Chapter 5

Fundamental Theorem of Calculus and Manipulation of Integrals

5.1 Fundamental theorem of calculus

We start with the fundamental theorem of calculus: If $F(t) = \int_a^t f(x)\,dx$ with $a < t < b$ and if $f(x)$ is continuous and monotonic for $a \leq x \leq b$, then $F'(t) = f(t)$. (Barrow)

To prove the fundamental theorem, we just compute the derivative

$$\lim_{t_1 \to t} \frac{F(t_1) - F(t)}{t_1 - t} = \lim_{t_1 \to t} \frac{\int_0^{t_1} f(x)\,dx - \int_0^t f(x)\,dx}{t_1 - t} = \lim_{t_1 \to t} \frac{\int_t^{t_1} f(x)\,dx}{t_1 - t} \tag{5.1}$$

Since $f(t) < f(x) < f(t_1)$ for $t < x < t_1$ because f is monotonic, we have

$$(t_1 - t) f(t) < \int_t^{t_1} f(x) dx < (t_1 - t) f(t_1) \tag{5.2}$$

$$\Longrightarrow f(t) < \frac{F(t_1) - F(t)}{t_1 - t} < f(t_1) \tag{5.3}$$

$$\Longrightarrow \lim_{t_1 \to t} \frac{F(t_1) - F(t)}{t_1 - t} = f(t) \tag{5.4}$$

since f is continuous.

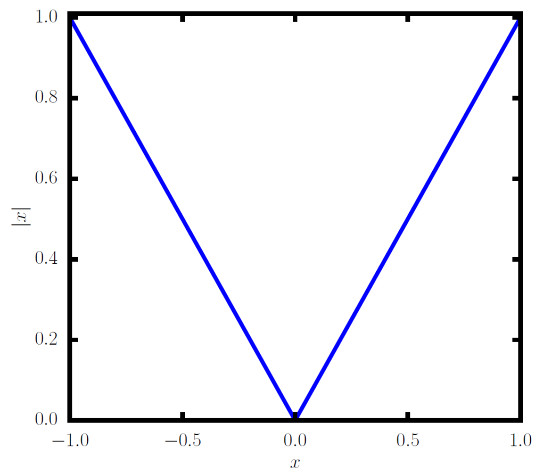

Figure 5.1: Absolute value function. Note how the derivative is -1 for $x < 1$ and 1 for $x > 1$. The limit as $x \to 0$ does not exists for the derivative because the left limit is not equal to the right limit.

The fundamental theorem of calculus has an obvious corollary as well. It is, in fact, the most common use of the theorem.

Corollary: If $\phi'(x) = f(x)$, $\phi(x) = F(x) + c$ where c is a constant.

If f is continuous at x, then the left and right limits match: $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x)$, but continuity also requires such a limit to exist in the first place (see figure 81 in the Toeplitz book for an example of a function with no limit). Note, however, that continuity does not imply that f is differentiable. $f(x) = |x|$ is a classic example: the function is continuous but is not differentiable at $x = 0$.

The converse is, however, true. If a function is differentiable at x, then it must also be continuous at x.

5.2 Product Rule

The product rule, sometimes called the Leibnitz rule, shows how one can calculate the derivative of a product of functions. The formula is well known to any student in a Calculus I class.

$$w(x) = u(x)v(x) \implies w'(x) = u'(x)v(x) + u(x)v'(x) \tag{5.5}$$

Proof:

$$
\begin{aligned}
w'(x) &= \lim_{x_1 \to x} \frac{u(x_1)v(x_1) - u(x)v(x)}{x_1 - x} && (5.6)\\
&= \lim_{x_1 \to x} \frac{u(x_1)v(x_1) - u(x)v(x_1) + u(x)v(x_1) - u(x)v(x)}{x_1 - x} && (5.7)\\
&= \lim_{x_1 \to x} \frac{u(x_1) - u(x)}{x_1 - x} v(x_1) + u(x) \frac{v(x_1) - v(x)}{x_1 - x} && (5.8)\\
&= u'(x)v(x) + u(x)v'(x) && (5.9)
\end{aligned}
$$

Note the use of the "add zero" trick in the second line (terms in red). Now, since v is differentiable, it is also continuous, so we can replace $v(x_1)$ by $v(x)$ in the last line.

