Chapter 37

Fourier Series and Separation of Variables

37.1 Formalism

Last time we showed "well behaved" functions of θ periodic on $0 \leq \theta \leq 2\pi$ can be written as a Fourier series

$$f(\theta) = \sum_{n=-\infty}^{\infty} A_n e^{in\theta}, \tag{37.1}$$

with the Fourier coefficients determined by a simple integral

$$A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta f(\theta) e^{-in\theta}. \tag{37.2}$$

One way to think of this is as an alternative expansion to the Taylor series expansion. Here we expand in a basis given by exponentials, not powers. The reason why is that the exponentials encompass *all* functions that share this periodicity (called completeness) and hence it is natural to believe such an expansion should exist. Since one can think of this as an expansion of a vector, given by the function $f(\theta)$, in terms of the basis, given by the exponentials, with the coordinates, or coefficients, given by the A_n values. Such expansions, generalized to arbitrary orthonormal and complete basis sets, are an important core concept in quantum mechanics, so it is well worth your time to learn these ideas now.

Now, we try to generalize these ideas a bit further, within the Fourier series concept. The first thing to note is that there is nothing sacred about the interval of length 2π. So if we have a function of x periodic on $-L \leq x \leq L$, then it can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{in\frac{\pi x}{L}} \tag{37.3}$$

with the Fourier coefficients given by

$$A_n = \frac{1}{2L}\int_{-L}^{L} dx f(x) e^{-in\frac{\pi x}{L}}. \tag{37.4}$$

We want to consider some special cases. Suppose $f(x)$ is an even function; that is, $f(-x) = f(x)$. Then we can rewrite the Fourier series expansion as

$$\begin{aligned}
A_n &= \frac{1}{2L}\int_{-L}^{L} dx f(x) e^{-in\frac{\pi x}{L}} \\
&= \frac{1}{2L}\int_{0}^{L} dx \left[f(x) e^{-in\frac{\pi x}{L}} + f(-x) e^{in\frac{\pi x}{L}}\right] \\
&= \frac{1}{2L}\int_{0}^{L} dx f(x) 2\cos\frac{n\pi x}{L} \\
&= \frac{1}{L}\int_{0}^{L} dx f(x) \cos\frac{n\pi x}{L}.
\end{aligned} \tag{37.5}$$

We immediately learn from the last form of the integral, that because cos is even, we have $A_n = A_{-n}$. Hence

$$\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} A_n e^{in\frac{\pi x}{L}} \\
&= A_0 + \sum_{n=1}^{\infty}\left(A_n e^{in\frac{\pi x}{L}} + A_n e^{-in\frac{\pi x}{L}}\right) \\
&= A_0 + 2\sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}.
\end{aligned} \tag{37.6}$$

Define

$$a_n = \frac{2}{L}\int_{0}^{L} dx f(x) \cos\frac{n\pi x}{L} = 2A_n, \tag{37.7}$$

then we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \tag{37.8}$$

for even f, called the *Fourier cosine series.*

Similarly, if f is odd, we obtain

$$\begin{aligned} A_n &= \frac{1}{2L} \int_0^L dx \left[f(x) e^{-in\frac{\pi x}{L}} + f(-x) e^{in\frac{\pi x}{L}} \right] \\ &= \frac{1}{2L} \int_0^L dx f(x)(-2i) \sin \frac{n\pi x}{L}. \end{aligned} \tag{37.9}$$

Since sin is odd, we have $A_{-n} = -A_n$ and

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \left(A_n e^{in\frac{\pi x}{L}} + A_{-n} e^{-in\frac{\pi x}{L}} \right) \\ &= \sum_{n=1}^{\infty} A_n (2i) \sin \frac{n\pi x}{L}. \end{aligned} \tag{37.10}$$

Define

$$b_n = \frac{2}{L} \int_0^L dx f(x) \sin \frac{n\pi x}{L}, \tag{37.11}$$

so that $b_{-n} = b_n$. Then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \tag{37.12}$$

which is called the *Fourier sine series.*

Since a general function can be written as the sum of an even function and an odd one, we have for the general case on $-L \leq x \leq L$, that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right). \tag{37.13}$$

This is called the *Fourier series.* It is written entirely in terms of real objects here.

