# Geometric and Categorical Approaches to Dynamical Representation Theory

**Dissertation**

**zur**

**Erlangung der naturwissenschaftlichen Doktorwürde**
**(Dr. sc. nat.)**

**vorgelegt der**

**Mathematisch-naturwissenschaftlichen Fakultät**

**der**

**Universität Zürich**

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**Zürich, 2021**

# Abstract

In the thesis, we present a geometric and categorical point of view on dynamical representation theory in terms of categories of Harish-Chandra bimodules and their shifted Poisson analogs. We prove Tannaka duality statements on forgetful functors to such categories in terms of a certain modification of the notion of a bialgebroid.

We study in detail the case of the standard dynamical quantum groups and $F_q(G)$. We relate them to the parabolic restriction functor for classical and quantum Harish-Chandra bimodules and exhibit a natural Weyl symmetry provided by the Zhelobenko operators. In the classical case, we also show for $\mathfrak{sl}_2$ its equivalence (understood in an appropriate sense) to the Kostant-Whittaker reduction functor.

As an application of such approach, we give geometric and categorical interpretations of the so-called vertex-IRF transformation between the standard dynamical $r$-matrix and a constant one. In particular, for $\mathfrak{g} = \mathfrak{gl}_n$ we interpret the constant $r$-matrix in terms of the Kostant-Whittaker reduction and show that the vertex-IRF transformation is given by an equivalence between the parabolic restriction and the Kostant-Whittaker reduction.

# Self-plagiarism

The content of chapter 2 except for section 2.7 and section 2.4, as well as of some parts of Introduction is taken, in a copy-paste manner, from the paper [KS20]. This paper is written in equal proportions by myself and Pavel Safronov. The remaining material is new and has not been published before.

# Acknowledgments

I would like to thank my advisor, Pavel Safronov, for his support and guidance over these years, as well as many helpful discussions and sharing his expertise.

I would also like to thank my co-advisor, Alberto Cattaneo, and my colleagues including, but not limited to: Satosh Kandel, Vincent Braunack-Mayer, Alessandro Valentino, Konstantin Wernli, Jonathan Lorand, Nima Moshayedi, Giovanni Canepa, Nicola Capacci, Ödül Tetik.

Last but not least, I wish to express my gratitude to my family for their support.

# Contents

# Introduction

This thesis is concerned with representation-theoretic structures that involve an additional parameter called *dynamical* one. Typically (but not always), the latter belongs to the dual space $\mathfrak{h}^*$ of the Cartan subalgebra $\mathfrak{h}$ of a reductive Lie algebra $\mathfrak{g}$.

One of the main motivations to study the dynamical structures (which, we should say, is not going to be covered in the thesis) is elliptic representation theory. Since Felder's introduction [Fel95] of the elliptic quantum group for $\mathfrak{g} = \mathfrak{sl}_n$ (or $\mathfrak{gl}_n$) defined via the dynamical analog of the RTT relation, there were numerous proposals for a general definition. For a brief history of the subject, see [GT19, Section 1.10-1.16]); let us mention only some of them:

- Geometric construction via elliptic stable envelopes [AO21] and its action on the equivariant elliptic cohomology of quiver varieties;
- Sheafifed version of elliptic quantum groups in [YZ17];
- Presentation in terms of currents in [GT19].

All these constructions are related, directly or not, to elliptic solutions of the so-called *quantum dynamical Yang-Baxter equation*

$$R_{12}(\lambda - h^{(3)})R_{13}(\lambda)R_{23}(\lambda - h^{(1)}) = R_{23}(\lambda)R_{13}(\lambda - h^{(2)})R_{12}(\lambda), \tag{1}$$

which generalizes the standard (non-dynamical) one:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{2}$$

As explained by Felder [Fel95], the equation (1) is closely related to the star-triangle relation for face-type statistical mechanical models [Bax89]. Moreover, it naturally appears in the description of the exchange algebra in Liouville and Toda conformal field theories [GN84].

The non-dynamical Yang-Baxter equation (2) has the following categorical interpretation. Let $G$ be an affine algebraic group over a field $k$. The Tannaka duality theorems [Saa72; Del90] imply that one can uniquely reconstruct $G$ from the data of a symmetric monoidal category $\mathrm{Rep}(G)$ of $G$-representations and the forgetful symmetric monoidal functor

$$F\colon \mathrm{Rep}(G) \longrightarrow \mathrm{Vect}. \tag{3}$$

Namely, $F$ admits a right adjoint $F^{\mathrm{R}}\colon \mathrm{Vect} \to \mathrm{Rep}(G)$ and the algebra $\mathcal{O}(G)$ of polynomial functions on $G$ can be reconstructed as

$$\mathcal{O}(G) \cong FF^{\mathrm{R}}(k),$$

where the Hopf algebra structure on $\mathcal{O}(G)$ is reconstructed from the monoidal structure on $F$.

Suppose $G$ is a reductive algebraic group, $q \in \mathbf{C}^\times$ and consider the category $\mathrm{Rep}_q(G)$ of representations of the quantum group with divided powers [Lus10; CP95]. Then $\mathrm{Rep}_q(G)$ carries a natural braided monoidal structure and the forgetful functor

$$F\colon \mathrm{Rep}_q(G) \longrightarrow \mathrm{Vect} \tag{4}$$

is merely monoidal. In the same way the Hopf algebra $\mathcal{O}_q(G)$ of functions on the quantum group is reconstructed as $FF^{\mathrm{R}}(k)$.

The failure of the forgetful functor to preserve the braiding is captured by the *R-matrix* (see definition 2.1.27), i.e. a collection of maps

$$R_{V,W}\colon V \otimes W \longrightarrow V \otimes W$$

for two representations $V, W \in \operatorname{Rep}_q(G)$. Moreover, for three representations $U, V, W \in \operatorname{Rep}_q(G)$ the $R$-matrix satisfies the Yang–Baxter equation (2) in $\operatorname{End}(U \otimes V \otimes W)$.

There is a close relation between non-dynamical $R$-matrices and *quantum groups*, see [FRT89]. Likewise, the study of the dynamical $R$-matrix gave rise to the theory of dynamical quantum groups; see [ES01; Eti02] for reviews. Ordinary quantum groups are Hopf algebras while dynamical quantum groups are bialgebroids or Hopf algebroids (see [Tak77] for the original definition of bialgebroids, [Lu96; Xu01] for Hopf algebroids and [EV98c] for the dynamical version).

One is naturally led to wonder if there is a categorical interpretation of dynamical quantum groups that generalizes the non-dynamical one. Our first goal is to develop such an approach (inspired by a previous work by Donin and Mudrov [DM05; DM06]) and prove Tannaka-type reconstruction statements.

**Semi-classical setting.** The inspiration for the approach is provided by interpretation of classical Poisson-Lie structures and, in general, various constructions in geometric representation theory via shifted Poisson geometry. The latter turns out to be well-suited (at least on the level of intuition) for quantization which can be roughly summarized in the following motto: "$n$-shifted Poisson structures define $\mathbb{E}_n$-deformations of the corresponding categories of quasi-coherent sheaves". For precise statements, we refer to [Cal+17] and [MS18b]; let us only give some examples:

(1) Let $G$ be a reductive group. Non-degenerate elements of $\operatorname{Sym}^2(\mathfrak{g})^G$ define the braided monoidal category $\operatorname{Rep}_q(G)$ which is an $\mathbb{E}_2$-deformation of the non-quantum category $\operatorname{Rep}(G)$. The latter is equivalent to the category of quasi-coherent sheaves on the classifying stack $\mathrm{B}G$, and the space of 2-shifted Poisson structures is equivalent to the set $\operatorname{Sym}^2(\mathfrak{g})^G$, see [Cal+17] and [Saf17b];

(2) Recall that the data of a monoidal functor $\operatorname{Rep}_q(G) \to \mathrm{Vect}$ (i.e. as of $\mathbb{E}_1$-algebras in an appropriate symmetric monoidal category) provides a solution to the quantum Yang-Baxter equation. The shifted Poisson counterpart of this statement is as follows: the space of 1-shifted Poisson morphisms $p\colon \mathrm{pt} \to \mathrm{B}G$, where the 1-shifted Poisson structure on $\mathrm{B}G$ is induced from a 2-shifted one, is equivalent to the set of quasi-triangular classical $r$-matrices, see [Saf17b]. Observe that the quantum functor can be thought of as a "deformation" of the classical functor $p^*$.

Analogously to the second example, the set of quasi-triangular dynamical $r$-matrices is "equivalent" to 1-shifted Poisson morphism $[\mathfrak{h}^*/H] \to \mathrm{B}G$, see [Saf17b] and recollection in section 1.3 for precise statements (the Poisson-Lie avatar of such interpretation is dynamical Poisson groupoid structures on the trivial groupoid $\mathfrak{h}^* \times G \times \mathfrak{h}^* \rightrightarrows \mathfrak{h}^*$ (see [LSX11] for Poisson groupoids and [EV98a] for the dynamical version)). Here, the stack $[\mathfrak{h}^*/H]$ for any algebraic group $H$ is the quotient stack of $\mathfrak{h}^*$ with respect to the coadjoint action. It has a natural 1-shifted symplectic structure, see section 1.1. Observe that $\operatorname{QCoh}([\mathfrak{h}^*/H]) \cong \operatorname{LMod}_{\operatorname{Sym}(\mathfrak{h})}(\operatorname{Rep}(H))$ and its monoidal deformation along this 1-shifted Poisson structure is given by the category $\operatorname{HC}(H)$ of *Harish-Chandra bimodules.*

**Dynamical quantum groups via Harish-Chandra bimodules.** The category $\operatorname{HC}(G)$ of Harish-Chandra bimodules is the monoidal category of $\mathrm{U}\mathfrak{g}$-bimodules with an integrable diagonal action. As the shifted Poisson interpretation suggests, the theory of dynamical quantum groups turns out to be closely related to the category $\operatorname{HC}(H)$ of Harish-Chandra bimodules for a torus $H$. In section 2.2.1, we present a general formalism which incorporates classical and quantum examples as well as non-abelian bases (following [Saf19]), but for simplicity here we stick to the case of $\operatorname{HC}(H)$.

First, we introduce the notion of a Harish-Chandra bialgebroid, which is an $\mathfrak{h}$-bialgebroid introduced in [EV98c, Section 4.1] with certain integrability assumptions, see definition 2.2.29 for the general definition and example 2.2.30 for the case of $\operatorname{HC}(H)$. Namely, it is a bigraded algebra $B = \oplus_{\alpha,\beta\in\Lambda} B_{\alpha\beta}$, where $\Lambda$ is the character lattice of $H$, together with two quantum moment maps $s, t\colon \mathcal{O}(\mathfrak{h}^*) \to B$, a coproduct $\Delta\colon B \to B \times_{\mathrm{U}\mathfrak{h}} B$, where the Takeuchi product introduced in [Tak77] is

$$(B \times_{\mathrm{U}\mathfrak{h}} B)_{\alpha\beta} = \bigoplus_{\delta\in\Lambda} B_{\alpha\delta} \otimes_{\mathcal{O}(\mathfrak{h}^*)} B_{\delta\beta},$$

and a counit $\epsilon\colon B \to \mathrm{D}(H)$ into the algebra of differential operators on $H$. We prove the following equivalent characterization of Harish-Chandra bialgebroids (see theorem 2.2.32).

Theorem. *A colimit-preserving lax monoidal comonad $\perp\colon \mathrm{HC}(H) \to \mathrm{HC}(H)$ is the same as a Harish-Chandra bialgebroid $B$, so that $\perp(M) = B \times_{\mathrm{U}\mathfrak{h}} M$.*

We may similarly define comodules over a Harish-Chandra bialgebroid in terms of a $\Lambda$-graded $\mathcal{O}(\mathfrak{h}^*)$-module $M = \oplus_{\alpha\in\Lambda} M_\alpha$ together with a coaction map $M \to B \times_{\mathrm{U}\mathfrak{h}} M$. We prove the following Tannaka reconstruction theorem (see theorem 2.2.35).

Theorem. *Suppose $\mathcal{D}$ is a monoidal category with a monoidal functor $F\colon \mathcal{D} \to \mathrm{HC}(H)$ which admits a colimit-preserving right adjoint $F^{\mathrm{R}}\colon \mathrm{HC}(H) \to \mathcal{D}$. Then there is a Harish-Chandra bialgebroid $B$, such that $(F \circ F^{\mathrm{R}})(-) \cong B \times_{\mathrm{U}\mathfrak{h}} (-)$ and $F$ factors through a monoidal functor*

$$\mathcal{D} \longrightarrow \mathrm{CoMod}_B(\mathrm{HC}(H)).$$

*If $F$ is conservative and preserves equalizers, the above functor is an equivalence.*

Let us now explain the origin of dynamical R-matrices. Assume that $\mathcal{D}$ in addition has a braided monoidal structure. Moreover, assume that the functor $F\colon \mathcal{D} \to \mathrm{HC}(H)$ lands in free Harish-Chandra bimodules, i.e. there is a functor $F'\colon \mathcal{D} \to \mathrm{Rep}(H)$ and an equivalence $F(x) \cong \mathrm{U}\mathfrak{h} \otimes F'(x)$ for any object $x \in \mathcal{D}$. The following is proposition 2.3.11.

Proposition. *Under the above assumptions the image of the braiding under $F\colon \mathcal{D} \to \mathrm{HC}(H)$ gives rise to dynamical R-matrices $R\colon \mathfrak{h}^* \to \mathrm{End}(F'(x) \otimes F'(y))$ satisfying the dynamical Yang–Baxter equation (1).*

Let us compare these results to Tannaka reconstruction results for bialgebroids proven in [Szl03; Shi19]. Suppose $R$ is a ring. It is shown in [Szl03, Theorem 5.4] that a colimit-preserving *oplax monoidal monad* on the category ${}_R\mathrm{BMod}_R$ of $R$-bimodules is the same as a bialgebroid over $R$. Comparing it to our theorem 2.2.32, the difference is that we work with *lax monoidal comonads* instead, replace ${}_{\mathrm{U}\mathfrak{h}}\mathrm{BMod}_{\mathrm{U}\mathfrak{h}}$ by the full subcategory $\mathrm{HC}(H)$ of Harish-Chandra bimodules and replace Takeuchi's bialgebroids by Harish-Chandra bialgebroids (i.e. adding an extra integrability assumption).

Szlachányi [Szl03, Theorem 3.6] has proven a Tannaka-type reconstruction result for monoidal functors $F\colon \mathcal{D} \to {}_R\mathrm{BMod}_R$ admitting *left adjoints* in terms of *modules* over the corresponding bialgebroid. Shimizu has also proven a version of such a Tannaka reconstruction result in terms of *comodules* over the bialgebroid (see [Shi19, Theorem 4.3, Lemma 4.18]).

**Parabolic reduction: semi-classical case.** Let $G$ be a split reductive algebraic group over a characteristic zero field $k$, $B \subset G$ a Borel subgroup and $H = B/[B, B]$ the abstract Cartan subgroup; we denote by $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}$ their Lie algebras. There are two standard solutions to the classical dynamical Yang-Baxter equation with respect to $H$ which are called the *standard rational* and *trigonometric* dynamical $r$-matrices correspondingly, see [ES01], also recollection in section 1.3. Recall that they give rise to 1-shifted Poisson morphisms $[\mathfrak{h}^*/H] \to \mathrm{B}G$. It turns out that the latter are related to familiar objects in geometric representation theory.

Consider the correspondence of algebraic stacks

$$\begin{tikzcd} & {[\mathfrak{b}/B]} \arrow[dl] \arrow[dr] & \\ {[\mathfrak{g}^*/G]} & & {[\mathfrak{h}^*/H].} \end{tikzcd} \tag{5}$$

It appears in many areas of symplectic geometry and geometric representation theory:

- Let $\tilde{\mathfrak{g}}$ be the variety parametrizing Borel subgroups of $G$ together with an element $x \in \mathfrak{g}$ contained in the Lie algebra of the corresponding Borel subgroup. The projection $\tilde{\mathfrak{g}} \to \mathfrak{g}$ is known as the Grothendieck–Springer resolution (see [CG10, Section 3.1.31]). We may identify $[\tilde{\mathfrak{g}}/G] \cong [\mathfrak{b}/B]$, so that the projection $[\mathfrak{b}/B] \to [\mathfrak{g}^*/G]$ is identified with the Grothendieck–Springer resolution $[\tilde{\mathfrak{g}}/G] \to [\mathfrak{g}/G]$. The study of the categories of D-modules on this correspondence is closely related to Springer theory (see [Gun18] and references there).
- Let $N \subset B$ be the unipotent radical. Then we may identify

$$[\mathfrak{b}/B] \cong [G\backslash \mathrm{T}^*(G/N)/H],$$

where $\mathrm{T}^*(G/N)/H \to G/B$ is the universal family of twisted cotangent bundles over the flag variety parametrized by $\lambda \in \mathfrak{h}^*$. In particular, quantization of this correspondence is closely related to the Beilinson–Bernstein localization theorem [BB81] (see [BN12]).

- Recall that the stacks $[\mathfrak{g}^*/G], [\mathfrak{h}^*/H]$ have 1-shifted symplectic structures; moreover, (5) is a 1-shifted Lagrangian correspondence. It is shown in [Cal15a, Section 2.2.1] that a Lagrangian $L$ in $[\mathfrak{g}^*/G]$ is the same as a Hamiltonian $G$-space, i.e. an algebraic symplectic variety $X$ equipped with a symplectic $G$-action and a moment map $X \to \mathfrak{g}^*$. Composing the Lagrangian $L \to [\mathfrak{g}^*/G]$ with the correspondence (5) we obtain a Lagrangian in $[\mathfrak{h}^*/H]$, i.e. a Hamiltonian $H$-space. It is shown in [Saf17c] that this procedure coincides with the procedure of symplectic implosion [GJS02; DKS13].
- One may replace Lie algebras by the corresponding groups, i.e. one may consider the correspondence $[G/G] \leftarrow [B/B] \to [H/H]$. It is shown in [Boa11, Theorem A] that this correspondence (and its analogue for a parabolic subgroup) appears in the description of logarithmic connections on a disk.

This Lagrangian correspondence provides a 1-shifted symplectic morphism $[U/H] \to [\mathfrak{g}^*/G]$, where $U \subset \mathfrak{h}^*$ is the open subset of the so-called *regular* elements. The composition with the 1-shifted Poisson morphism $[\mathfrak{g}^*/G] \to \mathrm{B}G$ gives a 1-shifted Poisson structure on the map $[U/H] \to \mathrm{B}G$ which turns out to be equivalent to the standard rational dynamical $r$-matrix, see section 1.3, originally due to [Saf17b]. Moreover, we have the classical analog of the ABRR theorem, see [ES01, Section 8]: up to a 2-form on $U$, there is a unique 1-shifted symplectic structure on the map $[U/H] \to [\mathfrak{g}^*/G]$. To obtain the standard trigonometric dynamical $r$-matrix, we need to consider the group version of this correspondence.

There is a natural map $U \to [U/H]$. The composition $U \to [\mathfrak{g}^*/G]$ is naturally 1-shifted Lagrangian and can be thought of as parametrization of the conjugacy classes of regular semisimple elements in $\mathfrak{g}^*$ (this is more or less the way how the standard dynamical $r$-matrices appeared in physics papers). There is a similar kind of object well-studied in representation theory called the *Kostant slice* $S\colon \mathfrak{h}/W \to [\mathfrak{g}^*/G]$ that parametrizes the conjugacy classes of regular elements in $\mathfrak{g}^*$. The map $S$ can be equipped with a 1-shifted Lagrangian structure, see [Saf20], also recollection in section 1.3, and so can the composition $U \to \mathfrak{h}/W \to [\mathfrak{g}^*/G]$. Therefore, we have a diagram

(6)
$$\begin{tikzcd} & {[U/H]} \arrow[dr] & \\ U \arrow[ur] \arrow[dr] & & {[\mathfrak{g}^*/G]} \\ & \mathfrak{h}/W \arrow[ur] & \end{tikzcd}$$

Proposition (Proposition 1.3.24). *Up to homotopy, this diagram of Lagrangian (hence coisotropic) maps is commutative.*

This is a semi-classical analog of the categorical equivalence of monoidal actions of $\mathrm{HC}(G)$ between the Kostant-Whittaker reduction and the parabolic restriction, see section 2.7.

**Parabolic restriction: categorical case.** Quantization of standard dynamical $r$-matrices are dynamical quantum groups $F(G)$ and $F_q(G)$ introduced in [EV99, Section 5] in terms of the so-called *exchange construction*. One of the main goals of the thesis is the interpretation of these objects in terms of parabolic restriction functors.

The diagram (5) provides the monoidal actions

(7)
$$\mathrm{QCoh}([\mathfrak{g}^*/G]) \curvearrowright \mathrm{QCoh}([\mathfrak{b}/B]) \curvearrowleft \mathrm{QCoh}([\mathfrak{h}^*/H]),$$

As we mentioned before, the natural quantization of $\mathrm{QCoh}([\mathfrak{g}^*/G])$ (resp. $\mathrm{QCoh}([\mathfrak{h}^*/H])$) is the monoidal category of Harish-Chandra bimodules $\mathrm{HC}(G)$ (resp. $\mathrm{HC}(G)$). A quantum analog of $\mathrm{QCoh}([\mathfrak{b}/B])$ is a universal version $\mathcal{O}^{\mathrm{univ}}$ of the category $\mathcal{O}$: it is the category of $\mathrm{U}\mathfrak{g}$-modules internal to the category $\mathrm{Rep}(H)$ whose $\mathfrak{n}$-action is locally nilpotent. Equivalently, it is the category of $(\mathrm{U}\mathfrak{g}, \mathrm{U}\mathfrak{h})$-bimodules whose diagonal

$B$-action is integrable. The quantum version of (7) becomes

$$\mathrm{HC}(G) \curvearrowright \mathcal{O}^{\mathrm{univ}} \curvearrowleft \mathrm{HC}(H). \tag{8}$$

The module structure on either side is given by the tensor product of bimodules using the latter description of $\mathcal{O}^{\mathrm{univ}}$. The universal Verma module $M^{\mathrm{univ}} = \mathrm{U}\mathfrak{g} \otimes_{\mathrm{U}\mathfrak{b}} \mathrm{U}\mathfrak{h}$ is naturally an object of $\mathcal{O}^{\mathrm{univ}}$.

Let us explain how it relates to the classical picture. The algebra $\mathrm{U}\mathfrak{g}$ has a natural PBW filtration; consider the corresponding Rees algebra over $k[\hbar]$. The above constructions can be repeated to produce $k[\hbar]$-linear categories, so that at $\hbar = 0$ the bimodule (8) reduces to the bimodule (7).

Passing to the right adjoint of the action functor $\mathrm{HC}(H) \to \mathcal{O}^{\mathrm{univ}}$ on the universal Verma module $M^{\mathrm{univ}}$, one obtains the parabolic restriction functor

$$\mathrm{res}\colon \mathrm{HC}(G) \longrightarrow \mathrm{HC}(H)$$

given by $\mathrm{res}(X) = (X/X\mathfrak{n})^N$, which is naturally lax monoidal. The following statement combines proposition 2.2.10 and remark 2.5.7 and provides a quantization of symplectic implosion.

Proposition. *An algebra in* $\mathrm{HC}(G)$ *is a* $G$*-equivariant algebra* $A$ *with a quantum moment map* $\mathrm{U}\mathfrak{g} \to A$. *We have an isomorphism of algebras* $\mathrm{res}(A) \cong A/\!\!/N$, *where* $A/\!\!/N$ *is the quantum Hamiltonian reduction by* $N$.

For a generic central character $\chi\colon \mathrm{Z}(\mathrm{U}\mathfrak{g}) \to \mathbf{C}$, the BGG category $\mathcal{O}_\chi$ with that central character is semisimple with simple objects given by Verma modules. We prove an analogous statement in the universal case. The following statement combines theorem 2.5.17 and corollary 2.5.18.

Theorem. *Consider the subcategories* $\mathrm{HC}(H)^{\mathrm{gen}} \subset \mathrm{HC}(H)$ *and* $\mathcal{O}^{\mathrm{univ,gen}} \subset \mathcal{O}^{\mathrm{univ}}$ *of modules with generic* $\mathfrak{h}$*-weights. Then the functor* $\mathrm{HC}(H)^{\mathrm{gen}} \to \mathcal{O}^{\mathrm{univ,gen}}$ *is an equivalence. In particular,*

$$\mathrm{res}^{\mathrm{gen}}\colon \mathrm{HC}(G) \longrightarrow \mathrm{HC}(H)^{\mathrm{gen}}$$

*is strongly monoidal and colimit-preserving.*

The key step in the above statement is to prove that the Verma module for generic highest weights is projective; in the universal setting this is captured by the existence of the *extremal projector* [AST71] (see theorem 2.5.14) which splits the projection $\mathrm{U}\mathfrak{g} \to M^{\mathrm{univ}}$ for generic $\mathfrak{h}$-weights.

There is a natural monoidal functor free: $\mathrm{Rep}(G) \to \mathrm{HC}(G)$ given by $V \mapsto \mathrm{U}\mathfrak{g} \otimes V$, so we get a monoidal functor

$$\mathrm{Rep}(G) \xrightarrow{\mathrm{free}} \mathrm{HC}(G) \xrightarrow{\mathrm{res}^{\mathrm{gen}}} \mathrm{HC}(H)^{\mathrm{gen}}.$$

We, moreover, show in theorem 2.5.22 that the Harish-Chandra bialgebroid reconstructed from $\mathrm{Rep}(G) \to \mathrm{HC}(H)^{\mathrm{gen}}$ is isomorphic to $F(G)$, so that $\mathrm{Rep}(G)$ is equivalent to $F(G)$-comodules. We also prove analogous statements in the setting of quantum groups in section 2.5.2.

We also expect that the approach to dynamical quantum groups $F(G)$ and $F_q(G)$ presented here in terms of the correspondence (5) might be useful to have an interpretation of Felder's dynamical quantum group [Fel95] in terms of the 1-shifted Lagrangian correspondence $\mathrm{Bun}_G(E) \leftarrow \mathrm{Bun}_B(E) \to \mathrm{Bun}_H(E)$ of moduli stacks of bundles on an elliptic curve $E$. It is interesting to note that the same correspondence is closely related to Feigin–Odesskii algebras [FO98] (in particular, Sklyanin algebras [Skl83]), see [Saf17b, Example 4.11] and [HP18].

It is shown in [BBJ18, Theorem 3.11] that $\mathrm{HC}_q(G)$-module categories are the same as $\mathrm{Rep}_q(G)$-braided module categories [Bro13, Section 5.1]. In particular, the monoidal functor $\mathrm{res}^{\mathrm{gen}}\colon \mathrm{HC}_q(G) \to \mathrm{HC}_q(H)^{\mathrm{gen}}$ allows one to transfer $\mathrm{Rep}_q(H)$-braided module categories to $\mathrm{Rep}_q(G)$-braided module categories.

**Dynamical Weyl group.** Let $W = \mathrm{N}(H)/H$ be the Weyl group and $\hat{W}$ the braid group covering $W$. The group $W$ naturally acts on the symmetric monoidal category $\mathrm{Rep}(H)$, so that we may consider the category of $W$-invariants $\mathrm{Rep}(H)^W$. Moreover, there exists a map $\hat{W} \to \mathrm{N}(H)$ lifting $\hat{W} \to W$ [Tit66], so that the forgetful functor $\mathrm{Rep}(G) \to \mathrm{Rep}(H)$ factors through a symmetric monoidal functor

$$\mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(H)^{\hat{W}}. \tag{9}$$

Our third goal of the paper is to exhibit Weyl symmetry of the parabolic restriction functor for Harish-Chandra bimodules. A similar setup works for quantum groups using the quantum Weyl group [Lus10; Soi90; KR90]. Note, however, that the resulting functor

$$\mathrm{Rep}_q(G) \longrightarrow \mathrm{Rep}_q(H)^{\hat{W}} \tag{10}$$

is *not* monoidal: in fact, the failure of the quantum Weyl group to be monoidal is related to the failure of the functor $\mathrm{Rep}_q(G) \to \mathrm{Rep}_q(H)$ to be braided; this can be encapsulated in the notion of a braided Coxeter category [AT19].

Zhelobenko [Zhe87] in the study of Mickelsson algebras has introduced a collection of *Zhelobenko operators* $q_w \colon \mathrm{U}\mathfrak{g} \to \mathrm{U}\mathfrak{g}$ for every element of the Weyl group $w \in W$ satisfying the braid relations (see theorem 2.6.1). It was realized in [KO08] that these operators give an action of the braid group $\hat{W}$ on a localized Mickelsson algebra.

Consider the $W$-action on $\mathrm{HC}(H)$, where $W$ acts on $\mathrm{U}\mathfrak{h}$ via the *dot action* (the usual $W$-action shifted by the half-sum of positive roots $\rho$) and on $H$ via the usual action. The above results directly imply the following statement (see theorem 2.6.5).

THEOREM. *The Zhelobenko operators define a monoidal functor*

$$\mathrm{res}^{\mathrm{gen}} \colon \mathrm{HC}(G) \longrightarrow \mathrm{HC}(H)^{\mathrm{gen},\hat{W}}$$

*lifting* $\mathrm{res}^{\mathrm{gen}} \colon \mathrm{HC}(G) \to \mathrm{HC}(H)^{\mathrm{gen}}$.

Suppose $V \in \mathrm{Rep}(G)$. Then $\mathrm{res}^{\mathrm{gen}}(\mathrm{U}\mathfrak{g} \otimes V) \cong (\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \otimes V$, where $(\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \supset \mathrm{U}\mathfrak{h}$ is a certain localization (see definition 2.5.12). In particular, the $\hat{W}$-symmetry is captured by certain rational maps $A_{w,V} \colon \mathfrak{h}^* \to \mathrm{End}(V)$ satisfying the braid relation. We prove in theorem 2.6.8 that these coincide with the *dynamical Weyl group* operators introduced in [TV00; EV02].

Let us mention a relationship between these results and the *generalized Harish–Chandra isomorphism* [KNV11]. Consider the functor $\widetilde{\mathrm{res}} \colon \mathrm{HC}(G) \to \mathrm{HC}(H)$ given by $\tilde{\mathrm{res}}(X) = \mathfrak{n}_- X \backslash X / X\mathfrak{n}_+$. There is a natural transformation $\mathrm{res}(X) \to \widetilde{\mathrm{res}}(X)$ which becomes an isomorphism in $\mathrm{HC}(H)^{\mathrm{gen}}$ (see proposition 2.5.16). We obtain a restriction map

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{HC}(G)}(\mathrm{U}\mathfrak{g}, \mathrm{U}\mathfrak{g} \otimes V) \arrow[r, "\widetilde{\mathrm{res}}"] \arrow[d, "\sim"] & \mathrm{Hom}_{\mathrm{HC}(H)}(\mathrm{U}\mathfrak{h}, \mathrm{U}\mathfrak{h} \otimes V) \arrow[d, "\sim"] \\ (\mathrm{U}\mathfrak{g} \otimes V)^G \arrow[r] & \mathrm{U}\mathfrak{h} \otimes V^H \end{tikzcd} \tag{11}$$

The object $(\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \in \mathrm{HC}(H)^{\mathrm{gen}}$ has a canonical $\hat{W}$-equivariance structure given by the dot action of $W$ on $\mathrm{U}\mathfrak{h}$. In particular, Zhelobenko operators define maps $\mathrm{U}\mathfrak{h} \otimes V^H \to (\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \otimes V^H$ and, in fact, the action factors through the action of the Weyl group. The resulting homomorphism

$$\widetilde{\mathrm{res}} \colon (\mathrm{U}\mathfrak{g} \otimes V)^G \longrightarrow (\mathrm{U}\mathfrak{h} \otimes V^H)^W$$

is shown in [KNV11] to be an isomorphism. It generalizes the usual Harish-Chandra isomorphism (see e.g. [Hum08, Theorem 1.10]) which is obtained for $V = k$ the trivial one-dimensional representation.

The papers [BF11; GR15] gave an interpretation of the dynamical Weyl group in terms of equivariant cohomology of the affine Grassmannian of the Langlands dual group, using the geometric Satake equivalence. It would be interesting to see the appearance of the Zhelobenko operators using the Langlands dual interpretation of Harish-Chandra bimodules from [BF08].

Let us mention a categorical point of view on the Weyl symmetry of the parabolic restriction functor $\mathrm{res}^{\mathrm{gen}} \colon \mathrm{HC}(G) \to \mathrm{HC}(H)$. By abstract reasons the action functor $\mathrm{HC}(G) \to \mathcal{O}^{\mathrm{univ}}$ factors through the category of coalgebras over a comonad $\mathrm{St} \colon \mathcal{O}^{\mathrm{univ}} \to \mathcal{O}^{\mathrm{univ}}$ obtained from the right adjoint of the action functor. In particular, for generic weights parabolic restriction factors through the category of St-coalgebras in $\mathrm{HC}(H)^{\mathrm{gen}}$. We expect that there is an equivalence between St and the comonad corresponding to the $W$-action on $\mathrm{HC}(H)^{\mathrm{gen}}$. We refer to [BN12], where it is called the Weyl comonad, and [Gun18, Theorem 4.6] for an analogous theorem in the setting of D-modules.

**Kostant-Whittaker reduction.** As we mentioned, the parabolic restriction functor is given by the "quantum Hamiltonian reduction" $X \mapsto (X/X\mathfrak{n})^N$. The latter, in general, can be performed with respect to any character $\psi$ of $\mathfrak{n}$. In particular, if $\psi$ is generic, then we obtain the so-called *Kostant-Whittaker reduction* functor $\mathrm{res}^\psi$ [BF08] that lands in the category of bimodules over the center $\mathrm{Z}(\mathrm{U}\mathfrak{g})$ of the universal enveloping algebra. In a certain sense, this functor is "nicer" than the parabolic restriction: for instance, it is exact and strongly monoidal even without genericity assumption. It appears, for instance, as the algebraic counterpart of the equivariant cohomology functor under the derived Satake equivalence: if $G$ is a reductive group and $\check{G}$ is the Langlands dual group, then there is an equivalence

$$\Phi\colon : D^b(\mathrm{HC}_\hbar(\check{G})) \to D_{\mathbf{G_O}\rtimes\mathbb{G}_m}(\mathrm{Gr}_G),$$

between the derived category of (asymptotic) Harish-Chandra bimodules of $\check{G}$ and the (graded version of) derived category of equivariant sheaves on the Grassmannian of $G$, such that there exists an isomorphism

$$\mathrm{res}^\psi_\hbar \cong \mathrm{H}^\bullet_{\mathbf{G_O}\rtimes\mathbb{G}_m} \circ \Phi$$

(for the notations and statement, we refer the reader to *loc. cit.*)

The semi-classical limit of the Kostant-Whittaker reduction is given by the Kostant slice $\mathfrak{h}/W \to [\mathfrak{g}^*/G]$ discussed earlier. In particular, it provides a map $U \to [\mathfrak{g}^*/G]$ such that the diagram (6) is commutative. An quantum analog of this statement is as follows. There is a functor

$$\otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})}\mathrm{U}\mathfrak{h}\colon \mathrm{RMod}_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \to \mathrm{RMod}_{\mathrm{U}\mathfrak{h}},$$

given by the Harish-Chandra homomorphism $\mathrm{Z}(\mathrm{U}\mathfrak{g}) \to \mathrm{U}\mathfrak{h}$. We have a diagram of categories

(12)
$$\begin{tikzcd}
 & \mathrm{HC}(H)^{\mathrm{gen}} \arrow[dr] & \\
\mathrm{HC}(G) \arrow[ur] \arrow[dr] & & \mathrm{RMod}_{(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}} \\
 & \mathrm{RMod}_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \arrow[ur] &
\end{tikzcd}$$

Proposition (Corollary 2.7.7). *The diagram* (12) *is commutative. In other words, for every Harish-Chandra bimodule $X$, there is a natural isomorphism*

$$\mathrm{res}^\psi(X) \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} (\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \to \mathrm{res}^{\mathrm{gen}}(X).$$

Moreover, the semi-classical diagram (6) is also commutative on the level of coisotropic structures. A quantum counterpart is as follows. There is a monoidal action of $\mathrm{HC}(G)$ on $\mathrm{RMod}_{(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}}$ induced from the one of $\mathrm{HC}(H)^{\mathrm{gen}}$. Likewise, the Kostant-Whittaker reduction can be upgraded to a monoidal action of $\mathrm{HC}(G)$ on $\mathrm{RMod}_{\mathrm{Z}(\mathrm{U}\mathfrak{g})}$.

