Chapter 7

Multivariable Integration: Cubic, Cylindrical, and Spherical Coordinates

7.1 Arc Length and Area

We start this lecture by discussing how to calculate the arc-length of a curve. Suppose we have a curve $y = f(x)$. The arc length along the curve (see Fig. 7.1) is

$$ds = \sqrt{dx^2 + dy^2}, \tag{7.1}$$

which arises from computing the length of the hypotenuse of a small right triangle with displacements dx and dy on the two legs. Factoring out a dx, then gives us that

$$ds = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\,\sqrt{1 + [f'(x)]^2}. \tag{7.2}$$

The arc-length of the curve S is found by adding all of these little ds lengths up, as follows:

$$S = \int_{x_{min}}^{x_{max}} ds = \int_{x_{min}}^{x_{max}} dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \int_{x_{min}}^{x_{max}} dx\,\sqrt{1 + [f'(x)]^2}. \tag{7.3}$$

Our first example will be an arc-length of a circle. We know the full circumference is $2\pi r$, but when we compute the arc-length, we can break the

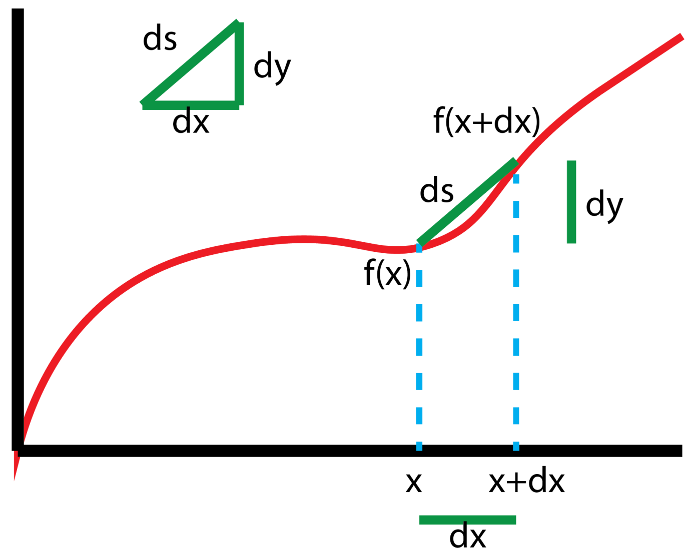

Figure 7.1: Geometry needed to construct the arc-length of a curve $y = f(x)$. One can see that the infinitesimal arc-length along the curve ds is found from computing the hypotenuse of the right triangle with legs dx and $dy = f'(x)dx$.

circle into four pieces. Then we have

$$y(x) = \sqrt{r^2 - x^2} \tag{7.4}$$

in the first quadrant. Taking the derivative, gives us

$$\frac{dy}{dx} = \frac{1}{2}\frac{-2x}{\sqrt{r^2 - x^2}}, \tag{7.5}$$

and then the term in the square-root of the infinitesimal arc-length becomes

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2}. \tag{7.6}$$

So

$$\text{arclength} = \frac{1}{4}\text{perimeter} = \int_0^r dx\sqrt{\frac{r^2}{r^2 - x^2}}. \tag{7.7}$$

To compute this integral, we recall the rules from Chapter 6 and let $x = r\sin\phi$, with $dx = r\cos\phi\, d\phi$. The arc-length becomes

$$\frac{1}{4}\text{perimeter} = \int_0^{\frac{\pi}{2}} r\cos\phi\, d\phi\left(\frac{r}{r\cos\phi}\right) = r\int_0^{\frac{\pi}{2}} d\phi = \frac{\pi r}{2}. \tag{7.8}$$

This then implies that the perimeter is $2\pi r$, which we already know to be true. But it is always nice to re-derive something familiar with our new and powerful calculus machinery.

Lets try it now with an ellipse. The ellipse satisfies

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \tag{7.9}$$

with $a \neq b$. We can solve for y in the first quadrant again, to learn that

$$y = b\sqrt{1 - \left(\frac{x}{a}\right)^2}. \tag{7.10}$$

Computing the derivative is next

$$\frac{dy}{dx} = \frac{b}{2}\left(\frac{-\frac{2x}{a^2}}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\right) = \frac{b}{a}\left(\frac{x}{\sqrt{a^2-x^2}}\right), \tag{7.11}$$

followed by squaring it and adding 1. This then gets put into the formula for the arc-length, which finally is

$$\frac{1}{4}\text{perimeter} = \int_0^a dx\sqrt{1 + \frac{b^2}{a^2}\left(\frac{x^2}{a^2-x^2}\right)} \tag{7.12}$$

$$= \int_0^a dx\frac{1}{a}\sqrt{\frac{a^4 + (b^2-a^2)x^2}{a^2-x^2}} \tag{7.13}$$

$$= \int_0^a dx\frac{1}{a}\left(\frac{a^4+(b^2-a^2)x^2}{\sqrt{(a^2-x^2)(a^4+(b^2-a^2)x^2)}}\right). \tag{7.14}$$

This has a square root of a quartic polynomial in the denominator. As we learned in Chapter 6, to integrate this requires introducing elliptic functions. We will not do that here.

