Chapter 1

Introduction to Chaos in the Lorenz System

1.1 Preamble

In higher dimensions, nonlinear systems of differential equations have the potential to act in ways not possible in one or two-dimensional systems. In two-dimensional systems, for instance, any trajectory that can be contained within a closed, bounded subset of the plane will either approach a fixed point (if it exists), or, if the fixed point is excluded from R, the trajectory is either a closed orbit or approaches one asymptotically. In effect, this proves that if we can 'trap' a trajectory within a bounded region, then we expect its long-term behavior to be periodic or fixed.

As we move to three dimensions this result no longer holds. A solution to a dynamical system can be shown to stay within a closed, bounded region yet never settle into any predictable long-term behavior. Furthermore, solutions that begin with similar initial conditions may actually fly apart in the system, so that it becomes difficult to extrapolate the motion of a trajectory from the starting point. These traits fall under the definition of chaos in a dynamic system. The term *chaos* in this context is distinguished from being nondeterministic or random. We reference the definition of chaos suggested by Strogatz[1]: "Chaos is aperiodic long-term behavior in a deterministc system that exhibits sensitive dependence on initial conditions." A chaotic system is aperiodic in that as $t \to \infty$, trajectories do not approach any fixed points, periodic orbits, or quasiperiodic orbits, deterministic in that no noise or random information is a part of the system, and sensitive in that neighboring trajectories tend to move far apart.

The goal of our report is to explore the basics of dynamical chaos through an analysis of the Lorenz equations, relying on computer simulation when deriving the behavior of the system by hand proves intractable.

1.2 Fixed Point Analysis

We first state the Lorenz equations:

$$\begin{aligned}\dot{x} &= \sigma y - \sigma x \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz\end{aligned}$$

where $\sigma, b, 0$and r are parameters greater than zero. Our first goal is to locate the fixed points of the system and identify their behavior. Setting $\dot{x}, \dot{y}.$ and $\dot{z}$ to zero, we have

$$\begin{aligned}x &= y \\ (r-z)x - y &= 0 \\ xy &= bz\end{aligned}$$

Replacing y with x as given in the first equation:

$$\begin{aligned}&x = y \\ &(r-z-1)x = 0 \to x = 0 \quad or \quad z = r-1 \\ &x^2 = bz\end{aligned}$$

We see that three fixed points are possible: $\boldsymbol{x}^* = (0,0,0)$, $(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$, and $(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$. The fixed point at the origin exists for all choices of r, while the symmetric pair of fixed points will only exist if $r > 1$. In fact, a supercritical pitchfork bifurcation occurs at the origin when $r = 1$.

The Jacobian matrix for the Lorenz system is

$$\begin{bmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{bmatrix}$$

For the fixed point at the origin, the linearization of the system becomes

$$\begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}$$

The z component decays exponentially, while the behavior of x and y depends on r: for $r < 1$, the determinant of the matrix containing only x and y is $\sigma - r\sigma > 0$, and the trace is negative. We find the discriminant $tr^2 - 4det = \sigma^2 + 2\sigma + 1 - 4(\sigma - r\sigma) = (\sigma-1)^2 + 4r\sigma$. Since r and $\sigma > 0$, we have $tr^2 - 4det > 0$, which means that the fixed point at the origin is a stable node. When $r > 1$, on the other hand, the origin changes stability and becomes a saddle node, and the fixed points C^+ and C^- appear. If this is a supercritical pitchfork bifurcation we expect these to be stable.

The linearization of C^+ and C^- is as follows:

$$\begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -d \\ d & d & -b \end{bmatrix}$$

Where $d = \pm\sqrt{b(r-1)}$. When we derive the characteristic polynomial for this linearization we find that they are identical, so we only show the process for positive d. Now we see that the behavior of z is no longer decoupled from x or y, so we cannot simplify the analysis to the 2-dimensional case. Instead, we find the eigenvalues directly:

$$\begin{aligned} det(A-\lambda I) &= (-\sigma-\lambda)((-1-\lambda)(-b-\lambda)+d^2)-\sigma(-b-\lambda+d^2) \\ &= (\lambda^2+(\sigma+1)\lambda+\sigma)(-b-\lambda)+d^2(-\sigma-\lambda)+b\sigma+\sigma\lambda-\sigma d^2 \\ &= (-1)(\lambda^3+(\sigma+b+1)\lambda^2+(b\sigma+\sigma+b)\lambda+b\sigma)+b\sigma+(\sigma-d^2)\lambda-2\sigma d^2 \end{aligned}$$

