Chapter 1

Irrational Numbers and Ratios

1.1 Introduction and Course Goals

The course will cover the following:

- Review of calculus, focusing on key concepts and ideas likely missed the first time through the material
- Multivariable calculus, Div, Grad, Curl, and multivariable integral theorems
- Complex variables and the calculus of residues
- Linear algebra, solving linear equations, eigenvectors and eigenvalues
- First and second order differential equations
- Fourier Series

Throughout the course we stress ideas in addition to learning the mechanics of how to do something. We also apply this to problems with a physics background.

1.2 Irrational Numbers

We begin with the proof of the existence of an irrational number. An irrational number is one that cannot be written as the ratio of two integers—$\frac{p}{q}$ with p and q relatively prime (have no common factors, or in "lowest-terms").

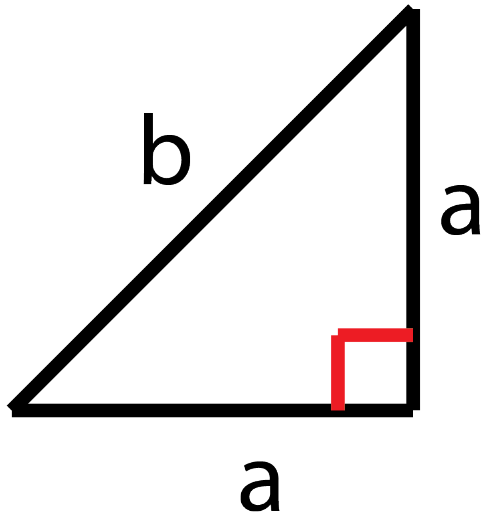

Figure 1.1: Right triangle employed in the proof. The length of each edge is a and the hypotenuse is $b = \sqrt{2}a$.

The argument is as follows: Consider the hypotenuse of an isosceles right triangle. Pythagoras says $a^2 + a^2 = b^2$ or $b^2 = 2a^2$.

Suppose we have an even integer $2n$. When we square it, it becomes $4n^2$, which is even and divisible by 4. Squaring the odd integer $2n + 1$ gives $(2n + 1)^2 = 4n^2 + 4n + 1$ which is an odd integer.

Now suppose $p \times e = b$ and $q \times e = a$ so that $\frac{a}{b} = \frac{q}{p}$ with p and q relatively prime and e the common factor in both a and b. Then $p^2 = 2q^2$, so p^2 is even. This means it must be the square of an even integer, so $p = 2n$. But then $2q^2$ is a multiple of 4, so $q = 2m$ for some m. Then p and q have a common factor of 2, which is not allowed, because we have set up the problem such that p and q have no common factors. Therefore the ratio $\frac{a}{b}$ cannot be written as a rational number ... it is irrational!

Irrational numbers can have odd properties. For example, if we take an irrational number and raise it to an irrational power, we can get a rational number. Here is a proof:

Consider $x = \sqrt{3}^{\sqrt{2}}$, which is an irrational raised to an irrational power. Now consider

$$x^{\sqrt{2}} = \left(\sqrt{3}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{3}\right)^{\sqrt{2}\times\sqrt{2}} = \left(\sqrt{3}\right)^2 = 3 \tag{1.1}$$

which is rational. Therefore either x is rational or $x^{\sqrt{2}}$ is rational. So an irrational number raised to an irrational power can yield a rational number! Note that this argument does not tell us whether the irrational is $\sqrt{3}$ or $x = \left(\sqrt{3}\right)^{\sqrt{2}}$, which when raised to an irrational power produces a rational number. (There are techniques that do answer this question, but we will not discuss further here.)

1.3 Zeno's Paradox

The book by Toeplitz discusses Zeno's Paradox — that if every step I take is half as large as the previous one, then I can never get from point A to point B because it would take an infinite number of steps.

The resolution is simple. I do need an infinite number of steps, but the total sum of all of these steps is *finite*, which is why we can arrive at point B. Next lecture, we will actually prove that the sum of all of those steps is just twice the length of the first one. This is because

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2. \tag{1.2}$$

1.4 Ratios

What is the definition of π? Think about this before I reveal the answer.

Be sure to give it a try.

Really. Think about it.

