Chapter 26

Change of Bases

26.1 Linear maps

We consider linear transformations from a vector space U (dimension n) to a vector space V (dimension m) that is linear (see Fig. 26.1 for a schematic).

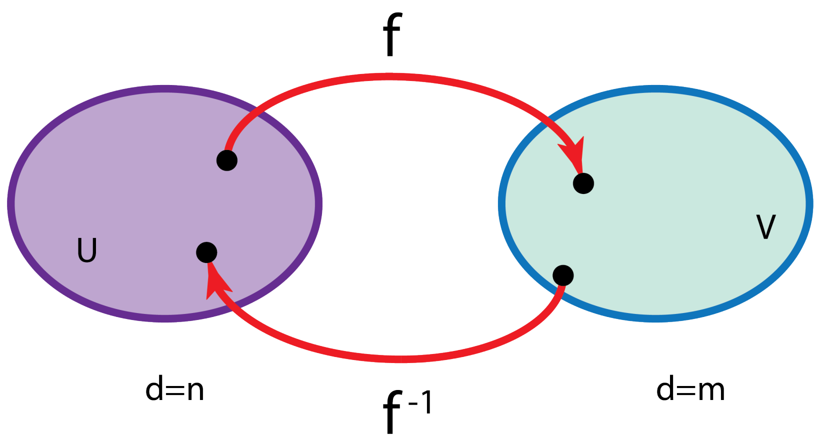

Figure 26.1: Schematic of a map f between two vector spaces U and V. The map produces an element in V for each element in U. In some cases, the map is invertible, so that one can find the inverse map (from V to U) such that $f^{-1}(f(u)) = u$. In this figure, the dimensions of the vector spaces are n and m, respectively. When an inverse exists, we necessarily have $n = m$.

The map f is *linear* if $f(a\vec{u}_1 + b\vec{u}_2) = af(\vec{u}_1) + bf(\vec{u}_2)$. The null space is defined to be the subspace of U consisting of all vectors in U that map to $\vec{0}$

in V. If the null space is just the identity element $\vec{0}$ in U, then the function or mapping is invertible and one can go backwards from V to U. (The fact that the map is linear requires the vector $\vec{0}$ in U to map to the vector $\vec{0}$ in V.)

Mappings are abstract functions. But with linear maps, we can always be concrete. If we use a set of basis vectors in U (and in V), then we can construct a matrix that faithfully represents the mapping f. The matrix is represented via a basis in U and one in V as follows: Suppose $\{\vec{e}_j^{\,u} : j = 1, ..., n\}$ = an orthonormal basis for U, and $\{\vec{e}_i^{\,v} : i = 1, ..., m\}$ = is an orthonormal basis for V. (Orthonormal means each basis vector has norm equal to 1 and they are mutually orthogonal with each other; this is the same as the conventional Cartesian basis we use in $\mathbb{R}^n$.) Then we define M_{ij} via

$$f(\vec{e}_j^{\,u}) = \sum_{j=1}^{m} M_{ij}\vec{e}_i^{\,v}, \tag{26.1}$$

with M_{ij} being numbers. Then, if $\vec{u} = u_1\vec{e}_1^{\,u} + u_2\vec{e}_2^{\,u} + ... + u_n\vec{e}_n^{\,u}$,

$$f(\vec{u}) = \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\vec{e}_i^{\,v}u_j = \sum_{i=1}^{m} v_i\vec{e}_i^{\,v}, \tag{26.2}$$

which we write as $M\vec{u} = \vec{v}$.

So, for the general case, we have that M is an $n \times m$ matrix. This is depicted schematically in Fig. 26.2. The inverse transformation, if it exists, is given by M^{-1} (which requires $m = n$).

26.2 Changing the bases of a map

What happens if we are working in a basis for a given map and we would like to use a new one? This might be because it simplifies the problem, or allows us to understand the relationship between two different ways of looking at things. Back in the late 1920's, it turns out that the relationship between matrix mechanics and wave mechanics was worked out when Schrödinger determined the transformation that connected the two. So this ends up being an important subject, especially for quantum mechanics.

Suppose we transform the bases of the vector spaces U and V as follows:

