

Reductio ad absurdum

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Outline

Reductio ad Absurdum

Balls in Boxes

Numbers in Tables

Pigeonhole Principle

An $(-1,0,1)$ Antimagic Square

Handshakes

Reductio ad absurdum

- How to prove that something is true?

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- Show that the opposite is impossible!

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- Show that the opposite is impossible!
- This method is called **reductio ad absurdum** or **proof by contradiction**
- One of the base methods of reasoning: is used everywhere
- Is often combined with other methods
- We will use constantly throughout our courses

Socratic Method

- Reductio ad absurdum is classic: used in Socratic method (Plato, ~400 BC)



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Socratic Method

- Reductio ad absurdum is classic: used in Socratic method (Plato, ~400 BC)
- Socrates revealed contradictions in his students' beliefs by asking them questions step by step



Example

Problem

There are boys and girls in the class. They are divided into two groups for the foreign language: there are students studying French, and there are students studying German. Each student picks one of the two languages. Show that there is a boy and a girl who study different languages.

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Seems impossible at first: we know basically nothing and we claim something nontrivial!

Solution

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Suppose the statement is **wrong**: there are **no** boy and girl studying different languages

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Everyone learns French!

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Puzzle

We have 10 boxes filled with lots of white balls. We additionally have 30 black balls. We want to distinguish boxes by placing different number of black balls in all of them. Can we do that?

Suppose we can do it and let's see what happens
Let us enumerate all boxes in the increasing order of the
number of black balls

[illegible]

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1	2	3	4	5	6	7	8	9	10
≥ 0	≥ 1								

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1	2	3	4	5	6	7	8	9	10
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$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \boxed{\geq 0} + \boxed{\geq 1} + \boxed{\geq 2} + \boxed{\geq 3} + \boxed{\geq 4} + \boxed{\geq 5} + \boxed{\geq 6} + \boxed{\geq 7} + \boxed{\geq 8} + \boxed{\geq 9} \geq 45 \end{array}$$

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Handshakes

Numbers in Boxes

Puzzle

There is a sequence of 10 cells, the leftmost contains number 1 and the rightmost cell contains 30. Is it possible to fill other cells with numbers in such a way that consecutive numbers differ by at most 3?

1									30
---	--	--	--	--	--	--	--	--	----

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1	4	7	10	13					30
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There is a sequence of 10 cells, the leftmost contains number 1 and the rightmost cell contains 30. Is it possible to fill other cells with numbers in such a way that consecutive numbers differ by at most 3?

1	4	7	10	13	16				30
---	---	---	----	----	----	--	--	--	----

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1	4	7	10	13	16	19			30
---	---	---	----	----	----	----	--	--	----

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1	4	7	10	13	16	19	22		30
---	---	---	----	----	----	----	----	--	----

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There is a sequence of 10 cells, the leftmost contains number 1 and the rightmost cell contains 30. Is it possible to fill other cells with numbers in such a way that consecutive numbers differ by at most 3?

1	4	7	10	13	16	19	22	25	30
---	---	---	----	----	----	----	----	----	----

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---	---	---	----	----	----	----	----	----	----

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Numbers in the cells grow too slow

