Chapter 16

Laplace's equation

16.1 A harder Laplace's equation problem

Last time, we worked out two simple examples of solving Laplace's equation: (i) the parallel plate capacitor and (ii) the spherical capacitor. One thing we did not mention is that these solutions to Laplace's equation are unique, so however one finds a solution (including guessing), if they work, they are the solution. Hence, the technique of "guess and check if it works" is a valid method to try. It does require sophisticated guessing in some cases.

Example 3: The cylinder between two parallel planes capacitor.

We use cylindrical coordinates (the angular variable will be θ, so we can use ϕ again for the scalar potential). The one important observation to make here is that when we are far away from the cylinder, but still between the parallel plates, the electric field must be pointing in the $\hat{\imath}$ direction. Hence, we know that the electric field satisfies $\vec{E} \to E_0\hat{\imath}$ far from cylinder. The symmetry of the problem also implies that $E_z = 0$, because the problem is translationally invariant in the z-direction.

The Laplacian in cylindrical coordinates is

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{16.1}$$

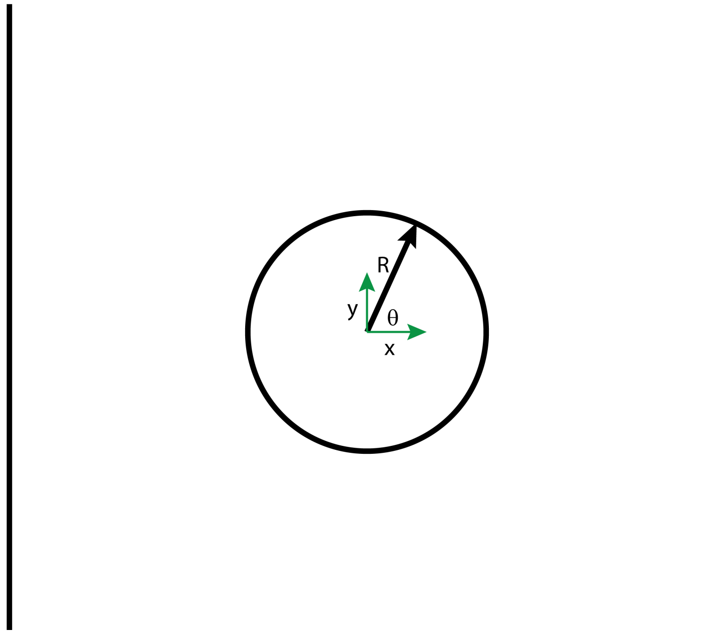

Figure 16.1: The "cylinder in the parallel plates" capacitor.

Since we have no z dependence, we obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0. \tag{16.2}$$

The boundary condition at $r = R$ is $\phi(r,\theta) = 0$ for $r = R$. Far from the cylinder, we know $\vec{E} \to E_0\hat{i} \implies \phi \to -E_0 x$ or $\phi \to -E_0 r\cos(\theta)$ for $r \gg R$. We choose this as the second boundary condition and motivate a guess:

$$\phi(r,\theta) = f(r)\cos(\theta) \tag{16.3}$$

with $f(r) = 0$ at $r = R$ and $f(r) \to -E_0 r$ as $r \to \infty$. Substitute into Laplace's equation to find that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f(r)}{\partial r}\right)+\frac{1}{r^2}(-f(r))=0. \tag{16.4}$$

We further guess that $f(r) = r^\lambda$ is a possible solution.

$$\frac{1}{r}\frac{\partial}{\partial r}\lambda r^\lambda - r^{\lambda-2}=0 \implies (\lambda^2-1)r^{\lambda-2}=0. \tag{16.5}$$

This means that $\lambda = \pm 1$ or $f(r) = Ar + \frac{B}{r} \implies \phi(r, \theta) = (Ar + \frac{B}{r})\cos(\theta)$. Now, examine the boundary conditions:

$$AR + \frac{B}{R} = 0, B = -AR^2, r \to \infty, \phi \to Ar\cos(\theta) \implies A = -E_0. \tag{16.6}$$

So,

$$\phi(r, \theta) = -E_0 r\left(1 - \left(\frac{R}{r}\right)^2\right)\cos(\theta). \tag{16.7}$$

This has determined the scalar potential. The electric field is found by taking the gradient:

$$\vec{E} = -\nabla\phi = -\frac{\partial\phi}{\partial r}\hat{e_r} - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e_\theta}, E_z = 0. \tag{16.8}$$

Evaluating, we find that

$$E_r = E_0\cos(\theta) + E_0\frac{R^2}{r^2}\cos(\theta) = E_0\left(1 + \left(\frac{R}{r}\right)^2\right)\cos(\theta) \tag{16.9}$$

and

$$E_\theta = -E_0\left(1 - \left(\frac{R}{r}\right)^2\right)\sin(\theta). \tag{16.10}$$

Note that as $r \to \infty$, we have $\vec{E} \to E_0(\cos(\theta)\hat{e_r} - \sin(\theta)\hat{e_\theta}) = E_0\hat{\imath}$, as expected.

