Gauged Linear Sigma Model for the Jacobians of Hyperelliptic Curves

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Abstract

In this paper we construct a gauged linear sigma model describing a geometric theory on the Jacobian of a hyperelliptic curve in one of its limits. In paricular, we establish its relation to the theory on the stacky orbifold $[C^g/S_g]$ reflecting a well-know connection between the derived category of coherent sheaves on the Jacobian and that of the symmetric power of the curve.

1 Introduction

Mirror symmetry (and string theory in general) is a powerful source of various mathematical predictions, the famous example being counting of rational curves on Calabi-Yau threefolds. Another one, with which we are concerned in this paper, is connected to derived categories of coherent sheaves. Namely, sometimes physical considerations predict at least an existence of a nontrivial equivalence between the derived categories of certain Calabi-Yau manifolds.

The first example was considered by Rødland [Rød00]. Roughly, the idea is as follows: if two Calabi-Yau manifolds X_1, X_2 have the same mirror Y, then by homological mirror symmetry conjecture their derived categories should be equivalent via the intermediate Fukaya category of Y: $D^bCoh(X_1) \simeq Fuk(Y) \simeq D^bCoh(X_2)$. It was observed in [Rød00] that if we take the variety given by intersection of Pfaffians of 7×7 skew-symmetric matrix (more precisely, a certain one-parameter family thereof), then its Picard-Fuchs equation is the *same* as that of a linear section of the Grassmannian $G(2, 7)$ in the Plücker embedding, but taken at a different point. Therefore, they should have the same mirror, hence equivalent derived categories. The actual equivalence was constructed and proven later by Borisov-Caldararu [BC09] and independently by Kuznetsov [Kuz06] using homological projective duality.

An important point of the example above is that the original Picard-Fuchs equation (of the Pfaffian variety, say) has *several* points of maximally unipotent monodromy. In mirror symmetry picture, this type of singularity should correspond to geometric theories. In principle, a natural idea would be to calculate the Picard-Fuchs operators of various Calabi-Yau varieties and check whether they have other points of maximally unipotent points; there are many lists of such operators (for instance [GJ16]). However, there is an obvious disadvantage of this method: it does not provide a derived partner, only an evidence of its existence.

There is another way which constitutes a part of the title: a gauged linear sigma model. It was introduced by Witten [Wit93] and provided a beautiful explanation of the equivalence between the geometric theory on a Calabi-Yau hypersurface in projective space and a certain orbifold Landau-Ginzburg theory. Very roughly, the idea is that one introduces an additional parameter called *the*

Fayet-Iliopoulos term limits of which give the theories mentioned earlier; since they are connected by a continuous deformation, they should be equivalent (for precise details, see Witten's paper). A categorical avatar of such an equivalence is, for example, the paper by Orlov [Orl09] where an equivalence between the derived category of coherent sheaves on a Calabi-Yau hypersurface and the category of singularities of the superpotential defining it was constructed.

The model above can be generalized *mutatis mutandis* to an arbitrary complete intersection in arbitrary toric variety. The only difference is that the other limit will give not an orbifold Landau-Ginzburg theory, but something more complicated, what is called *a hybrid Landau-Ginzburg* theory. Still, it is not a geometric model, so it does not provide equivalences we mentioned in the beginning. The point is that both limits arise as certain GIT quotients by the gauge group of a gauged linear sigma model (GLSM for short), and in those examples the gauge group is abelian. Certainly, the most natural guess is to take a nonabelian group, and sometimes it gives something geometric as a limit.

The first example, as the reader might suggest, is again the Pfaffian/Grassmannian correspodence considered by Rødland. Namely, in the paper [HT07] there was constructed a GLSM giving the Pfaffian variety in one limit and the linear section of the Grassmannian in another. Since then there was constructed many more GLSMs describing 'exotic' Calabi-Yau varieties as one of its phase (see, for instance, [CKS18]).

There are certain caveats here. First, the limits are not always geometric. Second, even from the physics point of view, nonabelian theories are usually 'not good' (presence of strong couplings as an example). However, it is still a very promising project to obtain nontrivial relations between Calabi-Yau varieties.

The present work deals with quite an unusual player in the GLSM picture: an abelian variety. On the one hand, they are well-undestood in various senses (for instance, autoequivalences of their derived categories [Orl02] or mirror symmetry for them [GLO01]), so one does not expect anything unusual. On the other hand, there is an observation by Reid [Rei72] that the Jacobian of a hyperelliptic curve C has a geometric realization as the space of planes on the intersection of two quadrics; stated differently, it is the zero locus of a section of a certain bundle over Grassmannian, and the latter can be obtained by GIT construction. With a little work one can introduce a GLSM describing the geometric theory on the Jacobian in one limit (Section 4), so it is natural to ask what kind of mathematical information the other limit brings.

Unfortunately, the result of this paper is negative in this regard, that is, the Landau-Ginzburg phase is nonmanageable and gives no mathematical information (at least to our knowledge). For instance, it also suffers from the strong coupling issues. However, what was noticed (and what we consider as the main result) is that if one quotients out by a smaller group of symmetries, then one obtains the theory which can be described as 'the symmetric power of the theory on the curve C'. The main ingredient of this description is a physical argument (introduced by [HHP⁺07]) stating (roughly) that an abelian GLSM with a quadratic potential gives rise to a geometric theory on a ramified double cover. In particular, if we consider the intersection of two quadrics in projective space of dimension $2g+1$, then the expected hybrid Landau-Ginzburg model is, in fact, equivalent to the geometric theory on a certain hyperelliptic curve of genus g.

