# Categorical approach to dynamical YBE

- Quantum Yang-Baxter equation:

$U, V, W$, $\quad R_{UV} : U \otimes V \to U \otimes V$, $R_{UW}$, $R_{VW}$.

QYBE: $\quad R_{UV} R_{UW} R_{VW} = R_{VW} R_{UW} R_{UV}$ on $U \otimes V \otimes W$

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

Assume $\mathcal{C}$ is a braided monoidal category

$F : \mathcal{C} \to \mathrm{Vect}$. $\quad R$, YB $\longleftrightarrow$ braiding on $\mathcal{C}$

$F$ monoidal: $\quad \forall X, Y \in \mathcal{C}$

$$R_{X,Y} : F(X) \otimes F(Y) \xrightarrow{J_{X,Y}} F(X \otimes Y) \xrightarrow{\beta_{X,Y}} F(Y \otimes X) \to$$

$$\xrightarrow{J_{Y,X}^{-1}} F(Y) \otimes F(X) \to F(X) \otimes F(Y)$$

Satisfy the QYBE ($\Leftarrow$ hexagon axiom $\beta_{X,Y}$)

Typically, $\quad \mathcal{C} = \mathrm{Comod}_H \quad (H = \mathcal{O}(G)$ or $\mathcal{O}_q(G))$

- DQYBE: let $H$ be a torus $(H = (\mathbb{C}^\times)^n)$

$\mathfrak{h} = \mathrm{Lie}(H)$

$$R_{UV}(\lambda - h^{(3)}) R_{UW}(\lambda) R_{VW}(\lambda - h^{(1)}) =$$

$$R_{uv}(\lambda)\, R_{uw}(\lambda - h^{(2)})\, R_{uv}(\lambda)$$

Here $\lambda \in \mathfrak{h}^*$, $U, V, W \in Rep(H)$, $R_{UV}: \mathfrak{h}^* \to End(U \otimes V)$

meromorphic function, $R_{UV}$ is $\mathfrak{h}$-inv.

Say $u \in U$: $h \cdot u := wt(u) \cdot u$

- Face-type models
- Exchange algebras in Liouville and Toda field theorie
- Felder: KZB eq's on elliptic curves

(in there, $\lambda \in Pic^0(K)$)

<u>Semiclassical version:</u> ...

[ standard solutions: $\cdot\, r(\lambda) = \sum_{\alpha \in \Delta_+} \frac{e_\alpha \wedge e_{-\alpha}}{(\lambda, \alpha)}$

$\mathfrak{g}$ a simple Lie alg.

$$\cdot\, r(\lambda) = \frac{\Omega}{2} + \sum \coth\left(\frac{(\alpha,\lambda)}{2}\right) e_\alpha \wedge e_{-\alpha}$$

Examples of solutions to QDYBE for any simple $\mathfrak{g}$:

<u>EV</u>: $M_\lambda$ = Verma module of weight $\lambda$

$\Phi: M_\lambda \to M_\mu \otimes V$, where $V$ finite-dim rep of $\mathfrak{g}$

$\Phi \rightsquigarrow \langle \Phi \rangle$: $M_\lambda^* := \bigoplus_{\nu \in \Lambda} M_\lambda[\nu]^* \ni x_\lambda^*$

$\phi \rightsquigarrow \langle \phi \rangle = \langle x_\mu^*, \phi(x_\lambda) \rangle$, $x_\lambda$ is the h.w.v. of $M_\lambda$
$$\underset{V[\lambda-\mu]}{\cap}$$

$\mathrm{Hom}_{\mathfrak{g}}(M_\lambda, M_\mu \otimes V) \to V[\lambda-\mu]$

<u>Prop</u>: If $M_\mu$ is irreducible, then it is an iso

<u>Construction</u>: → Etingof - Varchenko
let $V, W$ be f.-d. rep's of $\mathfrak{g}$

$v, w \in V, W$ weight vectors

Assume $\lambda \in \mathfrak{h}^*$ is generic, i.e. $(\lambda, \alpha^\vee) \notin \mathbb{Z}$ $\forall \alpha \in \Delta \Rightarrow M_{\lambda-\nu}$ is irred. $\forall \nu \in \Lambda$

$$M_\lambda \xrightarrow{\phi_\lambda^v} M_{\lambda-wt(v)} \otimes V \xrightarrow{\phi^w_{\lambda-wt(v)}} M_{\lambda-wt(v)-wt(w)} \otimes W \otimes V \Rightarrow$$