We let $det(A-\lambda I)=0$, which allows us to simplify the polynomial into its most useful form:

$$det(A-\lambda I) = \lambda^3+(\sigma+b+1)\lambda^2+b(\sigma+r)\lambda+2\sigma b(r-1)=0$$

Finding the eigenvalues for this linearization for certain parameter values is computationally laborious, but possible. When we set $\sigma = 10$, $b = 8/3$, and vary our choice of r, we find that for some r we get complex eigenvalues with negative real part, implying that C^+ and C^- are stable spirals at some point. To show this analytically, we will actually find the value of r at which a pair of purely imaginary solutions appear. We call this value r_H, and by deriving this we learn that C^+ and C^- undergo Hopf bifurcations.

Suppose we do have a pair of purely imaginary eigenvalues $\lambda = \pm i\omega$, where ω is real, and that $\sigma > b+1$. Then the characteristic polynomial is re-expressed as a complex number (it suffices just to use the positive imaginary eigenvalue):

$$\begin{aligned} -i\omega^3-(\sigma+b+1)\omega^2+ib(\sigma+r)\omega+2\sigma b(r-1)=0 \\ \rightarrow i(-\omega^3+\omega b(\sigma+r)+2\sigma b(r-1))-(\sigma+b+1)\omega^2=0 \end{aligned}$$

Since ω and the parameters are all real, the above implies both the real and imaginary parts of the equation must equal 0 as well. From this we find a pair of equations:

$$\begin{aligned} -\omega^3+\omega b(\sigma+r)=0 \\ -(\sigma+b+1)\omega^2+2\sigma b(r-1)=0 \end{aligned}$$

Our goal is to find an expression for r. Through some rearrangement we get

$$\begin{aligned} b(\sigma+r) &= \omega^2 \\ \frac{2\sigma b(r-1)}{\sigma+b+1} &= \omega^2 \end{aligned}$$

which we can merge into one equation that we rewrite to isolate r, and so obtain the Hopf bifurcation value:

$$\begin{aligned}2\sigma br - (\sigma + b + 1)br = b\sigma(\sigma + b + 1) + 2\sigma b \\ \rightarrow r = r_H = \sigma\frac{\sigma + b + 3}{\sigma - b - 1}\end{aligned}$$

Note that our assumption that $\sigma > b + 1$ was necessary, or else we either fail to derive the equation for r, or the equation implies r is negative, a contradiction to our constraints on the parameters. A symbolic solver in Mathematica confirms our derivation of r_H and also tells us that the third eigenvalue is $\lambda = -1 - b - \sigma$, which is negative. The computation can be found in the corresponding Mathematica notebook file.

Unfortunately this does not give any information as to whether C^+ and C^- at r_h are supercritical or subcritical, but knowing that they are Hopf bifurcations does confirm that they are locally stable spirals for $r < r_h$. Combined with the fact that the third eigenvalue is negative, we conclude that before the bifurcations C^+ and C^- are stable. It has been shown that the bifurcation is in fact subcritical[1] [2], which means that soon after the bifurcation, the (unstable) limit cycles disappear and all fixed points of the system are unstable. This sets up the conditions under which the Lorenz sytem behaves chaotically.

1.3 Properties of the Lorenz System Leading to the Strange Attractor

When r passes the Hopf bifurcation value r_H, it is unclear how trajectories in the system will behave. Do new limit cycles appear? Do trajectories approach infinity? We prove a couple of properties of the Lorenz system that effectively bound the long-term behavior of its trajectories.

First we show that the Lorenz equations form a volume-dissipating system. The derivation appears in Strogatz[1], but we reproduce it here. We let the Lorenz system be represented by a vector function $\boldsymbol{f}(\boldsymbol{x})$, and $V(t)$ a volume whose value changes with respect to the Lorenz system, with surface $S(t)$. Then

$$\dot{V} = \int_S \boldsymbol{f} \cdot \boldsymbol{n} dA$$

and by the divergence theorem:

$$\begin{aligned}\dot{V} &= \int_V \nabla \cdot \boldsymbol{f} dV \\ \nabla \cdot \boldsymbol{f} &= \frac{\partial}{\partial x}[\sigma(y - x)] + \frac{\partial}{\partial y}[rx - y - z] + \frac{\partial}{\partial z}[xy - bz] \\ &= -\sigma - 1 - b < 0\end{aligned}$$

The divergence here is constant and negative, which means volumes V shrink exponentially fast.