Translated into the world of derived categories, this result is actually well-known. More precisely, it simply reflects the fact that there is a fully faithful embedding of the derived category of the Jacobian into the symmetric power of the derived category (understood in an appropriate sense) of the curve. Such an embedding can be provided by the Abel-Jacobi map realizing the symmetric power of the curve as the blow-up of the Jacobian; however, as it is argued in the end of the paper, there could be other embedding more 'natural' from the point of view of this GLSM.

The paper is organized as follows: in Section 2, we review the construction of Reid realizing the Jacobian as the variety of planes on the intersection of two quadrics; in Section 3, we review the necessary facts about gauged linear sigma models in general (physical formulation, mathematical formulation, plus the quadratic potential argiment by [HHP+07] mentioned earlier); in Section 4, we provide the GLSM describing the Jacobian and relate it to the symmetric power of the curve theory; in Section 5, we translate this relation into the realm of derived categories, that is, we recall how the derived category of the Jacobian can be embedded into that of the symmetric power of the curve; in Section 6, we discuss why this embedding is not so natural from the point of view of theis GLSM, and how this work could possibly be connected to the previous one [Kal16].

2 Linear subspaces on quadrics

In this section we recall the construction of the isomorphism between the variety of linear subspaces on the intersection of two quadrics and the Jacobian of a curve. The main reference is [Rei72]; see also [DR76]. To avoid confusion, by *k-subspaces* we mean *linear* subspaces of dimension k, while *projective* subspaces are called *k-planes*; for instance, a k-subspace defines a $(k-1)$-plane, and vice versa.

Let Q_0, Q_1 be two quadrics in $\mathbb{P}^{2g+1} = \mathbb{P}(V)$, and $X := Q_0 \cap Q_1$ be their intersection. Denote by $\Phi \simeq \mathbb{P}^1$ the pencil generated by them; we will denote by Q_λ the quadric corresponding to the point $\lambda \in \mathbb{P}^1$ with the equation φ_λ.

Definition 2.0.1. The pencil Φ is called *nondegenerate* if all Q_λ are either nonsingular or simple cones; and for the values λ such that Q_λ is degenerate, if $\{e_\lambda\}$ is a basis of $\mathrm{Ker}(\varphi_\lambda)$, then $\varphi_\mu(\lambda)(\{e_\lambda\}) \neq 0$ for all the values $\mu \neq \lambda$.

By [Rei72, Proposition 2.1] the following are equivalent:

1. X is nonsingular of codimension 2 in $\mathbb{P}(V)$;
2. The pencil Φ is nondegenerate;
3. $\det(\varphi_\lambda)$ considered as a polynomial in λ is not identically zero and has distinct $2g+2$ roots;
4. There exists a basis of V, orthogonal for each λ, such that

$$\begin{aligned}\phi_0(\textstyle\sum x_i e_i) &= \textstyle\sum x_i^2;\\ \phi_1(\textstyle\sum x_i e_i) &= \textstyle\sum \lambda_i x_i^2,\end{aligned} \tag{2.1}$$

and all λ_i are distinct (such a basis is automatically unique up to sign).

So we assume that the pencil Φ satisfies one of the conditions above. To it we can canonically associate a hyperelliptic curve: there are $2g+2$ distinct points on $\mathbb{P}^1$ corresponding to the singular members of the pencil, so that the double cover ramified in these points defines a curve of genus g. Vice versa: given a curve C with a hyperelliptic cover $C \to \mathbb{P}^1$, one can find a coordinate system such that the ramification points are $\{[1:\lambda_i],\ i = 1\dots 2g+2\}$; we associate to it the quadrics in $\mathbb{P}^{2g+1}$ given by the equations (2.1).

There is a certain geometric meaning behind C. Namely, let $\mathrm{Gen}(\Phi)$ be the variety of *generators*, i.e. the incidence set $\{(\lambda, E) | \ E \subset Q_\lambda\} \subset \mathbb{P}^1 \times \mathrm{Gr}(g+1, 2g+2)$ (note that now the dimension of a linear subspace is $g+1$, not g). By definition, there is map $\mathrm{Gen}(\Phi) \to \mathbb{P}^1$; by [Rei72, Theorem 1.10], it factors through $\mathrm{Gen}(\Phi) \xrightarrow{p} C \xrightarrow{q} \mathbb{P}^1$, where q is the hyperelliptic cover. Roughly, for each nonsingular λ there are *two* connected components of planes on Q_λ, so p indicates the quadric *plus* the family to which the subspace belongs.

Consider the Grassmannian of $(g-1)$-planes $\mathrm{Gr}(g, 2g+2)$ and let J be the subvariety of planes lying on X. One can view it as follows: let U be the tautological bundle on $\mathrm{Gr}(g, 2g+2)$, then the equations φ_0, φ_1 being elements of S^2V^* give a section of the bundle $S^2U^* \oplus S^2U^*$ (by fiberwise restriction to a subspace), and J is just its zero locus. We have the following

Theorem 2.0.2. [Rei72, Theorem 4.8] *As a variety, J is (noncanonically) isomorphic to the Jacobian of the curve C.*

Here we give just the main ideas of the proof referring the reader to Chapter 4 of [Rei72] for details.

