Université de Genève

Faculté des sciences

Section de mathématiques

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Deformation quantization algebroid in the equivariant setting

Artem Kalmykov

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Pavol Ševera

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Abstract

In this work we show that non-equivariance of Kontsevich star-products is measured by the second non-abelian cohomology classes with some conditions on them.

1 Kontsevich's construction

For M a finite-dimensional manifold, denote by $A := C^\infty(M)$ its algebra of functions. *A star-product* on A is an associative product $\star$ with values in $A[[\hbar]]$ that can be written in the form

$$f \star g = fg + \hbar P_1(f,g) + \hbar^2 P_2(f,g) + \dots,$$

where all P_i are bidifferential operators. By linearity it can be extended to the product on $A[[\hbar]]$.

One can easily see that the bivector $\pi(df, dg) := P_1(f,g) - P_1(g,f)$ gives a Poisson structure on M (for instance, the Jacobi identity follows from the associativity of the product). So, given a star-product, we can construct a Poisson structure. Vice versa, is it true that given a Poisson structure on M we can construct a star-product such that its first-order term is the given Poisson bracket?

This problem was positively solved in the paper [Kona]. The proof goes as follows.

There are considered two differential graded Lie algebras: the polyvector fields T_{poly} with the Schouten-Nijenhuis bracket and zero differential and the part D_{poly} of the Hochschild complex corresponding to the polydifferential operators with the usual differetial and the Gerstenhaber bracket. The necessary data on both sides, i.e. Poisson brackets in T_{poly} and star-products in D_{poly}, is given just by the solutions to the Maurer-Cartan equations. Informally, we would like then to have a bijective (in a certain sense) map $T_{poly} \to D_{poly}$ commuting with dgLAs' structures.

There is an evident map between sending a polyvector to the corresponding polydifferential operator; by the Hochschild-Kostant-Rosenberg theorem, it is a quasi-isomorphism, however, it does not commute with the brackets. What Kontsevich proves is that it can be extended to a L_∞ morphism between those dgLAs. The higher-order corrections are given by explicit formulas *in coordinates* as the certain integrals over configuration spaces, see [Kona].

For general manifolds (other than $\mathbb{R}^d$) quantization is more subtle. In fact, there is a canonical Q-equivariant map

$$T[1]\mathsf{Conn_M} \times (T_{poly}(M)[1])_{formal} \to (D_{poly}(M)[1])_{formal}, \tag{1.1}$$

where $\mathsf{Conn_M}$ is the space of connections on M and $T[1]\mathsf{Conn_M}$ is the (graded) tangent bundle of $\mathsf{Conn_M}$ (see [Konb]). Informally, it can be understood as follows: any connection defines infinitesimal geodesics giving rise to a (formal) coordinate system up to $GL(d, \mathbb{R})$-action. Since the formulas are also $GL(d,\mathbb{R})$-invariant, we can apply the quantization procedure point-wise getting a well-defined *global* star-product. Therefore, Kontsevichs' star-products are parametrized (in the sense of the formula above) by the pairs (∇,π) for ∇ a connection and π a Poisson structure.

So canonically the described quantization procedure leads not to a single star-product, but to the bunch of them encoded in what is called *an algebroid* [Konb].

1.1 Quantization algebroid

The notion of an algebroid generalizes that of an algebra in a similiar way as "groupoid" does to "group". More precisely:

Definition 1.1.1. *An algebroid* over a commutative ring R is a small category $\mathcal{A}$ such that

- It is non-empty and all the objects of $\mathcal{A}$ are isomorphic;
- Morphism sets are endowed with a structure of R-modules;
- Composition are R-linear.

Similiar to groupoids, we can regard an algebroid as the data $\mathcal{A}_1 \rightrightarrows \mathcal{A}_0$, where $\mathcal{A}_1$ is the set of all morphisms, $\mathcal{A}_0$ is that of objects, and the arrows correspond to the source and target maps.