We will make use of the product rule to derive the important result called integration by parts.

5.3 Integration by Parts

Integating the Leibnitz rule (via the fundamental theorem of calculus) shows us that

$$u(x)v(x) = \int u'(y)v(y)\,dy + \int u(y)v'(y)\,dy. \tag{5.10}$$

Rearranging, we find

$$\int u'(y)v(y)\,dy = u(y)v(y) - \int u(y)v'(y)\,dx \tag{5.11}$$

Integration by parts is usually used for a definite integral, as follows:

$$\int_a^b u'(x)v(x)dx = u(x)v(x)\Big|_a^b - \int_a^b u(x)v'(x)dx. \tag{5.12}$$

Integration by parts is an extremely useful technique for evaluating integrals. For example, consider for $n \neq -1$: $\int x^n \ln(x)\,dx$. We take $u' = x^n$ and $v = \ln(x)$.

$$\int x^n \ln(x)\,dx = \frac{x^{n+1}}{n+1}\ln(x) - \int \frac{x^{n+1}}{n+1}\frac{1}{x}\,dx = \frac{x^{n+1}}{n+1}\ln(x) - \frac{x^{n+1}}{(n+1)^2} \tag{5.13}$$

In particular, for $n = 0$, we have $\int \ln(x)dx = x\ln(x) - x$ which would have been difficult to guess by any other means.

5.4 Inverse Chain Rule

The so-called chain rule for the derivatives of functions of functions is the following: if $g(u)$ is a differentiable function of u and $u(x)$ is a differentiable function of x, then

$$\frac{dg(u)}{dx} = \frac{dg(u)}{du}\frac{du}{dx}. \tag{5.14}$$

This is one of the most useful relations of differential calculus. It can simplify derivatives if you are clever. For example, consider $\frac{d}{d\theta}\left(\sin^2\theta + \frac{1}{\sin\theta}\right)$. It is easy to calculate as

$$\frac{d}{d\sin\theta}\left(\sin^2\theta + \frac{1}{\sin\theta}\right)\frac{d\sin\theta}{d\theta} = \left(2\sin\theta - \frac{1}{\sin^2\theta}\right)\cos\theta \tag{5.15}$$

In fact, we may be doing this subconsciously as we are following the rules for derivatives, but there are many situations where using the chain rule in this fashion can make calculations much easier to finish.

Related to this idea is the "inverse" of the chain rule

$$\int g(u)u'(x)\,dx = \int g(u)\frac{du}{dx}dx = \int g(u)du, \tag{5.16}$$

which is another valuable tool for integration. An example is $\int \frac{\ln x}{x}dx$. We have $g(u) = \ln x$ and $u'(x) = \frac{1}{x}$, so that

$$\int \frac{\ln x}{x}dx = \int uu'dx = \int udu = \frac{u^2}{2} = \frac{1}{2}(\ln x)^2. \tag{5.17}$$

Another example is

$$\int \frac{1}{x\ln x}dx = \int \frac{u'}{u}dx = \int \frac{du}{u} = \ln u = \ln(\ln x) \tag{5.18}$$

and so on.

5.5 Inverse Functions

Inverse functions are important throughout physics and math. One thing you must remember is that a function relates *one value* to each argument

x, which makes it single-valued: for each x, there is only one $f(x)$. If we want to compute the inverse of $f(x)$, for each y we must find the x such that $f(x) = y$, or $f^{-1}(y) = x$. We must have only one x value for each y value. This is, in general, where the complexity occurs, because the definition of a single-valued function does not guarantee that the inverse function is also single-valued. Indeed, many are not, which means we must restrict the domain for the inverse function to have it be single valued.