Let $s \in X$ be a g-dimensional subspace, in particular, $s \subset Q_\lambda$ for every quadric in the pencil Φ. Define

$$\tilde{C}_s := \{(\lambda, g) | \ g \supset s\} \subset \mathrm{Gen}(\Phi).$$

By [Rei72, Lemma 4.1] it is a nonsingular curve mapping isomorphically onto C by p. There is a natural map $r : \tilde{C}_s \to J$ defined as follows: if $g \in \tilde{C}_s$, then g is a linear $(g+1)$-subspace of some quadric Q_λ. Take $\mu \in \mathbb{P}^1$ such that Q_μ is smooth, then $X \cap g = (Q_\lambda \cap Q_\mu) \cap g = Q_\mu \cap g = s \cup s_g$ for some linear g-subspace s_g (for degree reasons say), and we define $r(g) := s_g \in J$.

Denote by C_s the image $r(\tilde{C}_s)$ in J. By definition, C_s is the closure (in J) of the subvariety of s' such that $\dim(s \cap s') = g-1$, and r is one-to-one on this set, hence birational; therefore, $C_s \simeq C$. Likewise, we can define the subvariety

$$C_s^{i'} = \{t \mid \dim(s \cap t) = g - i\}$$

(with some nondegeneracy condition, see [Rei72, Definition 4.3]) and its closure C_s^i. It turns out [Rei72, Proposition 4.4] that C_s^i is birationally isomorphic to the symmetric power $C^{(i)}$; therefore, a connected component of J is birational to $C^{(g)}$, hence to the Jacobian $J(C)$.

One can prove that J has only one connected component, and then we can apply Deligne's argument (see [Rei72, Lemma 4.7]): if there is a birational morhism $S \to A$ with A an abelian variety and $K_S = 0$ (for projective varieties the degree is enough), then it is an isomorphism. After some calculations [Rei72, Theorem 4.8] it is shown that it is indeed the case for J, hence J is isomorphic to the Jacobian $J(C)$.

3 Gauged Linear Sigma Model

3.1 $U(1)$-gauge

Let us recall the abelian GLSM with gauge group $U(1)$ giving Calabi-Yau/Landau-Ginzburg correspondence for hypersurfaces in projective spaces. The goal is to provide an intuition for what follows, so we refer the reader to the seminal paper of Witten [Wit93] or to a nice overview [Nie] for details (and notations which are not of importance to us).

Let $G = G(\varphi_1, \dots, \varphi_N)$ be a homogeneous polynomial of degree d in N variables defining a smooth hypersurface in $\mathbb{P}^{N-1}$. Consider a $U(1)$-gauge theory theory with $N+1$ chiral multiplets $\Phi_1, \dots, \Phi_N, P$, such that $(\Phi_1, \dots, \Phi_N)$ are of charge 1 and P is of charge $-d$. The (supersymmetric) Lagrangian is given by

$$\mathcal{L} = \int d^2x d^4\theta \Big(\sum_{i=1}^{N} \bar{\Phi}_i e^V \Phi_i + \bar{P} e^{-dV} P - \frac{1}{2e^2} \bar{\Sigma}\Sigma \Big) + \frac{1}{2} \Big(-t \int d^2x d^2\tilde{\theta} \Sigma + c.c. \Big) \tag{3.1}$$

$$+ \frac{1}{2} \Big(d^2x d^2\tilde{\theta} P \cdot G(\Phi_1, \dots, \Phi_N) + c.c. \Big), \tag{3.2}$$

where $t = r - i\vartheta$, r being called the *Fayet-Iliopoulos term.*

The potential term for scalar fields is given by

$$U = |\sigma^2| \sum_{i=1}^{N} |\varphi_i|^2 + |\sigma|^2 d^2 |p|^2 + \frac{e^2}{2} \Big(\sum_{i=1}^{N} |\varphi_i|^2 - d|p|^2 - r \Big)^2 + \frac{1}{4} |G(\varphi_1, \dots, \varphi_N)|^2 + \frac{1}{4} \sum_{i=1}^{N} |p|^2 |\partial_i G|^2.$$

Consider the vacuum manifold given by $U = 0$. It depends on the sign of r:

- Calabi-Yau phase $r >> 0$: we need $\varphi_i \neq 0$ for at least one i, hence $\sigma = 0$.

  Assume $p \neq 0$, then $G = \partial_1 G = \dots = \partial_N G = 0$; since the hypersurface is smooth, this equality is satisfied only for $\varphi = 0$ which is a contradiction. Therefore,

  $$p = \sigma = 0, \ \sum |\varphi_i|^2 = r, \ G(\varphi) = 0.$$

  The vacuum manifold satisfies the equations above modulo $U(1)$ action, so it is the hypersurface $G = 0$ in $\mathbb{P}^{N-1}$.

- Landau-Ginzburg phase $r << 0$: Again, $\sigma = 0$. Since $G = \partial_1 G = \dots = \partial_N G = 0$ and, consequently, $\varphi = 0$, we have

  $$|p| = \sqrt{\frac{|r|}{d}},$$

  so the theory has a unique classical vacuum, up to a gauge tranformation. In expansion around this vacuum, all φ_i's are massless (for $n \geq 3$) governed by the superpotential that can be determined by integrating out massive p. The latter means just setting p equal to its expectation value. Since $U(1)$-symmetry was not broken completely, but to the finite subgroup $\mathbb{Z}_d$ of roots of unity, we obtain an orbifold Landau-Ginzburg model on $[V/\mathbb{Z}_d]$.