Proposition (Proposition 2.7.12). *The functor* $\mathrm{RMod}_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \to \mathrm{RMod}_{(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}}$ *is a functor of* $\mathrm{HC}(G)$*-module categories.*

It turns out that in type A, this isomorphism is related to the notion of a *vertex-IRF transformation.*

**Vertex-IRF transformation.** On the set of dynamical $r$-matrices, there is a natural action of the group of gauge transformations, that is, functions on $\mathfrak{h}^*$ with values in the centralizer $G^H$ of the torus $H$. If we drop the assumption on the target, i.e. allow arbitrary $G$-valued functions, a gauge transformation is still defined, but it does not preserve the set of dynamical $r$-matrices. It turns out, however, that sometimes it is possible to eliminate dependence on the dynamical parameter with such transformation; the latter in this case is called a *vertex-IRF* transformation.

Originally, it appeared in physics papers as a transformation relating the vertex and IRF models, in particular, Felder's elliptic $R$-matrix to Belavin's one (for instance, see [Has94]). In non-spectral case, it was observed that for $\mathfrak{g} = \mathfrak{sl}_n$, it is possible to transform the standard solution of the dynamical quantum Yang-Baxter equation to a non-standard constant one which is now called the *Cremmer-Gervais* solution, see [Bab91] for $\mathfrak{g} = \mathfrak{sl}_2$ and [CGR94] for the general case. Quite surprisingly, it was shown in [BDF90] that a vertex-IRF transformation from the standard trigonometric dynamical $r$-matrix exists only in type A and to a unique constant one. In [BRT07], the authors constructed the vertex-IRF transformation in terms of explicit universal formulae.

We extend this result to an arbitrary trigonometric dynamical $r$-matrix, see theorem 1.4.7:

Theorem. *For non-standard trigonometric dynamical $r$-matrix, there is no vertex-IRF transformation.*

From the shifted Poisson point of view, a vertex-IRF transformation admits the following interpretation. Recall that the data of a dynamical $r$-matrix is equivalent to a 1-shifted Poisson map $[\mathfrak{h}^*/H] \to \mathrm{B}G$. Likewise, a constant $r$-matrix provides a 1-shifted coisotropic map $\mathrm{pt} \to \mathrm{B}G$. We have a commutative diagram of stacks

$$\begin{tikzcd} \mathfrak{h}^* \arrow[rr] \arrow[dr] & & \mathrm{pt} \arrow[dl] \\ & \mathrm{B}G & \end{tikzcd} \tag{13}$$

Proposition (Proposition 1.4.4). *The data of a vertex-IRF transformation is equivalent to the data of commutativity of the diagram (13) on the level of coisotropic structures.*

On the quantum level, a vertex-IRF transformation admits the following interpretation. There is a monoidal action of $\mathrm{HC}(H)$ on the category of modules $\mathrm{LMod}_{\mathrm{U}\mathfrak{h}}$ over $\mathrm{U}\mathfrak{h}$. If we have a monoidal functor $\mathrm{Rep}(G) \to \mathrm{HC}(H)$, it defines a monoidal action of $\mathrm{Rep}(G)$ on $\mathrm{LMod}_{\mathrm{U}\mathfrak{h}}$. Likewise, a monoidal functor $\mathrm{Rep}(G) \to \mathrm{Vect}$ defines a monoidal action of $\mathrm{Rep}(G)$ on Vect.

Observe that there is a functor $\mathrm{Vect} \to \mathrm{LMod}_{\mathrm{U}\mathfrak{h}}$ given by free $\mathrm{U}\mathfrak{h}$-modules. If we assume that there is a factorization $\mathrm{Rep}(G) \to \mathrm{Rep}(H) \to \mathrm{HC}(H)$ (thus giving rise to some dynamical $R$-matrix), we have a commutative diagram of categories

$$\begin{tikzcd} \mathrm{Vect} \arrow[rr] & & \mathrm{LMod}_{\mathrm{U}\mathfrak{h}} \\ & \mathrm{Rep}(G) \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{14}$$

Proposition (Proposition 2.4.3). *The data of a quantum vertex-IRF transformation is equivalent to a lift of the functor* $\mathrm{Vect} \to \mathrm{LMod}_{\mathrm{U}\mathfrak{h}}$ *to a functor between the* $\mathrm{Rep}(G)$*-module categories.*

Likewise, it can be formulated in terms of *right* $\mathrm{U}\mathfrak{h}$-modules. In fact, we show a more general version of these statements that involve the so-called *generalized* vertex-IRF transformations between a dynamical $r$ matrix with respect to $H$ and a (not necessarily constant) dynamical $r$-matrix with respect to some subgroup $L \subset H$.

We also provide a geometric and categorical interpretation of *the* vertex-IRF transformation. In particular, in the classical case, we show its relation to a special form of a transversal slice in type A called the *Mirković-Vybornov* slice that appeared in [MV03]. In the quantum case, we show the following result, see theorem 2.7.24:

Theorem. *For $\mathfrak{g} = \mathfrak{gl}_n$, there is a monoidal functor* $\mathrm{Rep}(G) \to \mathrm{Vect}$ *(i.e. there is a structure of a* $\mathrm{Rep}(G)$*-module category on* Vect*) such that the functor* $\mathrm{Vect} \to \mathrm{RMod}_{\mathrm{Z}(\mathrm{U}\mathfrak{g})}$ *given by* $V \mapsto V \otimes \mathrm{Z}(\mathrm{U}\mathfrak{g})$ *is a functor of* $\mathrm{Rep}(G)$*-module categories.*

Calculations in low ranks suggest that the monoidal structure on $\mathrm{Rep}(G) \to \mathrm{Vect}$ quantizes the rational version of the Cremmer-Gervais $r$-matrix. Since the functor $\mathrm{Rep}(G) \to \mathrm{HC}(G)$ given by free Harish-Chandra bimodules is monoidal, the functor $\mathrm{RMod}_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \to \mathrm{RMod}_{(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}}$ introduced earlier is a functor of $\mathrm{Rep}(G)$-module categories, and so is the composition $\mathrm{Vect} \to \mathrm{RMod}_{(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}}$. Therefore, the latter provides a vertex-IRF transformation between the standard dynamical $R$-matrix and a quantization of the rational Cremmer-Gervais $r$-matrix in the case of $\mathfrak{gl}_n$. In fact, it is in accordance with conjectures in [BRT12] where the authors constructed the universal vertex-IRF transformation using the so-called Sevostyanov characters which are closely related to the quantum analog of the Kostant-Whittaker reduction.

CHAPTER 1

# Dynamical representation theory: semi-classical picture

In this work, we will often be in the following setting:

**Definition 1.0.1.** The ***usual setup*** is: $G$ a simple group, $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra integrating to a subgroup $H \subset G$, $\Delta$ the associated set of roots, $\Pi$ the set of simple roots, and $\Delta^+ \subset \Delta$ the associated set of positive roots. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the corresponding polarization and $\mathfrak{g}_\alpha$ be the root subspace for $\alpha \in \Delta$. Let $\langle -, - \rangle$ be the invariant symmetric form on $\mathfrak{g}$ normalized by the condition $\langle \alpha, \alpha \rangle = 2$ for long roots, and $e_\alpha \in \mathfrak{g}_\alpha$ are chosen so that $\langle e_\alpha, e_{-\alpha} \rangle = 1$.

## 1.1. Recollection on shifted symplectic and Poisson structures

**1.1.1. Symplectic structures.** Let $X$ be a derived Artin stack and $\mathbb{L}_X$ its cotangent complex. Its de Rham algebra is

$$\mathrm{DR}(X) := \Gamma(X, \mathrm{Sym}(\mathbb{L}_X[-1])).$$

It is a commutative dg-algebra with an extra weight grading and a de Rham differential $\mathrm{d}_{\mathrm{dR}}$. We use the convention that the latter has weight 1 and degree 1.

A d-closed element of $\mathrm{DR}(X)$ of weight $p$ and degree $p+n$ is called a *$p$-form of degree $n$*. A $(\mathrm{d}+\mathrm{d}_{\mathrm{dR}})$-closed element of weight at least $p$ and degree $p+n$ lying in a completion of $\mathrm{DR}(X)$ is called a *closed $p$-form* (for precise definition, see [Pan+13]). Explicitly, it is given by a collection of elements $\omega_0, \omega_1, \ldots$, where $\omega_i$ has weight $p+i$ and degree $p+n$, satisfying

$$\mathrm{d}\omega_0 = 0,$$
$$\mathrm{d}\omega_{i+1} + \mathrm{d}_{\mathrm{dR}}\omega_i = 0.$$

Suppose $\omega_0$ is a 2-form of degree $n$. It defines a map

$$\omega_0 \colon \mathbb{T}_X \to \mathbb{L}_X[n].$$

We say that $\omega = \omega_0 + \omega_1 + \ldots$ is *nondegenerate* if $\omega_0$ is a quasi-isomorphism.

**Definition 1.1.1.** An ***$n$-shifted symplectic structure*** on $X$ is the data of a closed nondegenerate 2-form $\omega$ on $X$.

**Example 1.1.2.**
(1) Let $G$ be an affine group scheme and $\mathfrak{g}$ its Lie algebra. Consider the stack $X = [\mathfrak{g}^*/G]$, where $G$ acts on $\mathfrak{g}^*$ by the coadjoint action. Then $\mathbb{L}_X = \mathcal{O}_{\mathfrak{g}^*} \otimes (\mathfrak{g} \oplus \mathfrak{g}^*[-1])$ with the differential d given by the adjoint action $x \mapsto \mathrm{ad}_x$.

Observe that the canonical element in $\mathfrak{g}^* \otimes \mathfrak{g}$ gives an element $\omega$ in $(S^2(\mathfrak{g} \oplus \mathfrak{g}^*[-2]))^G$ of weight 2 and degree 3. It is $G$-invariant, hence $\mathrm{d}\omega = 0$, and also $\mathrm{d}_{\mathrm{dR}}$-closed. In particular, it defines a closed 2-form of degree 1 which is, in fact, nondegenerate; therefore, we have a 1-shifted symplectic structure on $[\mathfrak{g}^*/G]$.

(2) Let $G$ be a reductive group and $X = [G/G]$, where $G$ acts on itself by conjugation. Then $\mathbb{L}_X = \mathfrak{g}^* \otimes \mathcal{O}_G[-1] \oplus \mathbb{L}_G$. Denote by $\theta$ (corr. $\bar{\theta}$) the left (corr. right) Maurer-Cartan form in $\Omega^1(G) \otimes \mathfrak{g}$.

Let $\langle -, - \rangle$ be an invariant bilinear form on $\mathfrak{g}$. Define 2-form $\omega_0$ of degree 1 by $\omega_0(y) = -\langle \theta + \bar{\theta}, y \rangle / 2$. Although it is d-closed, it is not $\mathrm{d}_{\mathrm{dR}}$-closed on the nose, but up to homotopy provided by $\omega_1 = \langle \theta, [\theta, \theta] \rangle / 12$. Therefore, the form $\omega = \omega_0 + \omega_1$ defines a 1-shifted symplectic structure on $[G/G]$. (For the proofs, see [Saf16]).

Let $X$ be equipped with an $n$-shifted symplectic structure $\omega_X$ and $f \colon L \to X$ a morphism.

**Definition 1.1.3.** An ***isotropic structure*** on $f$ is a homotopy from $f^*\omega_X$ to 0.

In other words, an isotropic structure is a collection of forms $h = h_0 + h_1 + \ldots$, satisfying

$$\begin{aligned} f^*\omega_0 &= \mathrm{d}h_0 \\ f^*\omega_i &= \mathrm{d}_{\mathrm{dR}}h_i + \mathrm{d}h_{i+1}. \end{aligned}$$

The element $h_0$ defines a null-homotopy of the composition $\mathbb{T}_L \to f^*\mathbb{T}_X \xrightarrow{\omega_0} \mathbb{L}_X[n] \to \mathbb{L}_L[n]$, hence a morphism $\mathbb{T}_{L/X} \to \mathbb{L}_L[n-1]$, where $\mathbb{T}_{L/X}$ is a homotopy fiber of $\mathbb{T}_L \to f^*\mathbb{T}_X$.

**Definition 1.1.4.** An isotropic morphism $f\colon L \to X$ is ***Lagrangian*** if the induced morphism $\mathbb{T}_L \to f^*\mathbb{T}_X$ is an isomorphism.

**Example 1.1.5.**
(1) Let $M$ be a smooth scheme with a $G$-action. Let $\mu\colon [M/G] \to [\mathfrak{g}^*/G]$ be the morphism induced from some $G$-equivariant map $M \to \mathfrak{g}^*$. Then a Lagrangian structure on $\mu$ is equivalent to the moment map condition on $M \to \mathfrak{g}^*$, see [Saf16, Section 2] and [Cal15b, Section 2.2.1].
(2) Let $M$ be a smooth scheme with a $G$-action. Let $\mu\colon [M/G] \to [G/G]$ be the morphism induced from some $G$-equivariant map $M \to G$. There is a notion of a *quasi-Hamiltonian $G$-space* generalizing that of a moment map, see [AMM98, Definition 2.2] . Then a Lagrangian structure on $\mu$ is equivalent to the quasi-Hamiltonian $G$-space condition, see [Saf17c, Theorem 2.2], also [Cal15b, Section 2.2.2].

We have the following important theorem:

Theorem 1.1.6. *[Pan+13, Theorem 2.9] Let $X$ be $n$-shifted symplectic and $L_1 \to X$ and $L_2 \to X$ be two Lagrangian maps. Then the intersection $L_1 \times_X L_2$ carries an $(n-1)$-shifted symplectic structure.*

**Example 1.1.7.** The inclusion of the origin $\mathrm{pt} \to \mathfrak{g}^*$ provides a 1-shifted Lagrangian morphism $\mathrm{B}G \to [\mathfrak{g}^*/G]$. Let $X$ be a $G$-scheme with a moment map $\mu\colon X \to \mathfrak{g}^*$ providing a 1-shifted Lagrangian morphism $[X/G] \to [\mathfrak{g}^*/G]$. By theorem 1.1.6, we have a 0-shifted symplectic structure on $[X/G] \times_{[\mathfrak{g}^*/G]} \mathrm{B}G \cong [\mu^{-1}(0)/G]$. This is known as the *symplectic reduction* of $X$ by $G$.

For a derived $n$-shifted symplectic stack $X$ we denote by $\bar{X}$ the same stack equipped with the opposite symplectic structure.

**Definition 1.1.8.** An $n$-shifted ***Lagrangian correspondence*** is a triangle

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

where $X, Y$ are $n$-shifted symplectic stacks and $Z \to X \times \bar{Y}$ is a Lagrangian morphism.

**1.1.2. Poisson structures.** We will only give a flavor of shifted Poisson structures without going much into details. For precise definitions consult, for instance, in [Cal+17]; also see introduction to the subject in [Saf17a].

**Definition 1.1.9.** A $\mathbb{P}_n$***-algebra*** is a commutative dg-algebra $A$ over $k$ with a bracket $\{-,-\}\colon A \otimes_k A \to A[1-n]$ satisfying the graded Jacobi identity.

There is a also a homotopy version thereof, see, for instance, [Mel16].

It can be generalized to non-affine stacks as follows. Let $X$ be a derived Artin stack locally of finite presentation and $\mathbb{T}_X$ be its tangent complex. Consider the space of ***n-shifted polyvector fields***

$$\mathrm{Pol}(X, n) := \Gamma(X, \mathrm{Sym}(\mathbb{T}_X[-n-1])).$$

As in the symplectic case, this space has a weight grading. where $\mathbb{T}_X$ has degree 1, and an internal cohomological grading. One may define the Schouten bracket of weight $-1$ and degree $-n-1$.

**Definition 1.1.10.** An $n-$***shifted Poisson structure*** on $X$ is a formal power series $\pi = \pi_2 + \pi_3 + \ldots$, where $\pi_k$ is a polyvector of weight $k$ and degree $n+2$ such that $\pi$ satisfies the Maurer-Cartan equation $\mathrm{d}\pi + [\pi, \pi]/2 = 0$.

More rigorously, there is an equivalence of spaces

$$\mathrm{Pois}(X,n) = \mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathrm{Pol}(X,n)[n+1]),$$

where $k(2)[-1]$ is the trivial graded dg Lie algebra with weight grading 2, for instance, see [Saf17a].

The corresponding notion of a shifted coisotropic morphism can be formulated as follows. Recall the Poisson additivity theorem:

THEOREM 1.1.11. *[Saf18] There is an equivalence*

$$\mathrm{Alg}_{\mathbb{P}_{n+1}} \cong \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Alg}_{\mathbb{P}_n}).$$

Then the notion of a shifted coisotropic morphism can be formulated as follows:

**Definition 1.1.12.** [MS18a] Let $A, M$ be dg commutative algebras. An ***n-shifted coisotropic structure*** on an algebra morphism $f\colon A \to M$ is the data of a $\mathbb{P}_{n+1}$-structure on $A$ and an action of $A$ on $M$ in $\mathrm{Alg}_{\mathbb{P}_n}$ lifting $f$.

Equivalently, it is the data of the-so-called $\mathbb{P}_{[n+1,n]}$-algebra structure on the pair $(A, M)$, see [Saf18].

Let

$$\begin{tikzcd} M_1 \arrow[rr] & & M_2 \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

be a commutative diagram in cdgas. The data of ***morphism*** between $n$-shifted coisotropic structures is a lift of this diagram to a map $M_1 \to M_2$ of $A$-modules in $\mathrm{Alg}_{\mathbb{P}_n}$.

In terms of polyvectors, it can be generalized as follows. Let $f\colon L \to X$ be a morphism of derived Artin stacks locally of finite presentation and $\mathbb{T}_{L/X}$ be its relative tangent complex. Consider the algebra of relative polyvectors

$$\mathrm{Pol}(L/X, n-1) := \Gamma(L, \mathrm{Sym}(\mathbb{T}_{L/X}[-n])).$$

We have a morphism of graded cdgas

$$\mathrm{Pol}(X,n) \to \mathrm{Pol}(L/X, n-1). \tag{1.1.1}$$

It is shown in [MS18b] that it can be upgraded to a $\mathbb{P}_{[n+1,n]}$-structure on the pair $(\mathrm{Pol}(X,n), \mathrm{Pol}(L/X, n-1))$. It can be shown that the homotopy fiber $\mathrm{U}(\mathrm{Pol}(X,n), \mathrm{Pol}(L/X, n-1))$ can be upgraded to a non-unital $\mathbb{P}_{n+1}$-algebra of the so-called *relative n-shifted polyvector fields* $\mathrm{Pol}(f,n)$.

**Definition 1.1.13.** [MS18b, Theorem 2.7] The space of ***n-shifted coisotropic structures*** on $f\colon L \to X$ is the mapping space

$$\mathrm{Cois}(f,n) = \mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathrm{Pol}(f,n)[n+1]).$$

As in the case of Poisson structures, we can present it via the Maurer-Cartan elements in a suitable dg Lie algebra.

**Definition 1.1.14.** Let $f\colon A_1 \to A_2$ be a map between commutative dg algebras. The space of ***n-shifted Poisson structures*** $\mathrm{Pois}(f,n)$ on $f$ is a lift to a map between $\mathbb{P}_{n+1}$-algebras (notice that it is a data, not a property).

One can similarly formulate it for maps between derived Artin stacks using formal geometry as in [Cal+17].

Recall that in the nonderived case, the graph of a map between Poisson schemes is Poisson if and only if its graph is coisotropic.

THEOREM 1.1.15. *[MS18b, Theorem 2.9] For $f\colon X \to Y$ a morphism between derived Artin stacks, there is a Cartesian diagram of spaces*

$$\begin{tikzcd} \mathrm{Pois}(f,n) \arrow[r] \arrow[d] & \mathrm{Pois}(X,n) \times \mathrm{Pois}(Y,n) \arrow[d] \\ \mathrm{Cois}(\Gamma,n) \arrow[r] & \mathrm{Pois}(X \times Y, n), \end{tikzcd}$$

*where* $\Gamma\colon X \to X \times Y$ *is the graph morphism.*

There is also an analog of theorem 1.1.6:

THEOREM 1.1.16. *[MS18b, Theorem 3.6] Suppose* $L_1 \to X$ *and* $L_2 \to X$ *are* $n$*-shifted coisotropic morphisms. Then the intersection* $L_1 \times_X L_2$ *carries a natural* $(n-1)$*-shifted Poisson structure, such that the map* $L_1 \times_X L_2 \to L_1 \times \bar{L}_2$ *is Poisson.*

The notion of a morphism between coisotropic structures can be generalized to derived Artin stacks using the corresponding graded mixed commutative algebras [Cal+17]. In terms of polyvectors, the following statement can be proved by methods similar to [MS18a, Theorem 4.21]:

**Proposition 1.1.17.** *The data of a morphism between* $n$*-shifted coisotropic structures* $L_1 \to X$ *and* $L_2 \to X$ *is equivalent to the data of a coisotropic morphism* $L_1 \to L_1 \times_X L_2$ *such that the composition with the projection*

$$L_1 \to L_1 \times_X L_2 \to L_1$$

*is homotopic to identity.*

As in the non-shifted case, there is an equivalence between $n$-shifted symplectic and *non-degenerate* $n$-shifted Poisson structures, see [Cal+17]; for the relative version of coisotropic structures, see [MS18b].

## 1.2. Non-dynamical $r$-matrices

In this section, we recall some standard facts about the usual Yang-Baxter equation and give its shifted Poisson interpretation.

**1.2.1. Classical setting.** Let $G$ be an arbitrary affine algebraic group and $\mathfrak{g}$ its Lie algebra.

**Definition 1.2.1.** [KM04, Definition 1] A ***quasi-Poisson structure*** on $G$ is a pair $(\pi_G, \phi)$ of a bivector $\pi_G \in \Gamma(G, \Lambda^2 \mathrm{T}_G)$ and an element $\phi \in \Lambda^3\mathfrak{g}$ such that:
- $\pi_G$ is multiplicative;
- $\frac{1}{2}[\pi_G, \pi_G] + \phi^l - \phi^r = 0$, where the superscripts $l$ and $r$ mean the corresponding left- or right-invariant vector fields;
- $[\pi_G, \phi^r] = 0$.

A ***Poisson-Lie structure*** (or simply a Poisson structure on $G$ when it is clear from the context) is a quasi-Poisson structure on $G$ with $\phi = 0$.

**Definition 1.2.2.** [KM04, Definition 5] Let $(\pi_G, \phi)$ and $(\pi'_G, \phi')$ be two quasi-Poisson structures on $G$. A ***twisting*** between them is an element $\lambda \in \Lambda^2\mathfrak{g}$ satisfying

$$\pi'_G = \pi_G + \lambda^r - \lambda^l,$$
$$\phi' = \phi + \frac{1}{2}\mathrm{Alt}(\delta \otimes \mathrm{id})(\lambda) - \mathrm{CYBE}(\lambda),$$

where $\delta\colon \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is the map coming from $\pi_G$ (we refer the reader to *loc. cit.* for clarification of notations).

An important class of (honest) Poisson-Lie structures (essentially, all of them when $G$ is simple) is given by the so-called ***quasi-triangular r-matrices***.

**Definition 1.2.3.** An element $r \in \mathfrak{g} \otimes \mathfrak{g}$ is called a ***quasi-triangular classical r-matrix*** if
- It satisfies the so-called ***classical Yang-Baxter equation***:

$$\mathrm{CYBE}(r) := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0;$$

- Its symmetric part is $G$-invariant.

Given any element $a \in \Lambda^2\mathfrak{g}$, consider the following bivector field on $G$:

$$\Pi(a) := a^l - a^r, \tag{1.2.1}$$

where $a^l$ (resp. $a^r$) means the corresponding left-invariant (resp. right-invariant) bivector field. The following result is classical.

**Proposition 1.2.4.** *The bivector field* $\Pi(a)$ *defines a Poisson-Lie structure on* $G$ *if and only if*

$$\mathrm{CYBE}(a) \in (\Lambda^3\mathfrak{g})^{\mathfrak{g}}.$$

In particular, we can present a quasi-triangular $r$-matrix in the form $r = a + T/2$, where $a \in \Lambda^2\mathfrak{g}$ and $T \in (S^2\mathfrak{g})^{\mathfrak{g}}$; then

$$\mathrm{CYBE}(a) = \frac{1}{4}[T_{12}, T_{23}] \in (\Lambda^3\mathfrak{g})^{\mathfrak{g}},$$

hence $\Pi(a)$ defines a Poisson-Lie structure on $G$.

### 1.2.2. Shifted Poisson setting.

Here, the main player is the classifying stack $\mathrm{B}G$.

**Proposition 1.2.5.** *The space of* $n$*-shifted Poisson structure on* $\mathrm{B}G$ *is:*

- *[Saf17b, Proposition 2.6] Trivial for* $n > 2$*;*
- *[Saf17b, Proposition 2.6] Equivalent to the set* $(S^2\mathfrak{g})^{\mathfrak{g}}$ *for* $n = 2$*;*
- *[Saf17b, Theorem 2.9] Equivalent to the groupoid* $\mathrm{QPois}(G)$ *of quasi-Poisson structures on* $G$ *for* $n = 1$.

Inside the groupoid $\mathrm{QPois}(G)$, we have the set of honest Poisson-Lie structures on $G$. The latter, in fact, is also distinguished in the shifted world:

**Proposition 1.2.6.** *[Saf17b, Corollary 2.11] Let* $f\colon \mathrm{pt} \to \mathrm{B}G$ *be the natural morphism. Then the space of 1-shifted Poisson structures on* $f$ *(where* $\mathrm{pt}$ *is equipped with zero Poisson structure) is equivalent to the set of Poisson-Lie structures on* $G$*.*

The forgetful map $\mathrm{Pois}(\mathrm{B}G, 2) \to \mathrm{Pois}(\mathrm{B}G, 1)$, coming from the unique coisotropic structure on the identity morphism $\mathrm{B}G \to \mathrm{B}G$, is given as follows: it sends a Casimir element $c \in (S^2\mathfrak{g})^{\mathfrak{g}}$ to the quasi-Poisson structure $(0, \phi)$, where $\phi = -\frac{1}{6}[c_{12}, c_{23}]$, see [Saf17b, Proposition 2.16].

Theorem 1.2.7. *[Saf17b, Proposition 5.3] The space parametrizing the pairs of*

- *A 2-shifted Poisson structure* $\pi$ *on* $\mathrm{B}G$*,*
- *A 1-shifted Poisson morphism* $\mathrm{pt} \to \mathrm{B}G$ *with the 1-shifted Poisson structure on* $\mathrm{B}G$ *obtained from* $\pi$*,*

*is equivalent to the set of quasi-triangular* $r$*-matrices.*

Proof. A 2-shifted Poisson structure on $\mathrm{B}G$ is given by some element $c \in (S^2\mathfrak{g})^{\mathfrak{g}}$. It equips $\mathrm{B}G$ with the 1-shifted Poisson structure given by $(0, -\frac{1}{6}[c_{12}, c_{23}]) \in \mathrm{QPois}(G)$. Likewise, the data of a 1-shifted Poisson morphism $\mathrm{pt} \to \mathrm{B}G$ is given by $(\pi_G, 0) \in \mathrm{QPois}(G)$, where $\pi_G$ is a Poisson-Lie structure. Compatibility between these data is a morphism $\lambda \in \Lambda^2\mathfrak{g}$ in $\mathrm{QPois}(G)$ between them, i.e. we have an equation

$$\frac{1}{2}\mathrm{CYBE}(\lambda) = -\frac{1}{6}[c_{12}, c_{23}],$$

so that $r := 2\lambda + c$ is a quasi-triangular $r$-matrix. $\square$

It is related to the classical Poisson-Lie structure on $G$ as follows: the self-intersection $G = \mathrm{pt} \times_{\mathrm{B}G} \mathrm{pt}$ is equipped with a 0-shifted Poisson structure by theorem 1.1.16. One can see that, in fact, it is exactly the Poisson bivector (1.2.1).

## 1.3. Dynamical r-matrices

In this section, we generalize the results from the previous section to the dynamical case.

**1.3.1. Classical setting.** In what follows, we mainly follow [ES01].

Let $G$ be an arbitrary affine algebraic group and $H \subset G$ a subgroup with the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Let $U \subset \mathfrak{h}^*$ be an $H$-invariant subset, where $H$ acts by the coadjoint action.

**Definition 1.3.1.** An $\mathfrak{h}$-invariant function $r(\lambda)\colon U \to \mathfrak{g} \otimes \mathfrak{g}$ is a ***quasi-triangular dynamical r-matrix*** if it

- Satisfies the so-called ***classical dynamical Yang-Baxter equation***:
$$\mathrm{CDYBE}(r) := \sum_i \left( x_i^{(1)} \frac{\partial r_{23}(\lambda)}{\partial x^i} - x_i^{(2)} \frac{\partial r_{13}(\lambda)}{\partial x^i} + x_i^{(3)} \frac{\partial r_{12}(\lambda)}{\partial x^i} \right) + \\ + [r_{12}(\lambda), r_{13}(\lambda)] + [r_{12}(\lambda), r_{23}(\lambda)] + [r_{13}(\lambda), r_{23}(\lambda)] = 0,$$
where $(x_i)$ is a basis of $\mathfrak{h}$ and $(x^i)$ is a dual basis of $\mathfrak{h}^*$;
- Its symmetric part is constant and $G$-invariant.

**Remark 1.3.2.** By "function" we understand a holomorphic one that, however, is algebraic if we consider $U$ as an open subset of $\mathfrak{h}^*$ or $H$; the corresponding dynamical $r$-matrices will be called "rational" or "trigonometric". When $G$ is reductive as in the usual setup 1.0.1 and $H \subset G$ is a torus, we can also consider $U$ as an open subset of the coarse moduli space $\mathrm{Pic}^0_H(E)$ of degree 0 bundles on an elliptic curve $E$; in this case, a "function" means a section of some bundle on $\mathrm{Pic}^0_H(E)$ and the corresponding $r$-matrix is called "elliptic".

There are natural symmetries of dynamical $r$-matrices. Let $g\colon U \to G^H$ be a function and $\eta = g^{-1}\mathrm{d}g$ be the 1-form on $U$ with values in $\mathfrak{g}^H$. We can consider it as a function $\bar{\eta}\colon U \to \mathfrak{h} \otimes \mathfrak{g}$. Define $\tau\colon U \to \Lambda^2\mathfrak{g}$ by the formula $\tau(\lambda) = (\lambda \otimes 1 \otimes 1)[\eta_{12}, \eta_{13}]$.

**Definition 1.3.3.** A ***gauge transformation*** defined by a function $g\colon U \to G^H$ is the map $r(\lambda) \mapsto r(\lambda)^g$, where

$$r(\lambda)^g := (g \otimes g)(r(\lambda) - \bar{\eta} + \bar{\eta}^{21} + \tau)(g^{-1} \otimes g^{-1}). \tag{1.3.1}$$

Then one can check (see [EV98b, Proposition 1.2]) that $r(\lambda)^g$ is also a dynamical $r$-matrix.

**1.3.2. Geometry.** First, let us recall the following definition, for instance, see [Saf17b, Definition 2.4]:

**Definition 1.3.4.** Let $(G, \pi_G, \phi)$ be a quasi-Poisson group as in definition 1.2.1. A ***quasi-Poisson G-scheme*** is a $G$-scheme $X$ with a bivector $\pi_X \in \Gamma(X, \Lambda^2 \mathrm{T}_X)$ satisfying

$$\pi_X(xg) = R_{g,*}\pi_X(x) + L_{x,*}\pi_G(g),$$
$$\frac{1}{2}[\pi_X, \pi_X] + a(\phi) = 0, \tag{1.3.2}$$

where $X \times G \to X$ is the action morphism inducing point-wise maps $L_x\colon G \to X$ and $R_g\colon X \to X$ for $x \in X, g \in G$ and $a\colon \mathfrak{g} \to \Gamma(X, \mathrm{T}_X)$ is infinitesimal action map.

**Definition 1.3.5.** Let $G$ be a group and $X$ be a scheme. Let $\mathrm{QPois}(G, X)$ be the following groupoid:

- Objects are quasi-Poisson structures on $G$ and bivectors on $X$ making $X$ a quasi-Poisson $G$-scheme;
- Morphisms between $(\pi_G, \phi, \pi_X)$ and $(\pi'_G, \phi', \pi'_X)$ are given by the elements $\lambda \in \Lambda^2\mathfrak{g}$ defining a morphism in $\mathrm{QPois}(G)$ and satisfying
$$\pi'_X = \pi_X + a(\lambda).$$

The following example of a quasi-Poisson $G$-scheme provides one geometric interpretation of a dynamical $r$-matrices:

**Example 1.3.6.** Let $r(\lambda)$ a quasi-triangular dynamical $r$-matrix with symmetric part $c \in \mathrm{Sym}^2(\mathfrak{g})^G$. Denote $\phi = \frac{1}{6}[c_{12}, c_{23}]$ and equip $G$ with the quasi-Poisson structure $(0, \phi)$. Let $P^{\mathrm{act}} := U \times H \to H$ be the action bundle and $X = P^{\mathrm{act}} \times_H G$ be the induced $G$-bundle. The dynamical $r$-matrix $r(\lambda)$ equips $X$ with a structure of a quasi-Poisson $G$-scheme with $\pi_X$ given by

$$\begin{aligned}\{f_1(\lambda), f_2(\lambda)\} &= [\mathrm{d}f_1, \mathrm{d}f_2](\lambda),\\ \{a, f(g)\} &= (R(a)f)(g),\\ \{f_1(g), f_2(g)\} &= (\mathrm{d}f_1 \otimes \mathrm{d}f_2)(R(r(\lambda))),\end{aligned} \tag{1.3.3}$$

where $a \in \mathfrak{h}$ is a linear function on $U$ and $R$ denotes the right-invariant vector field, see [EE03, Section 2.4].