This is a very common trick to estimate the running time of some algorithm

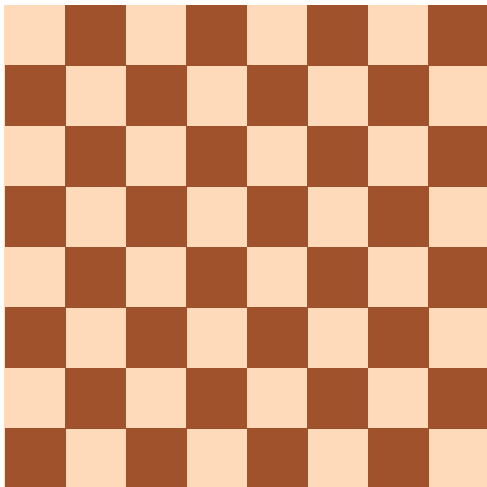
Numbers on the Chessboard

Puzzle

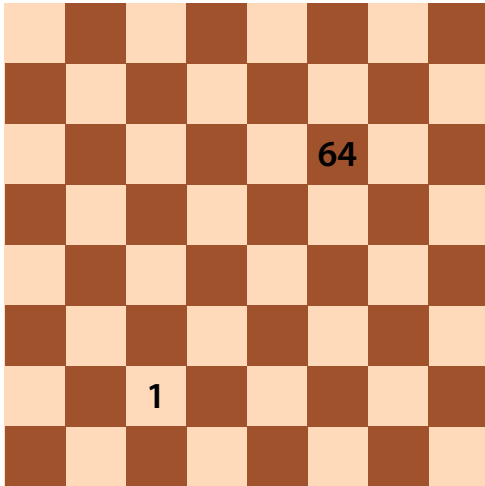
Is it possible to put numbers $1, 2, \dots, 64$ on the chessboard in such a way that neighbors (sharing a side) differ by at most 4?

Again, suppose we can do it and let's see what happens

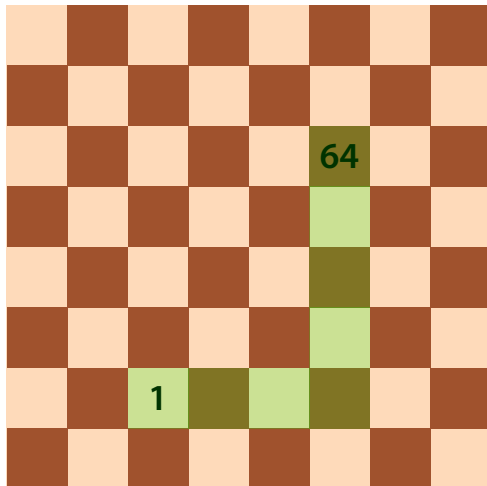
Numbers on the Chessboard



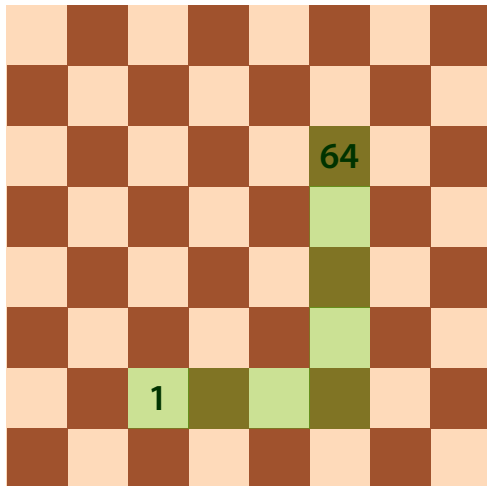
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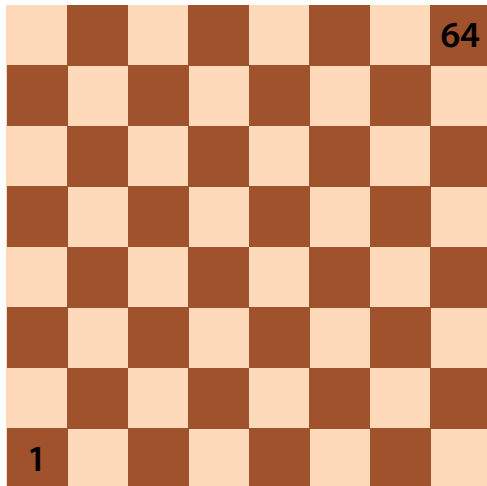


