Chapter 10

Vector-valued functions

10.1 Vector fields

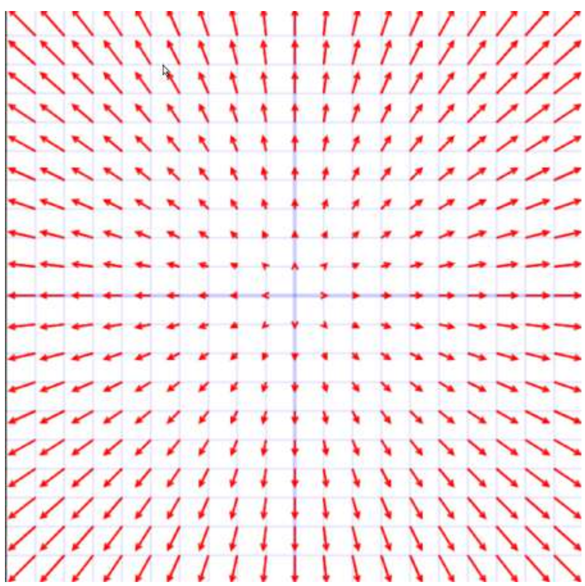

Figure 10.1: Vector field $x\hat{i} + y\hat{j}$.

You have been studying functions throughout calculus. In most cases the result of the function is a real number. Vector-valued functions, on the other hand map to a vector. The easiest way to visualize such functions is when they map a number on the plane to a vector. This is called a vector field.

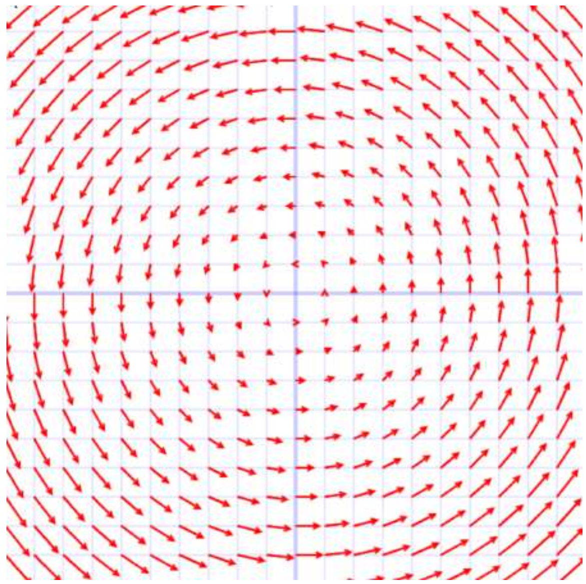

Figure 10.2: Vector field $-y\hat{i} + x\hat{j}$.

At each point, the point is mapped to a vector, which can be illustrated as a small arrow (of the size and direction of the vector value), located at the point.

There is an on-line internet tool that can be used to plot these vector fields. It is https://kevinmehall.net/p/equationexplorer/index.html .

Let's look at some examples of vector fields. These are all taken from the website above. We plot three vector fields: $x\hat{i} + y\hat{j}$ in Fig. 10.1; $-y\hat{i} + x\hat{j}$ in Fig. 10.2; and $-(x-2)\hat{i} - y\hat{j}$ in Fig. 10.3.

As we look at these different vector fields, we get sense of "motion" in them. In Fig. 10.1, we get the sense of stuff moving out from the origin. In Fig. 10.2, we get the feeling of rotation around the origin. In Fig. 10.3, we get a sense of stretching in the vertical and compression in the horizontal.

Be sure that you can describe in words exactly what a vector field is. Best if you use your own words. For me, I like to say a vector field is a little arrow that has a direction and length that I draw at each point in space.

One of the aspects of vector fields, that we saw in the three images, was that they convey different senses of "motion." There is a sense of expansion (almost like an explosion). The mathematical way to describe this is by using a concept called divergence. We also saw a sense of rotation. In

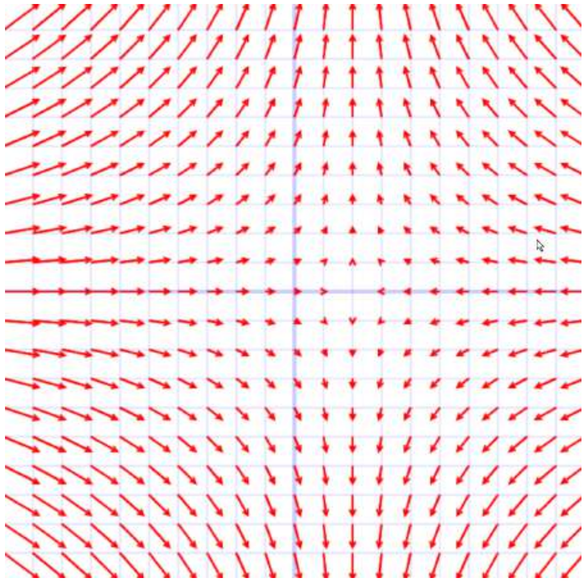

Figure 10.3: Vector field $-(x-2)\hat{\imath}+y\hat{\jmath}$.

mathematics, we use the concept of curl to describe it. We will define these concepts precisely in due course. For now, we will focus on some applications and examples of vector fields.