In this setting, gauge transformations as in definition 1.3.3 can be naturally interpreted as follows. Let $X$ be any $G$-scheme and assume we have an automorphism of $X$ commuting with the $G$-action. Let it act on $\pi_X$ in the usual way and on $\pi_G$ trivially. One can easily check that such action preserves equations (1.3.2), hence defines an action on the groupoid $\mathrm{QPois}(G, X)$. In particular, let $X = U \times G$ and $g(\lambda)$ be a gauge transformation. Its action can be explicitly described as

$$\begin{aligned}\frac{\partial}{\partial \lambda^i} &\mapsto \frac{\partial}{\partial \lambda^i} - \frac{\partial g}{\partial \lambda^i} g^{-1},\\ \xi &\mapsto \mathrm{Ad}_{g(\lambda)}\xi,\end{aligned} \tag{1.3.4}$$

where $\{\lambda^i\}$ are coordinates on $U$ and $\xi \in \mathfrak{g}$ is a right-invariant vector field on $G$. One can check that the transformed (quasi-)Poisson bivector $\pi_{U\times G}$ of (1.3.3) has the same form, but for the gauge transformed dynamical $r$-matrix $r(\lambda)^g$ as in (1.3.1).

There is another geometric interpretation which is closer in spirit to the non-dynamical case, now in terms of ***Poisson-Lie groupoids***. Let $X \to B$ be a Lie groupoid. We denote by $s, t, m$ correspondingly the source, the target, and the multiplication maps.

**Definition 1.3.7.** [Wei88] A Lie groupoid $X$ with a Poisson structure is called a ***Poisson-Lie groupoid*** if the graph of the multiplication map $\Gamma_m \subset X \times X \times \bar{X}$ is coisotropic (here, $\bar{X}$ means the same manifold with the opposite Poisson structure).

Consider the following groupoid: $X = U \times G \times U, B = U$ with

$$\begin{aligned}s(u_1, g, u_2) &= u_1,\\ t(u_1, g, u_2) &= u_2,\\ m((u_1, g, u_2), (u_3, g', u_4)) &= (u_1, gg', u_4),\end{aligned}$$

where the last composition is defined only when $u_2 = u_3$. We define the Poisson structure on $X$ as follows. Let $a(\lambda)\colon U \to \mathfrak{g} \otimes \mathfrak{g}$ be a regular function. Then we set

$$\begin{aligned}&\{a_1, b_1\} = -[a_1, b_1]_1, \qquad &&\{a_2, b_2\} = [a_2, b_2]_2, \qquad &&\{a_1, b_2\} = 0,\\ &\{a_1, f\} = R_a f, &&\{a_2, f\} = L_a f, &&\{f, g\} = (\mathrm{d}f \otimes \mathrm{d}g)(R_{a(u_1)} - L_{a(u_2)}).\end{aligned} \tag{1.3.5}$$

where $a, b$ are elements of $\mathfrak{h}$ considered as linear functions on $\mathfrak{h}^*$, $f$ is a function on $G$, $R_a$ (resp. $L_a$) is the right (resp. left) derivation with respect to $a \in \mathfrak{h}$, and subscripts mean the corresponding copy of $U$.

**Proposition 1.3.8.** *[EV98b, Theorem 1.1] Formulae (1.3.5) define a Poisson-Lie groupoid structure on $X \to B$ if and only if*

- $\mathrm{CDYBE}(a(\lambda))$ *is constant and belongs to* $(\Lambda^3\mathfrak{g})^{\mathfrak{g}}$*;*
- $a(\lambda)$ *is* $\mathfrak{h}$*-invariant.*

In particular, as in the non-dynamical case, a quasi-triangular dynamical $r$-matrix defines a Poisson-Lie structure on $X$.

### 1.3.3. Examples of dynamical $r$-matrices.

Let $H \subset G$ be a maximal torus in a simple group as in the usual setup 1.0.1. The following are examples of dynamical $r$-matrices with respect to $H \subset G$:

**Example 1.3.9.** (1) [ES01, (3.3)] Basic rational dynamical $r$-matrix

$$r(\lambda) = \sum_{\alpha \in \Delta^+} \frac{e_\alpha \wedge e_{-\alpha}}{\langle \alpha, \lambda \rangle}. \tag{1.3.6}$$

(2) [ES01, (3.4)] Let $\Omega_{\mathfrak{g}}$ be the quadratic Casimir element in $\mathfrak{g}\otimes\mathfrak{g}$. The basic trigonometric dynamical $r$-matrix is

$$r(\lambda) = \frac{\Omega}{2} + \sum_{\alpha\in\Delta^+} \coth\left(\frac{\langle\alpha,\lambda\rangle}{2}\right) e_\alpha \wedge e_{-\alpha} \tag{1.3.7}$$

Recall the group of gauge transformations of dynamical $r$-matrices. For $H$ a maximal torus, we have another kind of transformations:

- For $C_{ij}(\lambda)\mathrm{d}\lambda_i \wedge \mathrm{d}\lambda_j$ a closed 2-form on $U$, the transformation

$$r(\lambda) \mapsto r(\lambda) + C_{ij}(\lambda) h_i \wedge h_j; \tag{1.3.8}$$

- $r(\lambda) \mapsto \tilde{r}(\lambda) := r(\lambda-\nu)$ for a fixed $\nu \in \mathfrak{h}^*$;
- $r(\lambda) \mapsto (w\otimes w) r(w^*\lambda)$ for $w \in W$, the Weyl group of $\mathfrak{g}$,

(here, the transformed $r$-matrices are defined on an *a priori* different subset $U'$). In what follows, for dynamical $r$-matrices with respect to $H \subset G$, the term "gauge transformation" will also include the ones above.

The space $(S^2\mathfrak{g})^{\mathfrak{g}}$ is one-dimensional and generated by the quadratic Casimir element $\Omega_{\mathfrak{g}}$. Then, for a quasi-triangular dynamical $r$-matrix $r(\lambda)$, its ***coupling constant*** is defined as $\epsilon \in \mathbf{C}$ such that $r(\lambda) + r(\lambda)^{21} = \epsilon\Omega_{\mathfrak{g}}$. Up to rescaling, there are essentially two possibilities: $\epsilon = 0$ and $\epsilon = 1$.

Theorem 1.3.10. *[EV98b, Theorem 3.2, 3.10] In the assumptions above, any dynamical $r$-matrix with coupling constant $\epsilon = 0$ (resp. $\epsilon = 1$) is gauge equivalent to the standard dynamical $r$-matrix (1.3.6) (resp. (1.3.7)).*

**Remark 1.3.11.** There is a certain integrability issue which becomes especially clear when we pass to the geometric setting. Namely, observe that gauge transformations in the sense of Definition 1.3.3 act exactly by (1.3.8). Unfortunately, given a closed 2-form, it is not always possible in general to find the corresponding gauge transformation, and the reason is two-fold – first, exactness of the form, second, integration to the group $H$. A possible solution is to pass to appropriate formal completions, however, we will not address this issue in the future.

Non-dynamical quasi-triangular $r$-matrices with nonzero coupling constant can be classified in terms of the so-called ***Belavin-Drinfeld triples***, see [BD98]. For dynamical ones, there is a generalization of this result due to Schiffmann [Sch98].

Let $L \subset H$ be a subgroup such that restriction of the form $\langle,\rangle$ is non-degenerate on $\mathfrak{l} \subset \mathfrak{h}$.

**Definition 1.3.12.** (1) A ***generalized Belavin-Drinfeld triple*** is a triple $(\Gamma_1, \Gamma_2, \tau)$, where $\Gamma_1, \Gamma_2 \subset \Pi$ are subsets of simple roots and $\tau\colon \Gamma_1 \xrightarrow{\sim} \Gamma_2$ is a norm-preserving bijection.
(2) A generalized Belavin-Drinfeld triple is $\mathfrak{l}$-***graded*** if $\tau$ preserves the decomposition of $\mathfrak{g}$ in $\mathfrak{l}$-weight spaces.

Let $\Gamma_3 \subset \Gamma_1 \cap \Gamma_2$ be the largest subset stable under $\tau$. Observe that (usual) Belavin-Drinfeld triples are the generalized ones with $\Gamma_3 = \emptyset$.

Given a subset $\Gamma \subset \Pi$ of simple roots, denote by $\mathfrak{n}_+(\Gamma)$ the subalgebra of $\mathfrak{n}_+$ generated by simple root vectors $e_\alpha$ as in the usual setup 1.0.1 for $\alpha \in \Gamma$. Observe that a norm-preserving bijection $\tau\colon \Gamma_1 \to \Gamma_2$ defines the map $\tau\colon \mathfrak{n}_+(\Gamma_1) \to \mathfrak{n}_+(\Gamma_2)$ by $\tau(e_\alpha) = e_{\tau(\alpha)}$.

For $\lambda \in \mathfrak{l}^*$, consider the map

$$\begin{aligned} &K(\lambda)\colon \mathfrak{n}_+(\Gamma_1) \to \mathfrak{n}_+(\Gamma_2), \\ &e_\alpha \mapsto \frac{1}{2} + e^{-\langle\alpha,\lambda\rangle} \frac{\tau}{1 - e^{-\langle\alpha,\lambda\rangle}\tau}(e_\alpha) = \\ &= \frac{1}{2} e_\alpha + \sum_{n>0} e^{-n\langle\alpha,\lambda\rangle}\tau^n(e_\alpha). \end{aligned}$$

The sum is finite for $\alpha \notin \Gamma_3$.

Let $\mathfrak{h}_0$ be the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{h}$. Let $r_{\mathfrak{h}_0,\mathfrak{h}_0}$ satisfy the equation

$$(\tau(\alpha)\otimes 1)r_{\mathfrak{h}_0,\mathfrak{h}_0} + (1\otimes\alpha)r_{\mathfrak{h}_0,\mathfrak{h}_0} = \frac{1}{2}((\alpha+\tau(\alpha))\otimes 1)\Omega_{\mathfrak{h}_0},$$

where $\Omega_{\mathfrak{h}_0}$ is the restriction of $\Omega_{\mathfrak{g}}$ to $\mathfrak{h}_0$.

Theorem 1.3.13. *[Sch98, Theorem 4] Let $L \subset H$ be a subgroup such that restriction of the form $\langle , \rangle$ to $\mathfrak{l}$ is nondegenerate.*

(1) *The function*

$$r(\lambda) = \frac{1}{2}\Omega_{\mathfrak{g}} + \sum_{\alpha\in\langle\Gamma_1\rangle\cap\Delta_+} K(\lambda)(e_\alpha)\wedge e_{-\alpha} + \sum_{\alpha\in\Delta_+,\ \alpha\notin\langle\Gamma_1\rangle} \frac{1}{2}e_\alpha\wedge e_{-\alpha}$$

*is a dynamical $r$-matrix with respect to $L \subset G$;*

(2) *Any dynamical $r$-matrix with nonzero coupling constant is gauge equivalent to the one above.*

**1.3.4. Shifted Poisson setting.** In this section, $G$ is an arbitrary affine algebraic group and $H \subset G$ is a subgroup. For simplicity of exposition, we assume that $U$ is a subset of $\mathfrak{h}^*$; however, all the arguments translate *mutatis mutandis* to the trigonometric setting.

Recall (see example 1.1.2) that the quotient stack $[\mathfrak{h}^*/H]$ is equipped with a 1-shifted symplectic structure . The substack $[U/H] \subset [\mathfrak{h}^*/H]$ is open, therefore, it is also equipped with the 1-shifted symplectic and hence with a 1-shifted Poisson structure.

Consider the composition $[U/H] \to \mathrm{B}H \to \mathrm{B}G$.

Theorem 1.3.14. *[Saf17b, Proposition 5.6] The space parametrizing the pairs of*

- *A 2-shifted Poisson structure $\pi$ on $\mathrm{B}G$;*
- *A 1-shifted Poisson morphism $f\colon [U/H] \to \mathrm{B}G$ compatible with the Poisson structure on $[U/H]$ mentioned above and the 1-shifted Poisson structure on $\mathrm{B}G$ induced from $\pi$,*

*is equivalent to the set of quasi-triangular dynamical $r$-matrices.*

Some consequences from this point of view on the dynamical $r$-matrices. Fix a dynamical $r$-matrix giving a 1-shifted Poisson morphism $[U/H] \to \mathrm{B}G$.

(1) The natural map $U \to [U/H]$ is 1-shifted Lagrangian, in particular coisotropic. Therefore, the composition has $U \to [U/H] \to \mathrm{B}G$ is also coisotropic which, by [Saf17b, Proposition 2.14], is equivalent to the data of a quasi-Poisson Lie structure on $U \times G$. One can check that it is, in fact, provided by (1.3.3).

(2) Consider the coisotropic intersection

$$\begin{tikzcd} U\times G\times U \arrow[r] \arrow[d] & U \arrow[d] \\ U \arrow[r] & \mathrm{B}G \end{tikzcd}$$

which equips $U \times G \times U$ with a (0-shifted) Poisson structure by theorem 1.1.16. One can check that it is, in fact, provided by (1.3.5).

Recall the notion of a gauge transformation from Definition 1.3.3. In the shifted world, it can be interpreted as follows. Recall example 1.3.6. The bundle $U\times G \to U$ defines a map $U \to \mathrm{B}G$. Its homotopies (as a map between groupoids) naturally act on the space $\mathrm{Cois}(p,1)$ of 1-shifted coisotropic structures thereon; in our case, there is an identification $\mathrm{Cois}(p,1) \cong \mathrm{QPois}(G, U\times G)$, see [Saf17b, Proposition 2.14]. A homotopy of $p$ is a gauge transformation of the $G$-bundle $U \times G \to U$ (which should be thought of as the induced one from the action bundle $U \times H \to H$), and gauge transformations naturally act on the space $\mathrm{QPois}(G, U \times G)$ as in example 1.3.6.

**Proposition 1.3.15.** *The action of homotopies of $p$ on $\mathrm{Cois}(p,1)$ coincides with the ones induced from gauge transformations on $\mathrm{QPois}(G, U \times G)$.*

Proof. The space $\mathrm{Cois}(p, 1)$ can be identified with the Maurer-Cartan space of the relative polyvectors $\mathrm{Pol}(f, 1)^{\geq 2}[2]$. We have a fiber sequence of Lie algebras:

$$\mathrm{Pol}(U/\mathrm{B}G, 0)[1] \to \mathrm{Pol}(f, 1)[2] \to \mathrm{Pol}(\mathrm{B}G, 1)[2].$$

We have

$$\mathrm{Pol}(U/\mathrm{B}G, 0) = \Gamma(U, \mathrm{Sym}(\mathbb{T}_{U/\mathrm{B}G}[-1])$$

and

$$\mathbb{T}_{U/\mathrm{B}G} \cong \mathbb{T}_{U\times G} \cong \mathfrak{g} \otimes \mathcal{O}_U \oplus \mathbb{T}_U$$

with gauge transformations acting naturally on $\mathbb{T}_{U\times G}$. $\square$

**Corollary 1.3.16.** *The action of a homotopy of the 1-shifted Poisson morphism $[U/H] \to \mathrm{B}G$ given by a dynamical $r$-matrix is equivalent to a gauge transformation in the sense of Definition 1.3.3.*

Proof. A homotopy of $[U/H] \to \mathrm{B}G$ are gauge transformations of $U \times G$ compatible with reductions to the action bundle $P^{\mathrm{act}} := U \times H \to U$, i.e. functions $U \to G^H$. The result follows from (1) and arguments in example 1.3.6. $\square$

Let $H \subset G$ be a maximal torus in a simple group as in the usual setup 1.0.1 (in fact, the construction generalizes to reductive groups). Recall theorem 1.3.10. There is a geometric construction of the standard dynamical $r$-matrices in the shifted world. We will present the one for (1.3.6).

We have a $G$-equivariant isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ given by the bilinear form as in the usual setup 1.0.1; likewise, its restriction (understood in a suitable sense) provides identification $\mathfrak{h} \cong \mathfrak{h}^*$. Consider the correspondence of algebraic stacks

(1.3.9)
$$\begin{tikzcd} & {[\mathfrak{b}/B]} \arrow[dl] \arrow[dr] & \\ {[\mathfrak{g}^*/G]} & & {[\mathfrak{h}^*/H].} \end{tikzcd}$$

**Proposition 1.3.17.** *The map $p\colon [\mathfrak{b}/B] \to [\mathfrak{h}^*/H]$ is birational.*

Proof. Choosing a Cartan subalgebra gives rise to the morphism $\mathfrak{h}^* \cong \mathfrak{h} \hookrightarrow \mathfrak{b}$ inducing the map of stacks $i\colon [\mathfrak{h}^*/H] \to [\mathfrak{b}/B]$. Obviously, $p \circ i = \mathrm{id}_{[\mathfrak{h}^*/H]}$.

Let $U = \mathfrak{h}^{\mathrm{reg}} \subset \mathfrak{h}$ be the open subset defined as

$$\mathfrak{h}^{\mathrm{reg}} := \{\lambda \in \mathfrak{h} |\ \langle \alpha, \lambda\rangle \neq 0\ \forall \alpha \in \Delta\},$$

and $\mathfrak{b}' := \mathfrak{h}^{\mathrm{reg}} \oplus \mathfrak{n}$. We claim that $i \circ p$ is homotopic to identity when restricted to $\mathfrak{b}'$. Equivalently, it means that there exists a function $g(\lambda)\colon \mathfrak{b}^{\mathrm{reg}} \to B$ such that for every $b = \lambda + n \in \mathfrak{h}^{\mathrm{reg}} \oplus \mathfrak{n}$, we have $\mathrm{Ad}_{g(\lambda)}(b) = \lambda$. This is the content of [Sev13]. Observe that the construction from *loc. cit.* uses a semi-classical version of the so-called extremal projector which will be introduced later in the work. $\square$

Therefore, we have a Lagrangian map $[U/H] \cong [\mathfrak{b}'/B] \to [\mathfrak{g}^*/G] \times [U/H]$ which is identity on the second component, hence the graph of some symplectic map $f\colon [U/H] \to [\mathfrak{g}^*/G]$. As a plain morphism, it comes from the embedding $\mathfrak{h} \hookrightarrow \mathfrak{g}$ under identifications $\mathfrak{h}^* \cong \mathfrak{h}$ and $\mathfrak{g}^* \cong \mathfrak{g}$.

In fact, the map $f$ is an unramified $W$-cover of an open subset of $[\mathfrak{g}/G]$. Indeed, let $\tilde{\mathfrak{g}}$ be the variety parametrizing Borel subgroups of $G$ together with an element $x \in \mathfrak{g}$ contained in the Lie algebra of the corresponding Borel subgroup. The projection $\tilde{\mathfrak{g}} \to \mathfrak{g}$ is known as the Grothendieck-Springer resolution, see [CG10, Section 3.1.31]. We may identify $[\mathfrak{b}/B] \cong [\tilde{\mathfrak{g}}/G]$, so that the map $[\mathfrak{b}/B] \to [\mathfrak{g}/G]$ is identified with the Grothendieck-Springer resolution $[\tilde{\mathfrak{g}}/G] \to [\mathfrak{g}/G]$. Moreover, the open substack $[\mathfrak{b}'/B] \subset [\mathfrak{b}/B]$ is identified with the preimage $[\tilde{\mathfrak{g}}^{\mathrm{rs}}/G]$ of regular semisimple elements. Since the map $\tilde{\mathfrak{g}}^{\mathrm{rs}} \to \mathfrak{g}^{\mathrm{rs}}$ is a principal $W$-bundle (see [CG10, Proposition 3.1.36]), we conclude.

By [Saf17b, Lemma 5.4], the projection $[\mathfrak{g}^*/G] \to \mathrm{B}G$ is 1-shifted Poisson where $\mathrm{B}G$ is equipped with zero Poisson structure. Therefore, we have a 1-shifted Poisson morphism $[U/H] \to \mathrm{B}G$ which, by theorem 1.3.14, is equivalent to some dynamical $r$-matrix.

**Proposition 1.3.18.** *It is the standard rational one.*

(Sketch of) Proof. Recall that the skew-symmetric part of the $r$-matrix is provided by the element $\lambda^1$ from the proof of [Saf17b, Proposition 5.6]. In the case of the map $[U/H] \to [\mathfrak{g}^*/G] \to \mathrm{B}G$, we obtain this element as follows: the map $[U/H] \to [\mathfrak{g}^*/G]$ is étale, therefore, we have a quasi-isomorphism $\mathbb{T}_{[U/H]} \cong f^*\mathbb{T}_{[\mathfrak{g}^*/G]}$. The corresponding Poisson structures can be identified via the diagram

$$\begin{tikzcd} & \mathrm{Pol}([\mathfrak{g}^*/G],1) \arrow[d] \\ \mathrm{Pol}([U/H],1) \arrow[r] & \Gamma([U/H], \mathrm{Sym}(f^*\mathbb{T}_{[\mathfrak{g}^*/G]}[-2])) \end{tikzcd}$$

Observe that the Poisson structures are not equal on the nose, but up to homotopy given by the $r$-matrix $r(\lambda) \in \Lambda^2\mathfrak{g} \otimes \mathcal{O}(U)$. Indeed: let $\rho\colon \mathfrak{g} \otimes \mathcal{O}(U) \to \mathfrak{g}^* \otimes \mathcal{O}(U)$ be the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ restricted to $U$. Then

$$\rho\left(\sum_{\alpha\in\Delta_+} \frac{e_\alpha \wedge e_{-\alpha}}{\langle \lambda, \alpha\rangle}\right) = \sum_{\alpha\in\Delta} e_\alpha \otimes e_\alpha^*,$$

which is exactly the difference between the canonical elements for $\mathfrak{g}$ and $\mathfrak{h}$. □

In fact, we have the following result:

**Proposition 1.3.19.** *Up to a 2-form on $U$, there is a unique 2-form on $U \times G$ such that the $G$-equivariant map $\tilde{\mu}\colon U \times G \to \mathfrak{g}^*$, induced from the standard embedding $\mu\colon \mathfrak{h}^{\mathrm{reg}} \to \mathfrak{g}$, is a moment map.*

Proof. Denote by $\pi = \pi_U + \pi_{mix} + \pi_G$ the coordinate decomposition of the Poisson structure $\pi$ on $U \times G$. Recall the Poisson moment map equation: for every $\xi \in \mathfrak{g}$,

$$\pi(\tilde{\mu}^*\mathrm{d}\xi) = \xi^{vf}, \tag{1.3.10}$$

where $\mathrm{d}\xi$ is the differential of the linear function on $\mathfrak{g}^*$ and $\xi^{vf}$ is the corresponding action vector field. In matrix terms, after identification $\mathfrak{g} \cong \mathfrak{g}^*$ we have

$$\tilde{\mu}^*\mathrm{d}\xi = \langle \xi, [g^{-1}\mu(\lambda)g, g^{-1}\mathrm{d}g] + g^{-1}\mathrm{d}\mu g\rangle.$$

Consider (1.3.10) at points $(\lambda, e)$. Observe that if $\xi \in \mathfrak{n}_- \oplus \mathfrak{n}_+$, then the pairing $\langle \xi, \mathrm{d}\mu\rangle$ is zero, so that the form $\tilde{\mu}^*\xi$ acts on action vector fields $\mathfrak{g}$ by

$$\eta \mapsto \langle \xi, [\mu(\lambda), \eta]\rangle = \langle \mu(\lambda), [\eta, \xi]\rangle,$$

so that $\tilde{\mu}$ defines a linear map $\mathfrak{n}_- \oplus \mathfrak{n}_+ \to \mathfrak{g}^*$. The moment map equation (1.3.10) implies that the composition $\mathfrak{g}^* \xrightarrow{\pi_G} \mathfrak{g} \to \mathfrak{n}_- \oplus \mathfrak{n}_+$, where the second arrow is the projection, should be inverse of this map which is exactly (the negative of) the standard dynamical $r$-matrix $r(\lambda)$; by skew-symmetricity, we see that $\pi_G \in -r(\lambda) + \mathfrak{h} \wedge \mathfrak{h}$.

On the other hand, let $\xi \in \mathfrak{h}$. Then the pairing $\langle \xi, [g^{-1}\mu(\lambda), g^{-1}\mathrm{d}g]\rangle$ is zero (again, at points $(\lambda, e)$), so that $\tilde{\mu}^*\mathrm{d}\xi = \langle \xi, \mathrm{d}\mu\rangle$. In particular, if $\xi = h_i$, then $\tilde{\mu}^*\mathrm{d}\xi = \mathrm{d}\lambda_i$. Then the moment map equation implies that $\pi_U = 0$ and $\pi_{mix} = \sum \partial_{\lambda_i} \wedge h_i$. Therefore, the Poisson structure is unique up to a $\mathfrak{h} \wedge \mathfrak{h}$-part of the dynamical $r$-matrix. We also see that the Poisson structure is invertible, and the inverse of this part is, in fact, a closed 2-form on $U$. □

**Corollary 1.3.20.** *Up to a closed 2-form on $U$, there is a unique 1-shifted symplectic structure on the map $[U/H] \to [\mathfrak{g}^*/G]$, where the target is equipped with the standard 1-shifted symplectic structure on $[\mathfrak{g}^*/G]$.*

The standard trigonometric dynamical $r$-matrix can be obtained considering adjoint quotients of *groups*, i.e., for instance, taking the stack $[G/G]$ with its 1-shifted symplectic structure from example 1.1.2 instead of $[\mathfrak{g}^*/G]$.

**1.3.5. Relation to the Kostant-Whittaker reduction.** Let $\mathfrak{g}$ be a reductive Lie algebra and $\langle -, - \rangle$ be a non-degenerate invariant bilinear form (for instance, as in the usual setup 1.0.1). Let $e \in \mathfrak{g}$ be a principal nilpotent element. By the Jacobson-Morozov theorem [CG10, Theorem 3.7.1], we may extend it to an $\mathfrak{sl}_2$-triple $\{e, f, h\}$. Denote $\chi = \langle e, - \rangle \in \mathfrak{g}^*$ and $\mathfrak{g}^f$ the kernel of the operator $\mathrm{ad}(f)$ on $\mathfrak{g}$. Let $\kappa \colon \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ be the isomorphism given by the form $\langle -, - \rangle$.

**Definition 1.3.21.** The ***Kostant slice*** is the subvariety $S = \chi + \kappa(\mathfrak{g}^f) \subset \mathfrak{g}^*$.

By [Saf20, Proposition 4.18], there is a 1-shifted Lagrangian structure on the morphism $S \to [\mathfrak{g}^*/G]$.

**Definition 1.3.22.** The ***(semi-classical) Kostant-Whittaker reduction*** is the 1-shifted coisotropic morphism $S \to [\mathfrak{g}^*/G]$.

**Remark 1.3.23.** The name is motivated by the Kostant-Whittaker reduction of [BF08].

Also, by Kostant's theorem [Kos78], there is an isomorphism $S \cong \mathfrak{h}/W$, hence we have a morphism $\mathfrak{h}^{\mathrm{reg}} \to S \to [\mathfrak{g}^*/G]$.

**Proposition 1.3.24.** *The diagram*

(1.3.11)
$$\begin{tikzcd} & {[\mathfrak{h}^{\mathrm{reg}}/H]} \arrow[dr] & \\ \mathfrak{h}^{\mathrm{reg}} \arrow[ur] \arrow[dr] & & {[\mathfrak{g}^*/G]} \\ & S \arrow[ur] & \end{tikzcd}$$

*is commutative up to homotopy.*

Proof. We need to provide an algebraic function $g(\lambda) \colon \mathfrak{h}^{\mathrm{reg}} \to G$ such that the images of two maps are conjugated by $g(\lambda)$. By [Sev13, Lemma 1], we have an isomorphism $N_- \times \mathfrak{h}^{\mathrm{reg}} \to \mathfrak{b}'_-$ (notations are taken from Proposition 1.3.17); in particular, there is an algebraic function $u(\lambda) \colon \mathfrak{h}^{\mathrm{reg}} \to N_-$ such that $\mathrm{Ad}_{u(\lambda)}(\lambda) = \lambda + e$. The result follows from the isomorphism $N \times S \to e + \mathfrak{b}_-$, see [Kos78, Theorem 1.2], also [GG02, Lemma 2.1]. □

## 1.4. Vertex-IRF transformation

In this section, we denote by $G$ an arbitrary affine algebraic group.

Recall that on the set of dynamical $r$-matrices, there is an action of the gauge transformations as in definition 1.3.3 that are specified by functions $U \to G^H$.

**Definition 1.4.1.** Let $a(\lambda) \colon U \to \mathfrak{g} \otimes \mathfrak{g}$ be a function. A ***generalized gauge transformation*** is a function $g \colon U \to G$ acting on $a(\lambda)$ by (1.3.1).

Equivalently, a generalized gauge transformation is a self-homotopy of the composition $U \to [U/H] \to \mathrm{B}G$ acting on the space of 1-coisotropic structures thereon.

In the case of the standard dynamical $r$-matrix, there is an additional geometric structure on a generalized gauge transformation. Namely, recall that the former is obtained via the symplectic map $[\mathfrak{h}^{\mathrm{reg}}/H] \to [\mathfrak{g}^*/G]$ coming from the embedding $\mu(\lambda) \colon \mathfrak{h} \to \mathfrak{g}$. The morphism $U \cong [U \times G/G] \to [\mathfrak{g}^*/G]$ is given then by the $G$-equivariant morphism

$$U \times G \to \mathfrak{g}, \ (\lambda, h) \mapsto \mathrm{Ad}_{h^{-1}}(\mu(\lambda)).$$

Let $g(\lambda) \colon U \to G$ be a gauge transformation. Define

$$\mu^g \colon U \to \mathfrak{g}, \ \mu^g(\lambda) = \mathrm{Ad}_{g(\lambda)}(\mu(\lambda)).$$

It induces a homotopy of the map $U \cong [U \times G/G] \to [\mathfrak{g}^*/G]$ lifting the homotopy of $U \to \mathrm{B}G$ given by $g(\lambda)$. The former naturally acts on the space of 1-shifted Lagrangian structures on the map $U \to [\mathfrak{g}^*/G]$ which is equivalent to the set of moment map structures on $U \times G \to \mathfrak{g}^*$ (i.e. 2-forms on $U \times G$ such that this map is a moment map). One can check using [MS18b, Theorem 4.22] that the Poisson structure on $U \times G$ induced from such a Lagrangian (hence coisotropic) morphism coincides with the symplectic structure thereon coming from the moment map condition, and it is compatible with the action of homotopies of the map $U \to [\mathfrak{g}^*/G]$.

An example of such transformation is the diagram (1.3.11) relating the (semi-classical) Kostant-Whittaker reduction with the parabolic one. Recall that proposition 1.3.24 shows that it is commutative (up to homotopy) on the level of plain morphisms of stacks; however, as the discussion above shows, the following also holds:

**Proposition 1.4.2.** *Up to a 2-form on $U$, the diagram (1.3.11) of 1-shifted coisotropic morphisms to homotopy.*

Proof. Follows from proposition 1.3.19. $\square$

Let $r(\lambda)\colon U \to \mathfrak{g} \otimes \mathfrak{g}$ be a dynamical $r$-matrix with respect to $H$. Let $L \subset H$ be a subgroup with $\mathfrak{l} \subset \mathfrak{h}$ the Lie subalgebra, $U' \subset \mathfrak{l}^*$ be an open subset such that $U \subset \mathfrak{h}$ projects onto $U'$ under the natural morphism $\mathfrak{h}^* \to \mathfrak{l}^*$, and $r'(\lambda)\colon U' \to \mathfrak{g} \otimes \mathfrak{g}$ be another dynamical $r$-matrix, now with respect to $L$. Observe that $r'(\lambda)$ can be considered as a function on $U$.

**Definition 1.4.3.** (1) A ***generalized vertex-IRF transformation*** is a generalized gauge transformation $g(\lambda)\colon U \to G$ such that

$$r(\lambda)^g = r'(\lambda).$$

(2) A ***vertex-IRF transformation*** is a generalized vertex-IRF transformation with $L = \{e\}$ (i.e. $r' \in \mathfrak{g} \otimes \mathfrak{g}$ is a constant classical $r$-matrix).

The dynamical $r$-matrix $r'(\lambda)$ defines a 1-shifted coisotropic map $U' \to \mathrm{B}G$. Therefore, we have the diagram of coisotropic maps

$$\begin{tikzcd} U \arrow[dr] \arrow[dd] & \\ & \mathrm{B}G \\ U' \arrow[ur] & \end{tikzcd} \tag{1.4.1}$$

Observe that these two morphisms $U \to \mathrm{B}G$ are defined by the same principal bundle $U \times G \to U$.

Recall that by proposition 1.1.17, a lift of the diagram (1.4.1) to a morphism between coisotropic maps is equivalent to the data of a 0-shifted (i.e. classical) coisotropic map $U \to U \times_{\mathrm{B}G} U'$.

**Proposition 1.4.4.** *A generalized vertex-IRF transformation is equivalent to a gauge transformation of the coisotropic morphism $U \to \mathrm{B}G$ such that the diagram (1.4.1) lifts to a morphism of coisotropic structures.*

Proof. Observe that $U \times_{\mathrm{B}G} U' \cong U \times G \times U'$ with the obvious generalization of the Poisson structure (1.3.5). The map $U \to U \times G \times U'$ is given by

$$U \to U \times U' \xrightarrow{\mathrm{id}_U \times e \times \mathrm{id}_{U'}} U \times G \times U',$$

where the first arrow is the graph of $U \to U'$. Since $r(\lambda)^g = r'(\lambda)$, we obtain that this map is indeed coisotropic. $\square$

In the case of the standard dynamical $r$-matrix, as we discussed above, a generalized vertex-IRF transformation provides a moment map $U \times G \to \mathfrak{g}^*$ with the Poisson structure on $U \times G$ given by

$$\pi_{U \times G} = \sum_i \mathrm{Ad}_{g(\lambda)}(h_i) \wedge \partial_{\lambda_i} + r'(\lambda). \tag{1.4.2}$$

**1.4.1. Uniqueness.** What makes the notion of a vertex-IRF transformation interesting from the points of view of geometric and categorical interpretation of the existence and uniqueness result. In what follows, we denote by $G$ a simple group as in the usual setup 1.0.1.

Recall [BD98] that constant quasi-triangular $r$-matrices with nonzero coupling constant are classified by the Belavin-Drinfeld triples.

