Physics 155 HW # 8

1.) a.) If $z = x + iy$, find an expression for $\mathrm{Re}\,\frac{1}{z^3}$ and $\mathrm{Im}\,\frac{1}{z^3}$ in terms of x and y.

b.) Do the same for $\mathrm{Re}\,\frac{z-3}{2z+5}$ and $\mathrm{Im}\,\frac{z-3}{2z+5}$.

2.) a.) Find the n roots of unity.

$$z^n = 1 \tag{1}$$

once you have found the n roots, show that for each root ω:

$$1 + \omega + \omega^2 + ... + \omega^{n-1} = 0 \tag{2}$$

except of course for the root $\omega = 1$.

b.) Show Lagrange's identity

$$1 + \cos\theta + \cos 2\theta + ... + \cos n\theta = \frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta} \tag{3}$$

as long as $\sin\frac{1}{2}\theta \neq 0$. Hint: de Moivre's theorem and the geometric series will come in handy for this.

3.) Evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{2 - \sin\theta} \tag{4}$$

using residues. Hint: change variables to a unit circle and use $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ for $z = e^{i\theta}$ on the unit circle.

4.)

$$\text{Evaluate } \int_{-\infty}^{+\infty} \frac{\cos(bx)}{x^2+a^2}\,dx \qquad a>0 \quad b>0 \tag{5}$$

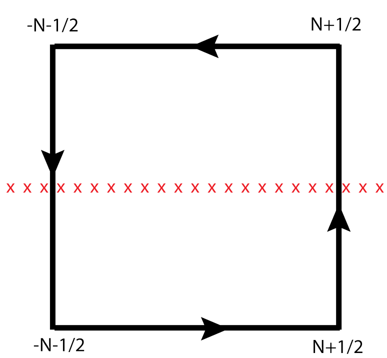
by residues. Hint: $\cos(bx) = \operatorname{Re} e^{ibx}$ is useful as well as the integral around a large semicircle in the upper half plane of $e^{ibz} \to 0$ since $\operatorname{Im} z > 0$.

5.) Infinite summations via residues

a.) Consider the function $\pi\cot(\pi z)$. Show that it has poles at all of the integers, and the residue at each pole is equal to one.

b.) Suppose we integrate

$$\oint dz\, \frac{\pi\cot(\pi z)}{z^2} \text{ over the contour} \tag{6}$$



poles marked by x at the integers

We will take the limit as $N \to \infty$. Once can show (I am not asking you to) that the integral $\to 0$ as $N \to \infty$ since it decays like $\dfrac{1}{z^2}$ for large z and the perimeter goes like z.

Given that this is true, show that

$$2\pi i \sum \text{res} \frac{\pi \cot(\pi z)}{z^2} = 0$$
$$= 2\pi i \ \text{res} \frac{\pi \cot(\pi z)}{z^2}\bigg|_{z=0} + 2\pi i \sum_{n=1}^{\infty} \frac{1}{n^2} + 2\pi i \sum_{n=1}^{-\infty} \frac{1}{n^2}$$

$$\text{Use this to show } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{7}$$

Hint: $\dfrac{\pi \cot(\pi z)}{z^2}$ has a Laurent expansion

$$\frac{b_{-3}}{3!\, z^3} + \frac{b_{-1}}{z} + a_0 + a_1 z + \ldots \text{ about } z = 0. \tag{8}$$

You need to find the residue given by the b_{-1} term to solve this problem. Work carefully and recall $\dfrac{1}{1-\delta} = 1 + \delta + \delta^2 + \ldots$ in getting the residue.