Chapter 23

Inverse of a Matrix

The inverse of a matrix A^{-1} satisfies

$$A \cdot A^{-1} = 1 = A^{-1} \cdot A, \tag{23.1}$$

where we assume square $n \times n$ matrices (in other words, the 1 stands for a diagonal $n \times n$ matrix whose diagonal elements are all ones; we also denote such a matrix with I). Writing out the matrix multiplication explicitly, we have

$$\sum_k a_{ik}\, a^{-1}_{kj} = \delta_{ij}, \qquad \sum_k a^{-1}_{ik} a_{kj} \ = \delta_{ij}. \tag{23.2}$$

The Kronecker delta function δ_{ij} is 1 if $i = j$ and 0 otherwise. If A is the inverse of a matrix, it is unique. Here is why. Suppose B was a different inverse. Then, $A \cdot B = I$. We multiply both sides of the equation on the left by A^{-1} to yield

$$A^{-1} \cdot A \cdot B = A^{-1} \cdot I. \tag{23.3}$$

Next, we recognize on the left hand side that $A^{-1} \cdot A = I$. Similarly, on the right-hand side, we have $A^{-1} \cdot I = A^{-1}$. In other words, we have

$$(A^{-1} \cdot A) \cdot B = A^{-1}, \tag{23.4}$$

$$I \cdot B = A^{-1}, \tag{23.5}$$

$$B = A^{-1}. \tag{23.6}$$

This implies that A^{-1} is unique!

Consider the following object

$$q_{ij} = \sum_k a_{ik} c_{jk}, \quad \text{where} \;\; c_{jk} = \text{ the } jk \text{ cofactor of } A. \tag{23.7}$$

If $i = j$, then $q_{ij} = \sum_k a_{ik} c_{ik} = \det A$ by the cofactor expansion for the determinant. If $i \neq j$, then $q_{ij} = \det$ of a matrix with the j^{th} row replaced by a_i. But then this matrix has two rows that are the same, so its determinant is 0. Hence,

$$q_{ij} = \det A \delta_{ij} = |A| \delta_{ij}, \tag{23.8}$$

where we introduced the shorthand notation $|A| = \det A$ for the determinant of A. If we define $(A^{-1})_{ij} = \frac{c_{ji}}{|A|}$ (this is the ijth element of the inverse of A; note the change in the order of the indices in c_{ji}), then

$$(A \cdot A^{-1})_{ij} = \sum_k a_{ik} \, a^{-1}_{kj} = \sum_k \frac{a_{ik} \cdot c_{jk}}{|A|} = \frac{|A| \delta_{ij}}{|A|} = \delta_{ij}. \tag{23.9}$$

So, we have that $A \cdot A^{-1} = I$, which implies A^{-1} is the unique inverse of A. This means that *we have an explicit formula for the inverse of the matrix A.* The problem is that it requires calculating many determinants so it is not so efficient to use for large matrices; but it is reasonable for 2×2 or 3×3 matrices. This formula is called *Cramer's rule for the inverse of a matrix.*

Note that since $\det AB = \det A \cdot \det B$, we have $\det A \cdot \det A^{-1} = \det I$ or $\det A \cdot \det A^{-1} = 1$. This then implies that $|A^{-1}| = \frac{1}{|A|}$. So, we see that an inverse will exist if and only if $|A| \neq 0$.

Example: Use this method to find the inverse of

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix} = A. \tag{23.10}$$

To start, we compute the determinant using the multiply along "diagonals" and add or subtract rule)

$$\det A = 1 \cdot 0 \cdot 3 + 2 \cdot 1 \cdot 1 + 1 \cdot (-1) \cdot 2 - 1 \cdot 0 \cdot 1 - 1 \cdot 1 \cdot 2 - 2 \cdot (-1) \cdot 3 = 0 + 2 - 2 - 0 - 2 + 6 = 4. \tag{23.11}$$

Next, we construct the matrix of cofactors (called the transpose of the adjugate matrix for jargon lovers), given by

$$C = \begin{pmatrix} \begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} -1 & 0 \\ 1 & 2 \end{vmatrix} \\ \\ -\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} \\ \\ \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} \end{pmatrix}, \tag{23.12}$$

which becomes

$$C = \begin{pmatrix} -2 & 4 & -2 \\ -4 & 2 & 0 \\ 2 & -2 & 2 \end{pmatrix}. \tag{23.13}$$

Taking the transpose, we obtain

$$C^T = \begin{pmatrix} -2 & -4 & 2 \\ 4 & 2 & -2 \\ -2 & 0 & 2 \end{pmatrix}, \tag{23.14}$$

which is the adjugate matrix. Dividing by the determinant, we find

$$A^{-1} = \frac{C^T}{|A|} = \begin{pmatrix} -\frac{1}{2} & -1 & \frac{1}{2} \\ \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}. \tag{23.15}$$

Now we check by matrix multiplication of $A \cdot A^{-1}$

$$A \cdot A^{-1} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & -1 & \frac{1}{2} \\ \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{23.16}$$

This verifies that $A \cdot A^{-1} = I$.

Because determinants are very difficult to calculate for large matrices, we now describe a more efficient way to compute the inverse via row reduction. The only change is that in this case the augmented matrix includes the identity matrix rather than just one column of b, and we row reduce to produce zeros *both above and below* the diagonal. We illustrate this methodology with the same example as before:

$$\left(\begin{array}{rrr|rrr} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \tag{23.17}$$

In the first step, we zero out the first column, as well as the second column:

$$\Longrightarrow \left(\begin{array}{rrr|rrr} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right) \tag{23.18}$$

We now normalize so that the diagonals are 1:

$$\Longrightarrow \left(\begin{array}{rrr|rrr} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right). \tag{23.19}$$

Now we remove the upper zeros. We do the third column first:

$$\Longrightarrow \left(\begin{array}{rrr|rrr} 1 & 2 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right), \tag{23.20}$$

and then the second

$$\Longrightarrow \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right). \tag{23.21}$$

The inverse is now read off from the augmented part:

$$A^{-1} = \begin{pmatrix} -\frac{1}{2} & -1 & \frac{1}{2} \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}. \tag{23.22}$$

This is the same inverse as we found before!

We can formally use the inverse found from cofactors to solve the original problem $Ax = b$ via a multiplication by A^{-1} on the left of both sides of the equation and then using the cofactor-based formula:

$$A^{-1}Ax = A^{-1}b \implies I \cdot x = A^{-1}b \implies x = A^{-1} \cdot b \tag{23.23}$$

$$A^{-1} = \frac{C^T}{|A|} \text{ so } x_i = (A^{-1} \cdot b)_i = \sum_k \frac{C_{ki}}{|A|} b_k \tag{23.24}$$

or $x_i = \frac{1}{|A|} \sum_k b_k c_{ki}$ - this is also called Cramer's Rule.

Note that this sum is the determinant of the matrix made by taking A and replacing the i^{th} column by the vector b.

This method is often not good for calculations unless a specific matrix element or a specific x_i is needed, then it can provide a fast way to directly obtain that result only.

Now for some nomenclature: An orthogonal matrix O satisfies $O^{-1} = O^T$ (its inverse is its transpose). Orthogonal matrices are made from unit vectors that are all perpendicular on the rows or the columns. A unitary matrix is a general complex matrix which satisfies the inverse is the Hermitian conjugate or $U^{-1} = (U^*)^T$ where U^* mean take the complex conjugate of all elements of the matrix U. These are formed by rows or columns of unit vectors using the complex norm which will be discussed in a future lecture.

If you know your matrix is orthogonal or unitary, then you can compute its inverse very easily. This can save huge time in calculations.