A second, analogous property is that an ellipsoid exists so that all trajectories eventually enter it and stay in there forever, represented by $rx^2 + \sigma y^2 + \sigma(z - 2r)^2 \leq C$. A complete derivation of C that defines the ellipsoid is found in [2]. For the report it suffices to know that whatever attracting objects exist in the system appear in this region.

1.4 Transition to Chaos

Given what we have learned about the Lorenz system, we know trajectories will not trend towards infinity, and for many values of $r > r_H$, no new limit cycles appear. (There are windows of r where in fact new period cycles appear, but we will cover that later.) In that case, numerical experiments show that trajectories begin to behave chaotically. In the analysis to follow, we use numerical computation to generate trajectories and their component graphs, and we select pairs of trajectories and plot the distance between them over time. The scope of our paper is limited, so we defer use of more analytical tools.

1.5 Changes in Phase Space over r

Using Mathematica, we generated diagrams that map out the changes in the Lorenz system as r changes. See that for $r < 1$ that only one fixed point exists and all trajectories funnel towards it, but as r increases two new fixed points are generated, and a distinct palmier shape is drawn out by the trajectories. Note that, for our calculations, we set $\sigma = 10$ and $b = 8/3$ fixed for all values of r, in accordance with Lorenz's original experiments. This means $r_H \approx 24.7$.

Figure 1.1 exhibits trajectories that move in in oblique arcs as the z component dissipates.

The next two figures illustrate what happens around the two new fixed points C^+ and C^-. We see that in Figure 1.2 the purple and orange trajectories begin on one side of the figure but both spiral into one fixed point, while the red and blue trajectories begin close enough to the other fixed point to remain there. By r = 20, some trajectories are progressing towards their fixed points but others are volleyed in between the two lobes. Current literature describes this phenomenon as "intermittent chaos": short term trajectories move around aperiodically before reaching a fixed point, and as we see, the time it may take before a trajectory is 'trapped' within the radius of a fixed point can vary.

Once $r > r_H \approx 24.7$, chaotic behavior begins to appear. By $r = 28$ the purple and orange trajectories, though coming close to the left fixed point, actually spiral outwards now (the orientation is subtle but clearer in the Mathematica animations). The other trajectories are further spread out on the attractor. By $r = 40$ the chaotic behavior of the attractor is abundantly clear.

Figures 1.6, 1.7 and 1.8 all illustrate the position of the z component of the red trajectory, for each r. Whereas for $r = 20$ the graph exhibits some irregular behvior before decaying in regular oscillation, for $r = 28$ and $r = 40$ no portion

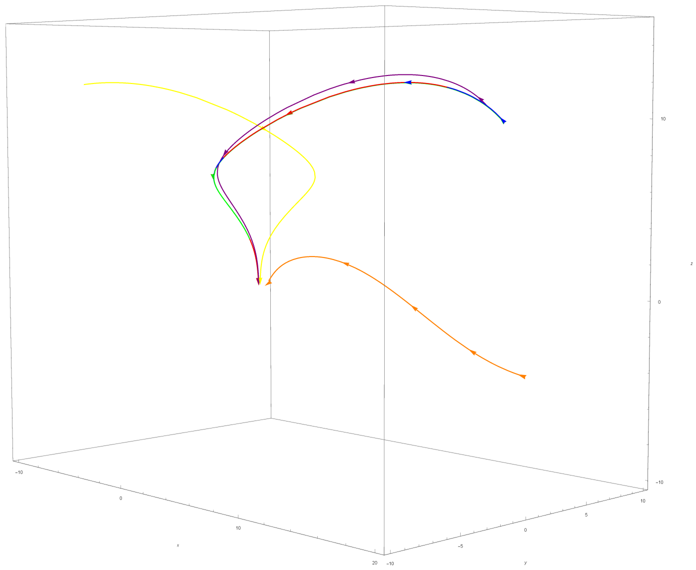

Figure 1.1: Plot of selected trajectories for r = 0.5

of the graph recurs periodically, at least for the range of t given. This fulfills one cornerstone of chaotic behavior, that the (long-term) behavior is aperiodic.

However, there are ranges of r where stable periodic cycles appear, and unlike the unstable cycles before the Hopf bifurcation, these approximate the palmier shape of the attractor itself. For our numerical example we chose $r > 300$, where the largest window of periodic behavior occurs.