In these terms, we have the following

Theorem 1.1.2. *There exists a natural (i.e. depending only on the manifold M) algebroid over* $\mathsf{Conn_M}$ *parametrizing Kontsevichs' star-products.*

Sketch of proof. According to [Konb], we have the following general procedure. Let X be a contractible manifold (maybe infinite-dimensional), and A a vector space with a distinguished vector $\mathbf{1}$. Consider the following dgLA:

$$\mathfrak{g} := \Omega^\bullet(X) \otimes C^\bullet(A,A)[1]$$

Here $C^\bullet(A,A)$ is the Hochschild complex of A endowed with zero multiplication (hence trivial differential) and with the usual bracket. Let $\gamma \in \mathfrak{g}^1$ be a solution to the Maurer-Cartan equation:

$$\mathrm{d}\gamma + \frac{[\gamma,\gamma]}{2} = 0.$$

Let us decompose it as $\gamma = \gamma_0 + \gamma_1 + \gamma_2$, where $\gamma_i \in \Omega^i(X) \otimes C^{2-i}(A,A)$. Then the equation decomposes (according to the form degree) into the system:

1. $[\gamma_0,\gamma_0] = 0$;
2. $[\gamma_1,\gamma_0] + \mathrm{d}\gamma_0 = 0$;
3. $[\gamma_2,\gamma_0] + (\mathrm{d}\gamma_1 + \frac{[\gamma_1,\gamma_1]}{2}) = 0$;
4. $[\gamma_2,\gamma_1] + \mathrm{d}\gamma_1 = 0$.

Also assume unitarity constraints:

(i) $\gamma_0|_x(\mathbf{1},f) = f$ for any $f \in A$, $x \in X$;

(ii) $\gamma_1(\mathbf{1}) = 0$.

Lemma 1.1.3. *Given the triple (X, A, γ) as above, there is a natural algebroid over X.*

Proof. Consider the trivial A-bundle over X with the connection $\nabla := \mathrm{d} + \gamma_1$. Equations (1) and (i) give a family of products with unit elements on A parametrized by X; for $p \in X$ denote by A_p the algebra with the product $\gamma_0|_p$. Equations (2) and (ii) mean that the holonomy of ∇ along any path preserves the algebra structure. As for the equation (3) and (4), they can be understood as follows: for any disk $D \subset X$ with a marked point p on its boundary, the monodromy along ∂D is a conjugation by some element $a_{D,p} \in A_p$; equation (4) assures that this element depends only on the boundary ∂D.

Under these considerations we construct the following algebroid:

- Objects: points of X;
- Morphisms ($x, y \in X$):
  1. $Hom(x, x)$ is identified with A_x as an algebra;
  2. $Hom(x, y)$: let I be a path between x and y. The holonomy along I provides an isomorphism between A_x and A_y. We identify $Hom(x, y)$ with the diagonal bimodule in $A_x \times A_y$. By construction, it is isomorphic to A_x though not canonically. Namely, for any other path I' let D be any disk with the (oriented) boundary $I \cup (-I')$, then two identifications given by I, I' differ by the right multiplication by $a_{D,x}$. Equation (4) assures that $Hom(x, y)$ is well-defined (i.e. depends only on the points x, y).

□

Now let us return to the quantization. We have the map (1.1). Take $\pi \in (T_{poly}(M)[1])^1$ a solution to the Maurer-Cartan equation (=Poisson structure). Then, putting $X := \mathbf{Conn_M}$ and $A := C^\infty(M)$, the restriction of the Kontsevich map

$$T[1]\mathsf{Conn_M} \times \{\pi\} \to (D_{poly}(M)[1])_{formal} \tag{1.2}$$

gives a solution γ to the Maurer-Cartan equation in the sense above. Applying the construction, we obtain a quantization algebroid.

□

2 Equivariant things

Now let (M, π) be a Poisson manifold, but this time with a Lie group G acting by Poisson diffeomorphisms. Does there exist a star-product equivariant with respect to the G-action?

In terms of the map (1.1), the problem can be solved in the following way: since the map is natural (i.e. depends only on a manifold), it is also $\mathrm{Diff}(M)$-equivariant. For instance, if $\star, \star'$ are star-products for the pairs $(\nabla, \gamma), (g^*\nabla, g_*\gamma)$ for $g \in \mathrm{Diff}(M)$, then for any two functions $a, b \in A$

$$g^*(a \star b) = (g^*a) \star' (g^*b). \tag{2.1}$$

Now consider the map (1.2) with an invariant π. If we could find a G-invariant connection, then by (2.1) we are done. Unfortunately, in general this can be guaranteed only for G a compact group.

Another useful point of view is to consider an algebroid over G.

2.1 G-equivariant algebroids over G

In what follows, the G-action on the manifold is assumed to be left, hence that on the space of connections is right:

$$(f \circ g)(m) = f(g(m)),\ but\ (\nabla)(f \circ g)^* = ((\nabla)f^*)g^*,$$

but for simplicity we make it left by taking inverses.