3.2 General GLSM

Mathematically, the input of a *gauged linear sigma-model* is the following data (see [FJR18]):

- A finite-dimensional vector space V;
- A reductive algebraic group $G \subset GL(V)$;
- A G-character θ such that $V_G^s(\theta) = V_G^{ss}(\theta)$; one says that it defines *a (strongly regular) phase* $\mathcal{X}_\theta = [V//_\theta G]$;

- A choice of $\mathbb{C}^*$-action called an *R-charge* which is compatible with the G-action and such that $G \cap \mathbb{C}^* = \langle J \rangle$ has finite order. We denote it by $\mathbb{C}^*_R$ and define Γ to be the subgroup of $GL(V)$ generated by G and $\mathbb{C}^*_R$;

- A G-invariant *superpotential* $W : V \to \mathbb{C}$ of degree d with respect to the $\mathbb{C}^*_R$ such that the GIT quotient of the critical locus is compact;

- A stability parameter $\epsilon > 0$ in $\mathbb{Q}$;

- A *good lift* ϑ of θ, i.e. a character of Γ such that $\vartheta|_G = \theta$ and $V^{ss}_\Gamma(\vartheta) = V^{ss}_G(\theta)$.

The output is a certain moduli space of *Landau-Ginzburg quasimaps* to the GIT quotient $[\mathrm{Crit}^{ss}_G(\theta)/G]$ of the critical locus of W, for precise definitions see [FJR18]. The space of characters θ is divided into chambers defining different theories, and the transition across the borders gives an analog of the Landau-Ginzburg/Calabi-Yau correspondence.

The example from the previous section is a GLSM in the sense above with the gauge group $U(1)$ and different choices of θ corresponding to the 'sign' of a homomorphism $U(1) \to U(1)$. Indeed, we take the vector space $V := \mathbb{C}^N \times \mathbb{C}$ on which $\mathbb{C}^*$ acts with the weights $(1, \dots, 1, -d)$ and the superpotential $W(p; \varphi_1, \dots, \varphi_N) := p \cdot G(\varphi_1, \dots, \varphi_N)$. For $\theta >> 0$ the quotient is

$$\frac{(\mathbb{C}^N \backslash \{0\}) \times \mathbb{C})}{\mathbb{C}^*} = \mathcal{O}_{\mathbb{P}^{N-1}}(-d),$$

which is a line bundle over $[\mathrm{Crit}^{ss}_G(\theta)/G] \simeq X$ the hypersurface $\{G = 0\} \subset \mathbb{P}^{N-1}$. The superpotential is just the corresponding section of this bundle defining X, so we get a geometric theory.

The phase $\theta << 0$, on the other hand, gives the quotient

$$\frac{\mathbb{C}^N \times (\mathbb{C} \backslash \{0\})}{\mathbb{C}^*} = [\mathbb{C}^N / \mathbb{Z}_d],$$

which is a bundle over $[\mathrm{Crit}^{ss}_G(\theta)/G] \simeq B\mathbb{Z}_d$. It is an orbifold Landau-Ginzburg theory.

The argument can be generalized to complete intersections in projective spaces (and further to toric varieties): say, we are given k polynomials $G_1, \dots, G_k$, then we may consider the superpotential $W := \sum_{i=1}^k p_i G_i$ of degrees $d_1, \dots, d_k$ on $V := \mathbb{C}^N \times \mathbb{C}^k$. A similar analysis shows that for $\theta >> 0$ we get the quotient

$$\frac{(\mathbb{C}^N \backslash \{0\}) \times \mathbb{C}^k}{\mathbb{C}^*} = \bigoplus_{i=1}^k \mathcal{O}_{\mathbb{P}^{N-1}}(-d_i),$$

corresponding to the geometric theory on the complete intersection of the hypersurfaces $\{G_i = 0\}$; the phase $\theta << 0$, however, corresponds to something more complicated: we get a *hybrid Landau-Ginzburg model* [Wit93], in a sense a 'family' of Landau-Ginzburg models over the base $\mathbb{P}[d_1, \dots, d_k]$. The latter is not necessarily a variety, it might be a stack.

In some cases, however, it is equivalent (via physical considerations) to a certain geometric model.

3.3 Intersection of quadrics

The main references for what follows are [HHP+07] and [CDH+10]. Here we briefly the main details.

Assume that the polynomials G_i defined earlier are of degree 2. Then the geometric phase $\theta >> 0$ corresponds to the complete intersection of k quadrics, and $\theta << 0$ is supposed to give rise to a hybrid Landau-Ginzburg model over $\mathbb{P}[2,\dots,2]$. However, the superpotential is now *quadratic*, say $W = \sum_{ij} \varphi_i A_{ij}(p)\varphi_j$, where $A_{ij}(p)$ are linear functions in $p = (p_1,\dots,p_k)$. Therefore, as long as the matrix $(A_{ij}(p))_{ij}$ is non-degenerate, fields φ acquire masses, hence can be integrated out, and locally we get a geometric theory on the gerbe $\mathbb{P}[2,\dots,2]$ which is equivalent (via T-duality, see [Sha08] for a review) to a disjoint union of two projective spaces. Globally we obtain a $2:1$-cover brached in the hypersurface $\{\det(A) = 0\} \subset \mathbb{P}^{k-1}$ of degree N.