1.6 Approximating the Large r Case

An approximation for the Lorenz system at large r exists when $r \to \infty$. We produce a new set of differential equations from the Lorenz system by transforming x, y, and z into functions X, Y,Z depending on τ:

$$\begin{aligned}
&X = \varepsilon x \\
&Y = \sigma \varepsilon^2 y \\
&Z = \sigma(\varepsilon^2 z - 1) \\
&t = \varepsilon \tau \\
&\varepsilon = r\text{-}1/2
\end{aligned}$$

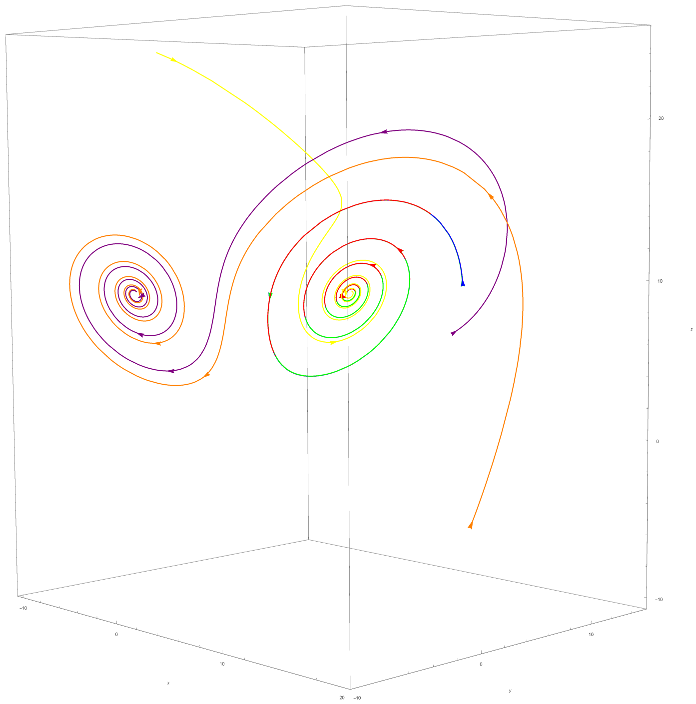

Figure 1.2: Trajectories for r = 10

By taking the derivative with respect to τ, we have

$$\begin{aligned}
X' &= \varepsilon^2(\sigma y - \sigma x) = Y - \varepsilon\sigma X \\
Y' &= \sigma\varepsilon^3(rx - y - xz) = \sigma X - \sigma\varepsilon^2 Xz - \varepsilon Y = -XZ - \varepsilon Y \\
Z' &= \sigma\varepsilon^3(xy - bz) = XY - \varepsilon\sigma(\varepsilon^2 z) = XY - \varepsilon(Z + \sigma)
\end{aligned}$$

so that when $\varepsilon \to 0$, corresponding to $r \to \infty$:

$$\begin{aligned}
X' &= Y \\
Y' &= -XZ \\
Z' &= XY
\end{aligned}$$

Unlike the Lorenz equations, the derived system of equations is volume preserving, due to the fact that the divergence$\nabla \cdot \boldsymbol{f} = 0$, and two conserved quantities exist: $E_1 = y^2 + z^2$ and $E_2 = x^2/2 - z$. Therefore, closed orbits certainly exist in this system and can give some information about the closed orbits that exist for large r in the Lorenz system.

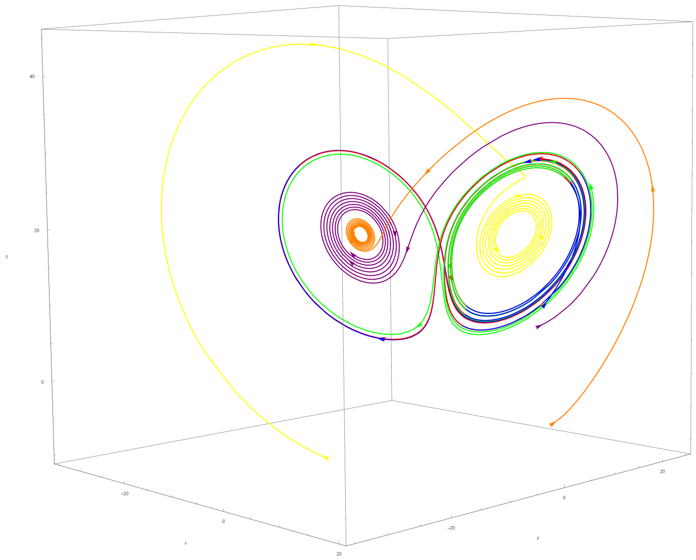

Figure 1.3: Trajectories for r = 20

Conclusion

The Lorenz system is an apt object study on the breadth of material that higher dimensional dynamic systems can produce. Even by fixing two of the parameters, varying r is still sufficient to find models of increasing unpredictability. It also presents one with the challenge to characterize a difficult to grasp attracting region, but one where the basic techniques for lower-dimensional systems provide a stepping stone to more eclectic analysis. Computational methods helped our research as well, giving us visual and numerical information about how actual trajectories of the Lorenz system move.