Definition 2.1.1. Let G be a group. *A G-equivariant algebroid* $\mathcal{A} : \mathcal{A}_1 \rightrightarrows \mathcal{A}_0$ is the following data:

- A left G-action on $\mathcal{A}_0$,
- A left G-action on $\mathcal{A}_1$ respecting compositions:
$$g.(a \cdot b) = g.a \cdot g.b,$$
- such that the source and the target maps are equivariant.

Let us take any connection and consider its G-orbit. It can regarded as a map $G \to \mathbf{Conn_M}$. Then we take the pull-back to G of the Kontsevich algebroid over $\mathbf{Conn_M}$.

It is not hard to see that

Proposition 2.1.2. *The constructed algebroid over G is equivariant with respect to the natural action.*

Proof. For any $g, x \in G$ the algebras A_x and A_{gx} are identified via (2.1).

Given a path I between x and y, we can consider the path gI between gx and gy, holonomy over which provides an isomorphism between A_{gx} and A_{gy}. Since $Hom(x,y)$ is identified with the diagonal bimodule in $A_x \times A_y$, for any element $f \in Hom(x,y)$ its translation $g.f$ is the image of all the isomorphisms described above.

Finally, for any two paths I, I' and a disk D between them, we can consider the translations $g.I, g.I', g.D$. Since the elements $a_{D,p}$ depend only on the boundary, the differences between identifications also behave well with respect to the G-action. □

This allows us to describe the algebroid in different terms.

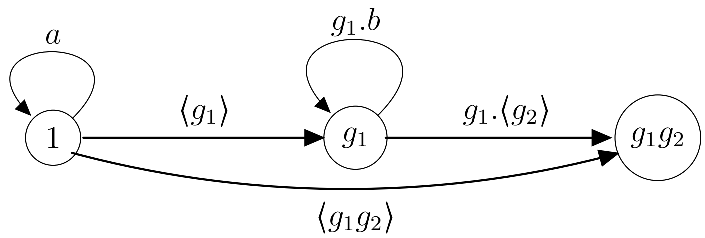
Proposition 2.1.3. *A G-equivariant algebroid over G is equivalent to an associative G-graded algebra such that: 1). There is an action of G on degree* 1 *component; 2). Each homogeneous component contains an invertible element.*

Proof.
- G-equivariant algebroid $\Rightarrow$ G-graded algebra:

  Consider a G-equivariant algebroid over G. Denote by $V_g := Hom(1,g)$ the corresponding morphism space. By G-equivariance any other morphism space is identified with V_g for some g.

  Consider the G-graded vector space $\mathcal{V} := \bigoplus_{g \in G} V_g$. We define the multiplication rule as follows: say, for $a \in Hom(1, g_1), b \in Hom(1, g_2)$, take $g_1.b \in Hom(g_1, g_1 g_2)$ and then define the product $a \cdot b$ as the composition $a \circ g_1.b$ (from left to right!). One can easily check that it indeed defines an associative product.

Now let us fix the isomorphisms between 1 and g for all $g \in G$ as objects of the category (for instance, in the quantization algebroid we fix the paths between 1 and g and consider the holonomies along them) and identify $V_g \cong A_1 \cdot \langle g \rangle$ by composition of paths (see picture). Then it is clear how we can describe the multiplication rule in terms of that of A_1. First, by symbol $\langle g_1 \rangle \langle g_2 \rangle$ we always understand $\langle g_1 \rangle \circ (g_1.\langle g_2 \rangle)$ Then, let $a\langle g_1 \rangle$, $b\langle g_2 \rangle$ be the elements of V_{g_1}, V_{g_2} respectively. Then their composition is $a \cdot (\langle g_1 \rangle^{-1} g_1.b)\langle g_1 \rangle \langle g_2 \rangle$ – we simply translate the loop $g_1.b$ along the arrow $\langle g_1 \rangle^{-1}$:



Now define $c_{g_1 g_2}$ by the following formula:

$$\langle g_1 \rangle \langle g_2 \rangle = c_{g_1, g_2} \langle g_1 g_2 \rangle.$$

One can see that c_{g_1,g_2} lies in A_1. Indeed, $\langle g_1 \rangle \langle g_2 \rangle$ and $\langle g_1 g_2 \rangle$ provide two different identifications of $V_{g_1 g_2}$ with A_1 hence they differ by the right multiplication by an element of A_1. Upgrading the formula above, the composition reads

$$a\langle g_1 \rangle \circ b\langle g_2 \rangle = a \cdot (\langle g_1 \rangle^{-1} g_1.b) c_{g_1,g_2} \langle g_1 g_2 \rangle$$