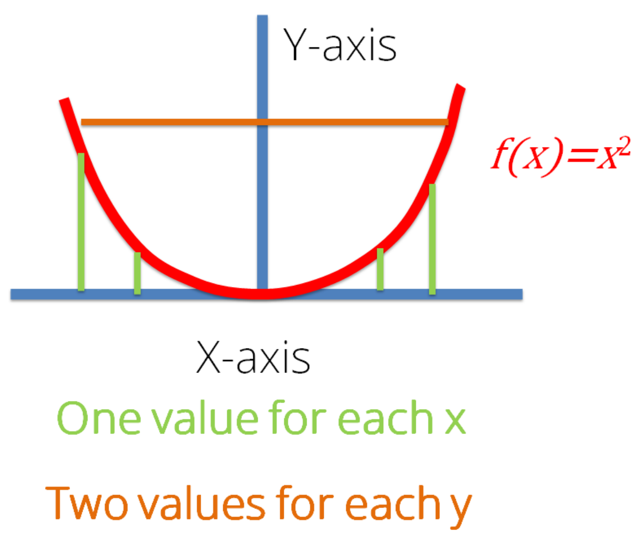

Figure 5.2: Schematic of functions and inverse functions for $y = x^2$.

Take a simple function like $f(x) = x^2$ for $-\infty < x < \infty$. Note how for every x, there is one and only one $y = x^2$ value (green lines in Fig. 5.2), so f is a single valued function. But for the inverse, if we set a value of y, there are two roots, one positive and one negative (red horizontal line in Fig.'5.2), so there are two inverse functions defined on different domains: $f^{-1}(y) = \sqrt{y}$ gives answers in the range 0 to ∞ and $\tilde{f}^{-1}(y) = -\sqrt{y}$ gives answers in the range $-\infty$ to 0. Both are valid inverse functions. For general functions that are not strictly monotonic, the inverse functions will be defined on different ranges. This is particularly true for trigonometric functions which we will treat in the next lecture.

5.6 Examples

We end the chapter with some examples.
1. Differentiate a^x.

Solution: $a^x = e^{x \ln a}$

$$\frac{d}{dx}a^x = e^{x\ln a}\frac{d}{dx}(x \ln a) = e^{x \ln a}\ln a = a^x \ln a. \tag{5.19}$$

So $\frac{d}{dx}a^x = a^x \ln a$. Note how $\frac{d}{dx}e^x = e^x$ since $\ln e = 1$.

2. Compute the following integral:

$$\int \frac{x^4 - a^4}{x^2 + a^2}dx \tag{5.20}$$

Solution: It looks impossible to get a simple answer at first, but recall that

$$x^4 - a^4 = \left(x^2 - a^2\right)\left(x^2 + a^2\right) \tag{5.21}$$

So the integral is

$$\int \left(x^2 - a^2\right)dx = \frac{x^3}{3} - xa^2 \tag{5.22}$$

3. Compute the following integral:

$$\frac{d}{dx}\ln\left(\frac{x^2+1}{x^2-1}\right) \tag{5.23}$$

Solution:

$$\frac{d}{dx}\ln\left(\frac{x^2+1}{x^2-1}\right) = \frac{d}{dx}\ln\left(x^2+1\right) - \frac{d}{dx}\ln\left(x^2-1\right) = \frac{2x}{x^2+1} - \frac{2x}{x^2-1} = \frac{-4x}{x^4-1}. \tag{5.24}$$

Could you show or recognize that

$$\int \frac{x}{1-x^4}dx = \frac{1}{4}\ln\left(\frac{x^2+1}{x^2-1}\right)? \tag{5.25}$$

To do this one would first expand by partial fractions, but one would then have to recognize that integrals of the form $\int \frac{2x}{x^2+1}dx$ are also of the form $\int \frac{du}{dx}dx = \int du = u$ for $u = \ln(x^2+1)$, which is often difficult to remember or recognize.