Don't peek yet

It is the ratio of the circumference of a circle to its diameter or the ratio of the area of a circle to the square of its radius. The idea of defining π as a *ratio* comes from the Greeks. Many math concepts have hidden definitions in terms of ratios. Examples include area, sine, cosine, tangent, hyperbolic tangent, and so on.

Next we will show amazing things you can do with ratios. We will assume a well known (and in many respects self-evident) fact that the ratio of the areas of any two pie slices of the same angle of two circles is proportional to the ratio of the squares of the radii of the two circles. Consider the circle with radius 1:

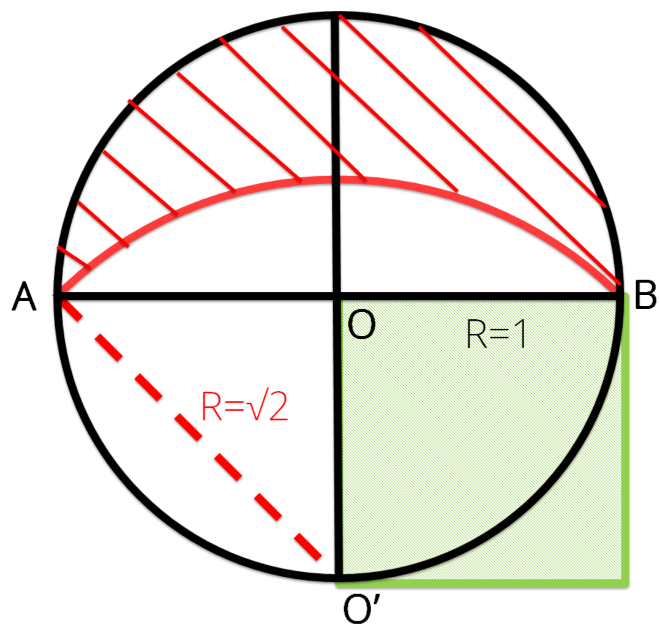

Figure 1.2: This figure has a circle of radius 1 (black) and the chord of a circle of radius $\sqrt{2}$ (red) drawn on it. The proof we will make is that the area between the two circles (hatched red) is equal to the area of the square (green).

We will prove that the red area and the green area are the same. This is close to the problem known as "squaring the circle," which has been proven to be impossible 2000 years after the Greeks gave up on their attempts. But this one can be proven, and perhaps inspired them on the bigger goal.

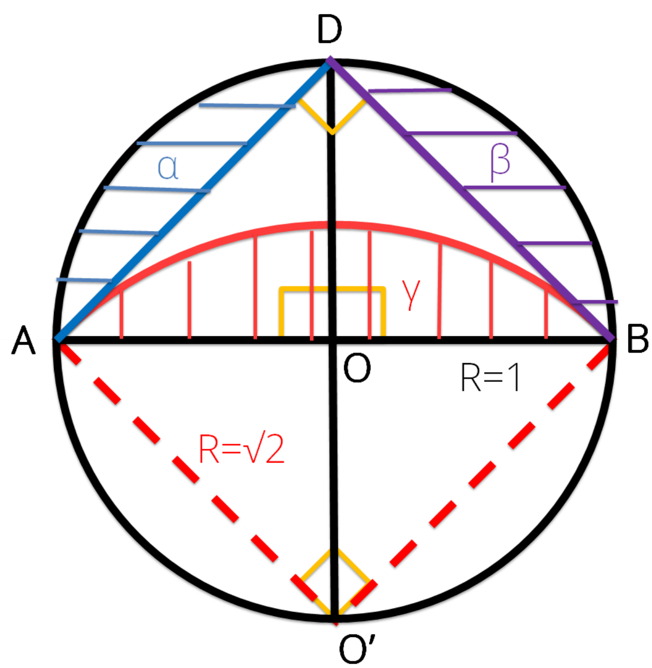

Figure 1.3: We start at the midpoint D of the upper half of the circle. Draw lines from D to A and B. This defines the hatched areas α (blue) and β (purple). We also have the hatched red area γ.