Tha main example for us is the intersection X of two quadrics Q_0, Q_1 in $\mathbb{P}^{2g+1}$. Let them be defined by the equations

$$\begin{aligned} Q_0(x_1,\dots,x_{2g+2}) &= x_1^2 + \dots x_{2g+2}^2; \\ Q_1(x_1,\dots,x_{2g+2}) &= \lambda_1 x_1^2 + \dots \lambda_{2g+2} x_{2g+2}^2 \end{aligned}$$

for some non-zero numbers $\lambda_1,\dots,\lambda_{2g+2} \in \mathbb{C}$. The space of fields is $\mathbb{C}^{2g+2} \times \mathbb{C}^2$ with the weights $(1,\dots,1;-2,-2)$, so $\theta >> 0$ gives a geometric model on the intersection of two quadrics defined above, while $\theta << 0$ should give a gerbe bundle

$$[\mathbb{C}^{2g+2} \times (\mathbb{C}^2 \backslash \{0\})/\mathbb{C}^*] \to \mathbb{P}^1[2,2].$$

However, from the considerations above we actually obtain the double cover of $\mathbb{P}^1$ ramified in $2g+2$ points. Indeed, $\det(p_0 Q_0 + p_1 Q_1) = 0$ precisely when $[p_0 : p_1] = [1 : \lambda_i]$ for $i = 1,\dots,2g+2$. Therefore, we obtain a geometric model on a hyperelliptic curve of genus g. Such correspondence is also observed on the level of branes: it is known [BO] that the derived category of coherent sheaves $D^b Coh(X)$ admits a semiorthogonal decomposition

$$D^b Coh(X) = \langle \mathcal{O}_X(-2g+3),\dots,\mathcal{O}_X, D^b Coh(C)\rangle$$

foe C the hyperelliptic curve above.

Remark 3.3.1. In [Orl09] it was proved that the category of quasi-coherent sheaves on a Fano hypersurface in a projective space admits a semiorthogonal decomposition with the (graded) category of B-branes of its a mirror as a direct summand.

4 GLSM for subspaces on quadrics

Recall that the Jacobian J of a hyperelliptic curve C can be realized as the zero locus of the bundle $S^2U^* \oplus S^2U^*$ on the Grassmannian $\mathrm{Gr}(g, 2g+2)$, where U is the tautological bundle.

We introduce the following GLSM due to S. Galkin [Gal17]. Let Φ be a polynomial functor from vector spaces to themselves; for instance, it might be a symmetric power, an exterior power, or, as we need in our case, the correspondence $V \mapsto \Phi(V) := S^2V \oplus S^2V$. Then the GLSM data is as follows:

- We take two vector spaces U, V and consider

  $$\mathcal{F} := Hom(U,V) \oplus \Phi(U)$$

  as the space of fields (what was previously denoted by V) with the corresponding coordinates (φ, p).

- Let f be an element of $\Phi(V)^*$. The superpotential is defined by the following diagramm:

$$\mathbb{C} \xrightarrow{p} \Phi(U) \xrightarrow{\Phi(\varphi)} \Phi(V) \xrightarrow{f} \mathbb{C}. \tag{4.1}$$

  The composition is a linear map from $\mathbb{C}$ to $\mathbb{C}$, hence canonically identified with some number, and we define $W(\varphi, p)$ to be this number (for every (φ, p)).

- The gauge group G is simply $GL(U)$ acting naturally on $\mathcal{F}$.

- The character $\theta(g) = \det(g)^k$ for some integer k. By $\theta >> 0$ we will mean $k >> 0$; the same for $\theta << 0$;

- The charges are equal to 1 for φ-fields and -2 for the field p.

In our case we take $\Phi(\cdot) := S^2(\cdot) \oplus S^2(\cdot)$, U a g-dimensional vector space, V of dimension $2g+2$, and $f \in S^2V^* \oplus S^2V^*$ be (Q_0, Q_1), where Q_i are quadratic polynomials such that their intersection is non-singular.

Let us check what happens in the case $g = 1$. Then $U = \mathbb{C}$, $V = \mathbb{C}^4$, hence

$$\mathcal{F} = \mathbb{C}^4 \oplus \mathbb{C} \oplus \mathbb{C} \ni (x_1, x_2, x_3, x_4; p_0, p_1),$$

with the weights $(1, 1, 1, 1; -2, -2)$. The superpotential is simply

$$W(\varphi; p_0, p_1) = p_0 Q_0(\varphi) + p_1 Q_1(\varphi), \tag{4.2}$$

and the gauge group $G = GL(U) = \mathbb{C}^*$. So one can see that we obtained the gauged linear sigma model from the previous section. The 'geometric' side corresponds to the intersection of two quadrics in $\mathbb{P}^3$, while the 'Landau-Ginzburg' phase gives the double cover of $\mathbb{P}^1$ ramified in four points. Both varieties are elliptic curves (in particular, Calabi-Yau's) the former ($\hat{C}$) being the Jacobian of the latter (C). In a sense, it is a geometrico-physical proof of the equivalence $D^bCoh(C) \simeq D^bCoh(\hat{C})$.