**Definition 1.4.5.** Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots constituting the $A_n$ diagram. Consider the subsets $\Gamma_1 := \{\alpha_1, \ldots, \alpha_{n-1}\}$ and $\Gamma_2 := \{\alpha_2, \ldots, \alpha_n\}$ and let $\tau \colon \Gamma_1 \to \Gamma_2$ be the morphism defined by $\tau(\alpha_i) = \alpha_{i+1}$. The ***Cremmer-Gervais r-matrix*** is the $r$-matrix corresponding to the Belavin-Drinfeld triple $(\Gamma_1, \Gamma_2, \tau)$.

THEOREM 1.4.6. *[BDF90] Let $\mathfrak{g}$ be a simple Lie algebra and $r(\lambda)$ be the standard dynamical $r$-matrix with nonzero coupling constant. Then*

(1) *A vertex-IRF transformation exists only in type A;*
(2) *For each $n$, a vertex-IRF transformation is unique and the resulting constant $r$-matrix is not standard, but the Cremmer-Gervais one.*

We extend this result:

THEOREM 1.4.7. *Let $L$ be a subgroup satisfying the conditions of theorem 1.3.10. Let $r(\lambda)$ be a quasi-triangular dynamical $r$-matrix with nonzero coupling constant. If $L \neq H$ (hence $r(\lambda)$ is not the standard dynamical $r$-matrix), then there exists no vertex-IRF transformation from $r(\lambda)$.*

The proof is pretty technical and it is given in the Appendix 3.1.

In fact, an easy generalization of the arguments of [BDF90] shows that the uniqueness result is also true in the rational case. Computer computations suggest the following ansatz:

$$\Omega(\lambda) = A(\lambda) S(\lambda) \tag{1.4.3}$$

where $A(\lambda)$ is the diagonal matrix

$$A_{kk}(\lambda) = \prod_{i \neq k} (\lambda_i - \lambda_j)^{-1/2}$$

and $S(\lambda)$ is the Vandermonde matrix

$$S(\lambda) = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{n-1} \\ & & & \ldots & \\ 1 & \lambda_n & \lambda_n^2 & \ldots & \lambda_n^{n-1} \end{pmatrix} \tag{1.4.4}$$

(checked in low ranks).

**Remark 1.4.8.** The matrix $A(\lambda)$ is not algebraic as it involves square roots. However, as section 2.7 shows, it may be possible to replace $A(\lambda)$ with another diagonal matrix depending rationally on $\lambda$. So, in what follows, we will not address this issue.

The resulting constant $r$-matrix is the following one. Let $M \subset GL_n$ be the maximal parabolic subgroup of invertible matrices preserving the functional $[e_n^*]$ in the projectivization $\mathbb{P}(V^*)$ of the dual representation and $\mathfrak{m} \subset \mathfrak{g}$ be its Lie algebra. Consider the functional $e^* = \sum_{i=1}^{n-1} e_{i,i+1}^*$ on $\mathfrak{m}$; observe that under identification $\mathfrak{g}^* \cong \mathfrak{g}$, it corresponds to the principal nilpotent element. Consider the map $\mathfrak{m} \wedge \mathfrak{m} \to \mathbf{C}$ defined by $\xi \wedge \eta \mapsto \langle e^*, [\xi, \eta] \rangle$. One can easily check that this map is invertible; define $b_{CG} \in \mathfrak{m} \wedge \mathfrak{m} \hookrightarrow \mathfrak{g} \wedge \mathfrak{g}$ to be its inverse. It is a standard fact (for instance, see [ES02, Section 3.5, Proposition 3.3]) that $b_{CG}$ is a triangular $r$-matrix.

**Definition 1.4.9.** The ***rational Cremmer-Gervais*** $r$-matrix is $b_{CG}$.

**Example 1.4.10.** Let $\mathfrak{g} = \mathfrak{sl}_2$. Then $\mathfrak{m} = \mathfrak{b}$ and the representation of $b_{CG}$ in $\mathbf{C}^2 \otimes \mathbf{C}^2$, where $\mathbf{C}^2$ is the defining representation, is

$$b_{CG} = \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & -0 \end{pmatrix} \tag{1.4.5}$$

Recall the following result:

THEOREM 1.4.11. *[[ES02](#), Section 3.5, Theorem 3.2] There is a 1-1 correspondence between triangular $r$-matrices on $\mathfrak{g}$ and the pairs $(\mathfrak{p}, B)$ where $\mathfrak{p} \subset \mathfrak{g}$ is a finite-dimensional subalgebra and $B$ is a nondegenerate 2-cocycle with values in the trivial representation.*

For such an $r$-matrix, we will call $\mathfrak{p}$ the ***carrier***.

**Proposition 1.4.12.** *If the vertex-IRF transform is given by the ansatz (1.4.3) for some diagonal matrix $A(\lambda)$, then the constant $r$-matrix is necessarily $b_{CG}$.*

PROOF. Recall the geometric meaning of a vertex-IRF transformation for the standard dynamical $r$-matrix. Namely, we have the data of the moment map $U \times G \to \mathfrak{g}^*$ such that $(\lambda, e) \mapsto \Omega(\lambda)^{-1}\lambda\Omega(\lambda)$. The diagonal part $A(\lambda)$ acts trivially on $\lambda$, and it is well-known that $S(\lambda)^{-1}\lambda S(\lambda)$ is the so-called companion matrix

$$K(\lambda) = \begin{pmatrix} 0 & & \dots & (-1)^n\lambda_1 \cdot \dots \cdot \lambda_n \\ 1 & & \dots & \dots \\ & & & \\ 0 & \dots & 1 & \lambda_1 + \dots + \lambda_n \end{pmatrix} \tag{1.4.6}$$

As in the proof of proposition 1.3.19, we study the moment map equation (1.3.10). By our assumption, the $G$-part $\pi_G$ of the Poisson structure is obtained from some constant $r$-matrix $r'$. We will show that it is necessarily $b_{CG}$.

Consider (1.3.10) at points $(\lambda, e)$. Observe that if $\xi \in \mathfrak{m}$, then the pairing $\langle \xi, \mathrm{d}\mu \rangle$ is zero, so that the form $\tilde{\mu}^*\xi$ acts on action vector fields $\mathfrak{g}$ by

$$\eta \mapsto \langle \xi, [\eta, \mu(\lambda)] \rangle = \langle \mu(\lambda), [\xi, \eta] \rangle.$$

In particular, if $\eta \in \mathfrak{m}$, then this form becomes $\eta \mapsto \langle e, [\xi, \eta] \rangle$. Under decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{u}_-$, equation (1.3.10) implies that $\pi_G$ restricted to $\mathfrak{m}$ should be the inverse of this form. Therefore, the carrier of $r'$ at least contains $\mathfrak{m}$.

Suppose that it is larger than $\mathfrak{m}$, i.e. it contains some nonzero element of $\mathfrak{u}_-$. Observe that the Levi subalgebra of $\mathfrak{m}$ is isomorphic to $\mathfrak{gl}_{n-1}$ and $\mathfrak{u}_-$ is isomorphic to the tautological representation of the latter. In particular, any nonzero element of $\mathfrak{u}_-$ generates the whole subalgebra under the Lie bracket with $\mathfrak{m}$. Therefore, if the carrier were larger than $\mathfrak{m}$, then it would necessarily be the whole algebra $\mathfrak{g}$. However, it is known that simple Lie algebras do not possess triangular structures. Hence, we conclude that $r'$ is necessarily the triangular $r$-matrix $b_{CG}$.

□

The map $U \to \mathfrak{g}^*$ defined by the matrix $K(\lambda)$ is, in fact, rather special: it is a transverse slice to $G$-orbits different from the Kostant one. A generalization to an arbitrary nilpotent element was introduced in [MV03] and was called in later works the ***Mirković-Vybornov*** slice. Therefore, the vertex-IRF transformation provides a homotopy of the diagram (1.3.11), just with another map $\mathfrak{h}/W \to \mathfrak{g}^*$.

**Remark 1.4.13.** Computer computations in low ranks suggest that formula (1.4.3) should provide the vertex-IRF transformation also in the case of $\mathfrak{g} = \mathfrak{gl}_n$. The resulting constant $r$-matrix is given by the same construction as for $\mathfrak{sl}_n$, but with the so-called ***mirabolic*** subgroup of invertible matrices preserving the *vector* $e_n^* \in V^*$ (instead of a line).

CHAPTER 2

# Dynamical representation theory: categorical picture

## 2.1. Background

In this section we recall some facts about locally presentable categories, cp-rigid monoidal categories and Tannaka reconstruction for bialgebras.

**2.1.1. Locally presentable categories.** Let $k$ be a field. All categories and functors we will consider are $k$-linear. Throughout this paper we work with locally presentable categories (we refer to [AR94] and [BCJ15, Section 2] for more details). Here are the main examples:

- If $\mathcal{C}$ is a small category, the category of presheaves $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Vect})$ is locally presentable. For instance, this applies to the category $\mathrm{LMod}_A$ of (left) modules over a $k$-algebra $A$.
- If $\mathcal{C}$ is a small category which admits finite colimits, the ind-completion $\mathrm{Ind}(\mathcal{C})$ (see [KS06, Chapter 6] for what it means) is locally presentable.
- If $C$ is a $k$-coalgebra, the category of $C$-comodules $\mathrm{CoMod}_C$ is locally presentable (see [Wis75, Corollary 9] noting that a Grothendieck category is locally presentable). In fact, $\mathrm{CoMod}_C$ is the ind-completion of the category of finite-dimensional $C$-comodules (see [Saa72, Corollaire 2.2.2.3]).
- If $\mathcal{C}, \mathcal{D}$ are locally presentable categories, the category $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ of colimit-preserving functors from $\mathcal{C}$ to $\mathcal{D}$ is locally presentable.

It turns out that many examples of locally presentable categories are, in fact, presheaf categories.

**Definition 2.1.1.** Let $\mathcal{C}$ be a locally presentable category. An object $x \in \mathcal{C}$ is ***compact projective*** if $\mathrm{Hom}_{\mathcal{C}}(x, -)\colon \mathcal{C} \to \mathrm{Vect}$ preserves colimits. $\mathcal{C}$ has ***enough compact projectives*** if every object receives a nonzero morphism from a compact projective.

We denote by $\mathcal{C}^{\mathrm{cp}} \subset \mathcal{C}$ the full subcategory of compact projective objects.

**Proposition 2.1.2.** *Suppose $\mathcal{C}$ has enough compact projectives. Then the functor*

$$\mathcal{C} \longrightarrow \mathrm{Fun}((\mathcal{C}^{\mathrm{cp}})^{\mathrm{op}}, \mathrm{Vect})$$

*given by $x \mapsto (y \mapsto \mathrm{Hom}_{\mathcal{C}}(y, x))$ is an equivalence.*

Locally presentable categories naturally form a symmetric monoidal 2-category $\mathrm{Pr}^{\mathrm{L}}$ [Bir84]:

- Its objects are locally presentable categories.
- Its 1-morphisms are colimit-preserving functors.
- Its 2-morphisms are natural transformations.
- The tensor product is uniquely determined by the following property: for $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Pr}^{\mathrm{L}}$ a colimit-preserving functor $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ is the same as a bifunctor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ preserving colimits in each variables.
- The unit is $\mathrm{Vect} \in \mathrm{Pr}^{\mathrm{L}}$.

An important fact about locally presentable categories is that a colimit-preserving functor between locally presentable categories admits a right adjoint. We will now write a formula for the adjoint assuming the source category has enough compact projectives. Let us first recall the notion of a coend (see [Lor19] for more details on coends).

**Definition 2.1.3.** Suppose $\mathcal{C}$ and $\mathcal{D}$ are locally presentable categories. The ***coend*** of a bifunctor $F\colon \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is the coequalizer

$$\int^{x \in \mathcal{C}} F(x, x) = \mathrm{coeq}\left( \textstyle\coprod_{x \to y} F(x, y) \rightrightarrows \coprod_x F(x, x) \right).$$

We will use the following Yoneda-like property of coends (see [Lor19, Proposition 2.2.1]).

**Proposition 2.1.4.** *For any functors $F\colon \mathcal{C} \to \mathcal{D}, G\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ we have natural isomorphisms*

$$\int^{x\in\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(x,y)\otimes F(x) \cong F(y), \qquad \int^{x\in\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(y,x)\otimes G(x) \cong G(y).$$

The following is an immediate corollary.

**Proposition 2.1.5.** *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a colimit-preserving functor of locally presentable categories, where $\mathcal{C}$ has enough compact projectives. Then the right adjoint is given by the coend*

$$F^{\mathrm{R}}(x) = \int^{y\in\mathcal{C}^{\mathrm{cp}}} \mathrm{Hom}_{\mathcal{D}}(F(y),x)\otimes y.$$

*The counit of the adjunction $FF^{\mathrm{R}}(x)\to x$ is given by the evaluation map $\mathrm{Hom}_{\mathcal{D}}(F(y),x)\otimes F(y)\to x$; the unit of the adjunction $z\to F^{\mathrm{R}}F(z)$ is given by including the identity map $\mathrm{id}\colon F(z)\to F(z)$ in the coend.*

**2.1.2. Cp-rigidity.** By convention all monoidal categories $\mathcal{C}$ we consider in this paper are locally presentable such that the tensor product bifunctor $\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ preserves colimits in each variable. So, by the universal property of the tensor product in $\mathrm{Pr}^{\mathrm{L}}$ it descends to a colimit-preserving functor

$$T\colon \mathcal{C}\otimes\mathcal{C}\longrightarrow\mathcal{C}.$$

We denote by $\mathcal{C}^{\otimes\mathrm{op}}$ the same category with the opposite monoidal structure.

We will consider rigid monoidal categories in the text. Since we work with large categories, we cannot expect all objects to be dualizable (as in the category of all vector spaces); instead, we will restrict our attention to compact projective objects.

**Definition 2.1.6.** Let $\mathcal{C}$ be a monoidal category with enough compact projectives. It is ***cp-rigid*** if every compact projective object admits left and right duals.

**Lemma 2.1.7.** *Suppose $\mathcal{C}$ is a cp-rigid monoidal category and $x,y\in\mathcal{C}$ are compact projective objects. Then $x\otimes y$ is also compact projective.*

Proof. We have

$$\mathrm{Hom}_{\mathcal{C}}(x\otimes y,-)\cong\mathrm{Hom}_{\mathcal{C}}(x,(-)\otimes y^{\vee}).$$

By assumption the tensor product preserves colimits in each variable, so $(-)\otimes y^{\vee}$ is colimit-preserving. Since $x$ is compact projective, $\mathrm{Hom}_{\mathcal{C}}(x,-)$ is colimit-preserving. Therefore, $\mathrm{Hom}_{\mathcal{C}}(x\otimes y,-)$ is also colimit-preserving. □

If $\mathcal{C}$ is cp-rigid, the tensor product functor $T\colon\mathcal{C}\otimes\mathcal{C}\to\mathcal{C}$ admits a colimit-preserving right adjoint $T^{\mathrm{R}}\colon\mathcal{C}\to\mathcal{C}\otimes\mathcal{C}$ (see e.g. [BJS18, Section 5.3]). It has the following explicit formula.

**Proposition 2.1.8.** *Suppose $\mathcal{C}$ is a cp-rigid monoidal category. Then*

$$T^{\mathrm{R}}(y)\cong\int^{x\in\mathcal{C}^{\mathrm{cp}}}(y\otimes x^{\vee})\boxtimes x.$$

Proof. By proposition 2.1.5 the right adjoint is

$$T^{\mathrm{R}}(y)\cong\int^{x_1,x_2\in\mathcal{C}^{\mathrm{cp}}}\mathrm{Hom}_{\mathcal{C}}(x_1\otimes x_2,y)\otimes(x_1\boxtimes x_2).$$

Since compact projective objects in $\mathcal{C}$ are dualizable, we can rewrite it as

$$\begin{aligned}T^{\mathrm{R}}(y)&\cong\int^{x_1,x_2\in\mathcal{C}^{\mathrm{cp}}}\mathrm{Hom}_{\mathcal{C}}(x_1,y\otimes x_2^{\vee})\otimes(x_1\boxtimes x_2)\\&\cong\int^{x_2\in\mathcal{C}^{\mathrm{cp}}}(y\otimes x_2^{\vee})\boxtimes x_2,\end{aligned}$$

where in the last isomorphism we have used proposition 2.1.4. □

Consider $\mathcal{C} \otimes \mathcal{C}$ as a $\mathcal{C} \otimes \mathcal{C}^{\otimes\mathrm{op}}$-module category via the left action on the first factor and the right action on the second factor. By [BJS18, Proposition 4.1] $T^{\mathrm{R}}$ is a functor of $\mathcal{C} \otimes \mathcal{C}^{\otimes\mathrm{op}}$-module categories. This can be expressed in the following isomorphism.

**Proposition 2.1.9.** *Suppose $\mathcal{C}$ is as before. Then $T^{\mathrm{R}} \colon \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ is a functor of $\mathcal{C} \otimes \mathcal{C}^{\otimes\mathrm{op}}$-module categories. Concretely, for any object $y \in \mathcal{C}$ there is a natural isomorphism*

$$\int^{x \in \mathcal{C}^{\mathrm{cp}}} (y \otimes x^{\vee}) \boxtimes x \cong \int^{x \in \mathcal{C}^{\mathrm{cp}}} x^{\vee} \boxtimes (x \otimes y)$$

*which is given for a compact projective $y \in \mathcal{C}$ by*

$$x^{\vee} \boxtimes (x \otimes y) \xrightarrow{\mathrm{coev}_y \otimes \mathrm{id}} (y \otimes y^{\vee} \otimes x^{\vee}) \boxtimes (x \otimes y) \xrightarrow{\pi_{x \otimes y}} \int^{x \in \mathcal{C}^{\mathrm{cp}}} (y \otimes x^{\vee}) \boxtimes x.$$

**Corollary 2.1.10.** *The object $T^{\mathrm{R}}(\mathbf{1}) \in \mathcal{C} \otimes \mathcal{C}^{\otimes\mathrm{op}}$ has a natural algebra structure.*

Proof. $T^{\mathrm{R}}T$ is naturally a monad on $\mathcal{C} \otimes \mathcal{C}^{\otimes\mathrm{op}}$. By definition $T \colon \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ is a functor of $\mathcal{C} \otimes \mathcal{C}^{\otimes\mathrm{op}}$-module categories. By proposition 2.1.9 $T^{\mathrm{R}} \colon \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ is also a functor of $\mathcal{C} \otimes \mathcal{C}^{\otimes\mathrm{op}}$-module categories. Therefore, $(T^{\mathrm{R}}T)(\mathbf{1}_{\mathcal{C} \otimes \mathcal{C}})$ has a natural algebra structure. ☐

The key property of cp-rigid monoidal categories is that they are canonically self-dual objects of $\mathrm{Pr}^{\mathrm{L}}$.

Theorem 2.1.11. *Let $\mathcal{C}$ be a cp-rigid monoidal category with a compact projective unit. The evaluation and coevaluation pairings*

$$\mathrm{ev} \colon \mathcal{C} \otimes \mathcal{C} \xrightarrow{T} \mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},-)} \mathrm{Vect} \tag{2.1.1}$$

$$\mathrm{coev} \colon \mathrm{Vect} \xrightarrow{(-)\otimes \mathbf{1}} \mathcal{C} \xrightarrow{T^{\mathrm{R}}} \mathcal{C} \otimes \mathcal{C}. \tag{2.1.2}$$

*establish self-duality of $\mathcal{C}$ as an object of the symmetric monoidal bicategory $\mathrm{Pr}^{\mathrm{L}}$.*

Proof. See [Hoy+18, Proposition 2.16] for an analogous statement on the level of $\infty$-categories. ☐

**Remark 2.1.12.** The conclusion of the theorem remains true if we drop the assumption that the unit of $\mathcal{C}$ is compact and projective and replace $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, -) \colon \mathcal{C} \to \mathrm{Vect}$ by the colimit-preserving functor which coincides with $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, -)$ on compact projective objects.

**Corollary 2.1.13.** *Let $\mathcal{C}$ be a cp-rigid monoidal category with a compact projective unit and $\mathcal{D}$ any monoidal category. Then the functor*

$$\mathcal{D} \otimes \mathcal{C} \longrightarrow \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}) \tag{2.1.3}$$

*given by*

$$d \boxtimes c \mapsto (c' \mapsto \mathrm{ev}(c, c') \otimes d)$$

*is an equivalence.*

**2.1.3. Duoidal categories.** Let us now study monoidal properties of the equivalence (2.1.3). The functor category $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ has a natural monoidal structure given by the Day convolution [Day70] defined by

$$(F \otimes_{\mathrm{Day}} G)(x) = \int^{x_1, x_2 \in \mathcal{C}^{\mathrm{cp}}} \mathrm{Hom}_{\mathcal{C}}(x_1 \otimes x_2, x) \otimes F(x_1) \otimes G(x_2) \tag{2.1.4}$$

with the unit functor

$$x \mapsto \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, x) \otimes \mathbf{1}_{\mathcal{D}}.$$

**Proposition 2.1.14.** *The equivalence (2.1.3) upgrades to a monoidal equivalence*

$$\mathcal{D} \otimes \mathcal{C}^{\otimes\mathrm{op}} \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}),$$

*where we equip $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ with the Day convolution monoidal structure.*

Proof. Clearly, the units are compatible since $\mathbf{1}_{\mathcal{D}} \boxtimes \mathbf{1}_{\mathcal{C}}$ is sent to the functor $(x \mapsto \mathrm{ev}(\mathbf{1}_{\mathcal{C}}, x) \otimes \mathbf{1}_{\mathcal{D}})$. Now consider two objects $d_1 \boxtimes c_1, d_2 \boxtimes c_2 \in \mathcal{D} \otimes \mathcal{C}$. Their Day convolution is computed by

$$\begin{aligned}&((d_1 \boxtimes c_1) \otimes_{\mathrm{Day}} (d_2 \boxtimes c_2))(x) \\ =& \int^{x_1, x_2 \in \mathcal{C}^{\mathrm{cp}}} \mathrm{Hom}_{\mathcal{C}}(x_1 \otimes x_2, x) \otimes (d_1 \otimes d_2) \otimes \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, c_1 \otimes x_1) \otimes \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, c_2 \otimes x_2).\end{aligned}$$

So, we have to exhibit a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, c_2 \otimes c_1 \otimes x) \cong \int^{x_1, x_2 \in \mathcal{C}^{\mathrm{cp}}} \mathrm{Hom}_{\mathcal{C}}(x_1 \otimes x_2, x) \otimes \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, c_1 \otimes x_1) \otimes \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, c_2 \otimes x_2).$$

By assumption $\mathcal{C}$ is generated by compact projectives, so it is enough to define this isomorphism on those.

The right-hand side is

$$\begin{aligned}&\int^{x_1, x_2} \mathrm{Hom}(x_1 \otimes x_2, x) \otimes \mathrm{Hom}(\mathbf{1}, c_1 \otimes x_1) \otimes \mathrm{Hom}(\mathbf{1}, c_2 \otimes x_2) \\ \cong& \int^{x_1, x_2} \mathrm{Hom}(x_1 \otimes x_2, x) \otimes \mathrm{Hom}(c_1^\vee, x_1) \otimes \mathrm{Hom}(c_2^\vee, x_2) \\ \cong& \mathrm{Hom}(c_1^\vee \otimes c_2^\vee, x) \\ \cong& \mathrm{Hom}(\mathbf{1}, c_2 \otimes c_1 \otimes x),\end{aligned}$$

where we have used proposition 2.1.4 in the third line. □

We will now examine monoidal properties of the self-duality pairings (2.1.1) and (2.1.2).

**Proposition 2.1.15.** *The functors*

$$\mathrm{ev}\colon \mathcal{C}^{\otimes\mathrm{op}} \otimes \mathcal{C} \longrightarrow \mathrm{Vect}, \qquad \mathrm{coev}\colon \mathrm{Vect} \to \mathcal{C} \otimes \mathcal{C}^{\otimes\mathrm{op}}$$

*have a natural lax monoidal structure.*

Proof. We begin with the evaluation functor. The unit map $k \to \mathrm{ev}(\mathbf{1}, \mathbf{1}) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$ is given by the inclusion of the identity. Suppose $c_1 \boxtimes c_2, d_1 \boxtimes d_2 \in \mathcal{C}^{\otimes\mathrm{op}} \otimes \mathcal{C}$ are two compact projective objects. Then we define $\mathrm{ev}(c_1, c_2) \otimes \mathrm{ev}(d_1, d_2) \to \mathrm{ev}(d_1 \otimes c_1, c_2 \otimes d_2)$ via the commutative diagram

$$\begin{tikzcd}
\mathrm{ev}(c_1, c_2) \otimes \mathrm{ev}(d_1, d_2) \arrow[r] \arrow[d, equal] & \mathrm{ev}(d_1 \otimes c_1, c_2 \otimes d_2) \arrow[d, equal] \\
\mathrm{Hom}(\mathbf{1}, c_1 \otimes c_2) \otimes \mathrm{Hom}(\mathbf{1}, d_1 \otimes d_2) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}(\mathbf{1}, d_1 \otimes c_1 \otimes c_2 \otimes d_2) \arrow[d, "\sim"] \\
\mathrm{Hom}(c_1^\vee, c_2) \otimes \mathrm{Hom}(d_1^\vee, d_2) \arrow[r] & \mathrm{Hom}(c_1^\vee \otimes d_1^\vee, c_2 \otimes d_2)
\end{tikzcd}$$

Next we consider the coevaluation functor. A lax monoidal structure on coev is the same as an algebra structure on $\mathrm{coev}(k) = T^{\mathrm{R}}(\mathbf{1})$, which, in turn, is provided by corollary 2.1.10. □

Now suppose $\mathcal{C}, \mathcal{D}$ are cp-rigid monoidal categories with compact projective units and $\mathcal{E}$ any monoidal category. Then the composition functor

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{D}, \mathcal{E}) \otimes \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{E})$$

has a natural lax monoidal structure with respect to the Day convolution.

**Proposition 2.1.16.** *Suppose $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are as above. The diagrams*

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{L}}(\mathcal{D}, \mathcal{E}) \otimes \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}) \arrow[r] & \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{E}) \\
\mathcal{E} \otimes \mathcal{D}^{\otimes\mathrm{op}} \otimes \mathcal{D} \otimes \mathcal{C}^{\otimes\mathrm{op}} \arrow[u, "\sim"] \arrow[r, "\mathrm{id}\otimes\mathrm{ev}\otimes\mathrm{id}"] & \mathcal{E} \otimes \mathcal{C}^{\otimes\mathrm{op}} \arrow[u, "\sim"]
\end{tikzcd}$$

*and*

$$\begin{tikzcd} \mathrm{Vect} \arrow[r, "\mathrm{id}"] \arrow[d, equal] & \mathrm{Fun}^{\mathrm{L}}(\mathcal{C},\mathcal{C}) \\ \mathrm{Vect} \arrow[r, "\mathrm{coev}"] & \mathcal{C}\otimes\mathcal{C}^{\otimes\mathrm{op}} \arrow[u, "\sim"'] \end{tikzcd}$$

*of lax monoidal functors with respect to the Day convolution commute up to a monoidal natural isomorphism.*

Recall the following notion (see [AM10, Definition 6.1] where it is called a 2-monoidal category).

**Definition 2.1.17.** A ***duoidal category*** is a category $\mathcal{C}$ equipped with two monoidal structures $(\mathcal{C}, \circ, I)$ and $(\mathcal{C}, \otimes, J)$, such that the functors $\circ\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and $I\colon \mathrm{Vect}\to\mathcal{C}$ are lax monoidal with respect to $(\otimes, J)$.

**Example 2.1.18.** Consider the category $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C},\mathcal{C})$. It carries a monoidal structure $\circ$ given by the composition of functors whose unit $I$ is the identity functor. It also carries the Day convolution monoidal structure $\otimes_{Day}$. It is shown in [GL16, Proposition 50] that the two are compatible so that $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C},\mathcal{C})$ is a duoidal category.

**Example 2.1.19.** Consider the category $\mathcal{C}\otimes\mathcal{C}$. It carries a convolution monoidal structure $\circ$ defined by

$$(M_1\boxtimes M_2)\circ(N_1\boxtimes N_2) = \mathrm{ev}(M_2, N_1)\otimes M_1\boxtimes N_2$$

whose unit is $I = \mathrm{coev}(k)\in\mathcal{C}\otimes\mathcal{C}$. It also carries a pointwise monoidal structure $\mathcal{C}\otimes\mathcal{C}^{\otimes\mathrm{op}}$ whose unit is $J = \mathbf{1}_{\mathcal{C}}\boxtimes\mathbf{1}_{\mathcal{C}}$. It follows from proposition 2.1.15 that the two monoidal structures are compatible, so that $\mathcal{C}\otimes\mathcal{C}$ becomes a duoidal category. Moreover, $\mathcal{C}$ is naturally a module category over $\mathcal{C}\otimes\mathcal{C}$ with respect to convolution:

$$(\mathcal{C}\otimes\mathcal{C})\otimes\mathcal{C}\longrightarrow\mathcal{C}$$

is given by

$$(c_1\boxtimes c_2)\boxtimes d\mapsto \mathrm{ev}(c_2, d)\otimes c_1.$$

We are now ready to relate the two duoidal structures. The following statement combines propositions 2.1.14 and 2.1.16.

Theorem 2.1.20. *Suppose $\mathcal{C}$ is a cp-rigid monoidal category with a compact projective unit. The equivalence* (2.1.3)

$$\mathcal{C}\otimes\mathcal{C}\longrightarrow\mathrm{Fun}^{\mathrm{L}}(\mathcal{C},\mathcal{C})$$

*given by $c_1\boxtimes c_2\mapsto (d\mapsto \mathrm{ev}(c_2,d)\otimes c_1)$ upgrades to an equivalence of duoidal categories, where the two monoidal structures are the convolution product and the pointwise monoidal structure on $\mathcal{C}\otimes\mathcal{C}^{\otimes\mathrm{op}}$ while the two monoidal structures on $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C},\mathcal{C})$ are the composition of functors and Day convolution. This equivalence intertwines $\mathcal{C}$ as a $\mathcal{C}\otimes\mathcal{C}$-module category with respect to convolution and $\mathcal{C}$ as a $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C},\mathcal{C})$-module category with respect to composition of functors.*

**2.1.4. Bimodules and lax monoidal functors.** Suppose $f\colon A\to B$ is a homomorphism of algebras. Then $B$ becomes an $(A,B)$-bimodule with a distinguished element given by $1\in B$. Conversely, the data of an $(A,B)$-bimodule $M$ with a distinguished element $1_M\in M$, such that the action map $B\to M$ is an isomorphism, is the same as the data of a homomorphism $A\to B$. In this section we will describe a similar construction on the categorical level. Recall from [Eti+15, Chapters 7.1, 7.2] the notion of a module category over a monoidal category.

Suppose $\mathcal{C}$ and $\mathcal{D}$ are monoidal categories and $\mathcal{M}$ a $(\mathcal{C},\mathcal{D})$ bimodule category together with a distinguished object $\mathrm{Dist}\in\mathcal{M}$. The action functors of $\mathcal{C}$ and $\mathcal{D}$ on $\mathrm{Dist}\in\mathcal{M}$ define colimit-preserving functors

$$\mathrm{act}_{\mathcal{C}}\colon\mathcal{C}\longrightarrow\mathcal{M},\qquad \mathrm{act}_{\mathcal{D}}\colon\mathcal{D}\longrightarrow\mathcal{M}$$

which we write as $x\mapsto x\otimes\mathrm{Dist}$ and $y\mapsto\mathrm{Dist}\otimes y$, respectively. By the adjoint functor theorem these admit right adjoints that we denote by $\mathrm{act}^{\mathrm{R}}_{\mathcal{C}}$ and $\mathrm{act}^{\mathrm{R}}_{\mathcal{D}}$. The counit of the adjunction defines a natural morphism

$$\epsilon\colon\mathrm{Dist}\otimes\mathrm{act}^{\mathrm{R}}_{\mathcal{D}}(m)\to m$$

for $m \in \mathcal{M}$. Moreover, $\mathrm{act}_{\mathcal{D}} \colon \mathcal{D} \to \mathcal{M}$ is a functor of right $\mathcal{D}$-module categories, so $\mathrm{act}_{\mathcal{D}}^{\mathrm{R}} \colon \mathcal{M} \to \mathcal{D}$ is a lax $\mathcal{D}$-module functor, i.e. we have a natural morphism

$$\phi \colon \mathrm{act}_{\mathcal{D}}^{\mathrm{R}}(m) \otimes y \longrightarrow \mathrm{act}_{\mathcal{D}}^{\mathrm{R}}(m \otimes y)$$

satisfying an associativity axiom.