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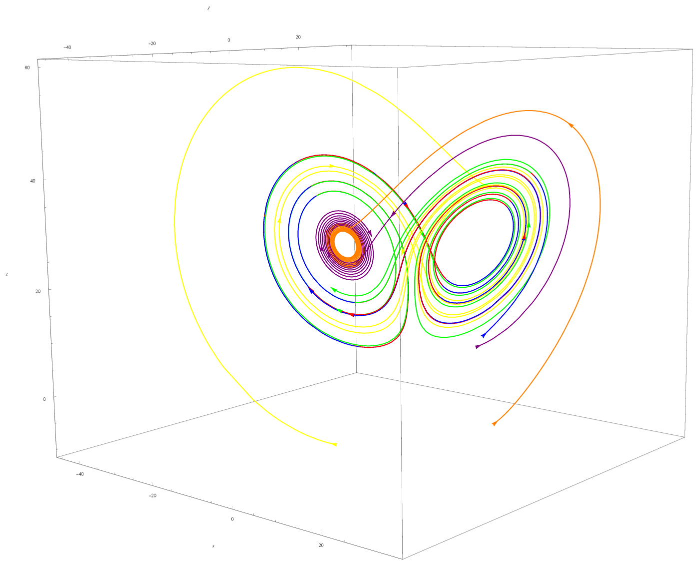

Figure 1.4: Trajectories for r = 28

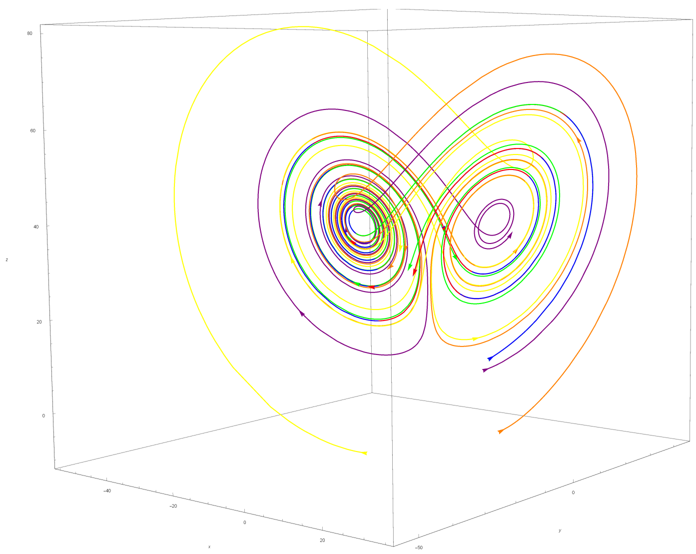

Figure 1.5: Trajectories for r = 40

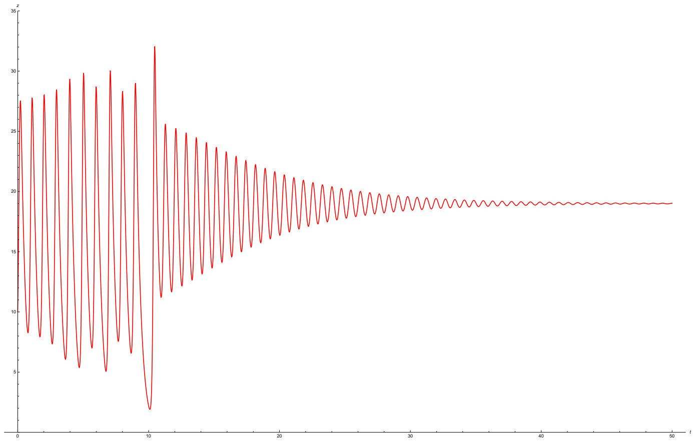

Figure 1.6: Plot of z(t) for r = 20 (intermittent chaos)

