infty} A_n e^{in\theta}. \tag{36.15}$$

This series is called the Fourier series. Obviously we can't make this work for arbitrary $f(\theta)$, but if $f(\theta)$ is continuous, then the answer turns out to be yes! It takes some work to show this.

Since we know the integral of the exponential function over 2π

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, e^{in\theta} = \begin{cases} 0, n \neq 0 \\ 1, n = 0 \end{cases} \tag{36.16}$$

We next interchange the summation and integration in the Fourier series to find

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta f(\theta) e^{-iN\theta} = \sum_{n=-\infty}^{+\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta A_n e^{i(n-N)\theta} = A_N. \tag{36.17}$$

So the coefficients are found via

$$A_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta f(\theta) e^{-iN\theta}. \tag{36.18}$$

But can this interchange be justified? We show how to establish this rigorously using Poisson's kernel. Recall the geometric series, given by $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z| < 1$. So,

$$\sum_{n=-\infty}^{+\infty} r^{|n|} e^{in\theta} = \sum_{n=0}^{\infty} r^n e^{in\theta} + \sum_{n=-\infty}^{-1} r^{-n} e^{in\theta}. \tag{36.19}$$

When $r < 1$, we have

$$\sum_{n=-\infty}^{+\infty} r^{|n|} e^{in\theta} = \frac{1}{1-re^{i\theta}} + \frac{1}{1-re^{-i\theta}} - 1 = \frac{2-2r\cos(\theta)}{1-2r\cos(\theta)+r^2} - 1 \tag{36.20}$$

(you need to recall that $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$). This sum is called Poisson's kernel $P(r,\theta)$ and is defined via

$$\frac{1-r^2}{1-2r\cos(\theta)+r^2} = 2\pi P(r,\theta). \tag{36.21}$$

Poisson's kernel satisfies the following properties:

1. $P(r,\theta+2\pi) = P(r,\theta)$
2. $P(r,-\theta) = P(r,\theta)$
3. The maximum of $P(r,\theta)$ is at $\theta = 0$ and equals $\frac{1}{2\pi}\frac{1+r}{1-r}$
4. $P(r,\theta)$ monotonically decreases from $\theta = 0$ to $\theta = \pi$. The minimum is $\frac{1}{2\pi}\frac{1-r}{1+r}$.
5. $\int_{-\pi}^{\pi} P(r,\theta)d\theta = 1$.

These results can be easily verified except for the last one, which requires significantly more work (it is probably easiest to carry out the integral using residues).

For our next step, observe that

$$P(r,\theta) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} r^{|n|}e^{in\theta} \tag{36.22}$$

so we have that

$$\begin{aligned}\int_{-\pi}^{\pi} d\theta\, P(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \sum_{n=-\infty}^{+\infty} r^{|n|}e^{in\theta} \\ &= \sum_{n=-\infty}^{+\infty} r^{|n|}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta e^{in\theta} = \sum_{n=-\infty}^{+\infty} r^{|n|}\delta_{n,0} = 1.\end{aligned} \tag{36.23}$$

These results are all rigorously valid for $r < 1$.

So we consider the summation with A_n and substitute in our guess in terms of an integral of $f(\theta)$

$$\sum_{n=-\infty}^{\infty} r^{|n|}A_n e^{in\theta} = \sum_{n=-\infty}^{+\infty} r^{|n|}e^{in\theta}\frac{1}{2\pi}\int_{-\pi}^{\pi} dt f(t)e^{-int} \tag{36.24}$$

For $r < 1$, we have

$$\sum_{n=-\infty}^{\infty} r^{|n|} A_n e^{in\theta} = \int_{-\pi}^{\pi} dt \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} r^{|n|} e^{in(\theta-t)} f(t). \tag{36.25}$$

Now, we recall that $\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} r^{|n|} e^{in(\theta-t)} = P(r, \theta - t)$. We then get

$$\sum_{n=-\infty}^{\infty} r^{|n|} A_n e^{in\theta} = \int_{-\pi}^{\pi} dt P(r, \theta - t) f(t). \tag{36.26}$$

Now we want to consider what happens as $r \to 1$. Note that as $r \to 1$ one can see that $P(r, \theta - t)$ becomes very strongly peaked about $\theta = t$ (the maximum goes to ∞, while the integral over t remains 1).

We now want to show that $U(r, \theta)$ does approach $f(\theta)$ as $r \to 1$. To do this, we compute the absolute value of the difference

$$\begin{aligned} |U(r,\theta) - f(\theta)| &= \left| \int_{\theta-\pi}^{\theta+\pi} dt P(r, \theta - t)[f(t) - f(\theta)] \right| \\ &\leq \int_{\theta-\pi}^{\theta+\pi} dt P(r, \theta - t)|f(t) - f(\theta)|. \end{aligned} \tag{36.27}$$

The bound arises because the integral of P is one, P is nonnegative, and it is periodic in θ.

Now consider an interval of size δ about $t = \theta$: $|\theta - t| \leq \delta$. In this interval, the integral is dominated by the maximum of P. The remainder of the integral is dominated by $P(r, \delta)$, because P is monotonic. Note that if $|\theta - t| < \delta$ then $|f(t) - f(\theta)| < \frac{\epsilon}{2}$ for some ϵ and we can make ϵ as small as desired by reducing δ. So,

$$\int_{|\theta-t|\leq\delta} dt P(r, \theta - t)|f(t) - f(\theta)| \leq \int_{|\theta-t|\leq\delta} dt P(r, \theta - t)\frac{\epsilon}{2} \leq \frac{\epsilon}{2} \tag{36.28}$$

because $P(r, \theta - t) \to P \geq 0$ and when Poisson's kernel is integrated over all t it equals 1. The other piece of the integral satisfies

$$\int_{\delta\leq|\theta-t|\leq\pi} dt P(r, \theta - t)|f(t) - f(\theta)| \leq \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos(\delta) + r^2} \int_{-\pi}^{\pi} |f(t) - f(\theta)| dt. \tag{36.29}$$

Note that the first term on the RHS vanishes as $r \to 1$, so for $r > r_0$ the RHS is bounded by $\frac{\epsilon}{2}$ and the total integral is then less than ϵ. Hence, we have shown that

$$|U(r,\theta) - f(\theta)| < \epsilon \text{ for } 1 \geq r \geq r_0 \implies \lim_{r \to 1} |U(r,\theta) - f(\theta)| = 0. \quad (36.30)$$

This establishes that we have indeed found the solution to Dirichlet's problem.

One can prove that this solution is unique. (See section 1-7 of Seeley). One can also show that the Fourier series converges pointwise if $f(\theta)$ is differentiable. This proof is a bit more complicated and it is given in section 1-8 of Seeley.

Note that there are discontinuous (or piecewise differentiable) functions that have the Fourier series converge to the wrong result at the point of discontinuity. This is called the Gibbs phenomenon and it corresponds to an approximate 10 percent error at the point of discontinuity (elsewhere the Fourier series converges as we showed above).

Many of the notions of continuity and piecewise continuous or differentiable functions comes from the results of studying this problem to try to determine when and how the Fourier series converges. We do not have the time to go through these in detail though. If you fin this interesting, you should examine these ideas further in either a differential topology course or a real analysis course.