Consider the functor

$$F_{\mathcal{C}\mathcal{D}} = \mathrm{act}_{\mathcal{D}}^{\mathrm{R}} \circ \mathrm{act}_{\mathcal{C}} \colon \mathcal{C} \longrightarrow \mathcal{D}.$$

**Proposition 2.1.21.** *The morphisms*

$$\mathbf{1}_{\mathcal{D}} \to \mathrm{act}_{\mathcal{D}}^{\mathrm{R}} \circ \mathrm{act}_{\mathcal{D}}(\mathbf{1}_{\mathcal{D}}) \cong \mathrm{act}_{\mathcal{D}}^{\mathrm{R}} \circ \mathrm{act}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}})$$

*and*

$$\begin{aligned} \mathrm{act}_{\mathcal{D}}^{\mathrm{R}}(x \otimes \mathrm{Dist}) \otimes \mathrm{act}_{\mathcal{D}}^{\mathrm{R}}(y \otimes \mathrm{Dist}) &\to \mathrm{act}_{\mathcal{D}}^{\mathrm{R}}(x \otimes \mathrm{Dist} \otimes \mathrm{act}_{\mathcal{D}}^{\mathrm{R}}(y \otimes \mathrm{Dist})) \\ &\to \mathrm{act}_{\mathcal{D}}^{\mathrm{R}}(x \otimes y \otimes \mathrm{Dist}) \end{aligned}$$

*define the structure of a lax monoidal functor on $F_{\mathcal{C}\mathcal{D}}$.*

Proof. Let us prove the associativity condition. For brevity denote $a^{\mathrm{R}} = \mathrm{act}_{\mathcal{D}}^{\mathrm{R}}$, $D = \mathrm{Dist}$. We have to show that the diagram

$$\begin{tikzcd}
(a^{\mathrm{R}}(x \otimes D) \otimes a^{\mathrm{R}}(y \otimes D)) \otimes a^{\mathrm{R}}(z \otimes D) \arrow[r, "\sim"] \arrow[d, "\phi"] & a^{\mathrm{R}}(x \otimes D) \otimes (a^{\mathrm{R}}(y \otimes D) \otimes a^{\mathrm{R}}(z \otimes D)) \arrow[d, "\phi"] \\
a^{\mathrm{R}}(x \otimes D \otimes a^{\mathrm{R}}(y \otimes D)) \otimes a^{\mathrm{R}}(z \otimes D) \arrow[d, "\epsilon"] & a^{\mathrm{R}}(x \otimes D) \otimes a^{\mathrm{R}}(y \otimes D \otimes a^{\mathrm{R}}(z \otimes D)) \arrow[d, "\epsilon"] \\
a^{\mathrm{R}}((x \otimes y) \otimes D) \otimes a^{\mathrm{R}}(z \otimes D) \arrow[d, "\phi"] & a^{R}(x \otimes D) \otimes a^{\mathrm{R}}((y \otimes z) \otimes D) \arrow[d, "\phi"] \\
a^{\mathrm{R}}((x \otimes y) \otimes D \otimes a^{\mathrm{R}}(z \otimes D)) \arrow[d, "\epsilon"] & a^{\mathrm{R}}(x \otimes D \otimes a^{\mathrm{R}}((y \otimes z) \otimes D)) \arrow[d, "\epsilon"] \\
a^{\mathrm{R}}((x \otimes y) \otimes z \otimes D) \arrow[r, "\sim"] & a^{\mathrm{R}}(x \otimes (y \otimes z) \otimes D)
\end{tikzcd}$$

is commutative. Using naturality and the associativity condition for the lax module structure on $\mathrm{act}^{\mathrm{R}}$, the above diagram is reduced to

$$\begin{tikzcd}
& a^{\mathrm{R}}(x \otimes D) \otimes (a^{\mathrm{R}}(y \otimes D) \otimes a^{\mathrm{R}}(z \otimes D)) \arrow[dl, "\phi"'] \arrow[dr, "\phi"] & \\
a^{\mathrm{R}}(x \otimes D \otimes a^{\mathrm{R}}(y \otimes D) \otimes a^{\mathrm{R}}(z \otimes D)) \arrow[d, "\epsilon"] \arrow[drr, "\phi"] & & a^{\mathrm{R}}(x \otimes D) \otimes a^{\mathrm{R}}(y \otimes D \otimes a^{\mathrm{R}}(z \otimes D)) \arrow[d, "\phi"] \\
a^{\mathrm{R}}((x \otimes y) \otimes D \otimes a^{\mathrm{R}}(z \otimes D)) \arrow[dd, "\epsilon"] & & a^{R}(x \otimes D \otimes a^{\mathrm{R}}(y \otimes D \otimes a^{\mathrm{R}}(z \otimes D))) \arrow[ll, "\epsilon"] \arrow[d, "\epsilon"] \\
& & a^{\mathrm{R}}(x \otimes D \otimes a^{\mathrm{R}}((y \otimes z) \otimes D)) \arrow[d, "\epsilon"] \\
a^{\mathrm{R}}((x \otimes y) \otimes z \otimes D) \arrow[rr, "\sim"] & & a^{\mathrm{R}}(x \otimes (y \otimes z) \otimes D)
\end{tikzcd}$$

The top segment commutes by naturality of $\phi$. The middle segment commutes since $\epsilon$ is a natural transformation of $\mathcal{D}$-module functors. The bottom segment commutes by naturality of $\epsilon$.

Unitality is proven analogously. $\square$

Note that in the above construction we may freely replace $\mathcal{C}$ and $\mathcal{D}$, so we similarly obtain a lax monoidal functor

$$F_{\mathcal{D}\mathcal{C}} \colon \mathcal{D} \longrightarrow \mathcal{C}.$$

**Definition 2.1.22.** Suppose $\mathcal{D}$ is a monoidal category and $\mathcal{M}$ a $\mathcal{D}$-module category with a distinguished object. $\mathcal{M}$ is ***free of rank 1*** if the action functor $\mathrm{act}_{\mathcal{D}} \colon \mathcal{D} \to \mathcal{M}$ is an equivalence.

**Proposition 2.1.23.** *Suppose $\mathcal{M}$ is free of rank 1 over $\mathcal{D}$. Then the lax monoidal functor $F_{\mathcal{C}\mathcal{D}} \colon \mathcal{C} \to \mathcal{D}$ is strongly monoidal and it preserves colimits.*

Proof. Since $\mathrm{act}_{\mathcal{D}}$ is an equivalence, both the counit $\epsilon \colon \mathrm{Dist} \otimes \mathrm{act}^{\mathrm{R}}_{\mathcal{D}}(m) \to m$ and the structure of a lax module functor $\phi \colon \mathrm{act}^{\mathrm{R}}_{\mathcal{D}}(m) \otimes y \to \mathrm{act}^{\mathrm{R}}_{\mathcal{D}}(m \otimes y)$ are isomorphisms. In particular, $F_{\mathcal{C}\mathcal{D}}$ is strongly monoidal.

Moreover, $\mathrm{act}^{\mathrm{R}}_{\mathcal{D}}$ is the inverse to $\mathrm{act}_{\mathcal{D}}$, so it preserves colimits. $\square$

**2.1.5. Tannaka reconstruction for bialgebras.** Recall Tannaka reconstruction results for bialgebras; we refer to [Del90; Saa72] for the commutative case and [Ulb89; Ulb90; Sch92] for the general case.

Let $B \in \mathrm{Vect}$ be a bialgebra. Then $\mathcal{C} = \mathrm{CoMod}_B$, the category of (left) $B$-comodules, is locally presentable. Moreover, it is equipped with a conservative and colimit-preserving monoidal forgetful functor $F \colon \mathcal{C} \to \mathrm{Vect}$ which admits a colimit-preserving right adjoint $F^{\mathrm{R}} \colon \mathrm{Vect} \to \mathcal{C}$ sending $V$ to the cofree $B$-comodule $B \otimes V$ cogenerated by $V$. There is a converse to this statement.

**Proposition 2.1.24.** *Suppose $\mathcal{C}$ is a monoidal category with a colimit-preserving monoidal forgetful functor $F \colon \mathcal{C} \to \mathrm{Vect}$ which admits a colimit-preserving right adjoint $F^{\mathrm{R}} \colon \mathrm{Vect} \to \mathcal{C}$. Then $B = FF^{\mathrm{R}}(k)$ is a bialgebra and $F$ factors as*

$$\mathcal{C} \longrightarrow \mathrm{CoMod}_B.$$

*Moreover, the latter functor is an equivalence if, and only if, $F$ is conservative and preserves equalizers.*

**Remark 2.1.25.** A more familiar statement of Tannaka reconstruction is obtained by passing to compact objects in the above statement. Namely, for a small abelian monoidal category $\mathcal{C}^{\mathrm{c}}$ with a biexact tensor product and a monoidal functor

$$F \colon \mathcal{C}^{\mathrm{c}} \longrightarrow \mathrm{Vec}$$

to the category of finite-dimensional vector spaces there is a canonical bialgebra $B$ (the bialgebra of coendomorphisms of $F$, see [Eti+15, Section 1.10]), such that $F$ factors through

$$\mathcal{C}^{\mathrm{c}} \longrightarrow \mathrm{CoMod}^{\mathrm{fd}}_B$$

through the category of finite-dimensional $B$-comodules. Moreover, the latter functor is an equivalence if, and only if, $F$ is exact and faithful. We refer to [Eti+15, Section 5.4] for more details.

Let us now be more explicit. Consider the setup of proposition 2.1.24, where $\mathcal{C}$ is a monoidal category with enough compact projectives and a compact projective unit. Since $F^{\mathrm{R}}$ preserves colimits, $F$ preserves compact projective objects. In particular, for $y \in \mathcal{C}^{\mathrm{cp}}$ the vector space $F(y)$ is finite-dimensional. So, by proposition 2.1.5 the bialgebra $B$ is

$$B = \int^{y \in \mathcal{C}^{\mathrm{cp}}} F(y)^\vee \otimes F(y). \tag{2.1.5}$$

For $y \in \mathcal{C}^{\mathrm{cp}}$ let us denote by

$$\pi_y \colon F(y)^\vee \otimes F(y) \to B$$

the natural projection. For $y, z \in \mathcal{C}$ denote by

$$J_{y,z} \colon F(y) \otimes F(z) \xrightarrow{\sim} F(y \otimes z)$$

the monoidal structure on $F$ (the unit isomorphism will be implicit). The bialgebra structure on $B$ is given on generators as follows:

- The coproduct is

$$F(y)^\vee \otimes F(y) \xrightarrow{\mathrm{id} \otimes \mathrm{coev}_{F(y)} \otimes \mathrm{id}} F(y)^\vee \otimes F(y) \otimes F(y)^\vee \otimes F(y) \xrightarrow{\pi_y \otimes \pi_y} B \otimes B.$$

- The counit is
$$F(y)^\vee \otimes F(y) \xrightarrow{\mathrm{ev}_{F(y)}} k.$$
- The product is
$$\begin{aligned}(F(y)^\vee \otimes F(y)) \otimes (F(z)^\vee \otimes F(z)) &\cong (F(y) \otimes F(z))^\vee \otimes F(y) \otimes F(z) \\ &\xrightarrow{(J_{y,z}^{-1})^\vee \otimes J_{y,z}} F(y\otimes z)^\vee \otimes F(y\otimes z) \\ &\xrightarrow{\pi_{y\otimes z}} B.\end{aligned}$$
- The unit is
$$k \cong F(\mathbf{1})^\vee \otimes F(\mathbf{1}) \xrightarrow{\pi_{\mathbf{1}}} B.$$

It will also be useful to think about $\pi_y$ as elements
$$T_y \in B \otimes \mathrm{End}(F(y)).$$
The following statement is immediate from the above formulas.

THEOREM 2.1.26. *The bialgebra $B$ is spanned, as a $k$-vector space, by the matrix coefficients of $T_y$ for $y \in \mathcal{C}^{\mathrm{cp}}$, subject to the relation*
$$F(f) \circ T_x = T_y \circ F(f) \tag{2.1.6}$$
*for every $f\colon x \to y$. Moreover:*
- *For $y \in \mathcal{C}^{\mathrm{cp}}$ we have*
$$\Delta(T_y) = T_y \otimes T_y. \tag{2.1.7}$$
- *For $y \in \mathcal{C}^{\mathrm{cp}}$ we have*
$$\epsilon(T_y) = \mathrm{id}_{F(y)} \in \mathrm{End}(F(y)). \tag{2.1.8}$$
- *Suppose $x, y \in \mathcal{C}^{\mathrm{cp}}$ are two objects. Then*
$$J_{x,y}^{-1} T_{x\otimes y} J_{x,y} = (T_x \otimes \mathrm{id}_{F(y)})(\mathrm{id}_{F(x)} \otimes T_y) \tag{2.1.9}$$
*as elements of $B \otimes \mathrm{End}(F(x\otimes y)) \cong B \otimes \mathrm{End}(F(x) \otimes F(y))$.*
- *$T_{\mathbf{1}} \in B \otimes \mathrm{End}(F(\mathbf{1})) \cong B$ is the unit.*

Let us now study what happens when $\mathcal{C}$ is in addition equipped with a braiding.

**Definition 2.1.27.** Suppose $\mathcal{C}$ is a braided monoidal category and $F\colon \mathcal{C} \to \mathrm{Vect}$ a monoidal functor. For $x, y \in \mathcal{C}$ the ***R-matrix*** is
$$R_{x,y}\colon F(x) \otimes F(y) \xrightarrow{J_{x,y}} F(x\otimes y) \xrightarrow{F(\sigma_{x,y})} F(y \otimes x) \xrightarrow{J_{y,x}^{-1}} F(y) \otimes F(x) \xrightarrow{\sigma^{-1}_{F(x),F(y)}} F(x) \otimes F(y).$$

It will be convenient to use the standard matrix notation for $R$-matrices acting on several variables: given $x, y, z \in \mathcal{C}$ we denote
$$R_{12} = R_{x,y} \otimes \mathrm{id}$$
as an element of $\mathrm{End}(F(x) \otimes F(y) \otimes F(z))$ and similarly for $R_{13}$ and $R_{23}$. We let the transposed $R$-matrix $R_{21}$ be
$$F(x) \otimes F(y) \xrightarrow{\sigma^{-1}} F(y) \otimes F(x) \xrightarrow{R_{y,x}} F(y) \otimes F(x) \xrightarrow{\sigma} F(x) \otimes F(y).$$
We also denote
$$T_1 = T_x \otimes \mathrm{id}, \qquad T_2 = \mathrm{id} \otimes T_y$$
as elements of $B \otimes \mathrm{End}(F(x) \otimes F(y))$.

**Proposition 2.1.28.** *Suppose $x, y, z \in \mathcal{C}$. Then the $R$-matrix satisfies the Yang–Baxter equation*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{2.1.10}$$

*in* $\mathrm{End}(F(x) \otimes F(y) \otimes F(z))$. *Moreover, $T$ satisfies the FRT relation*

$$R_{12}T_1T_2 = T_2T_1R_{12} \tag{2.1.11}$$

*in* $B \otimes \mathrm{End}(F(x) \otimes F(y))$.

Proof. Denote

$$\check{R}_{x,y} \colon F(x) \otimes F(y) \xrightarrow{J_{x,y}} F(x \otimes y) \xrightarrow{F(\sigma_{x,y})} F(y \otimes x) \xrightarrow{J_{y,x}^{-1}} F(y) \otimes F(x).$$

Then the Yang–Baxter equation (2.1.10) is equivalent to the braid equation

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}$$

which holds in any braided monoidal category.

By (2.1.6), we have $F(\sigma_{x,y})T_{x\otimes y} = T_{y\otimes x}F(\sigma_{x,y})$. Relation (2.1.9) and the equality

$$F(\sigma_{x,y}) = J_{y,x}\check{R}_{x,y}J_{x,y}^{-1}$$

imply (2.1.11). □

**Remark 2.1.29.** Quantum groups were originally introduced in [FST79; FRT89] as bialgebras as in theorem 2.1.26 satisfying the FRT relation (2.1.11). The above statements show, conversely, that this relation naturally follows from the categorical framework.

**2.1.6. Coend algebras and reflection equation.** Let $\mathcal{C}$ be a cp-rigid monoidal category. Recall the formula for the right adjoint $T^{\mathrm{R}} \colon \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ for the tensor product functor $\mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ from proposition 2.1.8.

**Definition 2.1.30.** The ***canonical coend*** is the object $\mathcal{F} \in \mathcal{C}$ defined by

$$\mathcal{F} = TT^{\mathrm{R}}(\mathbf{1}) = \int^{x \in \mathcal{C}^{\mathrm{cp}}} x^\vee \otimes x. \tag{2.1.12}$$

For $x \in \mathcal{C}^{\mathrm{cp}}$ let us denote by

$$\pi_x \colon x^\vee \otimes x \to \mathcal{F}$$

the natural projection.

Now, suppose in addition that $\mathcal{C}$ is braided monoidal. Then $\mathcal{F}$ admits a structure of a braided Hopf algebra (see e.g. [LM94; Lyu95; Shi20]). Explicitly, the algebra structure is given on generators as follows:

- The product is

$$\begin{aligned}(x^\vee \otimes x) \otimes (y^\vee \otimes y) &\xrightarrow{\sigma_{x^\vee \otimes x, y^\vee}} y^\vee \otimes x^\vee \otimes x \otimes y \\ &\cong (x \otimes y)^\vee \otimes x \otimes y \\ &\xrightarrow{\pi_{x \otimes y}} \mathcal{F}.\end{aligned}$$

- The unit is

$$\mathbf{1} \xrightarrow{\pi_{\mathbf{1}}} \mathcal{F}.$$

Consider a monoidal functor $F \colon \mathcal{C} \to \mathrm{Vect}$. The projections $\pi_x$ give rise to elements

$$K_x \in F(\mathcal{F}) \otimes \mathrm{End}(F(x)).$$

Comparing the formulas (2.1.5) and (2.1.12), we see that there is an isomorphism of vector spaces

$$F(\mathcal{F}) \cong B.$$

In particular, as before, $F(\mathcal{F})$ is spanned, as a $k$-vector space, by the matrix coefficients of $K_x$ for $x \in \mathcal{C}^{\mathrm{cp}}$ subject to the relation (2.1.6) for every $f \colon x \to y$. As before, $K_{\mathbf{1}} \in F(\mathcal{F})$ is the unit. However, the multiplication is different. The following was proved in [Maj95; DKM03].

**Proposition 2.1.31.** *Suppose $x, y \in \mathcal{C}^{\mathrm{cp}}$ are two objects. Then the reflection equation*

$$R_{21}K_1R_{12}K_2 = K_2R_{21}K_1R_{12} \tag{2.1.13}$$

*holds in $F(\mathcal{F}) \otimes \mathrm{End}(F(x) \otimes F(y))$.*

**Remark 2.1.32.** The reflection equation algebra in the theory of quantum groups was introduced in [KS92] as the algebra generated by the matrix elements of $K$ satisfying the reflection equation (2.1.13). We see that it coincides with $F(\mathcal{F})$. So, $\mathcal{F}$ is also sometimes known as the reflection equation algebra.

**Example 2.1.33.** Suppose $H$ is a Hopf algebra and consider $\mathcal{C} = \mathrm{LMod}_H$. Then the coend algebra $\mathcal{F}$ is a Drinfeld twist of the restricted dual Hopf algebra

$$H^\circ = \int^{V \in \mathrm{LMod}_H^{\mathrm{cp}}} V^\vee \otimes V,$$

see [DM03, Definition 4.12].

## 2.2. Harish-Chandra bimodules

In this section we study categories of classical and quantum Harish-Chandra bimodules as well as introduce Harish-Chandra bialgebroids.

**2.2.1. General definition.** We will now present a general categorical definition which encompasses categories of both classical and quantum Harish-Chandra bimodules. We refer to section 2.2.3 for a relationship to the usual Harish-Chandra bimodules. This formalism is closely related to the theory of dynamical extensions of monoidal categories introduced in [DM05], see remark 2.2.2.

Throughout this section we fix a cp-rigid monoidal category $\mathcal{C}$. Recall from [Eti+15, Definition 7.13.1] that the Drinfeld center $\mathrm{Z}_{\mathrm{Dr}}(\mathcal{C})$ is the braided monoidal category consisting of pairs $(z, \tau)$, where $z \in \mathcal{C}$ and

$$\tau_x \colon x \otimes z \xrightarrow{\sim} z \otimes x$$

is a natural isomorphism satisfying standard compatibilities. The monoidal structure is given by

$$(z, \tau) \otimes (z', \tau') = (z \otimes z', \tilde{\tau}),$$

where $\tilde{\tau}$ is the composite

$$x \otimes z \otimes z' \xrightarrow{\tau_x \otimes \mathrm{id}_{z'}} z \otimes x \otimes z' \xrightarrow{\mathrm{id} \otimes \tau'_x} z \otimes z' \otimes x,$$

where we omit associators. We refer to [Eti+15, Proposition 8.5.1] for the braided monoidal structure on $\mathrm{Z}_{\mathrm{Dr}}(\mathcal{C})$.

**Definition 2.2.1.** Let $(\mathcal{L}, \tau)$ be a commutative algebra in $\mathrm{Z}_{\mathrm{Dr}}(\mathcal{C})$. The ***category of Harish-Chandra bimodules*** is

$$\mathrm{HC}(\mathcal{C}, \mathcal{L}) = \mathrm{LMod}_{\mathcal{L}}(\mathcal{C}).$$

When there is no confusion, we simply denote $\mathrm{HC} = \mathrm{HC}(\mathcal{C}, \mathcal{L})$.

**Remark 2.2.2.** A commutative algebra in the Drinfeld center is called a *base algebra* in [DM05, Definition 4.1]. The full subcategory of $\mathrm{HC}(\mathcal{C}, \mathcal{L})$ consisting of free left $\mathcal{L}$-modules is called a *dynamical extension* of $\mathcal{C}$ over $\mathcal{L}$ in [DM05, Section 4.2].

If $H$ is a Hopf algebra, recall that the Drinfeld center $\mathrm{Z}_{\mathrm{Dr}}(\mathrm{LMod}_H)$ is equivalent to the category of Yetter–Drinfeld modules over $H$ (see [Kas95, Section XIII.5]). This gives rise to the following important example.

**Proposition 2.2.3.** *Suppose $H$ is a Hopf algebra and consider $\mathcal{C} = \mathrm{LMod}_H$. A commutative algebra $\mathcal{L}$ in $\mathrm{Z}_{\mathrm{Dr}}(\mathrm{LMod}_H)$ is the same as an $H$-algebra $\mathcal{L}$ equipped with a left $H$-coaction $\delta \colon \mathcal{L} \to H \otimes \mathcal{L}$, a map of $H$-algebras, denoted by $x \mapsto x_{(-1)} \otimes x_{(0)}$ satisfying*

$$xy = y_{(0)}(S^{-1}(y_{(-1)})) \triangleright x, \qquad x, y \in \mathcal{L}.$$

*The corresponding isomorphism $\tau_M \colon M \otimes \mathcal{L} \to \mathcal{L} \otimes M$ is given by*

$$m \otimes x \mapsto x_{(0)} \otimes (S^{-1}(x_{(-1)}) \triangleright m).$$

Proof. The compatibility of $\tau_M$ with the monoidal structure on $\mathrm{LMod}_H$ follows from the coassociativity and counitality of the $H$-coaction. The compatibility of $\tau_M$ with the algebra structure on $\mathcal{L}$ is equivalent to the equation

$$x_{(0)}y_{(0)} \otimes S^{-1}(y_{(-1)})S^{-1}(x_{(-1)}) \triangleright m = (xy)_{(0)} \otimes S^{-1}((xy)_{(-1)}) \triangleright m,$$

which follows from the condition that $\mathcal{L} \to H \otimes \mathcal{L}$ is an algebra map. The commutativity of the multiplication on $\mathcal{L} \in \mathrm{Z}_{\mathrm{Dr}}(\mathrm{LMod}_H)$ is

$$xy = y_{(0)}(S^{-1}(y_{(-1)})) \triangleright x.$$

□

**Remark 2.2.4.** The inverse morphism $\mathcal{L} \otimes M \to M \otimes \mathcal{L}$ is given by

$$x \otimes m \mapsto x_{(-1)} \triangleright m \otimes x_{(0)}.$$

**Example 2.2.5.** Consider a Hopf algebra $H$ and let $\mathcal{L} = H$. Consider the adjoint action of $H$ on $\mathcal{L}$:

$$h \otimes x \mapsto h_{(1)}xS(h_{(2)})$$

for $h \in H$ and $x \in \mathcal{L}$. Consider the $H$-coaction $\mathcal{L} \to H \otimes \mathcal{L}$ given by the coproduct on $H$. Then

$$S^{-1}(y) \triangleright x = S^{-1}(y_{(2)})xy_{(1)}.$$

In particular,

$$\begin{aligned} y_{(2)}(S^{-1}(y_{(1)})) \triangleright x &= y_{(3)}S^{-1}(y_{(2)})xy_{(1)} \\ &= \epsilon(y_{(2)})xy_{(1)} \\ &= xy, \end{aligned}$$

which shows that $(\mathcal{L}, \tau)$ is a commutative algebra in $\mathrm{Z}_{\mathrm{Dr}}(\mathrm{LMod}_H)$.

Since $\mathcal{L}$ is a commutative algebra in $\mathrm{Z}_{\mathrm{Dr}}(\mathcal{C})$, the category HC has a natural monoidal structure given by the relative tensor product: given left $\mathcal{L}$-modules $M, N \in \mathcal{C}$, we may turn $M$ into a right $\mathcal{L}$-module using $\tau_M$ and then the tensor product is given by $M \otimes_{\mathcal{L}} N$. We also have an adjunction

$$\mathrm{free}\colon \mathcal{C} \rightleftarrows \mathrm{HC}\colon \mathrm{forget},$$

where $\mathrm{free}\colon \mathcal{C} \to \mathrm{HC}$ is the monoidal functor $x \mapsto \mathcal{L} \otimes x$ given by the free left $\mathcal{L}$-module and $\mathrm{forget}\colon \mathrm{HC} \to \mathcal{C}$ is given by forgetting the $\mathcal{L}$-module structure.

Observe that $\mathcal{L}^{\mathrm{op}}$ is an algebra in $\mathcal{C}^{\otimes\mathrm{op}}$. Moreover, it lifts to a commutative algebra in $\mathrm{Z}_{\mathrm{Dr}}(\mathcal{C}^{\otimes\mathrm{op}})$ if we consider the inverse isomorphism $\tau_x$.

**Lemma 2.2.6.** *There is a natural monoidal equivalence* $\mathrm{HC}(\mathcal{C}, \mathcal{L})^{\otimes\mathrm{op}} \cong \mathrm{HC}(\mathcal{C}^{\otimes\mathrm{op}}, \mathcal{L}^{\mathrm{op}})$.

The following construction explains why HC deserves to be called the category of bimodules. There is a natural monoidal functor

$$\mathrm{bimod}\colon \mathrm{HC} \longrightarrow {}_{\mathcal{L}}\mathrm{BMod}_{\mathcal{L}}(\mathcal{C}) \tag{2.2.1}$$

given by sending a left $\mathcal{L}$-module $M$ to the $\mathcal{L}$-bimodule $M$, where the right $\mathcal{L}$-action is obtained via $\tau_M$. It realizes HC as a full subcategory of ${}_{\mathcal{L}}\mathrm{BMod}_{\mathcal{L}}(\mathcal{C})$ consisting of objects $M \in {}_{\mathcal{L}}\mathrm{BMod}_{\mathcal{L}}(\mathcal{C})$ such that the right and left actions are related by $\tau_M$.

Let us now analyze categorical properties of HC.

**Proposition 2.2.7.** *The category* HC *is cp-rigid. Moreover, we may take* $\mathrm{free}(V) \in \mathrm{HC}$ *for all* $V \in \mathcal{C}^{\mathrm{cp}}$ *as the generating set of compact projective objects. If the unit of* $\mathcal{C}$ *is compact projective, so is the unit in* HC.

Proof. The functor free: $\mathcal{C} \to \mathrm{HC}$ has a colimit-preserving right adjoint forget: $\mathrm{HC} \to \mathcal{C}$. So, $\mathrm{free}(V) \in \mathrm{HC}$ is compact projective if $V \in \mathcal{C}^{\mathrm{cp}}$.

The category HC is generated by $\mathrm{free}(V)$ for $V \in \mathcal{C}$ since forget is conservative. But since $\mathcal{C}$ has enough compact projectives, we may restrict to $V \in \mathcal{C}^{\mathrm{cp}}$.

Since $\mathcal{C}$ is cp-rigid, the objects $V \in \mathcal{C}^{\mathrm{cp}}$ are dualizable. Since free: $\mathcal{C} \to \mathrm{HC}$ is monoidal, the objects $\mathrm{free}(V) \in \mathrm{HC}$ are also dualizable. But we have just shown that such objects are the generating compact projective objects, while by [BJS18, Proposition 4.1] it is enough to check cp-rigidity on the generating compact projective objects.

The unit of HC is $\mathcal{L}$ viewed as a free left $\mathcal{L}$-module of rank 1, so

$$\mathrm{Hom}_{\mathrm{HC}}(\mathcal{L}, -) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, \mathrm{forget}(-))$$

which shows that $\mathcal{L}$ is compact projective if, and only if, $\mathbf{1}_{\mathcal{C}} \in \mathcal{C}$ is. □

**2.2.2. Quantum moment maps.** Recall that for an algebra $A \in \mathrm{Rep}(G)$ a quantum moment map is a map $\mu\colon U\mathfrak{g} \to A$ such that the infinitesimal $\mathfrak{g}$-action on $A$ is given by $[\mu(x), -]$ for $x \in \mathfrak{g}$. The following version of this definition in our setting was introduced in [Saf19, Definition 3.1].

**Definition 2.2.8.** Let $A \in \mathcal{C}$ be an algebra. A ***quantum moment map*** is an algebra map $\mu\colon \mathcal{L} \to A$ such that the diagram

$$\begin{tikzcd} \mathcal{L} \otimes A \arrow[r, "\mu\otimes\mathrm{id}"] & A \otimes A \arrow[dr, "m"] & \\ & & A \\ A \otimes \mathcal{L} \arrow[uu, "\tau_A"] \arrow[r, "\mathrm{id}\otimes\mu"] & A \otimes A \arrow[ur, "m"'] & \end{tikzcd} \tag{2.2.2}$$

commutes.

**Remark 2.2.9.** Recall that $\mathcal{L} \in \mathrm{Z}_{\mathrm{Dr}}(\mathcal{C})$ is a commutative algebra. The quantum moment map condition expressed by (2.2.2) says that $\mu\colon \mathcal{L} \to A$ is a central map.

**Proposition 2.2.10.** *An algebra in* HC *is an algebra in* $\mathcal{C}$ *equipped with a quantum moment map.*

Proof. Via the embedding bimod: $\mathrm{HC} \to {}_{\mathcal{L}}\mathrm{BMod}_{\mathcal{L}}(\mathcal{C})$ of (2.2.1) an algebra $A \in \mathrm{HC}$ gives rise to an algebra in ${}_{\mathcal{L}}\mathrm{BMod}_{\mathcal{L}}(\mathcal{C})$. An algebra in the category of bimodules is the same as an algebra $A \in \mathcal{C}$ equipped with an algebra map $\mu\colon \mathcal{L} \to A$. The condition that it lands in $\mathrm{HC} \subset {}_{\mathcal{L}}\mathrm{BMod}_{\mathcal{L}}(\mathcal{C})$ is precisely the quantum moment map equation (2.2.2). □

The following is [Saf19, Definition 3.10].

**Definition 2.2.11.** Suppose $\epsilon\colon \mathcal{L} \to \mathbf{1}_{\mathcal{C}}$ is a morphism of algebras in $\mathcal{C}$ and $A$ is an algebra equipped with a quantum moment map. The ***Hamiltonian reduction of*** $A$ is

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, A \otimes_{\mathcal{L}} \mathbf{1}) \cong \mathrm{Hom}_{\mathrm{LMod}_A(\mathcal{C})}(A \otimes_{\mathcal{L}} \mathbf{1}_{\mathcal{C}}, A \otimes_{\mathcal{L}} \mathbf{1}_{\mathcal{C}}).$$

A canonical example of an algebra with a quantum moment map we will use is the following. Let $T_{\mathrm{HC}}\colon \mathrm{HC}\otimes\mathrm{HC} \to \mathrm{HC}$ be the tensor product functor. By proposition 2.1.15 the object $T^{\mathrm{R}}(\mathbf{1}_{\mathrm{HC}}) \in \mathrm{HC}\otimes\mathrm{HC}^{\otimes\mathrm{op}}$ is an algebra. Identifying $\mathrm{HC}(\mathcal{C}, \mathcal{L})^{\otimes\mathrm{op}} \cong \mathrm{HC}(\mathcal{C}^{\otimes\mathrm{op}}, \mathcal{L}^{\mathrm{op}})$ using lemma 2.2.6, we see that

$$(\mathrm{forget} \otimes \mathrm{forget})(T^{\mathrm{R}}_{\mathrm{HC}}(\mathbf{1}_{\mathrm{HC}})) \in \mathcal{C} \otimes \mathcal{C}^{\otimes\mathrm{op}}$$

is an algebra equipped with a quantum moment map from $\mathcal{L} \boxtimes \mathcal{L}^{\mathrm{op}}$.

**Definition 2.2.12.** Let $\mathcal{C}, \mathrm{HC}$ be as before. The algebra $\mathcal{D} \in \mathcal{C} \otimes \mathcal{C}^{\otimes\mathrm{op}}$ is

$$\mathcal{D} = (\mathrm{forget} \otimes \mathrm{forget})(T^{\mathrm{R}}_{\mathrm{HC}}(\mathbf{1}_{\mathrm{HC}})).$$

We denote the canonical quantum moment map by

$$\mu\colon \mathcal{L} \boxtimes \mathcal{L}^{\mathrm{op}} \longrightarrow T^{\mathrm{R}}_{\mathcal{C}}(\mathcal{L}). \tag{2.2.3}$$

**Proposition 2.2.13.** *We have an equivalence*

$$\mathcal{D} \cong \int^{x\in\mathcal{C}^{\mathrm{cp}}} (\mathcal{L}\otimes x^\vee)\boxtimes x \cong \int^{x\in\mathcal{C}^{\mathrm{cp}}} x^\vee \boxtimes (x\otimes\mathcal{L}),$$

*where the latter isomorphism is provided by proposition 2.1.9. The algebra structure is given by*

$$\begin{aligned}((\mathcal{L}\otimes x^\vee)\boxtimes x)\otimes((\mathcal{L}\otimes y^\vee)\boxtimes y) &\cong (\mathcal{L}\otimes x^\vee\otimes\mathcal{L}\otimes y^\vee)\boxtimes(y\otimes x)\\ &\xrightarrow{\mathrm{id}\otimes\tau_{x^\vee}\otimes\mathrm{id}} (\mathcal{L}\otimes\mathcal{L}\otimes x^\vee\otimes y^\vee)\boxtimes(y\otimes x)\\ &\xrightarrow{m\otimes\mathrm{id}} (\mathcal{L}\otimes(y\otimes x)^\vee)\boxtimes(y\otimes x)\\ &\xrightarrow{\pi_{y\otimes x}} \int^{x\in\mathcal{C}^{\mathrm{cp}}}(\mathcal{L}\otimes x^\vee)\boxtimes x\end{aligned}$$

*The two quantum moment maps* $\mathcal{L}, \mathcal{L}^{\mathrm{op}} \to \mathcal{D}$ *are given by*

$$\mathcal{L} \cong (\mathcal{L}\otimes\mathbf{1})\boxtimes\mathbf{1} \xrightarrow{\pi_{\mathbf{1}}} \mathcal{D}$$

*and*

$$\mathcal{L}^{\mathrm{op}} \cong \mathbf{1}\boxtimes(\mathbf{1}\otimes\mathcal{L}) \xrightarrow{\pi_{\mathbf{1}}} \mathcal{D}.$$

Proof. Since free: $\mathcal{C}\to\mathrm{HC}$ is a monoidal functor, by adjunction we get a natural isomorphism

$$(\mathrm{forget}\otimes\mathrm{forget})\circ T^{\mathrm{R}}_{\mathrm{HC}} \cong T^{\mathrm{R}}_{\mathcal{C}}\circ\mathrm{forget}, \tag{2.2.4}$$

where $T_{\mathcal{C}}\colon \mathcal{C}\otimes\mathcal{C}\to\mathcal{C}$ is the tensor product functor. In particular, applying (2.2.4) to $\mathbf{1}_{\mathrm{HC}}$, we get isomorphisms

$$\mathcal{D} \cong \int^{x\in\mathcal{C}^{\mathrm{cp}}} (\mathcal{L}\otimes x^\vee)\boxtimes x \cong \int^{x\in\mathcal{C}^{\mathrm{cp}}} x^\vee \boxtimes (x\otimes\mathcal{L})$$

using proposition 2.1.8.