Draw straight lines from midpoint D to A and B (blue and purple). The angles AOD, DOB, and $AO'B$ are $90°$, so the the assumption we stated above implies that the ratio

$$\frac{\text{Area}\,\alpha}{\text{Area}\,\gamma} = \left(\frac{\text{rad}\,\alpha}{\text{rad}\,\gamma}\right)^2 = \frac{1}{2}. \tag{1.3}$$

A similar argument can be done for β, so we can immediately conclude that the areas satisfy $\alpha+\beta=\gamma$. Now consider $\triangle ADB$. Its area is $\frac{1}{2}\times\sqrt{2}\times\sqrt{2}=1$. So $1+\alpha+\beta=\frac{\pi}{2}$, which is the area of the semicircle with radius 1. Hence,

$$\frac{\pi}{2} - \gamma = 1 = \text{area of original red section in Figure 1.2} \tag{1.4}$$

But the area of the square is also 1 (since its side has length 1), so the area of the crescent and the square are the same!

I don't know about you, but I think this is *really cool*.

1.5 Archimedes

Archimedes is arguably one of the best mathematicians of all time. He discovered many important things. One was determining the size of π. Archimedes calculated π by the exhaustion method (now called the "squeeze method" by some for the methodology used).

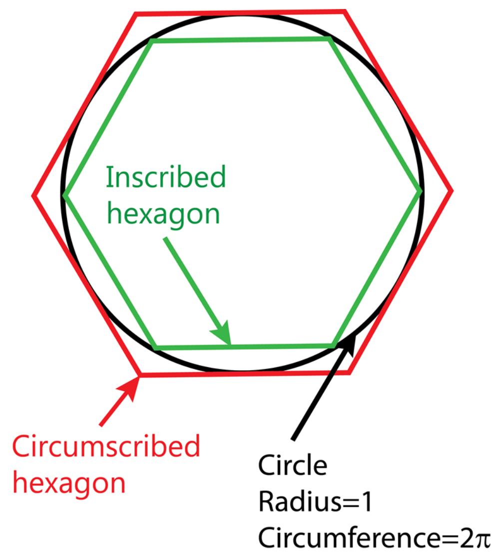

Figure 1.4: Schematic of the Archimedes exhaustion principle. The value of 2π lies in between the perimeter of the circumscribed polygon and the inscribed polygon. As the number of sides is made larger and larger the two perimeters approach each other, and the value of 2π.

Here is how the argument goes. Take a circle of unit radius. We will calculate the perimeter of an n-gon and compare it to the perimeter of a $2n$-gon both *inscribed inside* the circle.

The sides of the $2n$-gon and n-gon (see Fig. 1.5) can be written $s_{2n} = BD$, and $s_n = BC$. We can see that $\triangle ACP$, $\triangle ADB$, and $\triangle BDP$ are similar (this is because they are all right triangles with an acute angle of θ). One verifies this by comparing angles: we have $\angle ABC = 90^{\circ} - 2\theta = \angle ABP$, $\angle ABD = 90^{\circ} - \theta$, and $\angle PBD = \theta$. This follows because for any point on a circle the angle created from two diameter endpoints to the point is always $90°$, (sums of squares of chords)^2 = (diameter)^2. The proof of this is a homework problem.

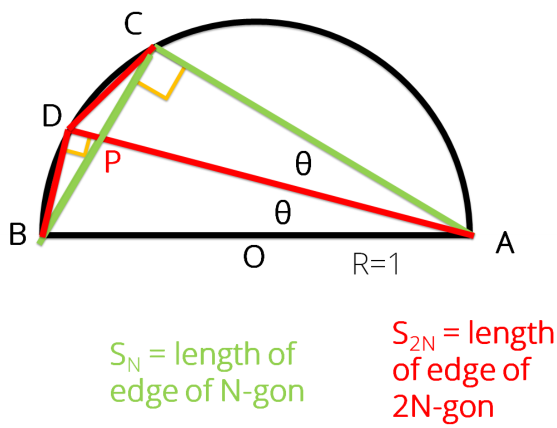

Figure 1.5: Triangles used to relate the side of one edge of the n-gon, given by $s_n = BC$ and the side of the $2n$-gon, given by $s_{2n} = BD$. The strategy is to first identify the similar triangles $\triangle ACP$, $\triangle ADB$, and $\triangle BDP$.

