Chapter 21

Gaussian Elimination

21.1 Solving m linear equations in n unknowns

In physics, we often have to solve simultaneous linear equations. These can be expressed in the following schematic form: $Ax = b$, where A is an $m \times n$ matrix, x is an n vector of unknowns, and b is a vector of m values. Written out, it looks like:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m. \end{aligned} \tag{21.1}$$

If $m \neq n$, there may be no solutions or infinitely many. The same turns out to be true when $m = n$ as well. In this lecture, we will focus on the square $m = n$ case. The book by Dettman shows an example of a rectangular case. The strategy to solve this problem is called Gaussian elimination. It is simply a systematic way of solving these equations as you might guess you would do it. We reduce the number of nonzero terms in each equation by one by adding or subtracting multiples of one equation from another. At the end, we can reverse substitute from the bottom up to solve the entire problem (if a solution exists). But the way of doing it is perhaps more strategic than how you would do it. It is foolproof in the sense that if a solution exists, you will always find it and you do so in an efficient fashion. Let's see how.

Start by multiplying row 1 by a_{21}/a_{11} and subtract it from row 2, assuming ($a_{11} \neq 0$). This is set up to cancel the leftmost coefficient in the second equation. Then repeat for $a_{31}, a_{41}...a_{n1}$ to get a new matrix with $n-1$ zeroes in the left most column. Then start with the second row and zero out all 2^{nd} column elements below it; then repeat for all subsequent columns. At this stage, the matrix looks like:

$$a'_{11}x_1 + a'_{12}x_2 + ... + a'_{1n}x_n = b'_1 \tag{21.2}$$

$$0 + a'_{22}x_2 + ... + a'_{2n}x_n = b'_2 \tag{21.3}$$

$$\vdots \tag{21.4}$$

$$0 + 0 + ... + a'_{nn}x_n = b'_n \tag{21.5}$$

Now, solve first for x_{n1}, then substitute into each row above in turn to get all solutions for $(x_1, ...x_n)$.

Some subtleties - if during row reduction, one of the first entries in the row that we subtract from all lower rows is equal to zero, then we cannot do the procedure. In this case, interchange that row with a lower one and continue to go with the algorithm. It is easiest to add the b column to the last column of the matrix when doing the calculations.

21.2 Concrete example of Gaussian elimination

It is easiest to see how to do this with an example:

$$\begin{aligned} x_1 \qquad\quad + 2x_3 - \ x_4 &= 0 \\ 2x_1 + \ x_2 - \ x_3 \qquad\ &= 5 \\ -x_1 + 2x_2 + \ x_3 + 2x_4 &= 3 \\ 3x_2 - 2x_3 + 5x_4 &= 1 \end{aligned} \tag{21.6}$$

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & 1 & -1 & 0 \\ -1 & 2 & 1 & 2 \\ 0 & 3 & -2 & 5 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ 5 \\ 3 \\ 1 \end{bmatrix}. \tag{21.7}$$

So our augmented matrix is:

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & \mathbf{0} \\ 2 & 1 & -1 & 0 & \mathbf{5} \\ -1 & 2 & 1 & 2 & \mathbf{3} \\ 0 & 3 & -2 & 5 & \mathbf{1} \end{array}\right], \tag{21.8}$$

Step 1: subtract 2× row 1 from row 2. Step 2: add row 1 to row 3. This yields

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & \mathbf{0} \\ 0 & 1 & -5 & 2 & \mathbf{5} \\ 0 & 2 & 3 & 1 & \mathbf{3} \\ 0 & 3 & -2 & 5 & \mathbf{1} \end{array}\right]. \tag{21.9}$$

Step 3: Subtract 2× row 2 from row 3. Step 4: Subtract 3× row 2 from row 4. This gives

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & \mathbf{0} \\ 0 & 1 & -5 & 2 & \mathbf{5} \\ 0 & 0 & 13 & -3 & \mathbf{-7} \\ 0 & 0 & 13 & -1 & \mathbf{-14} \end{array}\right]. \tag{21.10}$$

Step 5: Subtract row 3 from row 4. This finally gives

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & \mathbf{0} \\ 0 & 1 & -5 & 2 & \mathbf{5} \\ 0 & 0 & 13 & -3 & \mathbf{-7} \\ 0 & 0 & 0 & 2 & \mathbf{-7} \end{array}\right]. \tag{21.11}$$

Now we solve by reverse substitution. Starting from row 4, we find $x_4 = -\frac{7}{2}$. Then row 3 tells us that $13x_3 - 3x_4 = -7$

$$13x_3 = -\frac{21}{2} - 7 = -\frac{35}{2} \tag{21.12}$$

or

$$x_3 = -\frac{35}{26}. \tag{21.13}$$

Next, row 2 says $x_2 - 5x_3 + 2x_4 = 5$

$$x_2 = -\frac{175}{26} + 7 + 5 = \frac{-175 + 312}{26} = \frac{137}{26}. \tag{21.14}$$

Finally, row 1 implies that $x_1 + 2x_3 - x_4 = 0$

$$x_1 = \frac{35}{13} - \frac{7}{2} = \frac{70 - 91}{26} = -\frac{21}{26}. \tag{21.15}$$

So,

$$x_1 = -\frac{21}{26}, x_2 = \frac{137}{26}, x_3 = -\frac{35}{26}, \text{ and } x_4 = -\frac{7}{2}. \tag{21.16}$$

It can be seen that this procedure can be easily automated on a computer and it is also clear that it is a tedious approach that is easy to make an error while solving. So, using a computer is often better than using pencil and paper.

21.3 General considerations about Gaussian elimination

Uniqueness: The solution $Ax = b$ is unique unless one of the rows of the reduced matrix is all zeros. In that case, if the corresponding b value is not zero, then there is no solution. If it is zero, then there often are infinite solutions.

In essence, if $Ax = 0$ has solutions that are <u>*not*</u> $x = 0$, then any solution to $Ax = b$ can have any multiple of the solution of $Ax = 0$ added to it, and there is an infinite number of these. So, $Ax = b$ will have a unique solution if $Ax = 0$ has no solutions except $x = 0$.

We will find out in the next lecture that for square matrices, the system has no nontrivial solutions to $Ax = 0$ if and only if the determinant of A is nonzero. This is often the criterion used ($\det A = 0 \implies$ nontrivial solutions to $Ax = 0$) and you should become comfortable with it.