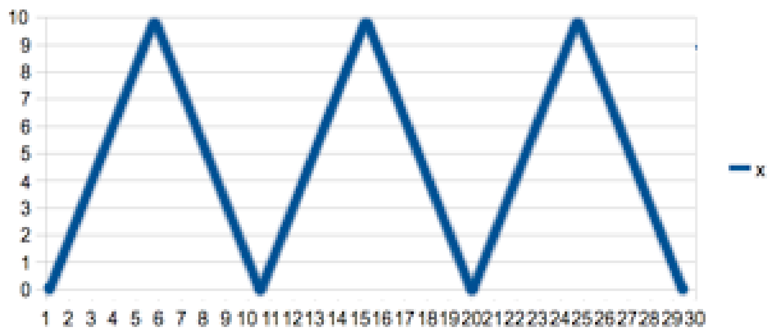

Figure 37.1: Schematic of the sawtooth wave (in this plot $L = 6$). This wave is given by an absolute value, so it always lies above zero. It is an even function. Over the half period from 0 to L, the function is linear and given simply by x.

37.2 Examples

We next consider two examples, to give you practice on the concepts.

The first example is what is called a sawtooth wave. The function is periodic, but has discontinuities at the end of each period. The sawtooth wave yields a Fourier cosine series. We can find the coefficients by integrating over a half period L. The integral is completed by integration by parts. For $n \neq 0$, we have

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L dx x \cos\frac{n\pi x}{L} \\ &= \frac{2}{L}\left[x\left(\sin\frac{n\pi x}{L}\frac{L}{n\pi}\right)\Big|_0^L - \int_0^L dx \frac{L}{n\pi}\sin\frac{n\pi x}{L}\right] \\ &= \frac{2}{L}\frac{L}{n\pi}\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L \qquad (37.14) \\ &= \frac{2L}{n^2\pi^2}\left[(-1)^n - 1\right] \\ &= \begin{cases} 0 & n \text{ even} \\ -\frac{4L}{n^2\pi^2} & n \text{ odd}, \end{cases} \qquad (37.15) \end{aligned}$$

while for $n = 0$, we find

$$\alpha_0 = \frac{2}{L} \int_0^L x dx = L. \tag{37.16}$$

So the Fourier series for the sawtooth wave becomes

$$f_{\text{sawtooth}} = \frac{L}{2} - \sum_{n=0}^{\infty} \frac{4L}{(2n+1)^2 \pi^2} \cos \frac{(2n+1)\pi x}{L}. \tag{37.17}$$

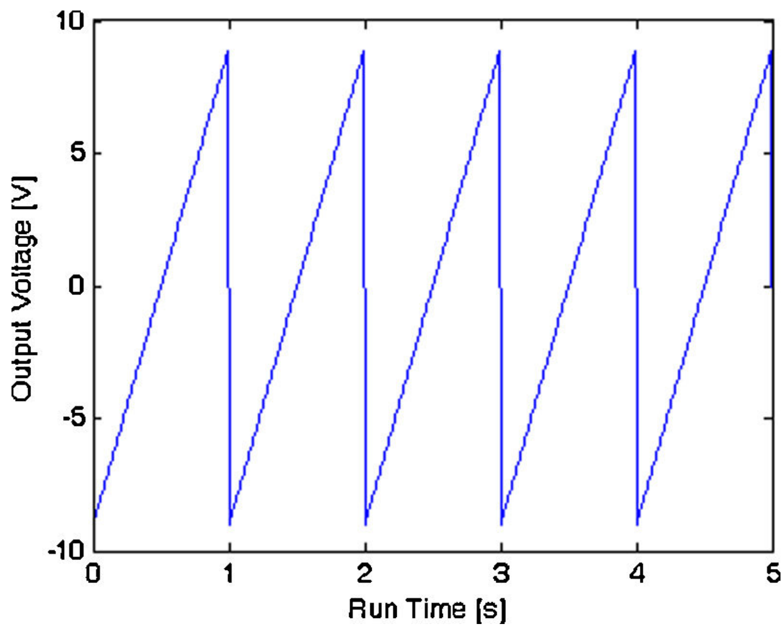

Figure 37.2: Schematic of the “modulo-line” wave. This wave is an odd function and here the period is 1.