Now consider a general case $g > 1$. When $\theta >> 0$, the stability condition implies that the rank of the matrix $\varphi \in Hom(U, V)$ is equal to g, hence the quotient $\mathcal{F} \backslash \{\text{rk} < g\}$ is the vector bundle $S^2\mathcal{U} \oplus S^2\mathcal{U}$ over the Grassmannian

$$\text{Gr}(g, 2g+2) = \frac{(Hom(U, V) \backslash \{\text{rk} < g\})}{GL(U)}.$$

As before, the superpotential gives a section of the dual bundle such that the corresponding zero locus parametrizes $(g-1)$-dimensional planes on the intersection of two quadrics which is precisely the Jacobian as was discussed earlier.

The negative case $\theta << 0$ is more interesting. We can think of an element $(q_1, q_2) \in S^2U \oplus S^2U$ as a pair of quadratic maps; under such identification, we have

Proposition 4.0.1. *The semistable locus consists of mutually diagonalizable quadratic forms* q_1, q_2.

Proof. Consider $f(\lambda) := \det(\lambda_1 q_1 + \lambda_2 q_2)$ as a polynomial on $\mathbb{P}^1$. Assume that it is not identically zero, then by a variant of [Rei72, Proposition 2.1] the matrices q_1, q_2 are mutually diagonalizable (probably with zero eigenvalues). Therefore, mutually diagonalizable quadratic maps form a dense subset of all pairs of quadratic functions.

By definition of the semistable locus, for each x we need a non-zero function f such that $f(g\cdot x) = \chi(g)^k f(x)$ for some $k > 0$. We claim that the only functions with such a property are $f(q_1, q_2) = \det(\lambda_1 q_1 + \lambda_2 q_2)^k$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$ (under our choice of charges). Indeed, this is easily observed on the diagonalizable forms; since the latter are dense and f is holomorphic (in fact, polynomial), we see that this is true in general. Therefore, the polynomial $f(\lambda) = \det(\lambda_1 q_1 + \lambda_2 q_2)$ is not zero for some $\lambda \in \mathbb{P}^1$, hence the forms (q_1, q_2) are mutually diagonalizable. □

So for each (q_1, q_2) in semistable locus there exists a basis such that $q_1 = \sum a_i e_i^2$, $q_2 = \sum b_i e_i^2$, and the pairs (a_i, b_i) are defined up to multiplication by a non-zero number. The stability condition implies that $(a_i, b_i) \neq (0,0)$ for each i (otherwise the determinant would be zero), thus defines a point in $\mathbb{P}^1$. Also in such a basis the space $Hom(U,V)$ is identified with the direct sum $V \oplus \ldots \oplus V$.

Such basis, however, is not unique. Assume that all the points (a_i, b_i) are different. Then the only freedom is to change the sign $e_i \mapsto -e_i$ and permute factors $(e_i, e_j) \mapsto (e_j, e_i)$. Let us forget about the latter for a moment. Then we claim that the 'Landau-Ginzburg' phase decomposes, roughly speaking, into the 'direct product' of the Landau-Ginzburg models associated with the hyperelliptic curve C (the one discussed earlier). Indeed, let us fix a basis as above, then the map p in (4.1) is given by the $2g$-tuple $((a_1, b_1), \ldots, (a_g, b_g))$; denote it by $((p_0^1, p_1^1), \ldots, (p_0^g, p_1^g)) \in (\mathbb{P}^1)^g$. The map φ is a g-tuple of vectors $e_i \mapsto v_i \in V$; therefore, one can easily observe that the composition (4.1) is simply

$$W(p,\varphi) = \sum_{i=1}^{g} p_0^i Q_0(v_i) + p_1^i Q_1(v_i) =: \sum_{i=1}^{g} W(p^i, \varphi^i),$$

where $W(P^i, \varphi^i)$ is the superpotential of the form (4.2) on the i-th copy of V in the direct sum $V \oplus \ldots \oplus V$. Ambiguity in the choice of the sign of e_i results in the gerby effects discussed earlier, so for each i we obtain the Landau-Ginzburg phase of the intersection of two quadrics in $\mathbb{P}^{2g+1}$ which, by physical considerations, is equivalent to the geometric theory on the hyperelliptic curve C of genus g; denote it by $\mathcal{M}_C$ whatever this means. Altogether they give the direct product $\mathcal{M}_C^{\times g}$ of g copies of such a model.

Now recall that the basis $\langle e_i \rangle$ was defined up to permutation, and so all the associated data introduced above: $2g$-tuple $((p_0^1, p_1^1), \ldots, (p_0^g, p_1^g))$, the direct sum $V \oplus \ldots \oplus V$ etc. In other words, there is a natural action of the symmetric group S_g on the model $\mathcal{M}_C^{\times g}$, and one needs to take the orbifold model with respect to this action. So outside of the small diagonals we have the 'g-th symmetric power' $S^g \mathcal{M}_C$ of the model $\mathcal{M}_C$.

