Chapter 19

Cauchy Theorem and Introduction to Residues

19.1 Contour Integrals in the Complex Plane

Recall that we derived the Cauchy-Riemann equations

$$f(x+iy) = u(x,y) + iv(x,y) \tag{19.1}$$

$$\frac{du}{dx} = \frac{dv}{dy} \tag{19.2}$$

$$\frac{du}{dy} = -\frac{dv}{dx} \tag{19.3}$$

in the previous chapter. We will use these results to help us understand integration in the complex plane. It turns out that this is quite similar to the Stokes' theorem we worked with for line integrals of vector fields.

Consider the integral of a complex-valued function $f(z)$ over a path in the complex plane starting at a and ending at b. Converting to a more standard form using the real and imaginary parts, we have

$$\begin{aligned}\int_a^b f(z)\,dz &= \int_a^b \left(u(x,y) + iv(x,y)\right)(dx + idy) && (19.4)\\ &= \int_a^b \left[u(x,y)dx - v(x,y)dy\right] + i\left[u(x,y)dy + v(x,y)dx\right]. && (19.5)\end{aligned}$$

Let us think of the first integral as a line integral of the vector field $\vec{F}_1 = u\hat{e}_x - v\hat{e}_y$ with $\hat{t} = \frac{dx\hat{e}_x + dy\hat{e}_y}{ds}$. The second as another vector field $\vec{F}_2 = u\hat{e}_x + v\hat{e}_y$ and same $\hat{t}$. Then

$$\int_a^b f(z)\,dz = \int \vec{F}_1 \cdot \hat{t}\,ds + i\int \vec{F}_2 \cdot \hat{t}\,ds \tag{19.6}$$

with $\vec{F}_1$ and $\vec{F}_2$ real-valued vector fields. Let's find the curl of each of them. Compute

$$\nabla \times \vec{F}_1 = \hat{e}_z\left(-\frac{dv}{dx} - \frac{du}{dy}\right) = 0 \tag{19.7}$$

and

$$\nabla \times \vec{F}_2 = \hat{e}_z\left(\frac{dv}{dx} - \frac{du}{dy}\right) = 0 \tag{19.8}$$

where they both vanish by the Cauchy-Riemann equations. So since the curl is zero, these integrals are independent of the path. From this, we immediately learn the first form of Cauchy's Theorem:

$$\oint f(z)\,dz = 0 \tag{19.9}$$

for a function f that is analytic on the interior of a closed curve. This means, for example, that there are no singularities in the interior of the closed curve (which is similar to the simply connected region requirement we had with Stokes' theorem).

Since the above integral is independent of the shape of the path, we can deform the path to any shape we want as long as f remains analytic in the region of the path.

Example: *Path independence for the integral along a circle and along the axes.*

Integrate e^z from 1 to i along a quarter of the unit circle: $\int_1^i e^z\,dz$. We let $z = \gamma(t) = \cos t + i\sin t$ for $0 \leq t \leq \frac{\pi}{2}$. Then we have $dz = \gamma'(t) =$

Figure 19.1: Two different paths for integration of the analytic function $f = e^z$ used to show path independence of a contour integral.

$(-\sin t + i\cos t)dt$ and the integral becomes

$$\begin{aligned}
\int_1^i e^z\,dz &= \int_0^{\frac{\pi}{2}} dt\, e^{\cos t + i\sin t}(-\sin t + i\cos t) \\
&= \int_0^{\frac{\pi}{2}} dte^{\cos t}\left(\cos(\sin t) + i\sin(\sin t)\right)(-\sin t + i\cos t) \\
&= \int_0^{\frac{\pi}{2}} dte^{\cos t}\,[] - \cos(\sin t)\sin t - \sin(\sin t)\cos t \\
&\qquad\qquad -i\sin(\sin t)\sin t + i\cos(\sin t)\cos t] \\
&= \int_0^{\frac{\pi}{2}} dt\,\frac{d}{dt}\left[e^{\cos t}\cos(\sin t) + ie^{\cos t}\sin(\sin t)\right] \\
&= 1\cos(1) + i\sin(1) - e^1\cos(0) - ie^1\sin(0) \\
&= e^i - e^1.
\end{aligned} \tag{19.10}$$

One can perform a similar integration around the “ell” shaped path in Fig. 19.1. The details are given in the video lecture. Here, we will be even more general and simply note that $e^z = \frac{d}{dz}e^z$ so $\int_1^i \frac{d}{dz}e^z\,dz = e^z\big|_1^i = e^i - e^1$ for *any path*, which shows again that the integral is independent of the path!

Now we consider the single most important contour integral in complex analysis. It is amazing how many interesting results derive from this one simple integral. It is also amazing how many students have trouble with analyzing and understanding this. So pay attention! We consider an integral around a unit circle centered at the point z_0 of $(z - z_0)^n$:

$$\oint (z - z_0)^n\,dz. \tag{19.11}$$

In this integral, n is an integer. The integration contour is illustrated schematically in Fig. 19.2.