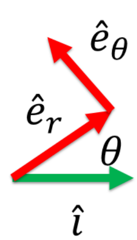

Figure 16.2: The combination of the unit vectors that we find for $r \gg R$ approach $\hat{\imath}$.

Fig. 16.2 clearly shows how the unit vectors add together to give $\hat{\imath}$ in the $r \gg R$ limit:

$$\hat{e_r}\cos(\theta) - \hat{e_\theta}\sin(\theta) = \hat{\imath}. \tag{16.11}$$

16.2 Relaxation method

Now we discuss a numerical method used to solve Laplace equation problems in general. It is called the relaxation method.

We focus on a two-dimensional problem and we employ Cartesian coordinates. Laplace's equation becomes

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{16.12}$$

Let's discretize this on a grid with spacing Δ in the x and y directions. Whenever we want to evaluate derivatives numerically, we must do so *approximately*, which we do by using the definition of the derivative, but evaluated at a finite "nudge" (Δ), rather than taking the limit as the "nudge" goes to zero. This procedure is called discretization.

For evaluating the Laplacian, we need to evaluate second derivatives, which require three discrete points in the x and in the y-directions. We use the so-called "cross" geometry, as shown in Fig. 16.3.

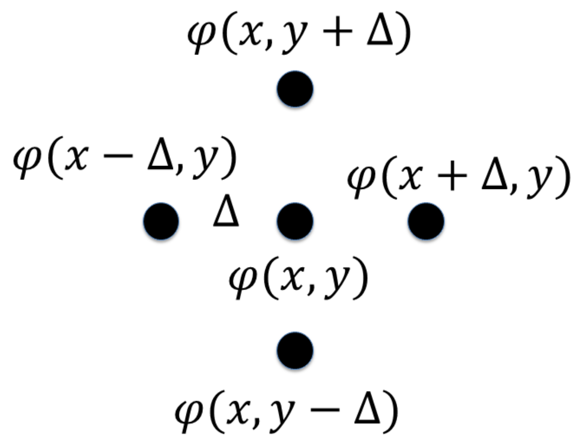

Figure 16.3: The "cross" employed in calculating approximate second derivatives in the x- and y-directions.

Following the geometry in Fig. 16.3, we find the Laplacian is approximated by

$$\nabla^2 \Phi \sim \frac{\Phi(x+\Delta, y) - 2\Phi(x, y) + \Phi(x-\Delta, y)}{\Delta^2} + \frac{\Phi(x, y+\Delta) - 2\Phi(x, y) + \Phi(x, y-\Delta)}{\Delta^2}. \tag{16.13}$$

Check, by evaluating the Taylor series expansion for the potential at the "nudged" positions:

$$\Phi(x+\Delta, y) = \Phi(x,y) + \Delta\frac{\partial}{\partial x}\Phi(x,y) + \frac{1}{2}\Delta^2\frac{\partial^2}{\partial x^2}\Phi(x,y) + ... \tag{16.14}$$

and

$$\Phi(x-\Delta, y) = \Phi(x,y) - \Delta\frac{\partial}{\partial x}\Phi(x,y) + \frac{1}{2}\Delta^2\frac{\partial^2}{\partial x^2}\Phi(x,y) + ... \tag{16.15}$$

Now, substitute into the expression on the "cross"

$$\begin{aligned}\Phi(x+\Delta,y) - 2\Phi(x,y) + \Phi(x-\Delta,y) &= \Phi(x,y) - 2\Phi(x,y) + \Phi(x,y)\\ &+ \Delta\frac{\partial}{\partial x}\Phi(x,y) - \Delta\frac{\partial}{\partial x}\Phi(x,y)\\ &+ \frac{1}{2}\Delta^2\frac{\partial^2}{\partial x^2}\Phi(x,y) + \frac{1}{2}\Delta^2\frac{\partial^2}{\partial x^2}\Phi(x,y)\\ &= \Delta^2\frac{\partial^2}{\partial x^2}\Phi(x,y),\end{aligned} \tag{16.16}$$

to lowest nonvanishing order in Δ^2, which checks out! Setting $\nabla^2\Phi = 0$ and solving for $\Phi(x,y)$ in terms of the surrounding points on the cross gives

$$\Phi(x,y) = \frac{1}{4}\left[\Phi(x+\Delta,y) + \Phi(x-\Delta,y) + \Phi(x,y+\Delta) + \Phi(x,y-\Delta)\right] \tag{16.17}$$

The relaxation method algorithm, then, is as follows:

1. Set up a uniformly-spaced grid in x and y with spacing Δ.
2. Set Φ equal to its values on the boundaries.
3. Guess Φ values in the interior (usually they vary linearly with respect to the boundary values).
4. Visit every point in the interior and update Φ via the above equation.
5. Repeat until Φ values stop changing.