Numbers on the Chessboard

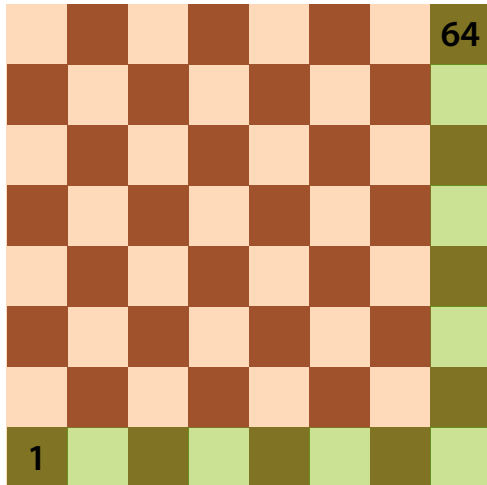


We need 7 steps to get from 1 to 64 in this example

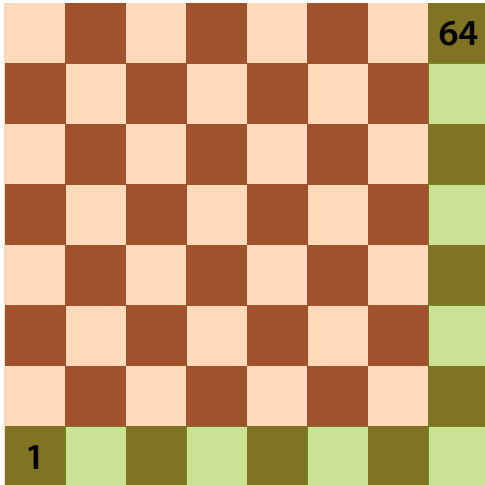
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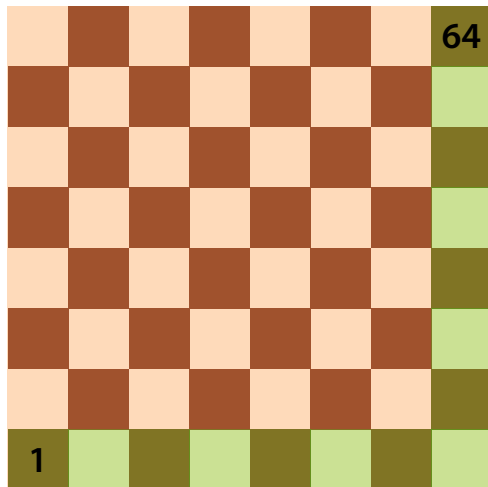


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Numbers on the Chessboard



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Numbers on the Chessboard

							64
							53
							49
							45
							41
							37
							33
1	5	9	13	17	21	25	29

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Handshakes

Same Number of Hairs

Problem

Show that there are two people in New York City with the same number of hairs.



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- There are way more people in NYC than possible numbers of hairs!

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- Wikipedia says 8,537,673 (as of 2016)
- How many hairs does a person have?
- Wikipedia says 150,000 at most
- There are way more people in NYC than possible numbers of hairs!
- Thus there should be people with the same number of hairs

Pigeonhole Principle

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Suppose there are n pigeonholes and $n + 1$ pigeon. Then one of pigeonholes must be occupied by at least two pigeons.



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- Proof by contradiction: if there is at most one pigeon in each hole, then summing up we have at most n pigeons in all holes in total

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- Simple, but very useful principle
- Proof by contradiction: if there is **at most one** pigeon in each hole, then summing up we have **at most n** pigeons in all holes in total
- In the previous example **people** of NYC are **"pigeons"** and **possible numbers of hairs** are **"holes"**

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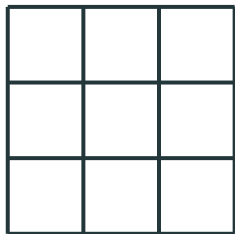
An $(-1,0,1)$ Antimagic Square

Handshakes

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Is it possible to fill a 3×3 table by numbers $-1, 0$ and 1 so that the sum of each row, each column and both diagonals produce different numbers?



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- The sum over each line ranges in between -3 and 3 , **7 values in total**

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- Thus there should be at least two equal sums

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- We are experienced: let's count

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- Each person could have shaken 0 to 29 hands. 30 options in total ("pigeonholes")
- It does not seem to work. Or does it?
- Note: 0 and 29 handshakes are impossible simultaneously! Now we have 29 pigeonholes!

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Conclusion

- **Proof by contradiction**
- A basic argument: used everywhere, usually combined with other ideas
- But sometimes can help a lot on its own
- **Pigeonhole principle**: one of the most basic proof ideas
- Basically amounts to counting