$\Rightarrow \exists\, J_{v,w}(\lambda)(v \otimes w) \in V \otimes W$ s.t. this composition is given by $\phi_\lambda^{J_{v,w}(\lambda)(v\otimes w)}$

$J_{v,w}(\lambda) : V \otimes W \to V \otimes W$

<u>Prop</u> (Etingof - Varchenko): $J_{v,w}(\lambda)$ is invertible, meromorphic, satisfying the <u>dynamical twist eq.</u>

<u>Cor</u>: $(J^{21}_{v,v})^{-1} J_{v,w} =: R_{v,w}(\lambda)$ satisfy the DYBE

**Q:** What is the categorical interpretation?

<u>Harish-Chandra bimodules</u>: Let $G$ be an affine alg. group, $\mathfrak{g} = \mathrm{Lie}(G)$

$[\mathfrak{g}^*/G]$ 1-shifted symplectic

<u>Thm (Calaque)</u>: 1-shifted Lagrangians in $[\mathfrak{g}^*/G]$

$\Longleftrightarrow$ Hamiltonian $G$-spaces

If $X$ is a $G$-scheme (smooth),

$\mu: X \longrightarrow \mathfrak{g}^*$ is a $G$-equiv. map,

$\rightsquigarrow [X/G] \to [\mathfrak{g}^*/G]$

$QCoh([\mathfrak{g}^*/G]) = \mathrm{Mod}_{S\mathfrak{g}}(\mathrm{Rep}\, G)$

$\rightsquigarrow HC(G) := \mathrm{Mod}_{U\mathfrak{g}}(\mathrm{Rep}\, G) \ni X \Longleftrightarrow$

$\Longleftrightarrow$ • $X$ is a $U\mathfrak{g}$-module

• $X$ is also a rep of $G$ s.t.

$\mathrm{act}: U\mathfrak{g} \otimes X \to X$ of $G$-representations

<u>Ex</u>: $\mathrm{free}: \mathrm{Rep}\, G \longrightarrow HC(G)$, $\mathrm{free}(V) = U\mathfrak{g} \otimes V$

$HC(G)$ is monoidal: $HC(G) \hookrightarrow \mathrm{Bimod}_{U\mathfrak{g}}$

Right $\mathfrak{g}$-action: $x \in X$, $\xi \in \mathfrak{g}$:

$$x\xi := \xi x - \underline{\xi \cdot x}$$

→ from $G$-action

<u>Ex</u>: $HC(H)$, $H$ a torus $(H \curvearrowright U\mathfrak{h}$ trivially$)$

$\quad \parallel$

$\Lambda$-graded $U\mathfrak{h}$-modules

$$X = \bigoplus X[\mu], \quad Y = \bigoplus Y[\nu]$$

$$X \otimes^{HC} Y = \bigoplus_{\lambda \in \Lambda} \lambda^* X \otimes_{U\mathfrak{h}} Y[\lambda], \quad \text{here}$$

$$\lambda : \mathfrak{h}^* \to \mathfrak{h}^*, \quad \lambda(\alpha) = \lambda \pm \alpha.$$

$$X \otimes^{HC_\hbar} Y = \bigoplus (\hbar\lambda)^* X \otimes_{U\mathfrak{h}} Y[\lambda]$$

<u>Thm</u>: (Ben-Zvi – Brochier – Jordan, Safronov): An algebra in $HC(G)$ $\Leftrightarrow$ an algebra $A$ in $\mathrm{Rep}(G)$ with a quantum moment map $U\mathfrak{g} \to A$.

<u>Thm (K.-Safronov)</u>: $\mathcal{C}$ a braided monoidal category

$$\mathcal{C} \longrightarrow HC(H) \quad \text{monoidal}$$

$\searrow$ $\mathrm{Rep}(H)$ $\nearrow$ → free modules

$\Rightarrow$ a solution to the DYBE

**Remark:** (Safronov) The dual of classical DYBE

"$\Longleftrightarrow$" 1-shifted Poisson morphism $[\mathfrak{h}^*/H] \to BG$

$[\,G$ is simple Lie group, $H \subset G\,]$

$r(\lambda) : \mathfrak{h}^* \to \mathfrak{g} \otimes \mathfrak{g}$

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**EV-construction:** $\tilde{\mathfrak{g}} := \{(x, \mathfrak{b}) \mid x \in \mathfrak{g},\ \mathfrak{b} \in G/B,\ x \in \mathfrak{b}\}$