The second example is the ”modulo line”. which is an odd function, so it is a sine series:

$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L dx x \sin\frac{n\pi x}{L} \\
&= \frac{2}{L}\left[x\left(\cos\frac{n\pi x}{L}\frac{L}{n\pi}\right)\Bigg|_0^L - \int_0^L dx\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right] \\
&= -\frac{2}{L}\frac{L}{n\pi}(-1)^n - \frac{2}{L}\frac{L}{n\pi}\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Bigg|_0^L \\
&= \frac{2}{n\pi}(-1)^{n+1}.
\end{aligned} \tag{37.18}$$

So we have

$$f(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{L} \tag{37.19}$$

as the Fourier sine series of the "modulo-line" wave.

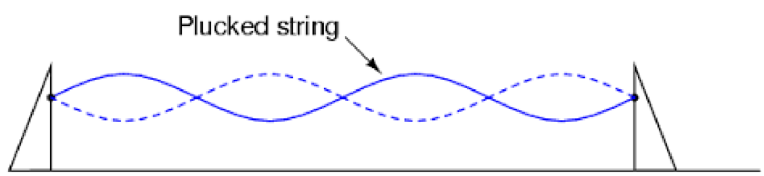

Figure 37.3: Schematic of a plucked string. The curved function is $U(x,t)$.

The third example, with more physics in it, is that of a plucked string that vibrates, as in a musical instrument. We have a demonstration of this in the videos as well, where we show a plucked guitar string.

The equation of motion for a taut string approximately satisfies the following differential equation:

$$\tau\frac{\delta^2 U}{\delta x^2} = \rho\frac{\delta^2 U}{\delta t^2} \tag{37.20}$$

where τ is the tension and ρ is the mass density of the string. Here, the left side of the equation can be thought of as acting like a potential energy, while the right side can be thought of as acting like a kinetic energy. So the equation can be interpreted as Total Energy = Constant. Note that a similar form holds in quantum mechanics for the Schrödinger equation.

The boundary conditions for the string correspond to it having "clamped" ends: $U(0,t) = 0$, and $U(L,t) = 0$ for all t. We also assume that the initial condition is $U(x,0) = f(x)$ and $\frac{\delta U(x,0)}{\delta t} = g(x)$. Our goal is to find $U(x,t)$ for all subsequent times.

The procedure we use is familiar—we employ separation of variables. To begin, we write the wavefunction as a product of two functions, which each depend on the independent variables x and t only:

$$U(x,t) = X(x)T(t). \tag{37.21}$$

Substituting into the differential equation and dividing by U, then yields

$$\frac{X''}{X} = \frac{\rho}{\tau}\frac{T''}{T} = -c \tag{37.22}$$

where c is a constant. The solution of each equation is given by exponentials if we choose $c > 0$, which is required to examine the oscillating behavior. This gives us

$$U(x,t) = X(x)T(t) = \left(Ae^{i\alpha x} + Be^{-i\alpha x}\right)\left(ae^{i\beta t} + be^{-i\beta t}\right) \tag{37.23}$$

with $\alpha = \sqrt{c}$ and $\beta = \sqrt{\frac{\tau c}{\rho}}$.

Now we need to solve the different boundary conditions. First, we find

$$X(0) = 0 \implies A + B = 0 \implies B = -A. \tag{37.24}$$

Then for the other edge, we have

$$X(L) = 0 \implies Ae^{i\alpha L} + Be^{-i\alpha L} = 0. \tag{37.25}$$

Combining these two together yields

$$A\left(e^{i\alpha L} - e^{-i\alpha L}\right) = 0 \implies \sin\alpha L = 0 \implies \alpha = \frac{n\pi}{L}. \tag{37.26}$$

So we have

$$U(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(a_n e^{itn\omega} + b_n e^{-itn\omega}\right), \tag{37.27}$$

where $\omega = \frac{\pi}{L}\sqrt{\frac{\tau}{\rho}}$. Assuming we can differentiate under the summation, we obtain

$$\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}(a_n + b_n) = f(x) \tag{37.28}$$

at $t = 0$ and

$$\sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} in\omega(a_n - b_n) = g(x) \tag{37.29}$$

for the derivative at $t = 0$. So to solve this, we compute the Fourier sine series for $f(x)$ with coefficients given by $B_n^{(f)}$ and for $g(x)$ with coefficients given by $B_n^{(g)}$. Plugging into the boundary conditions then shows that