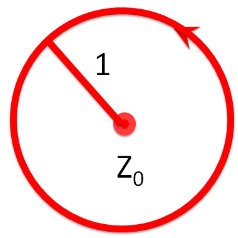

Figure 19.2: Contour around a unit circle centered at z_0 and traversed in the counterclockwise direction.

Note that for all n not equal to -1, we have that the integrand is a perfect differential, so that

$$(z-z_0)^n = \frac{d}{dz}\frac{(z-z_0)^n}{n+1} \tag{19.12}$$

and

$$\oint (z-z_0)^n\,dz = \left.\frac{z-z_0}{n+1}\right|_{\text{start}}^{\text{end}} = 0 \tag{19.13}$$

since the starting point and end point are the same for integration around a circle.

On the other hand, for $n = 1$, we use $\gamma(t) = z_0 + e^{it}$, $0 \le t \le 2\pi$. $dz = \gamma'(t)\,dt = ie^{it}\,dt$ and the integral becomes

$$\oint \frac{1}{z-z_0}\,dz = \int_0^{2\pi} dt\, ie^{it}\frac{1}{z_0+e^{it}-z_0} = \int_0^{2\pi} dt\, i\frac{e^{it}}{e^{it}} = 2\pi i. \tag{19.14}$$

Summarizing, we have

$$\oint (z-z_0)^n\,dz = \begin{cases} 2\pi i & n=-1 \\ 0 & \text{otherwise.} \end{cases} \tag{19.15}$$

Be sure you understand clearly why this integral is what it is and also be sure you can derive it yourself. It is that important.

19.2 Proof of Cauchy's Theorem

Now we will prove the full Cauchy's theorem. Let γ be a path that encircles z_0 once in the counterclockwise direction (the circle in Fig. 19.2 is an example of just such a path). Then, we claim that

$$f(z_0) = \int_\gamma \frac{f(z)}{z - z_0} \frac{dz}{2\pi i}. \tag{19.16}$$

To prove this, we expand $f(z)$ in a Taylor series expansion about z_0

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} (z - z_0)^n \frac{d^n}{dz^n} f(z) \Big|_{z=z_0} \tag{19.17}$$

$$= f(z_0) + (z - z_0) f'(z_0) + \frac{1}{2}(z - z_0)^2 f''(z_0) + ... \tag{19.18}$$

which holds for all analytic functions. So

$$\int_\gamma \frac{f(z)}{z - z_0} \frac{dz}{2\pi i} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{dz^n} f(z) \Big|_{z=z_0} \int_\gamma (z - z_0)^{n-1}. \tag{19.19}$$

Note that $\frac{d^n}{dz^n} f(z)\Big|_{z=z_0}$ can be removed from the integral because it is a number, not a function of z. Then, we find that only the $n = 0$ term contributes and integral becomes $2\pi i \frac{f(z_0)}{2\pi i} = f(z_0)$, which proves the result. Note, that if you are not happy with the rigor of our using a Taylor series expansion, or with our interchanging the order of the integration and the summation, congratulations, you are quite mathematically inclined. We will not go into the details as to why this is not an issue here. That is one of the major results you will learn in a complex analysis class. The goal here is just to make the result seem plausible, with a heuristic derivation and to show you the implications of it and how to use it. Rest assured, the result is perfectly rigorous.

Summarizing, we have that Cauchy's theorem says

$$f(z_0) = \int_\gamma \frac{f(z)}{z - z_0} dz \tag{19.20}$$

for any z_0 that lies *inside* the curve γ. Think about this for a moment. It is a truly remarkable result. It says that an analytic function is *completely* *determined in the interior of a region by its values on the boundary of that region*! This incredible result is true for all analytic functions.

We end the chapter with a discussion of a generalization of the Taylor series. Recall that the Taylor series was a best fit polynomial to a function. When you look carefully at Taylor series, you see they do not work well for functions that diverge at singular points. This is because the slope of a polynomial is always finite, but the slope at a singularity is always infinite. A Laurent series is a generalization of a Taylor series to include such singular behavior at z_0. If there are only a finite number of singular terms then we can expand

$$\begin{aligned} g(z) = \frac{1}{n!}\frac{b_{-n}}{(z-z_0)^n} + \frac{1}{(n-1)!}\frac{b_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{1}{2}\frac{b_{-2}}{(z-z_0)^2} + \frac{b_{-1}}{(z-z_0)} \\ + a_0 + a_1(z-z_0) + \frac{1}{2}a_2(z-z_0)^2 + \cdots + \frac{1}{n!}a_n(z-z_0)^n + \cdots \end{aligned} \tag{19.21}$$

The term b_{-1} is called the residue of the function g. For example, if f is analytic, and $g = \frac{f(z)}{z-z_0}$ then $f(z_0)$ is the residue of g at z_0. In many cases finding b_{-1} is simple. In some cases, it can be difficult. But the residue is also an extremely important quantity to know about a singular function. We will see why in the next chapter.