$\cong G \times_B \mathfrak{b}$

$$\mathfrak{g} \leftarrow \tilde{\mathfrak{g}} \to \mathfrak{h} \qquad G \times_B \mathfrak{b} \to G \times_B \mathfrak{h} \to \mathfrak{h}$$

moment map

(?) $(T^* G/U)/\!/H$

$$\begin{tikzcd} & {[\tilde{\mathfrak{g}}/G]} \arrow[dl] \arrow[dr] & \\ {[\mathfrak{g}/G]} & & {[\mathfrak{h}/H]} \end{tikzcd}$$

1-shifted $\qquad$ symplectic

$\mathfrak{g}$ is reductive

$\mathfrak{g} \simeq \mathfrak{g}^*$

**Thm** (Safronov): $[\tilde{\mathfrak{g}}/G] \to [\mathfrak{g}/G] \times \overline{[\mathfrak{h}/H]}$ is

1-shifted Lagrangian

**Quantization:** $HC(G) \curvearrowright \mathcal{O}^{univ} \curvearrowleft HC(H)$

$\qquad\qquad \| \sim$

$\qquad \mathrm{Mod}_{U\mathfrak{g}}(\mathrm{Rep}\, H)$

$\underline{\text{Ex}}$: $\quad M^{univ} := U\mathfrak{g} \otimes_{U\mathfrak{b}} U\mathfrak{h} \in \mathcal{O}^{univ}$

$$HC(G) \longrightarrow \mathcal{O}^{univ} \longleftarrow HC(H)$$

Taking adjoints: $\quad HC(G) \to HC(H)$

$$(*) : X \longmapsto \left(X/\mathfrak{n}\right)^{N}$$

$\underline{\text{Proof}}$: It is naturally lax monoidal

$$\left(X/\mathfrak{n}\right)^{N} \otimes_{U\mathfrak{h}} \left(Y/\mathfrak{n}\right)^{N} \longrightarrow \left(X \underset{U\mathfrak{g}}{\otimes} Y/\mathfrak{n}\right)^{N}$$

$$[x] \otimes [y] \longmapsto [x \otimes y]$$

$HC(G) \to HC(H)^{gen} = \mathrm{Mod}_{(U\mathfrak{h})^{gen}}(\mathrm{Rep}\, H)$, where

$(U\mathfrak{h})^{gen}$ is the alg. of meromorphic functions

with poles on $\{\lambda \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\}$ non-generic

$\underline{\text{Thm}}$:

$$\begin{tikzcd} \mathrm{Rep}\, G \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & HC(G) \arrow[d] \\ \mathrm{Rep}\, H \arrow[r] & HC(H)^{gen} \end{tikzcd}$$

$X = U\mathfrak{g} \otimes V$

$\left(U\mathfrak{g} \otimes V/\mathfrak{n}\right)^{N} \simeq$

$\simeq U\mathfrak{h} \otimes V$

$\rightsquigarrow$ standard solution of the DYBE

Proof uses extremal projector: $P \in \widehat{U\mathfrak{g}}$: $\quad P^2 = P$, $e_\alpha P = P f_\alpha = 0$

$\forall \alpha \in \Delta_+$

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$\underline{\text{Kostant-Whittaker reduction}}$: $\quad \psi : \mathfrak{n}_- \longrightarrow \mathbb{C}$, $\psi(f_{\alpha_i}) = 1$

$\forall \alpha_i \in \Pi$ simple

Thm: (Kostant): $\left(n_-^\psi \backslash U\mathfrak{g}\right)^{N_-} \simeq Z(U\mathfrak{g})$

$n_-^\psi$ = ideal gen. by $x - \psi(x)$, $x \in n$

$HC(G) \longrightarrow Z(U\mathfrak{g})$-bimod, $\quad X \mapsto \left(n_-^\psi \backslash X\right)^{N_-}$

Prop: (Bezrukavnikov-Finkelberg): Monoidal, exact, ...

$Z(U\mathfrak{g}) \longrightarrow U\mathfrak{h}$ HC homomorphism

$\mathrm{Bimod}_{Z(U\mathfrak{g})} \longrightarrow Z(U\mathfrak{g})$ - $U\mathfrak{h}$ - bimod,

$F \mapsto F \otimes_{Z(U\mathfrak{g})} U\mathfrak{h}$

Thm: (Ginzburg-Kazhdan, K. )
↳ for $sl_2$ precise form

$\exists$ a natural isomorphism

$$X \mapsto (X/n)^N$$
$$X \mapsto \left(n_-^\psi \backslash X\right)^{N_-} \otimes_{Z(U\mathfrak{g})} U\mathfrak{h}$$