Devious Math Tricks

0.1 Introduction

Before jumping into the main text, we start with a primer on mathematical tools that you can use to make your mathematical work easier, faster, and more accurate. I hope you enjoy these ideas and have a chance to implement them through the course. They take some time to master, but are well worth it.

Remember that kid from algebra class who sat across from you and seemed to know the answers to every question faster and more accurately than you? I am going to let you in on a secret—most likely that “curve buster” was not a genius or born to do math—instead he or she simply learned a number of devious tricks to get through the material faster and with fewer errors. After finishing this chapter, you will be able to do so too!

The biggest myth to dispel is that geniuses in algebra have photographic memories and can recall the relevant formulas and apply them at will. Such people may exist but the brainiac across the row from you was not one of these. The reality is you need to only know something like ten important equations, and by using them as I am about to describe to you, you will find that you too will just know them when you need them. You too can become an algebra master.

0.2 Devious trick number 1 (Add zero)

$x+0=x$. This is called the ”add zero” trick. It comes up so many times you cannot get through most problems without encountering it. The problem is no one has told you about it, so you often stumble onto it and fail to recognize its importance. We will see and use the add zero trick again and again. Right now, let me illustrate it for you. Consider the following problem: factor $x^2 - 25$.

Go ahead and give it a try, but do not just write down the answer if you have memorized it, see if you can derive it. Here is how the add zero trick works. An educated guess says we should add a number times x as one of the terms, the 25, suggests. We add 5x- 5x to get:

$$x^2 + 5x - 5x - 25 = (x^2 + 5x) - (5x + 25) \tag{1}$$

$$= x(x+5) - 5(x+5) \tag{2}$$

$$= (x-5)(x+5). \tag{3}$$

All we did was add zero! The creativity was to recognize what zero to add. Indeed this is how you apply the technique. You learn pretty quickly what ideas might work for what to add by practicing the method a number of times.

0.3 Devious trick number 2 (Multiply by 1)

$x \times 1 = x$. Just like the "add zero" trick, the "multiply by one" trick can be used to simplify many different expressions. It often is seen in cases where we want to rationalize a denominator, such as the following:
Rationalize the denominator of $\frac{1-\sqrt{3}}{1+2\sqrt{3}}$.

The strategy is to multiply the numerator and denominator by the same factor such that the denominator no longer has a square root. Recalling that $(a+b)(a-b) = a^2 - b^2$, our strategy is to use $1 = \frac{1-2\sqrt{3}}{1-2\sqrt{3}}$ as the term we multiply by. Now we compute:

$$\frac{1-\sqrt{3}}{1+2\sqrt{3}} \times \frac{1-2\sqrt{3}}{1-2\sqrt{3}} = \frac{1-\sqrt{3}-2\sqrt{3}+6}{1-12} = -\frac{7-3\sqrt{3}}{11}. \tag{4}$$

This "multiply by one" trick can also be used to simplify products of integers via factorials. For example:

$$(n+1)(n+2)...(2n-1)(2n) = \frac{1 \times 2 \times 3...n}{1 \times 2 \times 3...n}(n+1)(n+2)...(2n-1)(2n) = \frac{(2n)!}{n!} \tag{5}$$

0.4 Devious trick number 3 (Manipulate identities)

Remember simple identities and use them to make complicated identities. As I mentioned before, there are only a handful of identities worth remembering. Nearly everything else can be re-derived from these few. As an example, think about your trigonometry identities. There are loads upon loads of them. But they all follow from two main results—remembering the definitions of the trig functions in terms of sin and cos and using the fact that $\sin^2 + \cos^2 = 1$. Let's illustrate how. Suppose I want to express $\sec^2(\theta)$ in terms of $\tan(\theta)$. I simply recall that $\sec(\theta) = \frac{1}{\cos(\theta)}$. Then I use our "multiply by one" trick using $1 = \cos^2(\theta) + \sin^2(\theta)$, so

$$\begin{aligned}\sec^2(\theta) &= \frac{1}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}\left(\cos^2(\theta) + \sin^2(\theta)\right) = \frac{\cos^2(\theta)}{\cos^2(\theta)} + \frac{\sin^2(\theta)}{\cos^2(\theta)} \\ &= 1 + \tan^2(\theta). \end{aligned} \tag{6}$$

0.5 Devious trick number 4 (Recognize abstracted identities)

Recognize simple identities when expressed in a more complex form. This one takes a bit more practice, as it requires you to use abstraction to recognize the simple identity. Here is an example. We already saw that $x^2 - 25 = (x+5)(x-5)$. We need to recognize it in other contexts. So

$$\frac{\sin^2(\theta) - 25}{\cos^2(\theta)} = \tan^2(\theta) - 25\sec^2(\theta) = (\tan(\theta) + 5\sec(\theta))(\tan(\theta) - 5\sec(\theta)) \tag{7}$$

will also hold. A more complicated example is to factor $(x^4 + 2x^2 - 24)$. We use the add zero trick first:

$$\begin{aligned}\left(x^4 + 2x^2 + (1-1) - 24\right) &= \left((x^2+1)^2 - 25\right) = (x^2+1+5)(x^2+1-5) \\ &= (x^2+6)(x^2-4). \end{aligned} \tag{8}$$

Note how we had to recognize $x^4 + 2x^2 + 1 = (x^2+1)^2$ was the square in the expression.