Observing these similar triangles (see also Fig. 1.6), we see

$$\frac{AB}{AD} = \frac{BP}{BD} \tag{1.5}$$

and hence

$$\frac{AC}{PC} = \frac{AD}{BD} \implies \frac{AC}{AD} = \frac{PC}{BD}. \tag{1.6}$$

Adding the top equation to the right part of the second equation yields

$$\frac{AB + AC}{AD} = \frac{BP + PC}{BD} = \frac{BC}{BD}. \tag{1.7}$$

Cross multiply

$$\frac{AB + AC}{BC} = \frac{AD}{BD}. \tag{1.8}$$

Squaring this equation gives

$$\frac{(AB)^2 + 2(AB)(AC) + (AC)^2}{(BC)^2} = \frac{(AD)^2}{(BD)^2} \tag{1.9}$$

Now we add 1 to both sides:

$$\frac{(AB)^2 + 2(AB)(AC) + \mathbf{(AC)^2} + \mathbf{(BC)^2}}{(BC)^2} = \frac{\mathbf{(AD)^2} + \mathbf{(BD)^2}}{(BD)^2} \tag{1.10}$$

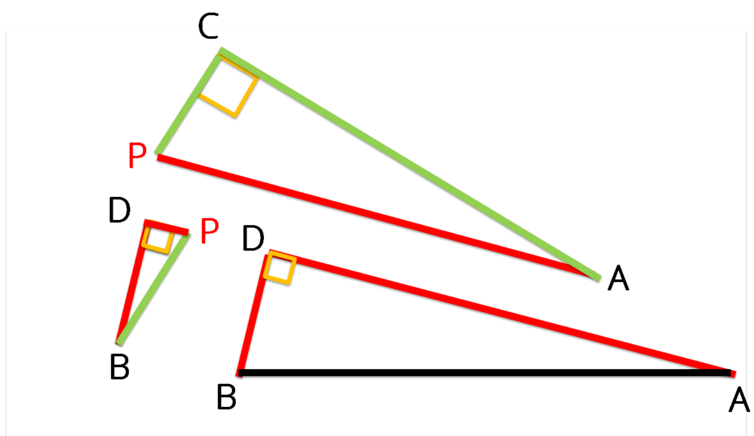

Figure 1.6: Closeup of the three triangles used in the proof. Matching the ratios of the lengths similar sides is used to finish the proof.

Note that each of the bold face terms are equal to $(AB)^2$. Substituting that result into the equation (in two places), then yields

$$\frac{2(AB)(AB+AC)}{(BC)^2} = \frac{(AB)^2}{(BD)^2} \tag{1.11}$$

We remember that $AB = 2 =$ diameter, $BD = s_{2n}$, and $BC = s_n$. We have

$$(AC)^2 = (AB)^2 - s_n^2 = 4 - s_n^2 \tag{1.12}$$

So

$$\frac{4(2+\sqrt{4-s_n^2})}{s_n^2} = \frac{4}{s_{2n}^2} \tag{1.13}$$

$$s_{2n}^2 = \frac{s_n^2}{2+\sqrt{4-s_n^2}}. \tag{1.14}$$

To start the recursion, We consider a hexagon, $n = 6$. The hexagon is made of six equilateral triangles, each with an edge equal to the radius of the circle, which is 1. So the initial perimeter is 6. (Be sure to draw picture and verify this is so.)

We solve the recursion numerically for the circumference $n \times s_n \to 2\pi r = 2\pi$ as n gets large. The results are given in the table (for the perimeter divided by 2).

N	S_N	Pi	T_N	Error width
6	3.0	3.14159265359	3.46410161514	0.464101615138
12	3.10582854123	3.14159265359	3.21539030917	0.109561767943
24	3.13262861328	3.14159265359	3.1596599421	0.0270313288163
48	3.13935020305	3.14159265359	3.14608621513	0.00673601208453
96	3.14103195089	3.14159265359	3.14271459965	0.0016826487548
192	3.14145247229	3.14159265359	3.14187304998	0.000420577694665
24576	3.14159264503	3.14159265359	3.14159267174	2.67075961347e-08

Figure 1.7: Table of the recursion results for increasing n. One can clearly see the convergence to π. This table is showing the perimeter divided by 2 for the inscribed and circumscribed polygons.

Archimedes also found the outer polygons to bound the value of π. The relationship between the n-gon and the $2n$-gon is

$$t_{2n} = \frac{2\sqrt{4+t_n^2}-4}{t_n} \tag{1.15}$$

Proving this is true will also be a homework problem.