We begin with Coulomb's Law: $\vec{F}_E = q_0\vec{E}(\vec{r})$, which expresses the electric force as the charge of the particle q_0 multiplied by the electric field $\vec{E}$. The electric field, in SI units, is the following for a charge q_1 sitting at the location $\vec{r}_1$: $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q_1}{|\vec{r}-\vec{r_1}|^2}$. Hence, if we have a charge distribution $\rho_E(\vec{r}\,')$, then the electric field is found by summing (really integrating) all of these little charges described by $\rho_e(\vec{r}\,')$, or

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r'\,\frac{\rho_e(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}\vec{e}_{\vec{r}-\vec{r}\,'}, \tag{10.1}$$

with $\vec{e}_{\vec{r}-\vec{r}\,'}$ the unit vector pointing in the direction of $\vec{r}-\vec{r}\,'$. Note that this integral is quite difficult to carry out because it is integrating *vectors*—we need to worry about where the unit vector $\vec{e}_{\vec{r}-\vec{r}\,'}$ is pointing when summing over all of the charges in the charge distribution.

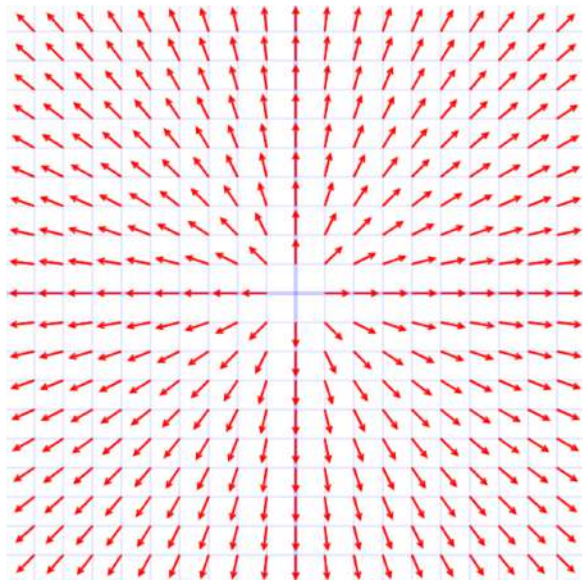

Figure 10.4: Radial vector field composed of unit vectors. Note how it is undefined at the origin, because we cannot tell what direction the vector points at when we are at the origin.

10.2 Examples of vector fields

Example 1: Find a formula for a vector field that is a unit vector in the radial direction in $2d$.

Recall that the vector $\vec{r} = x\hat{i} + y\hat{j}$. The length of the vector is $r = \sqrt{x^2 = y^2}$, so this radial unit vector field becomes

$$\vec{E}(x, y) = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}}, \tag{10.2}$$

which is plotted in Fig.10.4.

What happens as $x, y \to 0$? This point is a singular point, as shown in Fig. 10.4. Note that singular points do not need to have a vector diverging to infinity. Simply having a point where the vector field is undefined because we cannot choose its direction is enough. This is a real singularity.

Example 2: Find a vector field which points at $45°$ to the x-axis and whose magnitude is $(x + y)^2$ at x, y.

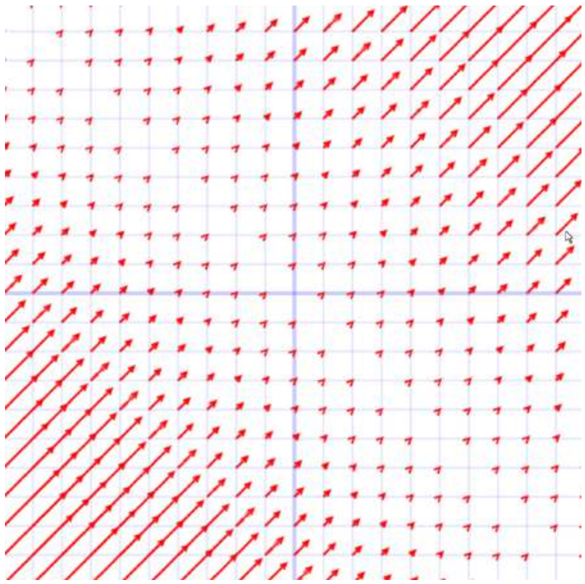

Figure 10.5: The vector field that points at 45° and has a magnitude $(x+y)^2$.