$$\{\vec{e}_j^{\,u}\} \xrightarrow{P} \{\vec{e}_j^{\,\prime u}\} \tag{26.3}$$

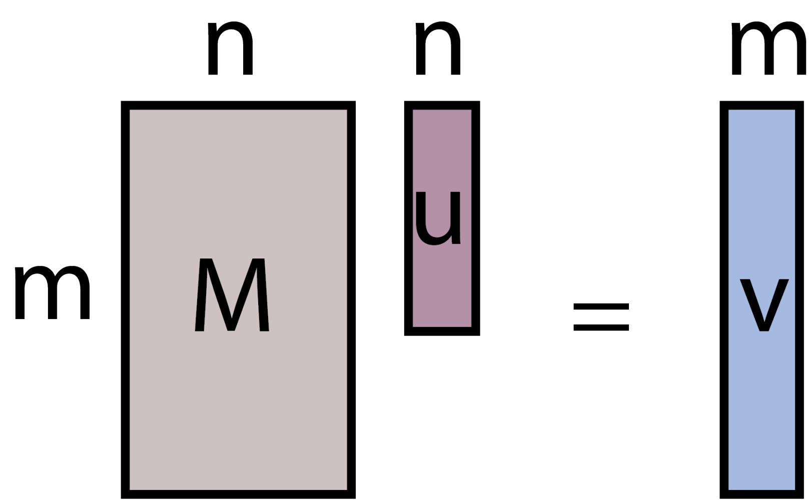

Figure 26.2: Schematic of the $n \times m$ matrix that maps vectors in U to V and represents the general linear map f. We can only have invertible maps when $n = m$ (but having $n = m$ does not guarantee that the map is invertible).

given by a matrix P_{ij}, which maps $U \to U$, and

$$\{\vec{e}_i^{\,v}\} \xrightarrow{Q} \{\vec{e}_i^{\,\prime v}\} \tag{26.4}$$

given by a matrix Q_{ij}, which maps $V \to V$. If a vector $\vec{u}$ is written as $(u_1, u_2, ..., u_n)$ in the basis $\{\vec{e}_j^{\,u}\}$ and as $(u_1', u_2', ..., u_n')$ in the basis $\{\vec{e}_j^{\,\prime u}\}$, then

$$P\vec{u} = \vec{u}\,' \text{ or } \vec{u} = P^{-1}\vec{u}\,' \tag{26.5}$$

or, more pictorially, as

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & \ddots & \\ \vdots & & p_{nn} \end{bmatrix}^{-1} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix}. \tag{26.6}$$

We have a similar result for $\vec{v}$ and $\vec{v}\,'$, with

$$Q\vec{v} = \vec{v}\,' \tag{26.7}$$

and Q an $m\times m$ matrix. But M denotes the transformation f in the original basis, so

$$\vec{v} = M\vec{u} \tag{26.8}$$

holds in general. Now we replace v and u in terms of v' and u' to find that

$$Q^{-1}\vec{v}\,' = MP^{-1}\vec{u}\,'. \tag{26.9}$$

We next multiply by Q on the left to obtain

$$\vec{v}\,' = QMP^{-1}\vec{u}\,'. \tag{26.10}$$

Hence, $M \to M' = QMP^{-1}$ is the transformed matrix when we change the bases of U and V. The case where we map $U \to U$, $(V = U)$ and $(m = n)$ is particularly interesting. In this case, we have

$$\vec{v} = \vec{u}, \quad \vec{v}\,' = \vec{u}\,', \quad \text{and} \quad Q = P, \text{ so} \tag{26.11}$$

$$M' = PMP^{-1}, \tag{26.12}$$

which is called a $\underline{similarity}$ $\underline{transformation}$, and we say that M is similar to M'.

Some properties of similarity transformations:
1) If A is similar to B and B is similar to C then A is similar to C. The proof of this is obvious, if you just write it out directly.
2) If A is similar to B, then $\det(A) = \det(B)$.
Proof: Since $\det(AB) = \det(A)\det(B)$, we have

$$\det(PP^{-1}) = \det(I) = 1 = \det(P)\det(P^{-1}) \tag{26.13}$$

$$\implies \det(P^{-1}) = \frac{1}{\det(P)}. \tag{26.14}$$

Hence, $\det(B) = \det(PAP^{-1}) = \det(P)\det(A)\det(P^{-1}) = \det(A)$.
3) If A is similar to B, then A^k is similar to B^k. Proof:

$$B = PAP^{-1}. \tag{26.15}$$

Now look at the square of B,

$$B^2 = (PAP^{-1})(PAP^{-1}) = PA^2P^{-1} \tag{26.16}$$

since $P^{-1}P = 1$. This means A^2 is similar to B^2 with the same matrix in the similarity transformation! We continue this pattern and find immediately that

$$B^k = PA^kP^{-1}. \tag{26.17}$$

When P is an orthogonal matrix (which is common for these similarity transformations), then its rows (or columns) are orthonormal vectors, so that $P^TP = I \implies P^{-1} = P^T$. Then we have

$$\begin{aligned} P^TP = \begin{bmatrix} \leftarrow e_1 \rightarrow \\ \leftarrow e_2 \rightarrow \\ \vdots \\ \leftarrow e_n \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ e_1 & e_2 & ... & e_n \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix} &= \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 & ... & \\ \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 & & \\ \vdots & & \ddots & \\ & & & \vec{e}_n \cdot \vec{e}_n \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}. \end{aligned} \tag{26.18}$$

If P is unitary, then its rows and columns are orthonormal basis vectors of a complex vector space and $PP^\dagger = I$, so $P^{-1} = P^\dagger$ where $\dagger$ denotes both the transpose and complex conjugation. These are two important types of similarity transformations that you will run into many times in your future physics courses.