- G-graded algebra $\Rightarrow$ G-equivariant algebroid:

  Essentially, everything is done in the first part. Consider a G-graded algebra $\mathcal{V} = \bigoplus_{g \in G} V_g$. Given invertible elements $\langle g \rangle \in V_g$ for any $g \in G$, we identify $V_g \cong V_1 \cdot \langle g \rangle$. We regard $\langle g \rangle$ as invertible arrows between 1 and g. The translated arrows $g_1.\langle g_2 \rangle$ are defined from the multiplication rule $\langle g_1 \rangle \cdot \langle g_2 \rangle$ (see the picture).

  An element $f \in Hom(g_1, g_3)$ is defined by the property that there exists an element $b \in V_{g_3}$ such that $b = \langle g_1 \rangle \circ f$ (see the picture; we need to take $g_2 = g_1^{-1} g_3$). Thanks to the invertibility of $\langle g \rangle$, it is well-defined. The G-action is specified by the isomorphisms $V_g \cong A_1 \cdot \langle g \rangle$ (apply G to A_1 and $\langle g \rangle$).

$\square$

The claim is that c represents a certain non-abelian cohomology class of G with values in the group $A_1^\times$ of invertible elements of A_1. Let me remind general formulae. Let G,H be groups.

Definition 2.1.4. ([Gir71]; see also the discussion at **nlab** "nonabelian group cohomology"). *The second nonabelian cohomology of the group G with H-coefficients* $\mathcal{H}^2(G; H)$ is defined as the factor $\mathcal{Z}^2(G;H)/\sim$, where $\mathcal{Z}^2(G;H)$ is the set of *degree 2 cocycles* defined by the following data:

1. a map $\psi : G \to \mathrm{Aut}(H)$,
2. a map $\chi : G \times G \to H$,

3. such that for all $g_1, g_2 \in G$

$$\psi(g_1)\psi(g_2)\psi(g_1g_2)^{-1} = \mathrm{Ad}(\chi(g_1,g_2)),$$

4. subject to the cocycle condition

$$\chi(g_1,g_2)\chi(g_1g_2,g_3) = \psi(g_1)(\chi(g_2,g_3))\chi(g_1,g_2g_3)$$

and the equivalence relation: $(\psi,\chi) \sim (\psi',\chi')$ if there is a map $h : G \to A_1^\times$ such that

(i) $\psi'(g) = \mathrm{Ad}(h(g))\psi(g)$,

(ii) $\chi'(g_1,g_2) = h(g_1)(\psi(g_1)(h(g_2)))\chi(g_1,g_2)h(g_1g_2)^{-1}$.

In our case we can define $\psi(g)(a) = \langle g\rangle^{-1}(g.a)$ for $a \in A_1$, $\chi(g_1,g_1) = c_{g_1,g_2}$.

Proposition 2.1.5. *The data (ψ,χ) defined above satisfies the cocycle conditions; moreover, two different choices of $\langle\bullet\rangle$ give equivalent cocycles.*

Proof. The LHS of Condition (3) applied to $a \in A_1$:

$$\psi(g_1)\psi(g_2)\psi(g_1g_2)^{-1}(a)$$

is by construction equal to the monodromy operator along the loop $\langle g_1\rangle - \langle g_2\rangle - \langle g_1g_2\rangle^{-1}$ (see the picture) applied to a; but, as it was discussed earlier, it is precisely the conjugation by the element c_{g_1,g_2}.

To prove relation (4), we consider the triple product $\langle g_1\rangle\langle g_2\rangle\langle g_3\rangle$. Namely,

$$\begin{aligned}\langle g_1\rangle\langle g_2\rangle\langle g_3\rangle &= c_{g_1,g_2}\langle g_1g_2\rangle\langle g_3\rangle = c_{g_1,g_2}c_{g_1g_2,g_3}\langle g_1g_2g_3\rangle = \\ &= \langle g_1\rangle c_{g_2,g_3}\langle g_2g_3\rangle = \psi(g_1)(c_{g_2,g_3})c_{g_1,g_2g_3}\langle g_1g_2g_3\rangle.\end{aligned} \tag{2.2}$$