We will explore this technique further in the lab.

16.3 Directional derivatives

In one dimension, the derivative is uniquely defined. But in higher dimensions, we can think of "nudging" in different directions, which gives rise to the so-called directional derivative. Consider a surface $z = f(x, y)$ that maps the x-y plane onto a surface in $3d$. An example is plotted in Fig. 16.4.

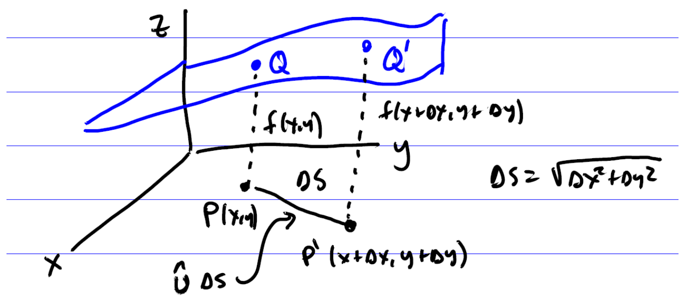

Figure 16.4: Example of a surface defined by a function $f(x, y)$ that we will use for determining directional derivatives.

The directional derivative requires us to compute the infinitesimal change in the arc length along a specific direction relative to how the function changes in the same direction (the two "nudges" Δ_x and Δ_y determine the direction of the derivative). So, we define $\Delta_s = \sqrt{\Delta_x^2 + \Delta_y^2}$ and determine Δf via a Taylor-series expansion:

$$\begin{aligned}
\Delta f &= f(x + \Delta_x, y + \Delta_y) - f(x, y) \\
&= f(x, y) + \Delta_x \frac{\partial}{\partial x} f(x, y) + \Delta_y \frac{\partial}{\partial y} f(x, y) - f(x, y) \\
&= \Delta_x \frac{\partial}{\partial x} f(x, y) + \Delta_y \frac{\partial}{\partial y} f(x, y) \\
&= \hat{u}\Delta_s \cdot \nabla f(x, y). \qquad (16.18)
\end{aligned}$$

Here, we used $\hat{u} = \frac{\Delta_x \hat{\imath} + \Delta_y \hat{\jmath}}{\Delta_s}$, and $\nabla f = \hat{\imath}\frac{\partial}{\partial x} f + \hat{\jmath}\frac{\partial}{\partial y} f$. This finally implies that

$$\frac{\Delta f}{\Delta_s} = \hat{u} \cdot \nabla f(x, y). \qquad (16.19)$$

This result is called the directional derivative of f ($\hat{u}$ is a unit vector). Note that the directional derivative is maximal when $\hat{u}$ is in the direction of ∇f, since then $\hat{u} \cdot \nabla f(x, y) = |\nabla f(x, y)|$. This says *the gradient points in the direction of maximal change to the function f*. This is useful for minimizing functions of multiple variables. Similar to Newton's method, we step in the direction of maximal change (called steepest descents).

One other property is of note. Suppose we plot iso-surfaces of f, which are given by the sets where $f(x, y)$ is equal to some constant. These are similar to contour plots on a map which plot lines of constant height. Since f does not change its value along an iso-surface, we must have the gradient is perpendicular to the iso-surface lines. This is a pictorial way of seeing the steepest descents method as a way to get to the maximum or minimum of a function. The situation is sketched in Fig. 16.5

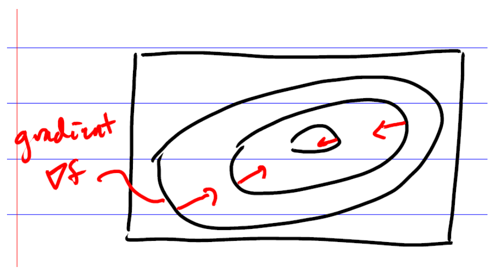

Figure 16.5: Example of a plot of different iso-surfaces of f. The gradient is perpendicular to the iso-surfaces and shows the direction of steepest descent (or ascent).