Remarkably though, the areas of the circle and ellipse can be calculated analytically. For the ellipse we have the geometry given in Fig. 7.2.

$$\text{area} = 4\int_0^a dx\, y \tag{7.15}$$

$$= 4\int_0^a dx\, b\sqrt{1-\left(\frac{x}{a}\right)^2} \tag{7.16}$$

$$= 4\,\frac{b}{a}\int_0^1 dx\,\sqrt{a^2-x^2} \tag{7.17}$$

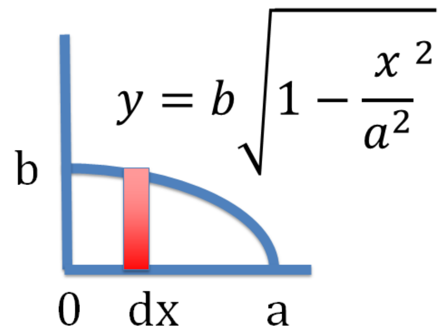

Figure 7.2: Set-up for an integral of the area of an ellipse. We add up the little rectangles of width dx and height $y(x)$.

Let $x = a\sin\phi$ and $dx = a\cos\phi\, d\phi$. Then $\sqrt{a^2 - x^2} = a\cos\phi$. Substituting in gives us

$$\text{area} = 4\,\frac{b}{a}\int_0^{\frac{\pi}{2}} a\cos\phi\, d\phi\, a\cos\phi \tag{7.18}$$

$$= 4ab\int_0^{\frac{\pi}{2}} \cos^2\phi\, d\phi. \tag{7.19}$$

But $\cos^2\phi = \frac{\cos(2\phi)+1}{2}$. So

$$\text{area} = \frac{4ab}{2}\int_0^{\frac{\pi}{2}} (\cos 2\phi + 1)\, d\phi \tag{7.20}$$

$$= 2ab\left(\frac{\sin 2\phi}{2}\bigg|_0^{\frac{\pi}{2}} + \frac{\pi}{2}\right) \tag{7.21}$$

$$= \pi ab. \tag{7.22}$$

For a circle, $a = b = r$, and area $= \pi r^2$ as we already know.

7.2 Differential volume elements

We now move on to integration in three dimensions. We start with volumes and the different three-dimensional coordinate systems (Cartesian, cylindrical, or spherical). Volume is the ratio of the amount of an object to the amount in a unit cube. In cubic coordinates, the volume "interval" is $dx\,dy\,dz$.

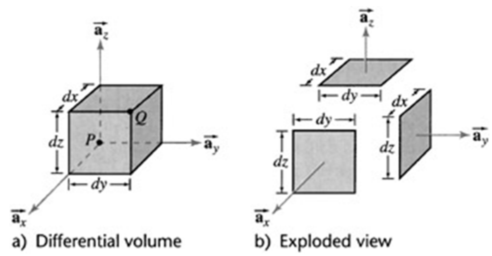

Figure 7.3: The Cartesian differential volume element, of size $dx \times dy \times dz$.

For cylindrical coordinates, we use polar coordinates for the plane and z for the third dimension, corresponding to the height. This yields a volume element given by $dr\, rd\theta\, dz$.

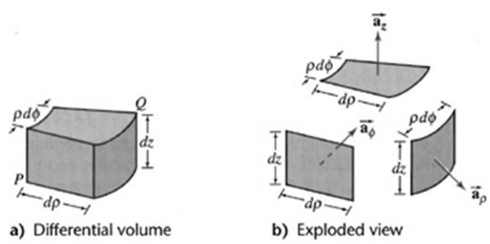

Figure 7.4: The differential volume element in cylindrical coordinates, given by $dr \times rd\phi \times dz$. (Note, we usually use r instead of ρ for the radial coordinate, but both are common.)

For spherical coordinates, the volume element is $dr\, rd\theta\, r\sin\theta\, d\phi = r^2\, dr\, d\cos\theta\, d\phi$. Learning how to evaluate integrals in this second form, integrating with respect to $d\cos\theta$, is the difference between a technician and a practitioner!