Unfortunately, this is not true in general. Namely, on the small diagonals of the product $(\mathbb{P}^1)^g$ the isotropy group becomes continuous. For instance, if $(a_1, b_1) = (a_2, b_2)$, then the group $O(2)$ acting on the first vectors e_1, e_2 preserves the diagonalized forms. In general, for the partition $(p^1 = \ldots, p^{k_1}, \ldots, p^{k_{l-1}} = \ldots = p^{k_l})$ the isotropy group is a subgroup of $O(k_1) \times \cdots \times O(k_l) \times S_g$ (we need to identify permutations inside each $O(k_i)$ with those of S_g). In a sense, we have a stratification of $(\mathbb{P}^1)^g$ corresponding to the partitions, and on each strata there is an orbifold vector bundle with non-discrete group of symmetries and with the superpotential given by its section.

However, the discrete group S_g is still contained in all the isotropy groups, so we can take the quotient with respect to S_g first and then by everything else. It gives a certain relation between the symmetric power $S^g \mathcal{M}_C$, discussed earlier, and our complicated Landau-Ginzburg model.

5 Derived categories

In principle, we would like to interpret the relation above in terms of B-branes, i.e. on the level of derived categories. The natural guess for the geometric regime $\theta >> 0$ is, of course, the derived category of coherent sheaves on the Jacobian J (though J is not strict Calabi-Yau manifold). On the other side things seem to be nonmanageable; however, it should equivalent to the geometric regime as it is the case for 'usual' GLSM-theories (say, complete intersection in toric varieties).

The symmetric product has a natural mathematical interpretation. Namely, we have the direct product $C^{\times g}$ and the natural action S_g on it, so a B-brane (=a complex of vector bundles) should be a B-brane on the product *equivariant* with respect to the action. Since every equivariant coherent sheaf has a resolution in locally free equivariant coherent sheaves (see [CG10, Section 5.1]), we claim that the appropriate category should be the equivariant derived category $D^b_{S_g}(C^g)$.

Remark 5.0.1. In particular, such a guess was inspired by [Gal17]. Namely, one considers there a similar GLSM giving the derived category of the variety of lines on a cubic in the 'geometric regime' and the symmetric square of the Kuznetsov category $\mathcal{A}_X$ [Kuz10] in the 'Landau-Ginzburg' phase. Conjecturally, they are equivalent; for instance, their classes in the K-group of dg-categories are equal [GS15].

Recall that the Landau-Ginzburg phase of our GLSM was obtained via quotiening the symmetric power theory by some larger group of symmetries. In particular, the B-branes of the former should be also B-branes of the latter (indeed, if something is equivariant with respect to the larger group, it is for the smaller one). Under a hypothetical equivalence with the derived category of the Jacobian, we see that one should get a fully faithful embedding of $D^bCoh(J)$ into $D^b_{S_g}(C^g)$. Indeed, such an embedding exists.

As an example, let us consider a curve of genus 2. By Abel's theorem the fibers of the morphism $S^2C \to \mathrm{Pic}^2(C)$ are projective spaces of dimension $h^0(D) - 1$; by Cliffords's theorem $h^0(D) - 1 \leq \deg/2 = 1$ and the equality attained iff D is the canonical divisor. Therefore, the Abel-Jacobi map $S^2C \to J_C$ is the blow-up at one point. In particular, by Orlov's blowup formula the derived category of the Jacobian embeds fully faithfully into the derived category of the symmetric square with the *non-trivial* orthogonal. By [PVdB15] the equivariant category $D^b_{S_2}(C^2)$ has an orthogonal decomposition with a summand equivalent to $D^bCoh(C^{(2)})$. Therefore, the derived category of the Jacobian can be embedded fully and faithfully into $D^b_{S_2}(C^2)$ as predicted by the constructed GLSM.

The same statement is true in general (the proof is more involved, however; see [ACGH85, Chapter IV]): the Abel-Jacobi map $S^gC \to J_C$ is the blow-up in the subvariety W^{g-2}, so that the derived category of J_C embeds fully and faithfully into that of the g-th symmetric power which is an orthogonal summand in the decomposition of the equivariant category $D^b_{S_g}(C^g)$. As predicted by the GLSM.

6 Further discussions

We have seen that the Landau-Ginzburg phase itself (not the symmetric power theory) seems to be nonmanageable and giving no mathematical information; however, the author hopes that it is entirely true. The reason is a rather strange coincidence between the direct summands of the derived category $D^b_{S_g}(C^g)$ and description of the strata of $(\mathbb{P}^1)^{(g)}$ corresponding to different isotropy groups. Namely, the latter are parametrized by the partitions of g (so that the group is the corresponding product $O(k_1) \times \ldots \times O(k_l))$, but so are the orthogonal summands of the category $D^b_{S_g}(C^g)$ by the result of [PVdB15]! In particular, the embedding of the derived category of the Jacobian predicted by this GLSM could be not the one coming from the Abel-Jacobi map, but rather with non-trivial intersections with all the orthogonal summands in the decomposition of $D^b_{S_g}(C^g)$.

Finally, there could be connections with the previous work [Kal16]. In a nutshell, it was observed that the Picard-Fuchs equation for the Jacobian of a hyperelliptic curve of genus g is the g-th symmetric power of a certain hypergeometric equation (the same for all the cases). Certainly, the 'symmetric power' pattern is rather common for Jacobians and it is not a convincing argument to connect things; however, the correspondence provided in this paper could be a nice explanation of the results of [Kal16].