We have a natural isomorphism

$$(\mathrm{forget}\circ\mathrm{free}\otimes\mathrm{id})\circ T^{\mathrm{R}}_{\mathcal{C}}(-) \cong T^{\mathrm{R}}_{\mathcal{C}}(\mathrm{forget}\circ\mathrm{free}(-))$$

given by proposition 2.1.9 which gives rise to an algebra isomorphism

$$\mathcal{D} \cong (\mathrm{forget}\circ\mathrm{free}\otimes\mathrm{id})\circ T^{\mathrm{R}}_{\mathcal{C}}(\mathbf{1})$$

which gives the required formula. □

**Example 2.2.14.** Suppose $H$ is a Hopf algebra, $\mathcal{L}$ is a commutative algebra in $\mathrm{Z}_{\mathrm{Dr}}(\mathrm{LMod}_H)$ (see proposition 2.2.3) and $\mathcal{C} = \mathrm{LMod}_H$. Let

$$H^\circ = \int^{V\in\mathrm{LMod}_H^{\mathrm{cp}}} V^\vee\otimes V$$

is the restricted dual Hopf algebra. By construction $\mathcal{L}$ is an $H$-comodule algebra and $H^\circ$ is an $H$-module algebra (via the left $H$-action). Then $\mathcal{D}$ is the smash product algebra generated by $\mathcal{L}$ and $H^\circ$ with the additional relation

$$hl = l_{(0)}(S^{-1}(l_{(-1)})\triangleright h)$$

for $h\in H^\circ$ and $l\in\mathcal{L}$.

**2.2.3. Classical Harish-Chandra bimodules.** Let $G$ be a reductive group over a characteristic zero field $k$ and denote by $\mathfrak{g}$ its Lie algebra. Let $\mathrm{Rep}(G)$ be the ind-completion of the category of finite-dimensional representations. The category $\mathrm{Rep}(G)$ is semisimple, so it has enough compact projectives and its unit is compact projective.

Suppose $V \in \mathrm{Rep}(G)$ is a $G$-representation. For $x \in \mathrm{U}\mathfrak{g}$ and $v \in V$ we denote by $x \triangleright v$ the induced $\mathrm{U}\mathfrak{g}$-action on $V$. Consider the natural isomorphism

$$\tau_V : V \otimes \mathrm{U}\mathfrak{g} \longrightarrow \mathrm{U}\mathfrak{g} \otimes V \tag{2.2.5}$$

given by

$$v \otimes x \mapsto x \otimes v - 1 \otimes xv$$

for $x \in \mathfrak{g}$. It follows from proposition 2.2.3 that $(\mathrm{U}\mathfrak{g}, \tau)$ defines a commutative algebra in $\mathrm{Z}_{\mathrm{Dr}}(\mathrm{Rep}(G))$.

**Definition 2.2.15.** The ***category of classical Harish-Chandra bimodules*** is

$$\mathrm{HC}(G) = \mathrm{HC}(\mathrm{Rep}(G), \mathrm{U}\mathfrak{g}).$$

**Remark 2.2.16.** The embedding (2.2.1) realizes $\mathrm{HC}(G)$ as the category of $\mathrm{U}\mathfrak{g}$-bimodules whose diagonal $\mathfrak{g}$-action is integrable (see [BG80, Definition 5.2] for the original definition of Harish-Chandra bimodules).

The following easy lemma (see [Saf19, Example 3.4]) shows that the definition of a quantum moment map we gave in definition 2.2.8 coincides with the classical notion of a quantum moment map.

**Lemma 2.2.17.** *Let $A \in \mathrm{Rep}(G)$ be an algebra. A quantum moment map $\mu\colon \mathrm{U}\mathfrak{g} \to A$ is the same as an algebra map such that for every $x \in \mathfrak{g}$ the commutator $[\mu(x), -]$ coincides with the differential of the $G$-action.*

In the same way, the quantum Hamiltonian reduction from definition 2.2.11 coincides with the usual definition

$$A/\!\!/G = (A/A\mu(\mathfrak{g}))^G$$

of the reduced algebra.

Given a variety $X$ equipped with a $G$-action, the algebra of differential operators $\mathrm{D}(X)$ carries a quantum moment map $\mu\colon \mathrm{U}\mathfrak{g} \to \mathrm{D}(X)$ which sends $\mathfrak{g} \subset \mathrm{U}\mathfrak{g}$ to vector fields on $X$ generating the infinitesimal action. For instance, $\mathrm{D}(G)$ carries a moment map

$$\mu\colon \mathrm{U}\mathfrak{g} \otimes \mathrm{U}\mathfrak{g}^{\mathrm{op}} \to \mathrm{D}(G) \tag{2.2.6}$$

coming from the left and right $G$-action on itself. Let us explain how it arises in our context.

By the Peter–Weyl theorem we have an isomorphism of algebras

$$\mathcal{O}(G) \cong \int^{V \in \mathrm{Rep}^{\mathrm{fd}}(G)} V^\vee \boxtimes V \in \mathrm{Rep}(G) \otimes \mathrm{Rep}(G),$$

where $\mathcal{O}(G)$ carries a $G \times G$-action coming from the left and right $G$-action on itself. Using this we can also describe the algebra $\mathcal{D}$ from definition 2.2.12.

**Proposition 2.2.18.** *The algebra $\mathcal{D} \in \mathrm{HC}(G) \otimes \mathrm{HC}(G)^{\otimes \mathrm{op}}$ is isomorphic to $\mathrm{D}(G) \cong \mathrm{U}\mathfrak{g} \otimes \mathcal{O}(G)$ equipped with the $G \times G$-action and the quantum moment map (2.2.6).*

In the abelian case the category of Harish-Chandra bimodules has a straightforward description. Suppose $H$ is a split torus; let $\mathfrak{h}$ be its Lie algebra and $\Lambda = \mathrm{Hom}(H, \mathbb{G}_{\mathrm{m}})$ the character lattice. Then $\mathrm{Rep}(H)$ is equivalent to the category of $\Lambda$-graded vector spaces and $\mathrm{HC}(H)$ is equivalent to the category of $\Lambda$-graded $\mathrm{Sym}(\mathfrak{h})$-modules $\oplus_{\lambda \in \Lambda} M(\lambda)$.

Given $\lambda \in \Lambda$ we consider the translation functor $\lambda^*\colon \mathrm{LMod}_{\mathrm{Sym}(\mathfrak{h})} \to \mathrm{LMod}_{\mathrm{Sym}(\mathfrak{h})}$. Then the monoidal structure $\otimes^{\mathrm{HC}}$ on $\mathrm{HC}(H)$ is given by

$$M \otimes^{\mathrm{HC}} N = \bigoplus_{\lambda \in \Lambda} \lambda^*(M) \otimes N(\lambda).$$

Suppose $V \in \mathrm{Rep}(H)$. Given a vector $v \in V$ of weight $\mu \in \Lambda$ and $f \in \mathcal{O}(\mathfrak{h}^*) \cong \mathrm{U}\mathfrak{h}$ the map (2.2.5) is given by

$$v \otimes f(\lambda) \mapsto f(\lambda - \mu) \otimes v$$

for $\lambda \in \mathfrak{h}^*$. It is convenient to write it as

$$v \otimes f(\lambda) \mapsto f(\lambda - h) \otimes v,$$

where $h$ is understood as acting on $v \in V$. Similarly, given a collection of representations $V_1, \dots, V_n \in \mathrm{Rep}(H)$ and vectors $v_i \in V_i$ we denote

$$f(\lambda - h^{(i)}) v_1 \otimes \dots v_n = f(\lambda - \mu_i) v_1 \otimes \dots v_n$$

if $v_i$ has weight $\mu_i \in \Lambda$.

**2.2.4. Quantum groups.** In this section we fix our conventions for quantum groups. Fix $k = \mathbf{C}$. Let $G$ be a connected reductive group, $B, B_- \subset G$ a pair of opposite Borel subgroups and $H = B \cap B_-$ a Cartan subgroup. Denote by $\Lambda = \mathrm{Hom}(H, \mathbb{G}_\mathrm{m})$ its weight lattice and $\Lambda^\vee = \mathrm{Hom}(\mathbb{G}_\mathrm{m}, H)$ the coweight lattice; we denote by $\langle -, - \rangle \colon \Lambda^\vee \times \Lambda \to \mathbf{Z}$ the canonical pairing. For two simple roots $\alpha_i, \alpha_j \in \Lambda$ denote by $\alpha_i \cdot \alpha_j \in \mathbf{Z}$ the corresponding entry of the symmetrized Cartan matrix. Choose an integer $d \in \mathbf{Z}$ and a symmetric bilinear form $(-,-) \colon \Lambda \times \Lambda \to \frac{1}{d}\mathbf{Z}$, such that $(\alpha_i, \alpha_j) = \alpha_i \cdot \alpha_j$. Given a complex number $q^{1/d} \in \mathbf{C}^\times$ we have the exponentiated pairing

$$\Pi \colon \Lambda \times \Lambda \longrightarrow \mathbf{C}^\times$$

given by $\lambda, \mu \mapsto q^{-(\lambda, \mu)}$. Our assumption is that $q^{1/d}$ is not a root of unity.

We denote by $\mathrm{U}_q(\mathfrak{g})$ the quantum group defined as in [Lus10] with a slight modification that its Cartan part is $\mathrm{U}_q(\mathfrak{h}) = k[\Lambda]$ with Cartan generators $K_\mu$ for $\mu \in \Lambda$ (note that the Cartan part in [Lus10] is $k[\Lambda^\vee]$). We denote by $\mathrm{U}_q(\mathfrak{b}) \subset \mathrm{U}_q(\mathfrak{g})$ the quantum Borel subalgebra, $\mathrm{U}_q(\mathfrak{n}), \mathrm{U}_q(\mathfrak{n}_-) \subset \mathrm{U}_q(\mathfrak{g})$ the quantum nilpotent subalgebras and $\mathrm{U}_q^{>0}(\mathfrak{n}), \mathrm{U}_q^{<0}(\mathfrak{n}_-)$ their augmentation ideals. For each simple root $\alpha$ we denote by $\{E_\alpha, K_\alpha, F_\alpha\}$ the corresponding generators of the $\mathrm{U}_q(\mathfrak{sl}_2)$-subalgebra (they are denoted by $E_i, \tilde{K}_i, F_i$ in [Lus10, Section 3.1.1]).

We have the corresponding categories obtained from this data:

- $\mathrm{Rep}_q(H)$ is the braided monoidal category of $\Lambda$-graded vector spaces with the braiding given by $\Pi\tau$, where $\tau$ is the map exchanging the tensor factors.
- $\mathrm{Rep}_q(G)$ is the ind-completion of the braided monoidal category of finite-dimensional $\Lambda$-graded vector spaces with a $\mathrm{U}_q(\mathfrak{g})$-module structure, such that for every vector $x_\lambda$ of weight $\lambda \in \Lambda$ we have $K_\mu x_\lambda = q^{(\mu, \lambda)} x_\lambda$. The braiding is given by $\Theta \circ \Pi \circ \tau$, where $\Theta \in \mathrm{U}_q(\mathfrak{n}_-) \widehat{\otimes} \mathrm{U}_q(\mathfrak{n})$ is the so-called quasi $R$-matrix. We refer to [Lus10, Section 32] for more details.
- $\mathrm{Rep}_q(B)$ is the ind-completion of the monoidal category of finite-dimensional $\Lambda$-graded vector spaces with a compatible $\mathrm{U}_q(\mathfrak{b})$-module structure.

**Definition 2.2.19.** A $\mathrm{U}_q(\mathfrak{g})$-module $M$ is ***integrable*** if it lies in the image of the forgetful functor

$$\mathrm{Rep}_q(G) \longrightarrow \mathrm{LMod}_{\mathrm{U}_q(\mathfrak{g})}.$$

Equivalently, an integrable $\mathrm{U}_q(\mathfrak{g})$-module is a locally finite type 1 $\mathrm{U}_q(\mathfrak{g})$-module. We introduce an analogous definition for $\mathrm{U}_q(\mathfrak{b})$-modules.

Denote by $\mathcal{O}_q(G) \in \mathrm{Rep}_q(G)$ the coend algebra from definition 2.1.30.

**Definition 2.2.20.** A $\mathrm{U}_q(\mathfrak{g})$-module $M$ is ***locally finite*** if for every $m \in M$ the vector space $\mathrm{U}_q(\mathfrak{g})m$ is finite-dimensional.

The algebra $\mathrm{U}_q(\mathfrak{g})$ with respect to the adjoint $\mathrm{U}_q(\mathfrak{g})$-action on itself $x, y \mapsto x_{(1)} y S(x_{(2)})$ is not locally finite and we denote by

$$\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}} \subset \mathrm{U}_q(\mathfrak{g})$$

the largest locally finite submodule.

**Example 2.2.21.** Consider $\mathrm{U}_q(\mathfrak{sl}_2)$ with the generators $E, K, F$ and relations

$$KE = q^2 EK, \qquad KF = q^{-2} FK, \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

Then $\mathrm{U}_q(\mathfrak{sl}_2)^{\mathrm{lf}}$ is the subalgebra generated by $EK^{-1}$, $F$ and $K^{-1}$.

It is easy to see that $\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}} \subset \mathrm{U}_q(\mathfrak{g})$ is a subalgebra, but note that it is not a subcoalgebra. Nevertheless, the following is shown in [Jos95, Theorem 7.1.6].

**Proposition 2.2.22.** $\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}} \subset \mathrm{U}_q(\mathfrak{g})$ *is a left coideal, i.e. the coproduct restricts to a map*

$$\Delta\colon \mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}} \longrightarrow \mathrm{U}_q(\mathfrak{g}) \otimes \mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}.$$

**Remark 2.2.23.** There is a close relationship between the algebras $\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}$ and $\mathcal{O}_q(G)$ which can be established using the quantum Killing form [Ros90]. If $G$ is semisimple simply-connected, $\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}} \cong \mathcal{O}_q(G)$, see [Jos95, Proposition 7.1.23] and [VY20, Theorem 2.113].

**2.2.5. Quantum Harish-Chandra bimodules.** For $V \in \mathrm{Rep}_q(G)$, $v \in V$ an $x \in \mathrm{U}_q(\mathfrak{g})$ we denote by $x \triangleright v$ the $\mathrm{U}_q(\mathfrak{g})$-action. For $x \in \mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}$ we denote by $\Delta(x) = x_{(1)} \otimes x_{(2)}$ the coproduct on $\mathrm{U}_q(\mathfrak{g})$, where we note that $x_{(2)} \in \mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}$ by proposition 2.2.22. We define the natural isomorphism

$$\tau_V : V \otimes \mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}} \longrightarrow \mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}} \otimes V \tag{2.2.7}$$

by

$$v \otimes x \mapsto x_{(2)} \otimes S^{-1}(x_{(1)}) \triangleright v.$$

Consider $\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}} \in \mathrm{Rep}_q(G)$ with respect to the adjoint action. It follows from proposition 2.2.3 that $(\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}, \tau)$ is a commutative algebra in $\mathrm{Z}_{\mathrm{Dr}}(\mathrm{Rep}_q(G))$.

**Definition 2.2.24.** The ***category of quantum Harish-Chandra bimodules*** is

$$\mathrm{HC}_q(G) = \mathrm{HC}(\mathrm{Rep}_q(G), \mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}).$$

**Remark 2.2.25.** A similar definition of the category of quantum Harish-Chandra bimodules is given in [VY20, Definition 5.26].

**Remark 2.2.26.** By [Saf19, Theorem 3.10] the notion of quantum moment maps in this setting coincides with the quantum moment maps for quantum group actions introduced in [VV10, Section 1.5].

In this case the algebra $\mathcal{D}$ from definition 2.2.12 is the algebra of quantum differential operators $\mathcal{D}_q(G)$ on $G$ (see [BK06a, Section 4.1] where it is denoted by $\mathcal{D}_q^{fin}$).

As in the case of classical Harish-Chandra bimodules, in the abelian case the category $\mathrm{HC}_q(G)$ has a straightforward description. Let $H$ be a torus and $\Lambda$ its weight lattice. Then $\mathrm{U}_q(\mathfrak{h})^{\mathrm{lf}} = \mathrm{U}_q(\mathfrak{h}) = \mathcal{O}(H)$ and $\mathrm{HC}_q(H)$ is equivalent to the category of $\Lambda$-graded $\mathcal{O}(H)$-modules. There is a homomorphism

$$\Lambda \longrightarrow H$$

whose dual map $\mathcal{O}(H) = k[\Lambda] \to \mathcal{O}(\Lambda)$ on the level of functions is

$$K_\mu \mapsto \left(\lambda \mapsto q^{(\mu,\lambda)}\right)$$

for $\mu, \lambda \in \Lambda$. In particular, $\Lambda$ acts by translations on $H$ an we denote the induced functor by

$$(q^\lambda)^* : \mathrm{LMod}_{\mathcal{O}(H)} \to \mathrm{LMod}_{\mathcal{O}(H)}.$$

The monoidal structure $\otimes^{\mathrm{HC}}$ on $\mathrm{HC}_q(H)$ is given by

$$M \otimes^{\mathrm{HC}} N = \bigoplus_{\lambda \in \Lambda} (q^\lambda)^*(M) \otimes N(\lambda).$$

Suppose $V \in \mathrm{Rep}(H)$, $v \in V$ and $f \in \mathcal{O}(H) \cong \mathrm{U}_q(\mathfrak{h})$. Then the map (2.2.7) is

$$v \otimes f(\lambda) \mapsto f(\lambda q^{-h}) \otimes v$$

for $\lambda \in H$.

**2.2.6. Harish-Chandra bimodules and bialgebroids.** Let us again consider the general setup of section 2.2.1, where $\mathcal{C}$ is a cp-rigid monoidal category with a compact projective unit. In particular, HC is also a cp-rigid monoidal category with a compact projective unit. Our goal in this section is to describe a Tannaka reconstruction result for monoidal forgetful functors to HC.

Recall from example 2.1.19 that the category $\mathrm{HC}\otimes\mathrm{HC}$ carries two monoidal structures: the pointwise monoidal structure on $\mathrm{HC}\otimes\mathrm{HC}^{\otimes\mathrm{op}}$ and the convolution product. We will call the latter the Takeuchi product in this setting.

**Definition 2.2.27.** The ***Takeuchi product*** $\times_{\mathcal{L}}$ is the monoidal structure on $\mathrm{HC}\otimes\mathrm{HC}$ given by

$$(M_1\boxtimes M_2)\times_{\mathcal{L}}(N_1\boxtimes N_2)=\mathrm{ev}(M_2,N_1)\otimes(M_1\boxtimes N_2)$$

with the unit $\mathrm{coev}(k)=\mathcal{D}\in\mathrm{HC}\otimes\mathrm{HC}$.

**Example 2.2.28.** Consider the setup of definition 2.2.15. An object of $\mathrm{HC}(G)\otimes\mathrm{HC}(G)\cong\mathrm{HC}(G\times G)$ is a $\mathrm{U}\mathfrak{g}\otimes(\mathrm{U}\mathfrak{g})^{\mathrm{op}}$-bimodule with a certain integrability condition. For a $(\mathrm{U}\mathfrak{g})^{\mathrm{op}}$-bimodule $M$ and a $\mathrm{U}\mathfrak{g}$-bimodule $N$ the Takeuchi product is the subspace

$$M\times_{\mathrm{U}\mathfrak{g}}N\subset M\otimes_{\mathrm{U}\mathfrak{g}}N$$

of elements $\sum_i m_i\otimes n_i$ satisfying

$$\sum_i m_i x\otimes n_i=m_i\otimes n_i x$$

for every $x\in\mathrm{U}\mathfrak{g}$, see [Tak77].

We will now formulate the notion of bialgebroids in the category of Harish-Chandra bimodules. Recall that the algebra $\mathcal{D}\cong T^{\mathrm{R}}(\mathcal{L})\in\mathcal{C}\otimes\mathcal{C}^{\mathrm{op}}$ carries a natural quantum moment map (2.2.3).

**Definition 2.2.29.** A ***Harish-Chandra bialgebroid*** is an algebra $B\in\mathcal{C}\otimes\mathcal{C}^{\otimes\mathrm{op}}$ equipped with a quantum moment map $s\otimes t\colon\mathcal{L}\boxtimes\mathcal{L}^{\mathrm{op}}\to B$, which allows us to regard $B$ as an algebra in $\mathrm{HC}\otimes\mathrm{HC}^{\otimes\mathrm{op}}$, together with a coassociative coproduct $\Delta\colon B\to B\times_{\mathcal{L}}B$, a map of algebras in $\mathrm{HC}\otimes\mathrm{HC}^{\otimes\mathrm{op}}$, and a counit map $\varepsilon\colon B\to\mathcal{D}$, a map of algebras in $\mathcal{C}\otimes\mathcal{C}^{\otimes\mathrm{op}}$ compatible with quantum moment maps.

**Example 2.2.30.** Let $H$ be a split torus, $\Lambda=\mathrm{Hom}(H,\mathbb{G}_{\mathrm{m}})$ its weight lattice and consider the category of Harish-Chandra bimodules $\mathrm{HC}(H)$. A bialgebroid in $\mathrm{HC}(H)$ is given by the following data:

- An algebra with a bigrading
$$B=\bigoplus_{\alpha,\beta\in\Lambda}B_{\alpha\beta}.$$
- Algebra maps
$$s,t\colon\mathcal{O}(\mathfrak{h}^*)\longrightarrow B$$
which satisfy the quantum moment map equations
$$s(f(\lambda))a=as(f(\lambda+\alpha)),\qquad t(f(\lambda))a=at(f(\lambda+\beta))$$
for $f\in\mathcal{O}(\mathfrak{h}^*)$ and $a\in B_{\alpha\beta}$.
- The coproduct $\Delta\colon B\to B\times_{\mathrm{U}\mathfrak{h}}B$, a map of algebras compatible with the grading and quantum moment maps. Here the Takeuchi product is
$$(B\times_{\mathrm{U}\mathfrak{h}}B)_{\alpha\beta}=\bigoplus_{\delta\in\Lambda}B_{\alpha\delta}\otimes_{\mathcal{O}(\mathfrak{h}^*)}B_{\delta\beta},$$
where the relative tensor product is the quotient of the $k$-linear tensor product modulo the relations $t(f)a\otimes b\sim a\otimes s(f)b$ for $a\otimes b\in B_{\alpha\delta}\otimes B_{\delta\beta}$ and $f\in\mathcal{O}(\mathfrak{h}^*)$.
- The counit $\epsilon\colon B\to\mathrm{D}(H)$, a map of algebras compatible with the grading and quantum moment maps.

**Remark 2.2.31.** Essentially, this data is an $\mathfrak{h}$-bialgebroid in the sense of [EV98c, Section 4.1]. The differences are:

- For an $\mathfrak{h}$-bialgebroid, the weights $\alpha,\beta$ are not necessarily integral;

- The counit of an $\mathfrak{h}$-bialgebroid takes values in the algebra of *difference operators* on $\mathfrak{h}^*$. However, it contains the subalgebra of difference operators with integral shifts (i.e. in $\Lambda \subset \mathfrak{h}^*$) which is equivalent to $\mathrm{D}(H)$ via the so-called *Mellin transform*, see, for instance, [BN18, Section 2.1].

So, a Harish-Chandra bialgebroid in $\mathrm{HC}(H)$ is an $\mathfrak{h}$-bialgebroid with certain integrability assumptions.

Theorem 2.2.32. *Suppose $B$ is a Harish-Chandra bialgebroid. The functor $\perp\colon \mathrm{HC} \to \mathrm{HC}$ given by*

$$\perp(M) = B \times_{\mathcal{L}} M$$

*defines a lax monoidal comonad. Conversely, let $\perp\colon \mathrm{HC} \to \mathrm{HC}$ be a colimit-preserving lax monoidal comonad on $\mathrm{HC}$. Then $\perp(-) \cong B \times_{\mathcal{L}} (-)$ for some Harish-Chandra bialgebroid $B$.*

Proof. Recall from [AM10, Definition 6.25] that a bimonoid in a duoidal category is an algebra with respect to one monoidal structure and a coalgebra with respect to the other monoidal structure, both compatible in a natural way. A coalgebra in $(\mathrm{Fun}^{\mathrm{L}}(\mathrm{HC}, \mathrm{HC}), \circ)$ is a colimit-preserving comonad on $\mathrm{HC}$ and a bimonoid in $\mathrm{Fun}^{\mathrm{L}}(\mathrm{HC}, \mathrm{HC})$ is the same as lax monoidal comonad on $\mathrm{HC}$.

A colimit-preserving lax monoidal comonad on $\mathrm{HC}$ is the same as a bimonoid in the duoidal category $\mathrm{Fun}^{\mathrm{L}}(\mathrm{HC}, \mathrm{HC})$. By theorem 2.1.20 we have an equivalence of duoidal categories $\mathrm{HC} \otimes \mathrm{HC} \cong \mathrm{Fun}^{\mathrm{L}}(\mathrm{HC}, \mathrm{HC})$. So, $\perp$ corresponds to an object $B \in \mathrm{HC} \otimes \mathrm{HC}$ which is both an algebra in $\mathrm{HC} \otimes \mathrm{HC}^{\otimes\mathrm{op}}$ as well as a coalgebra in $(\mathrm{HC} \otimes \mathrm{HC}, \times_{\mathcal{L}})$, both in a compatible way.

By lemma 2.2.6 we have an equivalence of monoidal categories

$$\mathrm{HC}(\mathcal{C}, \mathcal{L}) \otimes \mathrm{HC}(\mathcal{C}, \mathcal{L})^{\otimes\mathrm{op}} \cong \mathrm{HC}(\mathcal{C}, \mathcal{L}) \otimes \mathrm{HC}(\mathcal{C}^{\otimes\mathrm{op}}, \mathcal{L}^{\mathrm{op}}),$$

so by proposition 2.2.10 the data of an algebra $B \in \mathrm{HC} \otimes \mathrm{HC}^{\otimes\mathrm{op}}$ boils down to an algebra $B \in \mathcal{C} \otimes \mathcal{C}^{\otimes\mathrm{op}}$ equipped with a quantum moment map $\mathcal{L} \boxtimes \mathcal{L}^{\mathrm{op}} \to B$.

The data of a comonad boils down to a coalgebra $(B, \Delta, \epsilon)$ in $(\mathrm{HC} \otimes \mathrm{HC}, \times_{\mathcal{L}})$. The counit is given by a map of algebras $\epsilon\colon B \to \mathrm{coev}(k)$ in $\mathrm{HC} \otimes \mathrm{HC}^{\otimes\mathrm{op}}$. Identifying algebras in $\mathrm{HC} \otimes \mathrm{HC}^{\otimes\mathrm{op}}$ with algebras in $\mathcal{C} \otimes \mathcal{C}^{\otimes\mathrm{op}}$ equipped with quantum moment maps by proposition 2.2.10, the counit is the same as a map $\varepsilon\colon B \to T^{\mathrm{R}}(\mathcal{L}) \cong \mathcal{D}$ of algebras in $\mathcal{C} \otimes \mathcal{C}^{\otimes\mathrm{op}}$ compatible with quantum moment maps from $\mathcal{L} \boxtimes \mathcal{L}^{\mathrm{op}}$. $\square$

The definition of representations of Harish-Chandra bialgebroids is straightforward.

**Definition 2.2.33.** Suppose $B \in \mathrm{HC} \otimes \mathrm{HC}$ is a Harish-Chandra bialgebroid. A ***$B$-comodule*** is an object $M \in \mathrm{HC}$ together with a coassociative and counital coaction $M \to B \times_{\mathcal{L}} M$.

Equivalently, by theorem 2.2.32 a $B$-comodule is a coalgebra over the comonad $\perp(M) = B \times_{\mathcal{L}} M$.

**Example 2.2.34.** Consider the category of Harish-Chandra bimodules $\mathrm{HC}(H)$ for a split torus $H$ as in example 2.2.30 and let $B$ be a Harish-Chandra bialgebroid in $\mathrm{HC}(H)$. Then a $B$-comodule is a $\mathcal{O}(\mathfrak{h}^*)$-module

$$M = \bigoplus_{\alpha \in \Lambda} M_\alpha$$

together with a coaction map

$$M_\alpha \longrightarrow \bigoplus_{\beta \in \Lambda} B_{\alpha\beta} \otimes_{\mathcal{O}(\mathfrak{h}^*)} M_\beta,$$

where $B$ is considered as a right $\mathcal{O}(\mathfrak{h}^*)$-module via the *left* action of $t\colon \mathcal{O}(\mathfrak{h}^*) \to B$. We require this coaction map to be compatible with the $\mathcal{O}(\mathfrak{h}^*)$-actions on both sides, where the $\mathcal{O}(\mathfrak{h}^*)$-action on the right is via the left multiplication by $s\colon \mathcal{O}(\mathfrak{h}^*) \to B$ and be coassociative and counital in the obvious way.

We obtain a Tannaka recognition statement for Harish-Chandra bialgebroids.

Theorem 2.2.35. *Suppose $\mathcal{D}$ is a monoidal category with a monoidal functor $F\colon \mathcal{D} \to \mathrm{HC}$ which admits a colimit-preserving right adjoint $F^{\mathrm{R}}\colon \mathrm{HC} \to \mathcal{D}$. Then there is a Harish-Chandra bialgebroid $B$, such that $(F \circ F^{\mathrm{R}})(-) \cong B \times_{\mathcal{L}} (-)$ and $F$ factors through a monoidal functor*

$$\mathcal{D} \longrightarrow \mathrm{CoMod}_B(\mathrm{HC}).$$

*If $F$ is conservative and preserves equalizers, the above functor is an equivalence.*

Proof. Since $F$ is monoidal, $F^{\mathrm{R}}$ is lax monoidal. Therefore, $\perp = FF^{\mathrm{R}}$ is a colimit-preserving lax monoidal comonad on HC. By theorem 2.2.32 there is a Harish-Chandra bialgebroid $B$, such that $\perp(-) \cong B \times_{\mathcal{L}} (-)$. By the standard monadic arguments $F$ factors through

$$\mathcal{D} \longrightarrow \mathrm{CoMod}_{\perp}(\mathrm{HC}) \cong \mathrm{CoMod}_B(\mathrm{HC}),$$

which is monoidal (see [Szl03, Proposition 3.5] for the dual statement). If $F$ is conservative and preserves equalizers, by the Barr–Beck theorem [Mac71, Theorem VI.7.1] the above functor is an equivalence. □

## 2.3. Dynamical $R$-matrices

In this section we explain how the dynamical twists and dynamical $R$-matrices arise from the categorical formalism explained in this paper.

**2.3.1. Dynamical twists.** Consider a Hopf algebra $H$, a commutative algebra $\mathcal{L} \in \mathrm{Z}_{\mathrm{Dr}}(\mathrm{LMod}_H)$ (see proposition 2.2.3) with a coaction map $\delta\colon \mathcal{L} \to H \otimes \mathcal{L}$ (denoted by $x \mapsto x_{(-1)} \otimes x_{(0)}$) and a cp-rigid monoidal category $\mathcal{C}$ together with a forgetful functor $F\colon \mathcal{C} \to \mathrm{LMod}_H$ which we assume sends compact projective objects in $\mathcal{C}$ to finite-dimensional $H$-modules. It will be convenient to introduce the right $H$-coaction

$$\delta^R\colon \mathcal{L} \longrightarrow \mathcal{L} \otimes H$$

by

$$x \mapsto x_{(0)} \otimes S^{-1}(x_{(-1)}).$$

**Proposition 2.3.1.** *A monoidal structure with a strict unit map on the composite*

$$\mathcal{C} \longrightarrow \mathrm{LMod}_H \xrightarrow{\mathrm{free}} \mathrm{HC}(\mathrm{LMod}_H, \mathcal{L})$$

*is the same as a natural collection of elements $J_{X,Y} \in \mathcal{L} \otimes \mathrm{Hom}(F(X) \otimes F(Y), F(X \otimes Y))$ for $X, Y \in \mathcal{C}^{\mathrm{cp}}$ satisfying*

- *The elements $J_{X,Y}$ are $H$-invariant.*
- *For a triple $X, Y, Z \in \mathcal{C}^{\mathrm{cp}}$ the equation*

$$J_{X\otimes Y,Z} \circ (J_{X,Y} \otimes \mathrm{id}_Z) = J_{X,Y\otimes Z} \circ \delta^R_X(J_{Y,Z})$$

  *holds, where $\delta^R_X$ means the $H$-factor in $\delta^R$ acts on $X$.*
- *For any $X \in \mathcal{C}^{\mathrm{cp}}$ we have $J_{\mathbf{1},X} = J_{X,\mathbf{1}} = 1 \otimes \mathrm{id}_{F(X)}$.*

Proof. Recall that the monoidal structure on the functor free: $\mathrm{LMod}_H \to \mathrm{HC}(\mathrm{LMod}_H, \mathcal{L})$ is given by the natural isomorphism

$$\begin{aligned}(\mathcal{L} \otimes X) \otimes_{\mathcal{L}} (\mathcal{L} \otimes Y) &\xrightarrow{\sim} \mathcal{L} \otimes X \otimes Y \\ a \otimes x \otimes b \otimes y &\mapsto ab_{(0)} \otimes S^{-1}(b_{(-1)}) \triangleright x \otimes y\end{aligned}$$

for any $X, Y \in \mathrm{LMod}_H$. So, the monoidal structure on $\mathcal{C} \to \mathrm{HC}(\mathrm{LMod}_H, \mathcal{L})$ is given by

$$(\mathcal{L} \otimes F(X)) \otimes_{\mathcal{L}} (\mathcal{L} \otimes F(Y)) \cong \mathcal{L} \otimes F(X) \otimes F(Y) \xrightarrow{\sim} \mathcal{L} \otimes F(X \otimes Y),$$

where the first isomorphism is given by the monoidal structure on free and the second isomorphism is

$$l \otimes a \otimes b \mapsto (l \otimes \mathrm{id}_{F(X\otimes Y)}) J_{X,Y}(a \otimes b).$$

The composite is automatically a map of $\mathcal{L}$-modules and the compatibility with the $H$-action is the $H$-invariance condition on $J_{X,Y}$.

The associativity condition for the monoidal structure on $F$ is that for compact projective objects $X, Y, Z \in \mathcal{C}^{\mathrm{cp}}$ the diagram

$$\begin{tikzcd}
((\mathcal{L} \otimes F(X)) \otimes_{\mathcal{L}} (\mathcal{L} \otimes F(Y))) \otimes_{\mathcal{L}} (\mathcal{L} \otimes F(Z)) \arrow[rr, "\sim"] \arrow[d, "J_{X,Y} \otimes \mathrm{id}"] & & (\mathcal{L} \otimes F(X)) \otimes_{\mathcal{L}} ((\mathcal{L} \otimes F(Y)) \otimes_{\mathcal{L}} (\mathcal{L} \otimes F(Z))) \arrow[d, "\mathrm{id} \otimes J_{Y,Z}"] \\
(\mathcal{L} \otimes F(X \otimes Y)) \otimes_{\mathcal{L}} (\mathcal{L} \otimes F(Z)) \arrow[dr, "J_{X \otimes Y, Z}"'] & & (\mathcal{L} \otimes F(X)) \otimes_{\mathcal{L}} (\mathcal{L} \otimes F(Y \otimes Z)) \arrow[dl, "J_{X, Y \otimes Z}"] \\
& \mathcal{L} \otimes F(X \otimes Y \otimes Z) &
\end{tikzcd}$$

commutes. Considering the image of $(1 \otimes a) \otimes (1 \otimes b) \otimes (1 \otimes c)$ under these maps we get the second equation.