$$a_n + b_n = B_n^{(f)} \implies b_n = B_n^{(f)} - a_n \tag{37.30}$$

and

$$in\omega(a_n - b_n) = B_n^{(g)}. \tag{37.31}$$

Combining these says that

$$2a_n - B_n^{(f)} = \frac{1}{in\omega} B_n^{(g)} \tag{37.32}$$

or

$$a_n = \frac{1}{2} B_n^{(f)} + \frac{1}{2in\omega} B_n^{(g)} \tag{37.33}$$

and

$$b_n = \frac{1}{2} B_n^{(f)} - \frac{1}{2in\omega} B_n^{(g)}. \tag{37.34}$$

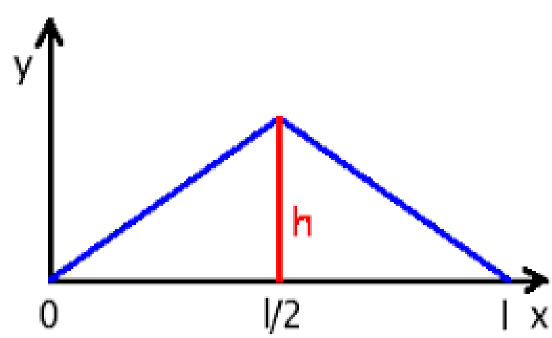

Figure 37.4: In this concrete example, the initial string is plucked at the midpoint, with a height given by $h = L/8$.

Lets work this up for a concrete example. Suppose the string is initially plucked as follows as indicated in Fig. 37.4. We assume it is not initially moving $\frac{\delta U}{\delta t}(x,0) = 0$. Then $B_n^{(g)} = 0$, and the solution preceeds as follows:

$$\begin{aligned}
B_n^{(f)} &= \frac{2}{L}\int_0^{\frac{L}{2}} dx \frac{x}{4}\sin\frac{n\pi x}{L} + \frac{2}{L}\int_{\frac{L}{2}}^{L} dx\left(\frac{L}{4} - \frac{x}{4}\sin\frac{n\pi x}{L}\right) \\
&= \frac{1}{2L}\left[\int_0^{\frac{L}{2}} dx x \sin\frac{n\pi x}{L} + \int_{\frac{L}{2}}^{L} dx (L-x)\sin\frac{n\pi x}{L}\right] \\
&= \frac{1}{2L}\left[-x\frac{L}{n\pi}\cos\frac{n\pi x}{L}\bigg|_0^{\frac{L}{2}} + \int_0^{\frac{L}{2}} dx \frac{L}{n\pi}\cos\frac{n\pi x}{L} - (L-x)\frac{L}{n\pi}\cos\frac{n\pi x}{L}\bigg|_{\frac{L}{2}}^{L}\right. \\
&\qquad \left. - \int_{\frac{L}{2}}^{L} dx \frac{L}{n\pi}\cos\frac{n\pi x}{L}\right] \\
&= \frac{1}{2L}\left[-\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{\pi^2 n^2}\sin\frac{n\pi}{2} + \frac{L^2}{2n\pi}\cos\frac{n\pi}{2} - \frac{L^2}{\pi^2 n^2}\left(\sin n\pi - \sin\frac{n\pi}{2}\right)\right] \\
&= \frac{L}{n^2\pi^2}\sin\frac{n\pi}{2} \\
&= \begin{cases} \frac{L}{(2n+1)^2\pi^2}(-1)^n & n = \text{odd} \\ 0 & n = \text{even}. \end{cases}
\end{aligned} \tag{37.35}$$

So we finally obtain

$$U(x,t) = \sum_{n=1}^{\infty} \sin\frac{(2n+1)\pi x}{L}\frac{L}{\pi^2(2n+1)^2}(-1)^n \sin(2n+1)\omega t. \tag{37.36}$$

Note how the amplitudes decay rapidly (like $1/n^2$) indicating the majority of the signal is in the fundamental harmonic. The overtones, however, are often what leads to the richness in sound of an instrument.

The procedure we use here is a standard approach for how to solve partial differential equations. As you can see it can get quite lengthy and complicated. But, because we can work out analytically the functions in the different expansions, we can actually carry the solution all the way to the end. More complicated partial differential equations must be solved numerically and these solutions require significant computational power. Most of the high performance computing resources are used for solving these types of differential equations in fluid dynamics. These methods are used to design airplanes and cars, among other things. The field is quite active and we do not yet have the most efficient ways to solve these types of problems. It is an area you can consider going into if you are inspired by this type of work.