Chapter 18

Introduction to complex numbers

18.1 Manipulations of complex numbers

The book, *Introduction to Linear Algebra and Differential Equations*, provides a review of a number of arithmetic manipulations with complex numbers in Chapter 1. Be sure to read and master that material. Here, we review the most critical facts and clarify points that are often confusing.

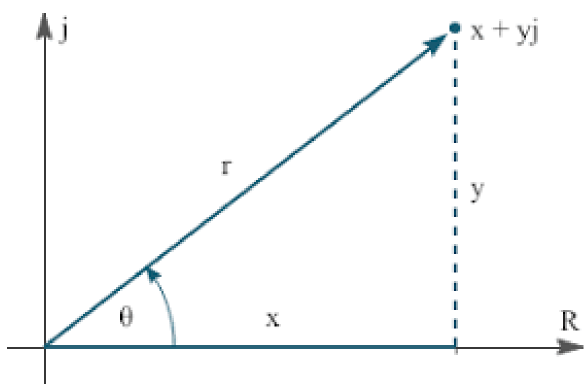

Figure 18.1: Polar representation of a complex number.

Let $z = x + iy$ be a complex number, with x and y being real. The imaginary number i satisfies $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$. Sometimes, we prefer to express the complex numbers in a polar representation where $x + iy = re^{i\theta}$, which implies that $x = r\cos\theta$ and $y = r\sin\theta$ (more on this later). We also have $r = \sqrt{x^2 + y^2}$ as usual. The complex arithmetic way of determining the polar radius is with the complex conjugate $\bar{z}$ or z^*. If $z = x + iy$, then $\bar{z} = x - iy$ and $z\bar{z} = |z|^2 = x^2 + y^2$ so $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2} = r$. Note how the complex conjugate is precisely what is needed to ensure that we get the sum of the squares of the Cartesian coordinates $[(x + iy)(x - iy) = x^2 + y^2]$.

This leads to an important exercise with complex arithmetic called *rationalizing the denominator*. Consider the inverse of a complex number. Use our "multiply by one" trick to insert the complex conjugate. This gives

$$\frac{1}{x + iy} = \frac{1}{x + iy} \times 1 = \frac{1}{x + iy}\frac{x - iy}{x - iy} = \frac{x - iy}{x^2 + y^2}, \tag{18.1}$$

or in complex notation: $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$. Often, people forget about this simple manipulation when working with complex numbers. Please be sure this is not you!

18.2 Complex exponentials

We will now develop De Moivre's theorem, which will ell us about roots of unity. The starting point is the so-called Euler relation

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{18.2}$$

which we used when discussing the polar form of a complex number. This can be proved geometrically (see Fig. 18.1), or it can be done algebraically, as we do now. We begin with

$$e^{i\theta} = 1 + i\theta + \frac{1}{2}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \dots, \tag{18.3}$$

which is the standard form of the Taylor series expansion of an exponential. Note that $\mathrm{Re}\, z = x = \frac{z+\bar{z}}{2}$, $\mathrm{Im}\, z = y = \frac{z-\bar{z}}{2i}$ so $\mathrm{Re}\, e^{i\theta} = \frac{e^{i\theta}+e^{-i\theta}}{2}$. Let's see how this works for the Taylor series. Note that

$$e^{i\theta} = 1 + i\theta - \frac{1}{2}\theta^2 - \frac{i}{3!}\theta^3 + \frac{1}{4!}\theta^4 + ... \tag{18.4}$$

and

$$e^{-i\theta} = 1 - i\theta - \frac{1}{2}\theta^2 + \frac{i}{3!}\theta^3 + \frac{1}{4!}\theta^4 + ... \tag{18.5}$$

So, we have

$$e^{i\theta} + e^{-i\theta} = 2 - \theta^2 + \frac{2}{4!}\theta^4 + ... \tag{18.6}$$

and therefore

$$\text{Re } e^{i\theta} = 1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 + ... = \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}. \tag{18.7}$$

(One verifies this either by comparing with the Taylor series for the cosine or by using the definition of the cosine as a way of determining the series.)

Similarly

$$\text{Im } e^{i\theta} = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(2i\theta - \frac{2i}{3!}\theta^3 + ...\right) = \theta - \frac{1}{3!}\theta^3 + ... = \sin(\theta). \tag{18.8}$$

So, we have

$$\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta. \tag{18.9}$$

Note that it is quite important for you to become facile in using both relations for cosine and sine—those from the power series and those in terms of the complex exponentials.

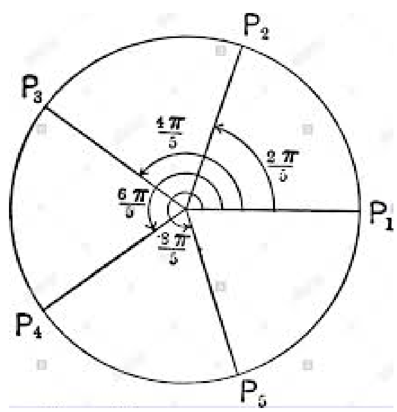

Figure 18.2: Schematic of the five roots of unity as found by de Moivre's theorem.