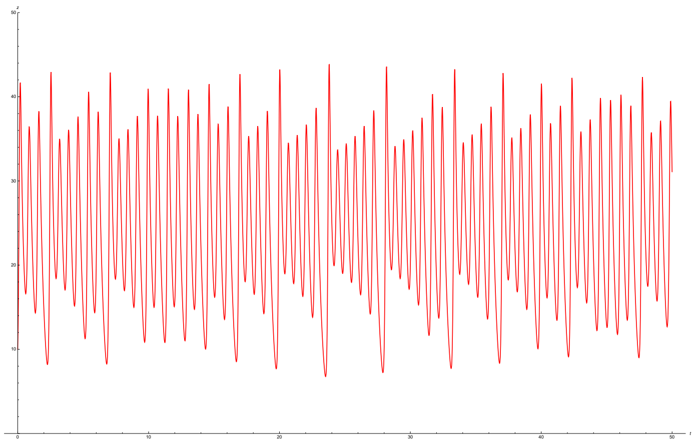

Figure 1.7: Plot of z(t) for r = 28 (chaotic)

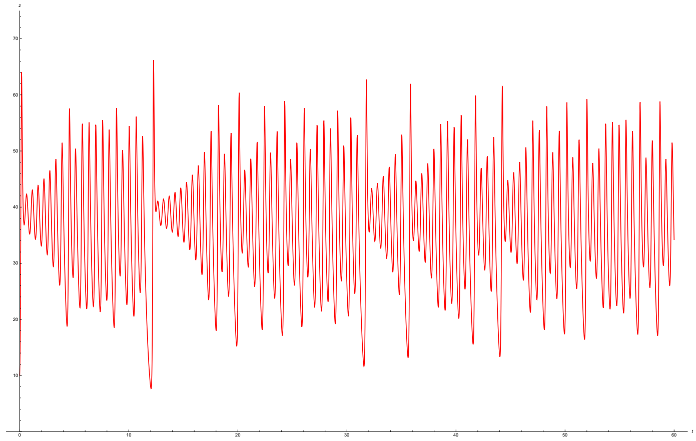

Figure 1.8: Trajectories for r = 40 (chaotic)

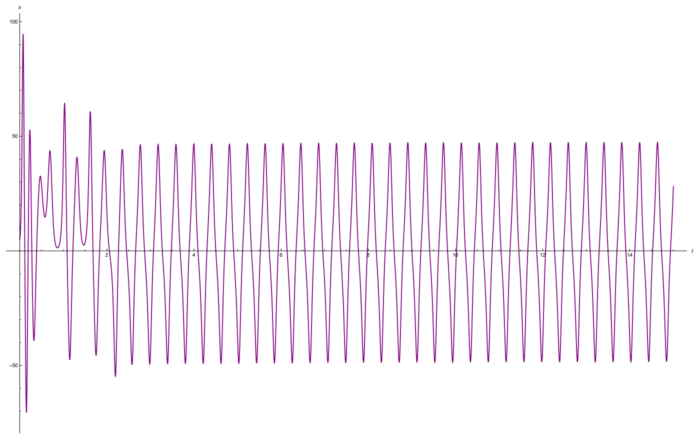

Figure 1.9: Plot of x(t) for $r > 300$. The oscillations appear to settle to a specific frequency.

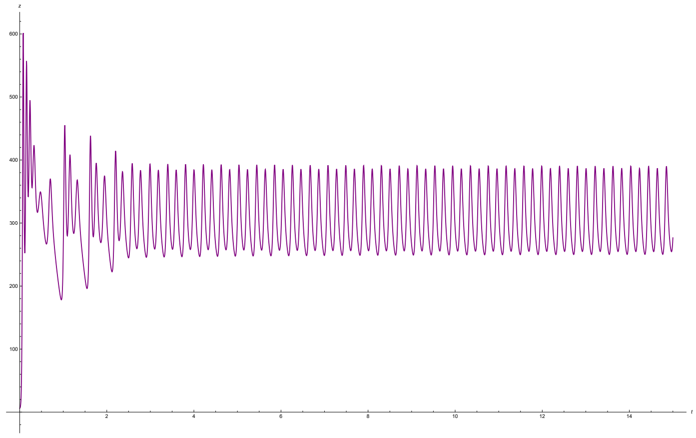

Figure 1.10: Plot of z(t) for $r > 300$. The interpolation of the plot is rougher for trajectories in this regime.

Bibliography

[1] Steven H. Strogatz, *Nonlinear Dynamics and Chaos: with Applications to Physics, Biology, Chemistry, and Engineering*. Westfield Press, Colorado, 2nd edition, 2015.

[2] Colin Sparrow, *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*. Springer-Verlag, New York, 1982.