0.6 Devious trick number 5 (subtract identities)

Use information you know and change expressions to those you and know and their difference. This one is a little more difficult to recognize but is quite important. We will illustrate by example. Consider evaluating the following:

$$x^2 + x^4 + x^6 + ... = \sum_{n=1}^{\infty} x^{2n}. \tag{9}$$

This looks like a geometric series, but for x^2, not x. In other words,

$$\sum_{n=1}^{\infty} x^{2n} = \sum_{n=1}^{\infty} (x^2)^n. \tag{10}$$

This would be the geometric series except it is missing the first term. Hence,

$$\sum_{n=1}^{\infty} x^{2n} = \sum_{n=0}^{\infty} (x^2)^n - 1 = \frac{1}{1-x^2} - 1 = \frac{x^2}{1-x^2}. \tag{11}$$

We also could find it another way, by recognizing that

$$x^2 + x^4 + x^6 + ... = x^2(1 + x^2 + x^4 + ...) = x^2 \sum_{n=0}^{\infty} (x^2)^n = x^2 \frac{1}{1-x^2} = \frac{x^2}{1-x^2}, \tag{12}$$

as before.

0.7 Devious trick number 6 (index gymnastics)

We end by discussing one other skill, which involves shifting indices in summations. Students often struggle with this skill. Yet it is fairly easy to master. Let's look at the last example:

$$x^2 + x^4 + x^6 + ... = \sum_{n=1}^{\infty} x^{2n}. \tag{13}$$

We know the geometric series starts with $n = 0$, not $n = 1$. So let's shift $n \to n' + 1$. Then n' runs from 0 to ∞. We obtain:

$$\sum_{n=1}^{\infty} x^{2n} = \sum_{n'=0}^{\infty} x^{2(n'+1)} = \sum_{n'=0}^{\infty} x^{2n'} x^2 = \frac{1}{1-x^2} x^2 = \frac{x^2}{1-x^2}, \tag{14}$$

just like before. We will see that this index shifting skill becomes critically important in mastering many of the identities we develop later. I will help you recognize and develop this skill as we move forward.

0.8 Important identities

We end this section with the important identities you should know already (some others will be developed in this chapter). The main ones are the following:

1) $ax^2 + bx + c$ has two roots given by $r_{\pm} = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$

2) $\cos^2(\theta) + \sin^2(\theta) = 1$

3) $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$, $\sec(\theta) = \frac{1}{\cos(\theta)}$, and $\csc(\theta) = \frac{1}{\sin(\theta)}$

4) $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$, $\quad \sin(2\theta) = 2\cos(\theta)\sin(\theta)$

5) $x^2 - a^2 = (x+a)(x-a)$

6) $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, $\quad \sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$

7) $\sum_{n=0}^{N} \binom{N}{n} x^{N-n} y^n = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} x^{N-n} y^n = (x+y)^N$ (the binomial theorem)

8) $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

Ok, now you are armed with the skills to tackle some tough algebra problems. Let's give it a try.

0.9 Problems

These problems are optional. But I encourage you to try them to hone your devious algebra trickery...

1. Factor $(x^2 + 2xy + y^2 - 9)$
2. Use the double angle formula, in the form $\cos(\theta) = \cos^2(\frac{\theta}{2}) - \sin^2(\frac{\theta}{2})$ to find the relation
$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{\cos(\theta) + 1}{2}}$$

Hint: $\cos^2(\frac{\theta}{2})+\sin^2(\frac{\theta}{2}) = 1$ will help. How do you pick the correct sign?

3. Use the same double angle formula to find $\sin(\frac{\theta}{2})$. Verify your results for 2) and 3) satisfy $\cos^2(\frac{\theta}{2}) + \sin^2(\frac{\theta}{2}) = 1$

4. Show that $\sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^n = \frac{e^x - 1}{x}$.
Hint: Use formula (8) above.

5. This is a challenging one. Show that

$$\sqrt{3 + 2\sqrt{2}} = \pm(1 + \sqrt{2})$$

by using the add zero trick to establish that the argument of the square root on the left hand side is a perfect square.

6. Use the double angle formulas for sine and cosine to show

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$$

7. Use multiplication by $(1 - x)$ (trick used in identity (6) above for the appropriate x) to show that:

$$1 + \sqrt[4]{2} + \sqrt{2} + \sqrt[4]{8} = \frac{1}{\sqrt[4]{2} - 1}$$

8. Show that

$$\frac{(2n)!}{2^n n!} = 1 \times 3 \times 5 \times 7 \cdots (2n-3) \times (2n-1) = (2n-1)!!$$

where the last equality defines the double factorial !! (the double factorial skips integers, so it is a product of all even or all odd integers only).

9. Show that

$$\sum_{n=1}^{\infty} (\cos(\theta))^{2n} = \frac{1}{\tan^2(\theta)}$$

Hint: Use the geometric series.