References

[ACGH85] E. Arbarello, M. Cornalba, P. Griffiths, and J.D. Harris, *Geometry of Algebraic Curves, Volume 1*, Springer-Verlag New York, 1985.

[BO] A. Bondal and D. Orlov, *Semiorthogonal decomposition for algebraic varieties*, available at `arXiv:alg-geom/9506012`.

[BC09] L. Borisov and A. Caldararu, *The Pfaffian-Grassmannian derived equivalence*, J. Algebraic Geom. **18** (2009), 201-222, available at `arXiv:math/0608404`.

[CDH+10] A. Caldararu, J. Distler, S. Hellerman, T. Pantev, and E. Sharpe, *Non-birational twisted derived equivalences in abelian GLSMs*, Communications in Mathematical Physics **294** (2010), 605-645, available at `arXiv:0709.3855`.

[CKS18] A. Caldararu, Knapp.J., and E. Sharpe, *GLSM realizations of maps and intersections of Grassmannians and Pfaffians*, Journal of High Energy Physics **2018** (2018), no. 4, 119.

[CG10] N. Chriss and V. Ginzburg, *Representation theory and complex geoemtry*, Birkhäuser Basel, 2010.

[DR76] U.V. Desale and S. Ramanan, *Classification of Vector Bundles of Rank 2 on Hyperelliptic Curves*, Inventiones mathematicae **38** (1976), 161-186.

[FJR18] H. Fan, T. Jarvis, and Y. Ruan, *A mathematical theory of gauged linear sigma model*, Geometry and Topology **22** (2018), 235-303, available at `arXiv:1506.02109`.

[Gal17] S. Galkin, *Beautiful formulae and gauged linear sigma models*, 2017. A talk (in Russian).

[GS15] S. Galkin and E Shinder, *On a zeta-function of a dg-category* (2015), available at `arXiv:1506.05831`.

[GK14] N. Ganter and M. Kapranov, *Symmetric and exterior powers of categories*, Transformation groups **19** (2014), 57-103.

[GJ16] A. Gerhardus and H. Jockers, *Quantum periods of Calabi-Yau fourfolds*, Nuclear Physics B **913** (2016), 425-474, available at `arXiv:1604.05325`.

[GLO01] V. Golyshev, V. Lunts, and D. Orlov, *Mirror symmetry for abelian varieties*, J. Algebraic Geom. **10** (2001), no. 3, 433-496.

[HHP+07] S. Hellerman, A. Henriques, T. Pantev, E. Sharpe, and M. Ando, *Cluster decomposition, T-duality, and gerby CFTs*, Adv. Theor. Math. Phys. **11** (2007), no. 5, 751 - 818, available at `arXiv:hep-th/0606034`.

[HT07] K. Hori and D. Tong, *Aspects of non-abelian gauge dynamics in two-dimensional $\mathcal{N} = (2,2)$ theories*, Journal of High Energy Physics **2007** (2007), available at `arXiv:hep-th/0609032`.

[HV] K. Hori and C. Vafa, *Mirror symmetry*, available at `arXiv:hep-th/0002222`.

[Kal16] A. Kalmykov, *Mirror symmetry for the Jacobians oh hyperelliptic curves* (2016). unpublished.

[Kuz10] A. Kuznetsov, *Derived Categories of Cubic Fourfolds*, Cohomological and Geometric Approaches to Rationality Problems, 2010, pp. 219-243.

[Kuz06] ______, *Homological projective duality for Grassmannians of lines* (2006), available at `arXiv:math/0610957`.

[LO10] V. Lunts and D. Orlov, *Uniqueness of enhancement for triangulated categories*, Journal of the American Mathematical Society **23** (2010), 853-908.

[Mil08] James S. Milne, *Abelian Varieties (v2.00)*, 2008. Available at www.jmilne.org/math/.

[Nie] M. Nietzscher, *Gauged Linear Sigma Model, Seminar – Supersymmetry in Geometry and Quantum Physics, WS 2015/16*, available at `https://www.mathi.uni-heidelberg.de/~walcher/teaching/wise1516/geo_phys/GLSM.pdf`.

[Orl09] D. Orlov, *Derived Categories of Coherent Sheaves and Triangulated Categories of Singularities*, Algebra, Arithmetic, and Geometry **2** (2009), 503-531, available at `arXiv:math/0503632`.

[Orl02] ______, *Derived categories of coherent sheaves on Abelian varieties and equivalences between them*, Izvestiya: Mathematics **3** (2002), available at `arXiv:alg-geom/9712017`.

[PVdB15] A. Polishchuk and M. Van der Bergh, *Semiorthogonal decompositions of the categories of equivariant coherent sheaves for some reflection groups* (2015), available at `arXiv:1503.04160`.

[Rei72] M. Reid, *The complete intersection of two or more quadrics*, Trinity College, Cambridge, 1972.

[Rød00] E.A. Rødland, *The Pfaffian Calabi–Yau, its Mirror, and their Link to the Grassmannian* $G(2,7)$, Compositio Mathematica **122** (2000), 135-149.

[Sha08] E. Sharpe, *Derived categories and stacks in physics*, Homological Mirror symmetry (2008), 1 - 24, available at `arXiv:hep-th/0608056`.

[Wit93] E. Witten, *Phases of* $N = 2$ *theories in two dimensions*, Nuclear Physics B **403** (1993), 159-222, available at `arXiv:hep-th/9301042`.