The unitality condition for the monoidal structure on $F$ is equivalent to the last equation. $\square$

Let us now introduce a universal version of the previous statement. Suppose $A$ is another Hopf algebra with a map of algebras $H \to A$. We assume $\mathcal{C} = \mathrm{LMod}_A$ and the forgetful functor

$$F\colon \mathrm{LMod}_A \longrightarrow \mathrm{LMod}_H$$

is given by restriction of modules. Denote by $(A, \Delta, \varepsilon)$ the coalgebra structure on $A$.

**Definition 2.3.2.** A ***dynamical twist*** is an invertible element

$$J = J^0 \otimes J^1 \otimes J^2 \in \mathcal{L} \otimes A \otimes A$$

satisfying

(1) The invariance condition

$$h_{(1)} \triangleright J^0 \otimes h_{(2)} J^1 \otimes h_{(3)} J^2 = J^0 \otimes J^1 h_{(1)} \otimes J^2 h_{(2)}$$

for every $h \in H$;

(2) The shifted cocycle equation

$$((\mathrm{id} \otimes \Delta \otimes \mathrm{id})J)(J \otimes 1) = ((\mathrm{id} \otimes \mathrm{id} \otimes \Delta)J)(J^0_{(0)} \otimes S^{-1}(J^0_{(-1)}) \otimes J^1 \otimes J^2);$$

(3) The normalization condition

$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})J = 1 \otimes 1 \otimes 1 = (\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)J.$$

**Example 2.3.3.** Consider the trivial pair $H = \mathcal{L} = k$ given by the ground field. The invariance condition is empty, while the cocycle equation and the normalization condition imply that $J \in A \otimes A$ is a (constant) twist for the Hopf algebra in the sense of [CP95, Proposition 4.2.13].

**Example 2.3.4.** Suppose $\mathfrak{h}$ is an abelian Lie algebras and consider $H = \mathcal{L} = \mathrm{U}\mathfrak{h}$ as in section 2.2.3. Then a dynamical twist is a function $J\colon \mathfrak{h}^* \to A \otimes A$. The invariance condition is that $J(\lambda)$ is $\mathfrak{h}$-invariant with respect to the adjoint action (the *zero-weight condition*). The shifted cocycle equation is

$$((\Delta \otimes \mathrm{id})J(\lambda))J_{12}(\lambda) = ((\mathrm{id} \otimes \Delta)J(\lambda))J_{23}(\lambda - h^{(1)}).$$

**Proposition 2.3.5.** *The data of a dynamical twist is equivalent to the data of a monoidal structure on* $\mathrm{LMod}_A \to \mathrm{HC}(\mathrm{LMod}_H, \mathcal{L})$ *with a strict unit map.*

Proof. By proposition 2.3.1 the monoidal structure is specified by a collection of elements

$$J_{X,Y} \in \mathcal{L} \otimes \mathrm{End}(X \otimes Y), \qquad X, Y \in \mathrm{LMod}_A.$$

By naturality these are uniquely determined by the elements

$$J = J_{A,A}(1_A \otimes 1_A) \in \mathcal{L} \otimes A \otimes A.$$

$\square$

Two dynamical twists may be related by a gauge transformation.

**Definition 2.3.6.** A ***gauge transformation*** is an invertible $H$-invariant element $G \in \mathcal{L} \otimes A$ satisfying the normalization condition

$$(\mathrm{id} \otimes \varepsilon)(G) = 1 \otimes 1.$$

Given a dynamical twist $J$ and a gauge transformation $G$ we obtain a new dynamical twist by the formula

$$J^G = (\mathrm{id} \otimes \Delta)G \cdot J \cdot ((\delta^R \otimes \mathrm{id})G)^{-1}(G \otimes 1)^{-1}. \tag{2.3.1}$$

**Example 2.3.7.** Consider the pair $H = \mathcal{L} = \mathrm{U}\mathfrak{h}$ as in example 2.3.4. Then a gauge transformation is a zero-weight function $G\colon \mathfrak{h}^* \to A^\times$ satisfying $\varepsilon(G(\lambda)) = 1$. Given a dynamical twist $J\colon \mathfrak{h}^* \to A \otimes A$, its gauge transformation is

$$J^G(\lambda) = (\mathrm{id} \otimes \Delta)G(\lambda) \cdot J(\lambda) \cdot (G_2(\lambda - h^{(1)}))^{-1}(G_1(\lambda))^{-1}.$$

**Proposition 2.3.8.** *Suppose $J_1, J_2$ are two dynamical twists which give rise to monoidal structures on the functor* $\mathrm{LMod}_A \to \mathrm{HC}(\mathrm{LMod}_H, \mathcal{L})$ *by proposition 2.3.5. The data of a gauge transformation between them is a monoidal natural isomorphism*

$$\begin{tikzcd} \mathrm{LMod}_A \arrow[r, bend left, "J_1"] \arrow[r, bend right, "J_2"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{HC}(\mathrm{LMod}_H, \mathcal{L}) \end{tikzcd}$$

**2.3.2. Dynamical FRT algebras.** Let us describe the Harish-Chandra bialgebroid $B$ from theorem 2.2.35 explicitly. Let $\mathcal{D}$ be a cp-rigid monoidal category and $F\colon \mathcal{D} \to \mathrm{HC}$ a monoidal functor which admits a colimit-preserving right adjoint $F^{\mathrm{R}}$.

By proposition 2.1.5 the functor $FF^{\mathrm{R}}$ can be calculated as

$$\begin{aligned} FF^{\mathrm{R}}(x) &= \int^{y \in \mathcal{D}^{\mathrm{cp}}} \mathrm{Hom}_{\mathrm{HC}}(F(y)^\vee, x) \otimes F(y)^\vee \\ &\cong \int^{y \in \mathcal{D}^{\mathrm{cp}}} \mathrm{Hom}_{\mathrm{HC}}(\mathcal{L}, F(y) \otimes_{\mathcal{L}} x) \otimes F(y)^\vee. \end{aligned}$$

Recalling the definition of the Takeuchi product from definition 2.2.27, we obtain that $FF^{\mathrm{R}}(x) \cong B \times_{\mathcal{L}} x$, where the Harish-Chandra bialgebroid $B$ is

$$B \cong \int^{y \in \mathcal{D}^{\mathrm{cp}}} F(y)^\vee \boxtimes F(y) \in \mathrm{HC} \otimes \mathrm{HC}.$$

As in section 2.1.5 denote by

$$\pi_y\colon F(y)^\vee \boxtimes F(y) \longrightarrow B$$

the natural projections. The Harish-Chandra bialgebroid structure is given on generators as follows:

(1) The coproduct

$$B \longrightarrow B \times_{\mathcal{L}} B \cong \int^{(y,z) \in \mathcal{D}^{\mathrm{cp}} \times \mathcal{D}^{\mathrm{cp}}} \mathrm{Hom}_{\mathrm{HC}}(\mathcal{L}, F(y) \otimes_{\mathcal{L}} F(z)^\vee) \otimes F(y)^\vee \boxtimes F(z)$$

is

$$F(y)^\vee \boxtimes F(y) \xrightarrow{\mathrm{coev} \otimes \mathrm{id}} \mathrm{Hom}_{\mathrm{HC}}(\mathcal{L}, F(y) \otimes_{\mathcal{L}} F(y)^\vee) \otimes F(y)^\vee \boxtimes F(y) \xrightarrow{\pi_{y,y}} B \times_{\mathcal{L}} B.$$

(2) The counit

$$B \longrightarrow T^{\mathrm{R}}_{\mathrm{HC}}(\mathcal{L}) = \int^{P \in \mathrm{HC}^{\mathrm{cp}}} P^\vee \boxtimes P$$

is the projection

$$F(y)^\vee \boxtimes F(y) \to T^{\mathrm{R}}_{\mathrm{HC}}(\mathcal{L}).$$

(3) The product is the composite

$$\begin{aligned}(F(y)^\vee \boxtimes F(y)) \otimes_{\mathcal{L}\otimes\mathcal{L}} (F(z)^\vee \boxtimes F(z)) &= (F(y)^\vee \otimes_{\mathcal{L}} F(z)^\vee) \boxtimes (F(z) \otimes_{\mathcal{L}} F(y)) \\ &\cong (F(z) \otimes_{\mathcal{L}} F(y))^\vee \boxtimes (F(z) \otimes_{\mathcal{L}} F(y)) \\ &\xrightarrow{(J_{z,y}^{-1})^\vee \boxtimes J_{z,y}} F(z\otimes y)^\vee \boxtimes F(z \otimes y) \\ &\xrightarrow{\pi_{z\otimes y}} B.\end{aligned}$$

(4) The quantum moment map is

$$\mathcal{L} \boxtimes \mathcal{L}^{\mathrm{op}} \cong F(\mathbf{1}) \boxtimes F(\mathbf{1}) \xrightarrow{\pi_{\mathbf{1}}} B.$$

We will now concentrate on the case $\mathcal{C} = \mathrm{Rep}(H)$ for $H$ a split torus with weight lattice $\Lambda$ and $\mathcal{L} = \mathrm{U}\mathfrak{h}$, so that $\mathrm{HC} = \mathrm{HC}(H)$. Moreover, we assume that the functor $F\colon \mathcal{D} \to \mathrm{HC}(H)$ factors as the composite

$$\mathcal{D} \longrightarrow \mathrm{Rep}(H) \xrightarrow{\text{free}} \mathrm{HC}(H).$$

For an object $y \in \mathcal{D}$ we denote its image in $\mathrm{Rep}(H)$ by the same letter. In this case the monoidal structure is given by a dynamical twist

$$J_{y,z}(\lambda)\colon \mathfrak{h}^* \to \mathrm{End}(y \otimes z)$$

as in proposition 2.3.1.

The projections

$$\pi_y\colon (\mathrm{U}\mathfrak{h} \otimes y^\vee) \boxtimes (\mathrm{U}\mathfrak{h} \otimes y) \longrightarrow B$$

in $\mathrm{HC}(H \times H)$ may be encoded in elements

$$T_y \in B \otimes \mathrm{End}(y).$$

Analogously to theorem 2.1.26 we obtain the following explicit description of the bialgebroid $B$.

Theorem 2.3.9. *The bialgebroid $B$ is spanned, as an $\mathcal{O}(\mathfrak{h}^*)$-bimodule, by the matrix coefficients of $T_y$ for $y \in \mathcal{D}^{\mathrm{cp}}$ subject to the relation*

$$F(j) \circ T_x = T_y \circ F(j)$$

*for every $j\colon x \to y$. Moreover, we have:*

(1) $\Delta(T_y) = T_y \otimes T_y$ *for every* $y \in \mathcal{D}^{cp}$.
(2) $\epsilon(T_y) = 1 \otimes \mathrm{id}_y \in \mathrm{D}(H) \otimes \mathrm{End}(y)$ *for every* $y \in \mathcal{D}^{\mathrm{cp}}$.
(3) *For every* $f \in \mathcal{O}(\mathfrak{h}^*)$ *and* $y \in \mathcal{D}^{\mathrm{cp}}$

$$\begin{aligned} s(f(\lambda))T_y &= T_y s(f(\lambda+h)) \\ t(f(\lambda+h))T_y &= T_y t(f(\lambda)).\end{aligned}$$

(4) $J^t_{y,z}(\lambda)^{-1} T_{y\otimes z} J^s_{y,z}(\lambda) = (T_y \otimes \mathrm{id})(\mathrm{id}\otimes T_z)$, *where by the superscripts we mean the* left *multiplication with the $\mathcal{O}(\mathfrak{h}^*)$-part by either the source $(s)$ or the target $(t)$ map.*
(5) $T_{\mathbf{1}} \in B$ *is the unit.*

**Definition 2.3.10.** Suppose $\mathcal{D}$ is a braided monoidal category together with a forgetful functor $\mathcal{D} \to \mathrm{Rep}(H)$ and a monoidal structure on the composite $\mathcal{D} \to \mathrm{Rep}(H) \xrightarrow{\text{free}} \mathrm{HC}(H)$. For $x, y \in \mathcal{D}$ define the morphism $\mathrm{U}\mathfrak{h} \otimes x \otimes y \to \mathrm{U}\mathfrak{h} \otimes y \otimes x$ by

$$\check{R}_{x,y}\colon F(x) \otimes_{\mathrm{U}\mathfrak{h}} F(y) \xrightarrow{J_{x,y}} F(x \otimes y) \xrightarrow{F(\sigma_{x,y})} F(y \otimes x) \xrightarrow{J_{x,y}^{-1}} F(y) \otimes_{\mathrm{U}\mathfrak{h}} F(x).$$

The ***dynamical $R$-matrix*** is the map $R_{x,y}\colon \mathfrak{h}^* \to \mathrm{End}(x \otimes y)$ given by

$$R_{x,y} = (\mathrm{id}_{\mathrm{U}\mathfrak{h}} \otimes \sigma_{x,y}^{-1}) \circ \check{R}_{x,y}.$$

As in section 2.1.5, we use the standard notation $T_1 = T_x \otimes \mathrm{id}$ and $T_2 = \mathrm{id} \otimes T_y$ and similarly for the $R$-matrix.

**Proposition 2.3.11.** *Let $x, y, z \in \mathcal{D}^{\mathrm{cp}}$.*

(1) *The dynamical R-matrix satisfies the dynamical Yang-Baxter equation*

$$R_{23}(\lambda)R_{13}(\lambda - h^{(2)})R_{12}(\lambda) = R_{12}(\lambda - h^{(3)})R_{13}(\lambda)R_{23}(\lambda - h^{(1)})$$

*in* $\mathrm{End}(x \otimes y \otimes z)$.

(2) *The element* $T$ *satisfies the dynamical FRT relation*

$$R^t(\lambda)T_1T_2 = T_2T_1R^s(\lambda)$$

*in* $B \otimes \mathrm{End}(x \otimes y)$.

Proof. As in the proof of proposition 2.1.28, the element $\check{R}$ satisfies the braid relation

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}$$

in $\mathrm{U}\mathfrak{h} \otimes \mathrm{End}(x \otimes y \otimes z)$. Observing that $\check{R} = \sigma \circ R$, we get the dynamical Yang–Baxter equation.

To show the second part, recall from theorem 2.3.9 that

$$F(\sigma_{x,y})T_{x\otimes y} = T_{y\otimes x}F(\sigma_{x,y}).$$

Decomposing $T_{x\otimes y}$ and $T_{y\otimes x}$ into $T_x$ and $T_y$ using property (4) of the same theorem, we get the result. $\square$

## 2.4. Vertex-IRF transformation

Recall proposition 1.4.4: in the semi-classical case, a generalized vertex-IRF transformation is equivalent to the data of a morphism between coisotropic structures

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & U' \arrow[dl] \\ & \mathrm{B}G & \end{tikzcd}$$

In this section, we provide a categorical counterpart of this statement.

To fix the notations, let us recall the notion of a module category over a monoidal category from [Eti+15, Chapters 7.1, 7.2]. Let $\mathcal{C}$ be a monoidal category. A $\mathcal{C}$-module category is a category $\mathcal{M}$ together with an action $\mathcal{C} \otimes \mathcal{M} \to \mathcal{M}$ which we denote by $X \otimes V$ for $X \in \mathcal{C}, V \in \mathcal{M}$ and a natural isomorphism

$$\Psi_{X,Y,V} \colon (X \otimes Y) \otimes V \to X \otimes (Y \otimes V),$$

satisfying a pentagon axiom.

A functor $F \colon \mathcal{M}_1 \to \mathcal{M}_2$ of $\mathcal{C}$-module categories is a functor of plain categories together with a natural isomorphism

$$\alpha_{X,V} \colon X \otimes F(V) \to F(X \otimes V)$$

satisfying the unit and pentagon axioms.

Consider a pair $(H, \mathcal{L})$ as in the previous section. Then we have a natural $\mathrm{HC}(\mathrm{LMod}_H, \mathcal{L})$-module structure on $\mathrm{LMod}_{\mathcal{L}}$ given by $P \otimes_{\mathcal{L}} M$ for $P \in \mathrm{HC}(\mathrm{LMod}_H, \mathcal{L})$ and $M \in \mathrm{LMod}_{\mathcal{L}}$.

Now suppose we have a pair $(H_1, \mathcal{L}_1)$ and $(H_2, \mathcal{L}_2)$, both equipped with maps of Hopf algebras $H_1 \to A$ and $H_2 \to A$ and dynamical twists $J_{H_1}, J_{H_2}$, respectively.

**Definition 2.4.1.** Suppose $(H_1, \mathcal{L}_1)$ and $(H_2, \mathcal{L}_2)$ are pairs of a Hopf algebra and a commutative algebra in Yetter–Drinfeld modules, both equipped with maps of Hopf algebras $H_1 \to A$ and $H_2 \to A$ and dynamical twists $J_{H_1}, J_{H_2}$, respectively. Moreover, suppose $\mathcal{L}_1 \to \mathcal{L}_2$ is an algebra map. A ***generalized vertex-IRF transformation*** is an invertible element $S \in \mathcal{L}_2 \otimes A$ satisfying

(1) For every $l \in \mathcal{L}_1$, the invariance condition

$$\delta^R_{H_1}(l)S = S\delta^R_{H_2}(l);$$

(2) The coboundary equation

$$(\delta^R_{H_2} \otimes \mathrm{id})(S)(1 \otimes S)J^{-1}_{H_1} = J^{-1}_{H_2}(1 \otimes \Delta_A)S;$$

(3) The normalization condition

$$(\mathrm{id} \otimes \varepsilon)S = 1.$$

**Example 2.4.2.** Consider the pair $H_1 = \mathcal{L}_1 = k$ with a constant twist $J$ and $H_2 = \mathcal{L}_2 = \mathrm{U}\mathfrak{h}$ with a dynamical twist $F(\lambda)$. Then $S$ is a function $\mathfrak{h}^* \to A$ satisfying $\varepsilon(S(\lambda)) = 1$ and the quantum dynamical coboundary equation

$$F(\lambda) = \Delta(S(\lambda))JS_2(\lambda - h^{(1)})^{-1}S_1(\lambda)^{-1}.$$

The version for right $\mathcal{L}$-modules is as in [BRT07, Section 3.3]:

$$F(\lambda) = \Delta(S(\lambda))JS_2(\lambda)^{-1}S_1(\lambda + h^{(2)})^{-1}.$$

**Proposition 2.4.3.** *The data of a generalized vertex-IRF transformation is equivalent to the data of an* $\mathrm{LMod}_A$*-module structure on the functor* $\mathrm{LMod}_{\mathcal{L}_1} \to \mathrm{LMod}_{\mathcal{L}_2}$ *given by* $M \mapsto \mathcal{L}_2 \otimes_{\mathcal{L}_1} M$*.*

Proof. Such a data is given by a natural isomorphism of $\mathcal{L}_2$-modules

$$\alpha_{X,V} \colon (\mathcal{L}_2 \otimes X) \otimes_{\mathcal{L}_2} (\mathcal{L}_2 \otimes_{\mathcal{L}_1} V) \to \mathcal{L}_2 \otimes_{\mathcal{L}_1} ((\mathcal{L}_1 \otimes X) \otimes_{\mathcal{L}_1} V)$$

for $X \in \mathrm{LMod}_A$ and $V \in \mathrm{LMod}_{\mathcal{L}_1}$. Since $\mathrm{LMod}_A$ is generated by $A$ and $\mathrm{LMod}_{\mathcal{L}_1}$ is generated by $L_1$, such an isomorphism is uniquely specified by the value of the map

$$\alpha_{A,\mathcal{L}_1} \colon \mathcal{L}_2 \otimes A \cong (\mathcal{L}_2 \otimes A) \otimes_{\mathcal{L}_2} (\mathcal{L}_2 \otimes_{\mathcal{L}_1} \mathcal{L}_1) \to \mathcal{L}_2 \otimes_{\mathcal{L}_1} ((\mathcal{L}_1 \otimes A) \otimes_{\mathcal{L}_1} \mathcal{L}_1) \cong \mathcal{L}_2 \otimes A$$

on $1_{\mathcal{L}_1} \otimes 1_A$. Naturality implies that it commutes with arbitrary morphisms $\mathcal{L}_1 \to \mathcal{L}_1$ of left $\mathcal{L}_1$-modules which are parametrized by $l \in \mathcal{L}_1$, and it gives the invariance condition. The pentagon axiom boils down to the coboundary equation. The unit axiom gives the normalization condition.

□

Likewise, it can be formulated in terms of *right* modules.

## 2.5. Fusion of Verma modules

In this section we construct standard dynamical twists for $\mathrm{U}\mathfrak{g}$ and $\mathrm{U}_q(\mathfrak{g})$ using the so-called exchange construction introduced in [EV99].

**2.5.1. Classical parabolic restriction.** Let $G$ be a reductive group over an algebraically closed field $k$ of characteristic zero and $\mathfrak{g}$ its Lie algebra. Fix a Borel subgroup $B \subset G$ and denote $H = B/[B, B]$; their Lie algebras are denoted by $\mathfrak{b}$ and $\mathfrak{h}$. We denote by $N$ the kernel of $B \to H$ with Lie algebra $\mathfrak{n}$. Let $W$ be the Weyl group.

We will later use the Harish-Chandra isomorphism, see [Hum08, Theorem 1.10].

Theorem 2.5.1. *There is a unique homomorphism of algebras*

$$\mathrm{hc} \colon \mathrm{Z}(\mathrm{U}\mathfrak{g}) \longrightarrow \mathrm{U}\mathfrak{h},$$

*the* **Harish-Chandra homomorphism***, such that for any* $z \in \mathrm{Z}(\mathrm{U}\mathfrak{g})$ *and* $m \in M^{\mathrm{univ}}$ *we have*

$$zm = m\mathrm{hc}(z).$$

**Definition 2.5.2.** The ***universal category*** $\mathcal{O}$ is the category $\mathcal{O}^{\mathrm{univ}}$ of $(\mathrm{U}\mathfrak{g}, \mathrm{U}\mathfrak{h})$-bimodules whose diagonal $\mathfrak{b}$-action integrates to a $B$-action. The ***universal Verma module*** is

$$M^{\mathrm{univ}} = \mathrm{U}\mathfrak{g} \otimes_{\mathrm{U}\mathfrak{b}} \mathrm{U}\mathfrak{h} \in \mathcal{O}^{\mathrm{univ}}.$$

**Remark 2.5.3.** Just like the usual category $\mathcal{O}$ is constructed to contain objects like Verma modules, we define $\mathcal{O}^{\mathrm{univ}}$ to contain objects like universal Verma modules.

**Remark 2.5.4.** We may identify $\mathcal{O}^{\mathrm{univ}}$ with the category of $\mathrm{U}\mathfrak{g}$-modules in the category $\mathrm{Rep}(H)$ whose $\mathfrak{n}$-action is locally nilpotent.

We will now define an important bimodule structure on $\mathcal{O}^{\mathrm{univ}}$:

$$\mathrm{HC}(G) \curvearrowright \mathcal{O}^{\mathrm{univ}} \curvearrowleft \mathrm{HC}(H). \tag{2.5.1}$$

Both actions are given by the relative tensor products of bimodules. Given a $\mathrm{U}\mathfrak{g}$-bimodule $X \in \mathrm{HC}(G)$ and a $(\mathrm{U}\mathfrak{g}, \mathrm{U}\mathfrak{h})$-bimodule $M \in \mathcal{O}^{\mathrm{univ}}$, $X \otimes_{\mathrm{U}\mathfrak{g}} M$ is an $(\mathrm{U}\mathfrak{g}, \mathrm{U}\mathfrak{h})$-bimodule. Since the diagonal $\mathfrak{g}$-action on $X$ is integrable, so is the diagonal $\mathfrak{b}$-action. Therefore, the diagonal $\mathfrak{b}$-action on $X \otimes_{\mathrm{U}\mathfrak{g}} M$ is integrable. The $\mathrm{HC}(H)$ action is defined similarly.

Let

$$\mathrm{act}_G \colon \mathrm{HC}(G) \longrightarrow \mathcal{O}^{\mathrm{univ}}, \qquad \mathrm{act}_H \colon \mathrm{HC}(H) \longrightarrow \mathcal{O}^{\mathrm{univ}}$$

be the actions of $\mathrm{HC}(G)$ and $\mathrm{HC}(H)$ on the universal Verma module $M^{\mathrm{univ}} \in \mathcal{O}^{\mathrm{univ}}$. Using proposition 2.1.21 we obtain the following lax monoidal functors.

**Definition 2.5.5.** The ***parabolic restriction*** is the lax monoidal functor

$$\mathrm{res} = \mathrm{act}_H^{\mathrm{R}} \circ \mathrm{act}_G \colon \mathrm{HC}(G) \longrightarrow \mathrm{HC}(H).$$

The ***parabolic induction*** is the lax monoidal functor

$$\mathrm{ind} = \mathrm{act}_G^{\mathrm{R}} \circ \mathrm{act}_H \colon \mathrm{HC}(H) \longrightarrow \mathrm{HC}(G).$$

Let us now make these functors more explicit. Consider the functor

$$(-)^N \colon \mathcal{O}^{\mathrm{univ}} \longrightarrow \mathrm{HC}(H)$$

which sends a $(\mathrm{U}\mathfrak{g}, \mathrm{U}\mathfrak{h})$-bimodule to the subspace of highest weight vectors with respect to the $\mathrm{U}\mathfrak{g}$-action. It still has a remaining $\mathrm{U}\mathfrak{h}$-bimodule structure and so it defines an object of $\mathrm{HC}(H)$.

**Proposition 2.5.6.** *The functor $(-)^N \colon \mathcal{O}^{\mathrm{univ}} \to \mathrm{HC}(H)$ is right adjoint to $\mathrm{act}_H \colon \mathrm{HC}(H) \to \mathcal{O}^{\mathrm{univ}}$.*

Proof. Identify $\mathcal{O}^{\mathrm{univ}}$ with highest-weight $\mathrm{U}\mathfrak{g}$-modules in the category $\mathrm{Rep}(H)$ following remark 2.5.4. For $M \in \mathcal{O}^{\mathrm{univ}}$ and $X \in \mathrm{HC}(H)$ we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}^{\mathrm{univ}}}(\mathrm{act}_H(X), M) &= \mathrm{Hom}_{\mathcal{O}^{\mathrm{univ}}}(\mathrm{U}\mathfrak{g} \otimes_{\mathrm{U}\mathfrak{b}} X, M) \\ &\cong \mathrm{Hom}_{{}_{\mathrm{U}\mathfrak{b}}\mathrm{BMod}_{\mathrm{U}\mathfrak{h}}}(X, M) \\ &\cong \mathrm{Hom}_{\mathrm{HC}(H)}(X, M^N). \end{aligned}$$

∎

So,

$$\mathrm{res}(X) \cong (X/X\mathfrak{n})^N. \tag{2.5.2}$$

The lax monoidal structure on res can be described explicitly as follows. For $X, Y \in \mathrm{HC}(G)$ the morphism

$$(X/X\mathfrak{n})^N \otimes_{\mathrm{U}\mathfrak{h}} (Y/Y\mathfrak{n})^N \longrightarrow (X \otimes_{\mathrm{U}\mathfrak{g}} Y/(X \otimes_{\mathrm{U}\mathfrak{g}} Y)\mathfrak{n})^N \tag{2.5.3}$$

is given by $[x] \otimes [y] \mapsto [x \otimes y]$. This assignment is independent of the choice of a representative of $[x]$ since $[y]$ is $N$-invariant.

**Remark 2.5.7.** Since $\mathrm{res} \colon \mathrm{HC}(G) \to \mathrm{HC}(H)$ is lax monoidal, it sends algebras in $\mathrm{HC}(G)$ to algebras in $\mathrm{HC}(H)$. By proposition 2.2.10, an algebra in $\mathrm{HC}(G)$ is a $G$-algebra equipped with a quantum moment map $\mu \colon \mathrm{U}\mathfrak{g} \to A$. It is easy to see that $\mathrm{res}(A)$ is the quantum Hamiltonian reduction $A /\!\!/ N$. This algebra is known as the Mickelsson algebra [Mic73], we refer to [Zhe90] for more details.

Recall that the coinduction functor

$$\mathrm{coind}_B^G \colon \mathrm{Rep}(B) \longrightarrow \mathrm{Rep}(G)$$

is right adjoint to the obvious restriction functor $\mathrm{Rep}(G) \to \mathrm{Rep}(B)$. Denote in the same way the functor

$$\mathrm{coind}_B^G \colon \mathcal{O}^{\mathrm{univ}} \longrightarrow \mathrm{HC}(G)$$

of coinduction from $B$ to $G$ using the diagonal $B$-action.

**Proposition 2.5.8.** *The functor $\mathrm{coind}_B^G \colon \mathcal{O}^{\mathrm{univ}} \to \mathrm{HC}(G)$ is right adjoint to $\mathrm{act}_G \colon \mathrm{HC}(G) \to \mathcal{O}^{\mathrm{univ}}$.*

Proof. For $M \in \mathcal{O}^{\mathrm{univ}}$ and $X \in \mathrm{HC}(G)$ we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}^{\mathrm{univ}}}(\mathrm{act}_G(X), M) &= \mathrm{Hom}_{\mathcal{O}^{\mathrm{univ}}}(X \otimes_{\mathrm{U}\mathfrak{b}} \mathrm{U}\mathfrak{h}, M) \\ &\cong \mathrm{Hom}_{\mathrm{U}\mathfrak{g}\,\mathrm{BMod}_{\mathrm{U}\mathfrak{b}}}(X, M).\end{aligned}$$

Both $X$ and $M$ are $(\mathrm{U}\mathfrak{g}, \mathrm{U}\mathfrak{b})$-bimodules whose diagonal $\mathfrak{b}$-action integrates to a $B$-action, i.e. they are objects of $\mathrm{LMod}_{\mathrm{U}\mathfrak{g}}(\mathrm{Rep}\, B)$. Moreover, $X$ lies in the image of the forgetful functor

$$\mathrm{HC}(G) = \mathrm{LMod}_{\mathrm{U}\mathfrak{g}}(\mathrm{Rep}\, G) \to \mathrm{LMod}_{\mathrm{U}\mathfrak{g}}(\mathrm{Rep}\, B).$$

But by definition $\mathrm{coind}_B^G$ is the right adjoint to the forgetful functor $\mathrm{Rep}\, G \to \mathrm{Rep}\, B$. □

Let us now compute the values of res and ind on the units.

**Proposition 2.5.9.** *The natural morphism* $\mathrm{U}\mathfrak{h} \to \mathrm{res}(\mathrm{U}\mathfrak{g})$ *is an isomorphism.*

Proof. By proposition 2.5.6 $\mathrm{res}(\mathrm{U}\mathfrak{g}) \cong (M^{\mathrm{univ}})^N$ and we have to show that

$$\mathrm{U}\mathfrak{h} \longrightarrow (M^{\mathrm{univ}})^N$$

is an isomorphism. Let

$$M^{\mathrm{univ,mer}} = \mathrm{U}\mathfrak{g} \otimes_{\mathrm{U}\mathfrak{b}} \mathrm{Frac}(\mathrm{U}\mathfrak{h}),$$

where $\mathrm{Frac}(\mathrm{U}\mathfrak{h})$ is the fraction field of $\mathrm{U}\mathfrak{h}$. The map $M^{\mathrm{univ}} \to M^{\mathrm{univ,mer}}$ is injective and $(-)^N$ is left exact, so $(M^{\mathrm{univ}})^N \longrightarrow (M^{\mathrm{univ,mer}})^N$ is injective. But the Verma module for generic highest weights is irreducible (see [Hum08, Theorem 4.4]), so

$$\mathrm{Frac}(\mathrm{U}\mathfrak{h}) \longrightarrow (M^{\mathrm{univ,mer}})^N$$

is an isomorphism. This implies the claim. □

**Corollary 2.5.10.** *The induced map*

$$\mathrm{res}\colon \mathrm{Z}(\mathrm{U}\mathfrak{g}) = \mathrm{End}_{\mathrm{HC}(G)}(\mathrm{U}\mathfrak{g}) \longrightarrow \mathrm{U}\mathfrak{h} = \mathrm{End}_{\mathrm{HC}(H)}(\mathrm{U}\mathfrak{h})$$

*coincides with the Harish-Chandra homomorphism* $\mathrm{hc}\colon \mathrm{Z}(\mathrm{U}\mathfrak{g}) \to \mathrm{U}\mathfrak{h}$.

Proof. The map

$$\mathrm{act}_G\colon \mathrm{Z}(\mathrm{U}\mathfrak{g}) = \mathrm{End}_{\mathrm{HC}(G)}(\mathrm{U}\mathfrak{g}) \longrightarrow \mathrm{End}_{\mathcal{O}^{\mathrm{univ}}}(M^{\mathrm{univ}})$$

sends a central element $z \in \mathrm{Z}(\mathrm{U}\mathfrak{g})$ to the left action of $z \in \mathrm{Z}(\mathrm{U}\mathfrak{g})$ on $M^{\mathrm{univ}}$. By theorem 2.5.1 it is equal to the right action of $\mathrm{hc}(z) \in \mathrm{U}\mathfrak{h}$ on $M^{\mathrm{univ}}$. To conclude, observe that the map

$$\mathrm{U}\mathfrak{h} \longrightarrow (M^{\mathrm{univ}})^N$$

is an isomorphism of right $\mathrm{U}\mathfrak{h}$-modules. □

**Proposition 2.5.11.** *Suppose $G$ is connected and simply-connected. Then there is an isomorphism*

$$\mathrm{ind}(\mathrm{U}\mathfrak{h}) \cong \mathrm{U}\mathfrak{g} \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \mathrm{U}\mathfrak{h},$$

*where the* $\mathrm{Z}(\mathrm{U}\mathfrak{g})$*-action on* $\mathrm{U}\mathfrak{h}$ *is via the Harish-Chandra homomorphism* hc.

Proof. By proposition 2.5.8 $\mathrm{ind}(\mathrm{U}\mathfrak{h}) \cong \mathrm{coind}_B^G(M^{\mathrm{univ}})$. Identifying $B$-representations with $G$-equivariant quasi-coherent sheaves on $G/B$, $M^{\mathrm{univ}}$ is sent to $(\pi_* \mathrm{D}_{G/N})^H$, where $\pi\colon G/N \to G/B$. Therefore,

$$\mathrm{coind}_B^G(M^{\mathrm{univ}}) \cong \mathrm{D}(G/N)^H.$$

The claim then follows from [Šap73; HV78], see also [Mil93, Lemma 3.1]. □

Note that the functor res preserves neither limits nor colimits and it is merely lax monoidal. We will now show that after a localization it becomes exact and monoidal.