7.3 Examples of volume and surface area integrals

Our first example is the volume of a sphere of radius a. The reason why we use spherical coordinates is because the equation of a sphere is so simple in

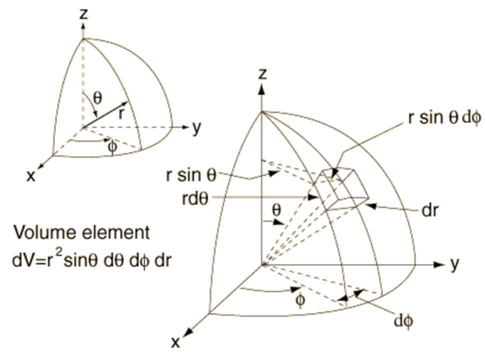

Figure 7.5: The differential volume element in spherical coordinates. Note that the angle θ is the rotation down from the z-axis and the angle ϕ is the angle in the $x-y$ plane. This convention is the opposite to what mathematicians use. But it is the one adopted by physicists.

these coordinates. It is just $r = a$. So

$$\begin{aligned}
\text{Volume} &= \int_0^a r^2\,dr \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi && (7.23)\\
&= 2\pi \int_0^a r^2\,dr \int_{-1}^1 d\cos\theta && (7.24)\\
&= 2\times 2\pi \int_0^a r^2\,dr && (7.25)\\
&= 4\pi \left(\frac{r^3}{3}\right)\Big|_0^a && (7.26)\\
&= \frac{4}{3}\pi a^3 && (7.27)
\end{aligned}$$

which we know to be the volume of a sphere.

Our second example is the volume of a cylinder of radius a and height h. The cylinder lives in $0 \le z \le h$ and $r \le a$. We compute the volume using cylindrical coordinates:

$$\begin{aligned}
\text{Volume} &= \int_0^h dz \int_0^a dr\, r \int_0^{2\pi} d\theta && (7.28)\\
&= h \times \frac{a^2}{2} \times 2\pi && (7.29)\\
&= \pi a^2 h && (7.30)
\end{aligned}$$

as expected.

Now we move on to surface areas, starting with the surface area of a cylinder. We know from a previous calculation that the surface area of each of the circular endcaps is πa^2. We add the area of the two endcaps to the area of the sides of the cylinder, which is the perimeter of the circle integrated over the height of the cylinder.

$$\text{Area} = 2\pi a^2 + \int_0^h dz 2\pi a \tag{7.31}$$

$$= 2\pi a^2 + 2\pi a h. \tag{7.32}$$

The final simple example is the surface area of a sphere. The area element is $r d\theta \, r \sin\theta d\phi$ with $r = a$.

$$\text{Area} = a^2 \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi = 4\pi a^2. \tag{7.33}$$

Our final topic is surfaces of revolution. It is critical that you understand surfaces of revolution to be comfortable with the ideas of this chapter. Students often have trouble with them because the curves can be revolved around the x or y axes. One needs to always pay attention.

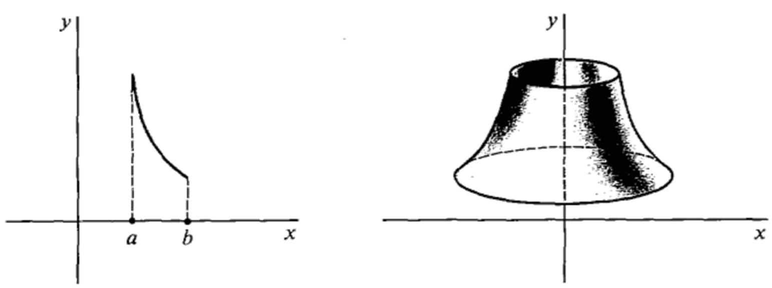

Figure 7.6: In this example, we have the surface of revolution about the y-axis.

The volume of a surface of revolution is the sum of the volumes of different disks, each with radius x.

$$\text{Volume} = \int_{y_{min}}^{y_{max}} dy \, \pi x^2 \tag{7.34}$$

We know the equation for $y(x)$, so $dy = \frac{dy}{dx}dx$. The volume integral then becomes

$$\text{Volume} = \pi \int_0^{x_{max}} dx \frac{dy}{dx} x^2. \tag{7.35}$$

Now we go onto the final topic, the surface area of a surface of revolution. Each area element was just an arc-length ds times $2\pi x$ the perimeter of the circle.

$$\begin{aligned} \text{Surface Area} &= \int ds\, 2\pi x \qquad (7.36) \\ &= \int_0^{x_{max}} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, 2\pi x \qquad (7.37) \\ &= 2\pi \int_0^{x_{max}} dx\, x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \qquad (7.38) \end{aligned}$$

You will have some examples of this on the homework.