Since $\langle g_1g_2g_3\rangle$ is invertible, we get

$$c_{g_1,g_2}c_{g_1g_2,g_3} = \psi(g_1)(c_{g_2,g_3})c_{g_1,g_2g_3}.$$

For any other choice $\langle g\rangle'$ there exist an element a_g such that $\langle g\rangle' = a_g\langle g\rangle$ (it corresponds to the choice of a path), so condition (i) is fulfilled by the same reasons as condition (3). Now we just use the definition of c:

$$\begin{aligned}\langle g_1\rangle'\langle g_2\rangle' &= a_{g_1}\langle g_1\rangle a_{g_2}\langle g_2\rangle = a_{g_1}\psi(g_1)(a_{g_2})\langle g_1\rangle\langle g_2\rangle = a_{g_1}\psi(g_1)(a_{g_2})c_{g_1,g_2}\langle g_1g_2\rangle = \\ &= c'_{g_1,g_2}\langle g_1g_2\rangle' = c'_{g_1,g_2}a_{g_1g_2}\langle g_1g_2\rangle.\end{aligned} \tag{2.3}$$

Again, since $\langle g_1g_2\rangle$ is invertible, we get

$$a_{g_1}\psi(g_1)(a_{g_2})c_{g_1,g_2}a_{g_1g_2}^{-1} = c'_{g_1,g_2},$$

as (ii) requires. ☐

Therefore, to every G-equivariant algebroid over G we can associate a cohomology class in $\mathcal{H}^2(G; A_1^\times)$. Vice versa, if we are given an action of G on A_1 (just as a vector space), then for every cohomology class we can define the G-equivariant algebroid over G. Indeed, all we need is to specify isomorphisms $\langle g\rangle$; let us take a representative given by the maps ψ, c as above. Then we simply put $\langle g\rangle(a) := g^{-1}.(\psi(g)^{-1}(a))$. It is well-defined by all the considerations above.

2.2 G-equivariant algebroids over G/K

Now we return back to the question of existence a G-invariant star-product. As we mentioned, it can be guaranteed just for compact Lie groups. So what we can do is to gain a G-invariance "as much as possible". Namely, let choose a maximal compact subgroup $K \subset G$. We can find a K-invariant connection and then take its G-orbit. It can considered as a map $G/K \to \mathsf{Conn_M}$. We may take pull-back algebroid.

Proposition 2.2.1. *The constructed algebroid over G/K is equivariant with respect to the natural G-action.*

Proof. See the proof of Proposition 2.1.2. □

To describe it in more algebraic terms as we did in the previous subsection we can pull it back to G under the natural map $G \to G/K$. Again, we obtain a G-equivariant algebroid over G, but now with some triviality conditions on the K-action. Consider the following toy example.

Example 2.2.2. Let us take the Kontsevich algebroid $\mathcal{A}$ with compact G, i.e. $G = K$. Then $\mathcal{A}_0$ is just a point, and the pull-back algebroid over K is trivial. However, the group still acts on the space of morphisms $\mathcal{A}_1$ by pull-backs:

$$k.f = (k^{-1})^* f,\ \ k \in K, f \in A_1.$$

Therefore, after the pull-back to K, the isomorphisms $\langle k \rangle$ are trivial, but the automorphisms $\psi(k)$ are not. Moreover, they are *canonical* (depend only on the action on the algebroid) and satisfy $\psi(k_1)\psi(k_2) = \psi(k_1 k_2)$. Therefore, as expected, the corresponding cohomology class is trivial.

The general case (G not compact) does not differ that much.

Proposition 2.2.3. *The cohomology class representing a G-equivariant algebroid over G/K is characterized by the following conditions: 1). $c_{g,k} = 1$ for any $g \in G, k \in K$; 2). The coboundary condition map is trivial on K.*

Proof. More or less obvious:

- $c_{g,k} = 1$: since all the points of the form gk, $g \in G$ fixed, $k \in K$ varies, are identified in G/K, the isomorphisms $\langle gk \rangle$ are equal. However, the automorphisms ψ differ by the action of k: $\psi(gk)(a) = \psi(g)(k.a) = \psi(g)\psi(k)(a)$ for $a \in A_1$. Therefore, the cocyles $c_{g,k}$ are trivial for any $g \in G, k \in K$.

- $h|_K = 1$: the choice of $\langle k \rangle$ is unique, so the coboundary condition map $h : G \to A_1^\times$ (see (2.1.4)) is indeed trivial on K.

□

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