Now we complete de Moivre's theorem (which involves finding the complex roots of unity). So if $z = e^{i\theta}$ then $z^n = e^{in\theta}$ and we find

$$\text{Re } e^{in\theta} = \cos(n\theta) + i\sin(n\theta). \tag{18.10}$$

This allows us to find the N roots of unity $1 = e^{i2\pi} \implies 1^{\frac{1}{N}} = e^{\frac{i2\pi}{N}}$ and the N roots are given by the expression $e^{\frac{i2n\pi}{N}}$, for $0 \leq n \leq N-1$. Note how the N roots of unity are evenly distributed around a unit circle (see Fig. 18.2 for an example with $N = 5$). Note how we used the "multiply by one" trick again, replacing 1 by $e^{2i\pi}$. Remember this. It comes up often.

Since we know

$$z = re^{i\theta} = r\cos\theta + ir\sin\theta, \tag{18.11}$$

one can determine the logarithm in a straightforward fashion. The logarithm function is often confusing in complex analysis. While it is the inverse of the exponential, the weird looking real and imaginary parts are often mishandled. Be sure to work carefully with logarithms. From the above Euler relation, we immediately have

$$\ln z = \ln\left(re^{i\theta}\right) = \ln r + \ln e^{i\theta} = \frac{1}{2}\ln z\bar{z} + i\theta. \tag{18.12}$$

This implies that the real part of the logarithm is the ordinary logarithm of the modulus of z and the imaginary part is i times the phase of z. So as we wrap around a circle, the phase changes by 2π, *and the logarithm does not return to itself*. This is one reason why we have to be very careful in working with logarithms in complex analysis.

18.3 Cauchy-Riemann equations

We end with an introduction to the theory of analytic functions—complex-valued functions of $z = x + iy$ only (no dependence on $\bar{z} = x - iy$). This restriction ends up providing lots of constraints on f, which subsequently result in lots of interesting properties, such as analytic functions being infinitely differentiable, always having Taylor series expansions, being continuous with all derivatives continuous, etc. They are *super well-behaved functions.* Let's see how some of these properties arise.

A function is analytic when it depends only on z (and not $\bar{z}$)

$$f(z) = f(x + iy) = \text{ analytic}. \tag{18.13}$$

We can evaluate derivatives with respect to the real and imaginary parts of z, respectively:

$$\frac{df}{dx} = f' \quad \text{and} \quad \frac{df}{dy} = if'. \tag{18.14}$$

So, if we think of the analytic function as a two-dimensional vector field (since the complex numbers live in a plane), we have $f(z) = u(z) + iv(z) = u(x, y) + iv(x, y)$ with u and v being real-valued functions. Then, we have

$$\frac{df}{dx} = \frac{du}{dx} + i\frac{dv}{dx} \quad \text{and} \quad \frac{df}{dy} = \frac{du}{dy} + i\frac{dv}{dy}. \tag{18.15}$$

But we saw above that $\partial_x f = f'$ and $\partial_y f = if'$, so

$$-i\frac{df}{dy} = \frac{df}{dx} \implies \frac{du}{dx} + i\frac{dv}{dx} = \frac{dv}{dy} - i\frac{du}{dy}. \tag{18.16}$$

Simplifying, by equating the real and imaginary parts, we have

$$\frac{du}{dx} - \frac{dv}{dy} = 0 \quad \text{and} \quad \frac{dv}{dx} + \frac{du}{dy} = 0. \tag{18.17}$$

These two relations are called the Cauchy-Riemann equations. They resemble $\nabla \cdot f = 0$ and $\nabla \times f = 0$, except there seems to be a sign difference. It turns out these relations are closely related and there is a minus sign issue. We will discuss this and find the right way to think about this in the next lecture.

Any function that satisfied the Cauchy-Riemann equations is an analytic function.

We end by checking the Cauchy-Riemann equations for some common known functions next. First, the exponential:

$$e^z = e^{x+iy} = e^x(\cos(y) + i\sin(y)). \tag{18.18}$$

We have

$$u = e^x\cos(y), v = e^x\sin(y). \tag{18.19}$$

The Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = e^x\cos(y), \quad \frac{\partial v}{\partial y} = e^x\cos(y) \implies \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{18.20}$$

which checks out! In addition,

$$\frac{\partial u}{\partial y} = -e^x\sin(y), \ \frac{\partial v}{\partial x} = e^x\sin(y) \implies \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{18.21}$$

which also checks out!

Now, try the logarithm:

$$\ln z = \frac{1}{2}\ln(x^2+y^2) + i\arctan\frac{y}{x}, \tag{18.22}$$

so that

$$u = \frac{1}{2}\ln(x^2+y^2),\ v = \arctan\frac{y}{x}. \tag{18.23}$$

Next, we check the Cauchy-Riemann relations

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2},\quad \frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}. \tag{18.24}$$

Rearranging, we have

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2},\quad \frac{\partial v}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x} = \frac{x}{x^2+y^2}, \tag{18.25}$$

so $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ again.

You will find many of the “good” functions you know will also be analytic when they are extended to be defined on the complex plane. Analytic functions are not only good. You should think of them as being truly “super.”