The vector field has each component the same to have them point in a 45° direction. We pick the magnitude to yield the required $(x+y)^2$. This gives us

$$\vec{E}(x,y) = \frac{(x+y)^2}{\sqrt{2}}\hat{i} + \frac{(x+y)^2}{\sqrt{2}}\hat{j}, \tag{10.3}$$

which is plotted in Fig. 10.5. Note how the vector field goes to zero along the line $x = -y$. But there is no singularity here, because a zero vector has no direction.

Example 3: Construct a vector field that points tangential to a circle and has a magnitude given by the distance from the origin.

The vector field is pointing in a tangential direction. This is given by a vector field that behaves as $-y\hat{i} + x\hat{j}$. Computing the length of the vector at that point, we find its length is r. So this is the vector field we are after. It is already plotted in Fig. 10.2.

$$\vec{E}(x,y) = -y\hat{i} + x\hat{j}. \tag{10.4}$$

We are not yet going to define the mathematical terms of divergence and of curl, but we will depict them in images. The divergence is like an "explosion". The arrows all point outwards as if the origin is a source and

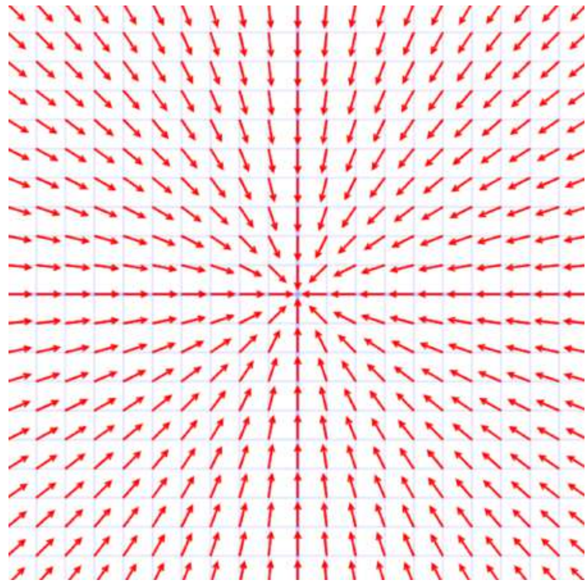

Figure 10.6: Example of a drain. This case is a singular case because the arrows all have the same magnitude. The image of water flowing into a drain at the origin is clear.

everything moves outwards away from it. We already have seen two images of this, in Fig. 10.1 and Fig. 10.4. The difference between these two is just the magnitudes of the arrows. In one case they approach zero at the origin (and there is no singularity), while in the other case they are the same magnitude as we approach the origin (and there is a singularity).

The opposite of a source is a drain. Here, everything flows into the origin, just like water flowing down a drain. We show an example in Fig. 10.6 of the singular version of this vector fields, where every term has a unit magnitude.

The concept of a curl is a bit more complicated. We have one example already in Fig. 10.2 that clearly rotates and has a curl that is nonzero. But there are some vector fields that look like they rotate, but actually have no curl. So, once we define curl, one needs to carefully use the definition to determine whether a vector field actually has a nonzero curl.

We do have another example of a curl, which may not seem obvious at all. This is shown in Fig. 10.7. While there is no clear rotation of these vectors, if we look at the horizontal row of vectors and we move down the page, we clearly see an over all rotation in a counter-clockwise direction by a total of

180°. It turns out that this vector field also has a nonzero curl.

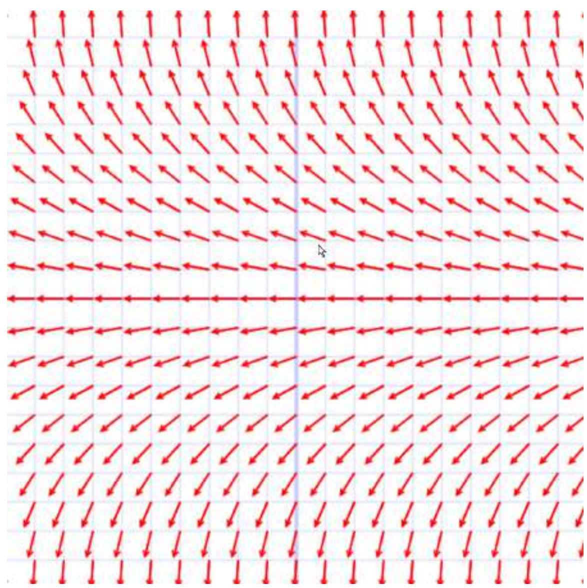

Figure 10.7: A vector field with nonzero curl but no obvious global rotation to it. The rotation is from one horizontal row to another.

We end with a simple summary of our terms:
Vector fields—a vector-valued function at each point in space.
Electric field—a force on an infinitesimal test charge divided by the magnitude of the test charge.
Electric force—the electric field times the charge.
One can compute the electric field from the charge density of a set of charged objects. But the integral is not easy to evaluate, as we mentioned above.