**Definition 2.5.12.** A weight $\lambda \in \mathfrak{h}^*$ is ***generic*** if $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}$ for every root $\alpha$. Denote by $\mathfrak{h}^{*,\mathrm{gen}} \subset \mathfrak{h}^*$ the subset of generic weights. Let $\mathrm{HC}(H)^{\mathrm{gen}} \subset \mathrm{HC}(H)$ and $\mathcal{O}^{\mathrm{univ,gen}} \subset \mathcal{O}^{\mathrm{univ}}$ be the full subcategories of right $\mathrm{U}\mathfrak{h}$-modules supported on generic weights. Let $(\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \subset \mathrm{Frac}(\mathrm{U}\mathfrak{h})$ be the subspace of rational functions on $\mathfrak{h}^*$ regular on $\mathfrak{h}^{*,\mathrm{gen}}$.

By construction

$$\mathrm{HC}(H)^{\mathrm{gen}} = \mathrm{HC}(\mathrm{Rep}(H), (\mathrm{U}\mathfrak{h})^{\mathrm{gen}})$$

and similarly for $\mathcal{O}^{\mathrm{univ,gen}}$. Moreover, both $\mathrm{HC}(H)^{\mathrm{gen}} \subset \mathrm{HC}(H)$ and $\mathcal{O}^{\mathrm{univ,gen}} \subset \mathcal{O}^{\mathrm{univ}}$ admit left adjoints given by localization. Let

$$M^{\mathrm{univ,gen}} = \mathrm{U}\mathfrak{g} \otimes_{\mathrm{U}\mathfrak{b}} (\mathrm{U}\mathfrak{h})^{\mathrm{gen}}$$

be the universal Verma module with generic highest weights.

Choose a Borel subgroup $B_- \subset G$ opposite to $B$, with Lie algebra $\mathfrak{b}_-$. Let

$$M^{\mathrm{univ}}_- = \mathrm{U}\mathfrak{h} \otimes_{\mathrm{U}\mathfrak{b}_-} \mathrm{U}\mathfrak{g}$$

be the opposite universal Verma module.

**Definition 2.5.13.** The functor of $\mathfrak{n}_-$***-coinvariants***

$$(-)_{\mathfrak{n}_-} \colon \mathcal{O}^{\mathrm{univ}} \longrightarrow \mathrm{HC}(H)$$

is $M_{\mathfrak{n}_-} = M^{\mathrm{univ}}_- \otimes_{\mathrm{U}\mathfrak{g}} M$.

We will now recall the ***extremal projector*** introduced in [AST71], see also [Zhe90].

Theorem 2.5.14. *There is an extension $T(\mathfrak{g})$ of $\mathrm{U}\mathfrak{g}$ obtained by replacing $\mathrm{U}\mathfrak{h} \subset \mathrm{U}\mathfrak{g}$ with $(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}$ and considering certain power series. There is an element $P \in T(\mathfrak{g})$ satisfying the following properties:*

(1) $\mathfrak{n}P = P\mathfrak{n}_- = 0$.
(2) $P - 1 \in T(\mathfrak{g})\mathfrak{n} \cap \mathfrak{n}_- T(\mathfrak{g})$.

*The action of $P$ is well-defined on left $\mathrm{U}\mathfrak{g}$-modules whose $\mathfrak{n}$-action is locally nilpotent and which have generic $\mathfrak{h}$-weights.*

**Example 2.5.15.** Suppose $\mathfrak{g} = \mathfrak{sl}_2$. The extremal projector in this case is (see e.g. [KO08])

$$P = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} g_n^{-1} f^n e^n,$$

where

$$g_n = \prod_{j=1}^{n} (h + j + 1).$$

We will now describe some applications of extremal projectors.

**Proposition 2.5.16.** *There is a natural isomorphism of functors $(-)_{\mathfrak{n}_-} \cong (-)^N \colon \mathcal{O}^{\mathrm{univ,gen}} \to \mathrm{HC}(H)^{\mathrm{gen}}$. In particular, they are exact.*

Proof. Take $M \in \mathcal{O}^{\mathrm{univ,gen}}$ and consider the composite

$$\pi \colon M^N \longrightarrow M \longrightarrow M_{\mathfrak{n}_-}.$$

We will prove that it is an isomorphism.

Since the weights of the right $\mathrm{U}\mathfrak{h}$-action on $M$ are generic and the weights of the diagonal $\mathrm{U}\mathfrak{h}$-action are integral, the weights of the left $\mathrm{U}\mathfrak{h}$-action are also generic. Moreover, the left $\mathrm{U}\mathfrak{n}$-action is locally nilpotent. In particular, the action of the extremal projector from theorem 2.5.14

$$P \colon M \longrightarrow M$$

is well-defined. It lands in $N$-invariants by the property $\mathfrak{n}P = 0$. It factors through $\mathfrak{n}_-$-coinvariants by the property $P\mathfrak{n}_- = 0$. So, it gives a map

$$P \colon M_{\mathfrak{n}_-} \longrightarrow M^N.$$

For $m \in M^N$ we have $Pm = m$ since $P - 1 \in T(\mathfrak{g})\mathfrak{n}$. In particular, $P \circ \pi = \mathrm{id}$. For $m \in M$, $[m] = [Pm]$ in $M_{\mathfrak{n}_-}$ since $P - 1 \in \mathfrak{n}_- T(\mathfrak{g})$. In particular, $\pi \circ P = \mathrm{id}$. □

Theorem 2.5.17. *The category $\mathcal{O}^{\mathrm{univ,gen}}$ is free of rank 1 as a $\mathrm{HC}(H)^{\mathrm{gen}}$-module category in the sense of definition 2.1.22.*

Proof. The unit of the adjunction $\mathrm{act}_H \dashv (-)^N$ between $\mathrm{HC}(H)^{\mathrm{gen}}$ and $\mathcal{O}^{\mathrm{univ,gen}}$ is

$$X \longrightarrow (\mathrm{act}_H(X))^N \xrightarrow{\sim} (M^{\mathrm{univ}} \otimes_{\mathrm{U}\mathfrak{h}} X)_{\mathfrak{n}_-}.$$

By the PBW isomorphism this map is an isomorphism. In particular, $\mathrm{act}_H \colon \mathrm{HC}(H)^{\mathrm{gen}} \to \mathcal{O}^{\mathrm{univ,gen}}$ is fully faithful.

Since the $\mathfrak{n}$-action on $M \in \mathcal{O}^{\mathrm{univ,gen}}$ is locally nilpotent, $M^N = 0$ if, and only if, $M = 0$. But $(-)^N \colon \mathcal{O}^{\mathrm{univ,gen}} \to \mathrm{HC}(H)^{\mathrm{gen}}$ is exact by proposition 2.5.16. Therefore, it is conservative. Since its left adjoint $\mathrm{act}_H$ is fully faithful, it is an equivalence. □

**Corollary 2.5.18.** *The composite*

$$\mathrm{res}^{\mathrm{gen}} \colon \mathrm{HC}(G) \xrightarrow{\mathrm{res}} \mathrm{HC}(H) \longrightarrow \mathrm{HC}(H)^{\mathrm{gen}}$$

*is strongly monoidal and colimit-preserving.*

Proof. By theorem 2.5.17 $\mathcal{O}^{\mathrm{univ,gen}}$ is free of rank 1 as a $\mathrm{HC}(H)^{\mathrm{gen}}$-module category. The claim then follows from proposition 2.1.23. □

We will now show that $\mathrm{res}^{\mathrm{gen}}$ gives rise to a dynamical twist. For this, according to proposition 2.3.1, we have to show that $\mathrm{res}^{\mathrm{gen}}$ of a free Harish–Chandra bimodule is free, i.e. we have to establish an isomorphism between $(V \otimes M^{\mathrm{univ}})^N$ and $V \otimes (\mathrm{U}\mathfrak{h})^{\mathrm{gen}}$ in $\mathrm{HC}(H)^{\mathrm{gen}}$, for every $V \in \mathrm{Rep}(G)$.

Theorem 2.5.19. *The morphism*

$$(V \otimes M^{\mathrm{univ,gen}})^N \subset V \otimes M^{\mathrm{univ,gen}} \longrightarrow V \otimes (\mathrm{U}\mathfrak{h})^{\mathrm{gen}},$$

*where the second morphism is induced by the projection $M^{\mathrm{univ,gen}} \to (\mathrm{U}\mathfrak{h})^{\mathrm{gen}}$ onto highest weights, defines a natural isomorphism witnessing commutativity of the diagram*

$$\begin{tikzcd}
\mathrm{Rep}(G) \arrow[r, "\mathrm{free}_G"] \arrow[d] & \mathrm{HC}(G) \arrow[d, "\mathrm{res}^{\mathrm{gen}}"] \\
\mathrm{Rep}(H) \arrow[r, "\mathrm{free}_H"] & \mathrm{HC}(H)^{\mathrm{gen}}
\end{tikzcd}$$

Proof. Let $M_\lambda$ be the Verma module of a generic highest weight $\lambda \in \mathfrak{h}^*$ and denote by $x_\lambda \in M_\lambda$ the highest weight vector. We have to show that the map $(V \otimes M_\lambda)^N \to V$ given by

$$v \otimes x_\lambda + \cdots \mapsto v \otimes 1,$$

where $\ldots$ contain elements of $M_\lambda$ of weight less than $\lambda$, is an isomorphism. This is the content of [EV99, Theorem 8]. □

**Remark 2.5.20.** The $(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}$-module $\mathrm{res}^{\mathrm{gen}}(V \otimes \mathrm{U}\mathfrak{g})$ admits *another* natural basis constructed in [Kho04].

Consider $V, W \in \mathrm{Rep}(G)$. Let us recall that Etingof and Varchenko [EV99] have introduced the ***fusion matrix***

$$J^{EV}_{V,W}(\lambda) \colon V \otimes W \to V \otimes W$$

depending rationally on a parameter $\lambda \in \mathfrak{h}^*$ as follows. Consider the Verma module $M_\lambda$ with highest weight $\lambda \in \mathfrak{h}^*$. For $V \in \mathrm{Rep}(G)$ denote by $V = \oplus_{\lambda \in \Lambda} V[\lambda]$ its weight decomposition. Consider a morphism $M_\lambda \to M_\mu \otimes V$. The image of a highest-weight vector $x_\lambda \in M_\lambda$ has the form

$$x_\mu \otimes v + \ldots,$$

where $\ldots$ denote terms containing elements of $M_\mu$ of lower weight. This determines a morphism

$$\mathrm{Hom}_{\mathrm{U}\mathfrak{g}}(M_\lambda, M_\mu \otimes V) \longrightarrow V[\lambda - \mu]. \tag{2.5.4}$$

For generic $\mu$ it is an isomorphism and for $v \in V[\lambda - \mu]$ we denote by $\Phi^v_\lambda \in \mathrm{Hom}_{\mathrm{U}\mathfrak{g}}(M_\lambda, M_\mu \otimes V)$ the preimage of $v$ under this map.

For $v \in V$ and $w \in W$ of weights $\mathrm{wt}(v)$ and $\mathrm{wt}(w)$ consider the composite

$$M_\lambda \xrightarrow{\Phi^v_\lambda} M_{\lambda-\mathrm{wt}(v)} \otimes V \xrightarrow{\Phi^w_{\lambda-\mathrm{wt}(v)}\otimes \mathrm{id}} M_{\lambda-\mathrm{wt}(v)-\mathrm{wt}(w)} \otimes W \otimes V. \tag{2.5.5}$$

The fusion matrix is defined so that this composite is $\Phi_\lambda^{J^{EV}_{W,V}(\lambda)(w\otimes v)}$. By [EV99, Theorem 48] $J^{EV}_{W,V}(\lambda)$ quantizes the standard rational solution of the dynamical Yang–Baxter equation (see [EV98a, Theorem 3.2]).

Combining corollary 2.5.18 and theorem 2.5.19 we obtain a monoidal structure on the composite

$$\mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(H) \longrightarrow \mathrm{HC}(H)^{\mathrm{gen}}.$$

In particular, as in proposition 2.3.1 this gives rise to linear maps

$$J_{V,W}(\lambda)\colon V\otimes W \to V\otimes W$$

depending rationally on $\lambda \in \mathfrak{h}^*$.

**Proposition 2.5.21.** *Let $V, W \in \mathrm{Rep}(G)$. The map $J_{V,W}(\lambda)\colon V\otimes W \to V\otimes W$ coincides with a permutation of the fusion matrix:*

$$J_{V,W}(\lambda) = \tau J^{EV}_{W,V}(\lambda)\tau,$$

*where $\tau$ is the flip of tensor factors.*

Proof. Let $x^{\mathrm{univ}} \in M^{\mathrm{univ}}$ be the generator of the universal Verma module and $x_\lambda \in M_\lambda$ be the generator of the Verma module of highest weight $\lambda$. Using the PBW identification $M^{\mathrm{univ}} \cong \mathrm{U}\mathfrak{n}_- \otimes \mathrm{U}\mathfrak{h}$ we identify elements of $M^{\mathrm{univ}}$ with functions $\mathfrak{h}^* \to \mathrm{U}\mathfrak{n}_-$.

For $v \in V$ we denote by $\sum v_i \otimes a_i x^{\mathrm{univ}}$ the unique highest-weight element of $V \otimes M^{\mathrm{univ}}$ which has an expansion $v \otimes x^{\mathrm{univ}} + \dots$. Similarly, for $w \in W$ we denote by $\sum w_i \otimes b_i x^{\mathrm{univ}} = w \otimes x^{\mathrm{univ}} + \dots$ the highest-weight element of $W \otimes M^{\mathrm{univ}}$.

Under the morphism (2.5.3)

$$(V\otimes M^{\mathrm{univ}})^N \otimes_{\mathrm{U}\mathfrak{h}} (W \otimes M^{\mathrm{univ}})^N \longrightarrow (V\otimes W\otimes M^{\mathrm{univ}})^N$$

we have

$$\sum_{i,j} (v_i \otimes a_i x^{\mathrm{univ}}) \otimes (w_j \otimes b_j x^{\mathrm{univ}}) \mapsto \sum_{i,j} v_i \otimes (a_i)_{(1)} w_j \otimes (a_i)_{(2)} b_j x^{\mathrm{univ}}.$$

It is then easy to see that

$$J_{V,W}(\lambda)(v\otimes w) = \sum_i v_i \otimes a_i(\lambda - \mathrm{wt}(v)) w.$$

Using the same notations, the map $\Phi^v_\lambda\colon M_\lambda \to M_{\lambda-\mathrm{wt}(v)} \otimes V$ is

$$x_\lambda \mapsto \sum_i a_i(\lambda - \mathrm{wt}(v)) x_{\lambda - \mathrm{wt}(v)} \otimes v_i.$$

Therefore, the composite (2.5.5) is

$$\begin{aligned} x_\lambda &\mapsto \sum_i a_i(\lambda - \mathrm{wt}(v)) x_{\lambda-\mathrm{wt}(v)} \otimes v_i \\ &\mapsto \sum_{i,j} a_i(\lambda - \mathrm{wt}(v))_{(1)} b_j(\lambda - \mathrm{wt}(v) - \mathrm{wt}(w)) x_{\lambda-\mathrm{wt}(v)-\mathrm{wt}(w)} \otimes a_i(\lambda-\mathrm{wt}(v))_{(2)} w_j \otimes v_i. \end{aligned}$$

The resulting element of $M_{\lambda-\mathrm{wt}(v)-\mathrm{wt}(w)} \otimes W \otimes V$ is

$$\sum_i x_{\lambda-\mathrm{wt}(v)-\mathrm{wt}(w)} \otimes a_i(\lambda - \mathrm{wt}(v)) w \otimes v_i + \dots$$

which proves the claim. ☐

Moreover, in [EV99, Section 5] Etingof and Varchenko have introduced an $\mathfrak{h}$-bialgebroid $F(G)$. Recall that Harish-Chandra bialgebroids are examples of $\mathfrak{h}$-bialgebroids.

Theorem 2.5.22. *Consider the monoidal functor*

$$\mathrm{Rep}(G) \xrightarrow{\mathrm{free}_G} \mathrm{HC}(G) \xrightarrow{\mathrm{res}^{\mathrm{gen}}} \mathrm{HC}(H)^{\mathrm{gen}}.$$

*It admits a colimit-preserving right adjoint; denote by $B \in \mathrm{HC}(H)^{\mathrm{gen}} \otimes \mathrm{HC}(H)^{\mathrm{gen}}$ the Harish-Chandra bialgebroid corresponding to this monoidal functor constructed in theorem 2.2.35. Then we have an isomorphism of $\mathfrak{h}$-bialgebroids*

$$B \otimes_{(\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \otimes (\mathrm{U}\mathfrak{h})^{\mathrm{gen}}} (\mathrm{Frac}(\mathrm{U}\mathfrak{h}) \otimes \mathrm{Frac}(\mathrm{U}\mathfrak{h})) \cong F(G).$$

Proof. By theorem 2.5.19 the functor $\mathrm{Rep}(G) \to \mathrm{HC}(H)^{\mathrm{gen}}$ factors as

$$\mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(H) \longrightarrow \mathrm{HC}(H)^{\mathrm{gen}}.$$

Under this composite a finite-dimensional $G$-representation $V \in \mathrm{Rep}(G)$ is sent to a compact projective object $(\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \otimes V \in \mathrm{HC}(H)^{\mathrm{gen}}$, so this functor admits a colimit-preserving right adjoint.

Since $G$ is semisimple, by theorem 2.3.9 the Harish-Chandra bialgebroid $B$ is isomorphic to

$$\bigoplus_{V \in \mathrm{Irr}(G)} ((\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \otimes V^\vee) \boxtimes ((\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \otimes V) \in \mathrm{HC}(H)^{\mathrm{gen}} \otimes \mathrm{HC}(H)^{\mathrm{gen}},$$

where the sum is over isomorphism classes of irreducible finite-dimensional $G$-representations. In particular, we get an isomorphism of $(\mathrm{Frac}(\mathrm{U}\mathfrak{h}), \mathrm{Frac}(\mathrm{U}\mathfrak{h}))$-bimodules

$$B \otimes_{(\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \otimes (\mathrm{U}\mathfrak{h})^{\mathrm{gen}}} (\mathrm{Frac}(\mathrm{U}\mathfrak{h}) \otimes \mathrm{Frac}(\mathrm{U}\mathfrak{h})) \cong F(G).$$

In the notations of theorem 2.3.9 and [EV99, Section 5], the isomorphism is given by

$$\begin{aligned} T_V &\mapsto L^V, \\ t(f(\lambda)) &\mapsto f(\lambda^1), \\ s(f(\lambda)) &\mapsto f(\lambda^2). \end{aligned}$$

It is clear that this isomorphism preserves coproduct, counit and unit and the only nontrivial check is that the product is preserved as well. The relations (18), (19) in *loc. cit.* are clearly satisfied. For (20), the claim follows from proposition 2.5.21. ☐

**2.5.2. Quantum parabolic restriction.** In this section we define parabolic restriction in the setting of quantum groups; we use the notation from section 2.2.4.

**Definition 2.5.23.** The ***universal quantum category*** $\mathcal{O}$ is the category $\mathcal{O}_q^{\mathrm{univ}}$ of $(\mathrm{U}_q(\mathfrak{g}), \mathrm{U}_q(\mathfrak{h}))$-bimodules whose diagonal $\mathrm{U}_q(\mathfrak{b})$-action is integrable. The ***universal quantum Verma module*** is the object

$$M^{\mathrm{univ}} = \mathrm{U}_q(\mathfrak{g}) \otimes_{\mathrm{U}_q(\mathfrak{b})} \mathrm{U}_q(\mathfrak{h}) \in \mathcal{O}_q^{\mathrm{univ}}.$$

**Remark 2.5.24.** As in the classical case, we may identify $\mathcal{O}_q^{\mathrm{univ}}$ with the full subcategory of $\mathrm{LMod}_{\mathrm{U}_q(\mathfrak{g})}(\mathrm{Rep}_q(H))$ of $\mathrm{U}_q(\mathfrak{g})$-modules whose $\mathrm{U}_q(\mathfrak{n})$-action is locally finite.

We will now define a quantum analog of the bimodules (2.5.1):

$$\mathrm{HC}_q(G) \curvearrowright \mathcal{O}_q^{\mathrm{univ}} \curvearrowleft \mathrm{HC}_q(H). \tag{2.5.6}$$

**Lemma 2.5.25.** *Suppose $X \in \mathrm{HC}_q(G)$. The left $\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}$-module structure on $X \otimes_{\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}} \mathrm{U}_q(\mathfrak{g})$ has a canonically extension to a $\mathrm{U}_q(\mathfrak{g})$-module structure. Moreover, the left $\mathrm{U}_q(\mathfrak{n})$-action on $X \otimes_{\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}} M^{\mathrm{univ}}$ is locally finite.*

Proof. Recall from remark 2.2.4 that the left action of $a \in \mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}$ on $x \in X$ is

$$a \triangleright x = (\mathrm{ad}\, a_{(1)})(x) \triangleleft a_{(2)},$$

where ad refers to the diagonal $\mathrm{U}_q(\mathfrak{g})$-action on $X$. So, we may extend the left $\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}$-action on the relative tensor product $X \otimes_{\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}} \mathrm{U}_q(\mathfrak{g})$ to a $\mathrm{U}_q(\mathfrak{g})$-action by the formula

$$a \triangleright (x \otimes h) = (\mathrm{ad}\, a_{(1)})(x) \otimes a_{(2)} h$$

for $a \in \mathrm{U}_q(\mathfrak{g})$ an $x \otimes h \in X \otimes_{\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}} \mathrm{U}_q(\mathfrak{g})$. It is well-defined (i.e. descends to the relative tensor product) using the formula $(\mathrm{ad}\, a_{(1)})(l)a_{(2)} = al$ for any $a \in \mathrm{U}_q(\mathfrak{g})$ and $l \in \mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}$.

The diagonal $\mathrm{U}_q(\mathfrak{n})$-action on $X \otimes_{\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}} M^{\mathrm{univ}}$ is locally finite since it is so on $X$ and $M^{\mathrm{univ}}$. □

A $\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}$-bimodule $X \in \mathrm{HC}_q(G)$ acts on a $(\mathrm{U}_q(\mathfrak{g}), \mathrm{U}_q(\mathfrak{h}))$-bimodule $M \in \mathcal{O}_q^{\mathrm{univ}}$ via

$$X, M \mapsto X \otimes_{\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}} M.$$

By construction it is a $(\mathrm{U}_q(\mathfrak{g}), \mathrm{U}_q(\mathfrak{h}))$-bimodule. Since the diagonal $\mathrm{U}_q(\mathfrak{n})$-action on $X$ and the left $\mathrm{U}_q(\mathfrak{n})$-action on $M$ are locally finite, so is the left $\mathrm{U}_q(\mathfrak{n})$-action on this bimodule. In particular, it lies in $\mathcal{O}_q^{\mathrm{univ}}$.

For a $\mathrm{U}_q(\mathfrak{h})$-bimodule $X \in \mathrm{HC}_q(H)$ and a $(\mathrm{U}_q(\mathfrak{g}), \mathrm{U}_q(\mathfrak{h}))$-bimodule $M \in \mathcal{O}_q^{\mathrm{univ}}$ the action is

$$M, X \mapsto M \otimes_{\mathrm{U}_q(\mathfrak{h})} X.$$

Let

$$\mathrm{act}_G \colon \mathrm{HC}_q(G) \longrightarrow \mathcal{O}_q^{\mathrm{univ}}, \qquad \mathrm{act}_H \colon \mathrm{HC}_q(H) \longrightarrow \mathcal{O}_q^{\mathrm{univ}}$$

be the actions of $\mathrm{HC}_q(G)$ and $\mathrm{HC}_q(H)$ on the universal Verma module $M^{\mathrm{univ}}$.

**Definition 2.5.26.** The ***parabolic restriction*** and ***parabolic induction*** are the lax monoidal functors

$$\mathrm{res} = \mathrm{act}_H^{\mathrm{R}} \circ \mathrm{act}_G \colon \mathrm{HC}_q(G) \longrightarrow \mathrm{HC}_q(H)$$
$$\mathrm{ind} = \mathrm{act}_G^{\mathrm{R}} \circ \mathrm{act}_H \colon \mathrm{HC}_q(H) \longrightarrow \mathrm{HC}_q(G).$$

We have a functor

$$(-)^{\mathrm{U}_q(\mathfrak{n})} \colon \mathcal{O}_q^{\mathrm{univ}} \longrightarrow \mathrm{HC}_q(H)$$

of $\mathrm{U}_q(\mathfrak{n})$-invariants.

**Proposition 2.5.27.** *The functor* $(-)^{\mathrm{U}_q(\mathfrak{n})} \colon \mathcal{O}_q^{\mathrm{univ}} \to \mathrm{HC}_q(H)$ *is right adjoint to* $\mathrm{act}_H \colon \mathrm{HC}_q(H) \to \mathcal{O}_q^{\mathrm{univ}}$*.*

Proof. For $M \in \mathcal{O}_q^{\mathrm{univ}}$ and $X \in \mathrm{HC}_q(H)$ we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_q^{\mathrm{univ}}}(\mathrm{act}_H(X), M) &= \mathrm{Hom}_{\mathcal{O}_q^{\mathrm{univ}}}(\mathrm{U}_q(\mathfrak{g}) \otimes_{\mathrm{U}_q(\mathfrak{b})} X, M)\\ &\cong \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{b})\mathrm{BMod}_{\mathrm{U}_q(\mathfrak{h})}}(X, M)\\ &\cong \mathrm{Hom}_{\mathrm{HC}_q(H)}(X, M^{\mathrm{U}_q(n)}).\end{aligned}$$

□

**Proposition 2.5.28.** *The natural morphism* $\mathrm{U}_q(\mathfrak{h}) \to \mathrm{res}(\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}})$ *is an isomorphism.*

Proof. The proof is similar to the proof of proposition 2.5.9, where we again use the fact that the quantum Verma module is irreducible for generic parameters [VY20, Theorem 4.15]. □

A weight for a $\mathrm{U}_q(\mathfrak{g})$-module is specified by an element of $H(k) \cong \mathrm{Hom}(\Lambda, k^\times)$. We will use an additive notation for weights, so that a vector $v$ of weight $\lambda$ satisfies $K_\mu v = q^{(\lambda,\mu)}v$. For a root $\alpha$ we denote $q_\alpha = q^{(\alpha,\alpha)/2}$.

**Definition 2.5.29.** A weight $\lambda$ is ***generic*** if $q^{(\alpha,\lambda)} \notin \pm q_\alpha^{\mathbf{Z}}$ for every root $\alpha$. Denote by $H^{\mathrm{gen}} \subset H$ the subset of generic weights. We denote by $\mathrm{HC}_q^{\mathrm{gen}}(H) \subset \mathrm{HC}_q(H)$ and $\mathcal{O}_q^{\mathrm{univ,gen}} \subset \mathcal{O}_q^{\mathrm{univ}}$ the full subcategories of modules with generic $\mathrm{U}_q(\mathfrak{h})$-weights. Let $\mathrm{U}_q(\mathfrak{h})^{\mathrm{gen}} \subset \mathrm{Frac}(\mathrm{U}_q(\mathfrak{h}))$ be the subspace of rational functions on $H$ regular on $H^{\mathrm{gen}}$.

We denote by

$$M^{\mathrm{univ,gen}} = \mathrm{U}_q(\mathfrak{g}) \otimes_{\mathrm{U}_q(\mathfrak{b})} \mathrm{U}_q(\mathfrak{h})^{\mathrm{gen}}$$

the universal quantum Verma module with generic highest weights.

A generalization of the extremal projector to quantum groups was introduced in [KT92].

Theorem 2.5.30. *There is an extension* $T_q(\mathfrak{g})$ *of* $\mathrm{U}_q(\mathfrak{g})$ *obtained by replacing* $\mathrm{U}_q(\mathfrak{h}) \subset \mathrm{U}_q(\mathfrak{g})$ *with* $\mathrm{Frac}(\mathrm{U}_q(\mathfrak{h}))$ *and considering certain power series. There is an element* $P \in T_q(\mathfrak{g})$ *satisfying the following properties:*

(1) $\mathrm{U}_q^{>0}(\mathfrak{n})P = P\mathrm{U}_q^{>0}(\mathfrak{n}_-) = 0.$

(2) $P-1\in T_q(\mathfrak{g})\mathrm{U}_q^{>0}(\mathfrak{n})\cap \mathrm{U}_q^{>0}(\mathfrak{n}_-)T_q(\mathfrak{g})$.

*The action of $P$ is well-defined on left $\mathrm{U}_q(\mathfrak{g})$-modules whose $\mathrm{U}_q(\mathfrak{n})$-action is locally nilpotent and which have generic $\mathrm{U}_q(\mathfrak{h})$-weights.*

**Example 2.5.31.** Consider $\mathrm{U}_q(\mathfrak{sl}_2)$ with generators $E, K, F$ as in example 2.2.21. Let $[n]=\frac{q^n-q^{-n}}{q-q^{-1}}$ be the quantum integer, $[n]!=\prod_{j=1}^n [j]!$ the quantum factorial and

$$[h+n]=\frac{Kq^n-K^{-1}q^{-n}}{q-q^{-1}}\in \mathrm{U}_q(\mathfrak{sl}_2)$$

for $n\in\mathbf{Z}$. Then the extremal projector is (see e.g. [KO08, Section 9])

$$P=\sum_{n=0}^\infty \frac{(-1)^n}{[n]!}g_n^{-1}F^nE^n,$$

where $g_n=\prod_{j=1}^n [h+j+1]$.

Completely analogously to the proof of theorem 2.5.17, one proves the following statement.

Theorem 2.5.32. *The category $\mathcal{O}_q^{\mathrm{univ},\mathrm{gen}}$ is free of rank 1 as a $\mathrm{HC}_q(H)^{\mathrm{gen}}$-module category.*

**Corollary 2.5.33.** *The functor $\mathrm{res}^{\mathrm{gen}}\colon \mathrm{HC}_q(G)\to \mathrm{HC}_q(H)^{\mathrm{gen}}$ is strongly monoidal and colimit-preserving.*

Similarly to the classical case, parabolic restriction of a free Harish-Chandra bimodule is free.

Theorem 2.5.34. *For every $V\in \mathrm{Rep}_q(G)$ the morphism*

$$(V\otimes M^{\mathrm{univ},\mathrm{gen}})^{\mathrm{U}_q(\mathfrak{n})}\subset V\otimes M^{\mathrm{univ},\mathrm{gen}}\longrightarrow V\otimes \mathrm{U}_q(\mathfrak{h})^{\mathrm{gen}},$$

*where the second morphism is induced by the projection $M^{\mathrm{univ},\mathrm{gen}}\to \mathrm{U}_q(\mathfrak{h})^{\mathrm{gen}}$ onto highest weights, defines a natural isomorphism witnessing commutativity of the diagram*

$$\begin{tikzcd} \mathrm{Rep}_q(G) \arrow[r, "\mathrm{free}_G"] \arrow[d] & \mathrm{HC}_q(G) \arrow[d, "\mathrm{res}^{\mathrm{gen}}"] \\ \mathrm{Rep}_q(H) \arrow[r, "\mathrm{free}_H"] & \mathrm{HC}_q(H)^{\mathrm{gen}} \end{tikzcd}$$

Combining corollary 2.5.33 and theorem 2.5.34 we obtain a monoidal structure on the composite

$$\mathrm{Rep}_q(G)\longrightarrow \mathrm{Rep}_q(H)\longrightarrow \mathrm{HC}_q(H)^{\mathrm{gen}}.$$

In particular, by proposition 2.3.1 this gives rise to linear maps

$$J_{V,W}(\lambda)\colon V\otimes W\to V\otimes W,$$

rational functions on $H$.

**Example 2.5.35.** Consider $G=\mathrm{SL}_2$ and $V\in\mathrm{Rep}_q(\mathrm{SL}_2)$ the irreducible two-dimensional representation with the basis $\{v_+,v_-\}$ such that

$$Kv_+=qv_+,\qquad Kv_-=q^{-1}v_+,\qquad Fv_+=v_-.$$

The isomorphism $\mathrm{U}_q(\mathfrak{h})^{\mathrm{gen}}\otimes V\to (V\otimes M^{\mathrm{univ},\mathrm{gen}})^{\mathrm{U}_q(\mathfrak{n})}$ is given by

$$\begin{aligned} 1\otimes v_+ &\mapsto v_+\otimes 1,\\ 1\otimes v_- &\mapsto v_-\otimes 1-q^{-1}v_+\otimes F\cdot [h]^{-1}x^{\mathrm{univ}}. \end{aligned}$$

Then the matrix of $J_{V,V}(\lambda)$ in the basis $\{v_+\otimes v_+, v_+\otimes v_-, v_-\otimes v_+, v_-\otimes v_-\}$ is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & -q^{-\lambda-1}[\lambda+1]^{-1} & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Our convention for the coproduct on $\mathrm{U}_q(\mathfrak{g})$ follows [Lus10, Lemma 3.1.4]. For two $\mathrm{U}_q(\mathfrak{g})$-modules $V, W$ we denote by $V\overline{\otimes}W$ the vector space $V \otimes W$ equipped with the $\mathrm{U}_q(\mathfrak{g})$-module structure via the opposite coproduct:

$$h \triangleright (v \otimes w) = h_{(2)} \triangleright v \otimes h_{(1)} \triangleright w.$$

Consider $V, W \in \mathrm{Rep}_q(G)$. Similarly to the classical case, Etingof and Varchenko [EV99] have introduced the ***fusion matrix*** $J^{EV}_{V,W}(\lambda)\colon V \otimes W \to V \otimes W$, a rational function on $H$, using intertwiners of quantum Verma modules. Note, however, that in our notations they are considering maps

$$\Phi^v_\lambda\colon M_\lambda \longrightarrow M_{\lambda-\mathrm{wt}(v)}\overline{\otimes}V$$

with the property that $\Phi^v_\lambda(x_\lambda) = x_\lambda \otimes v + \dots$ Analogously to proposition 2.5.21 we have the following statement.

**Proposition 2.5.36.** *Let $V, W \in \mathrm{Rep}_q(G)$. The maps $J_{V,W}$ and $J^{EV}_{W,V}$ are related as follows:*

$$J_{V,W}(\lambda) = \tau J^{EV}_{W,V} \tau.$$

In [EV99, Section 5] Etingof and Varchenko have introduced an $\mathfrak{h}$-bialgebroid $F_q(G)$. Analogously to theorem 2.5.22 we obtain the following statement.

THEOREM 2.5.37. *Consider the monoidal functor*

$$\mathrm{Rep}_q(G) \xrightarrow{\mathrm{free}_G} \mathrm{HC}_q(G) \xrightarrow{\mathrm{res}^{\mathrm{gen}}} \mathrm{HC}_q(H)^{\mathrm{gen}}.$$

*It admits a colimit-preserving right adjoint; denote by $B \in \mathrm{HC}_q(H)^{\mathrm{gen}} \otimes \mathrm{HC}_q(H)^{\mathrm{gen}}$ the Harish-Chandra bialgebroid corresponding to this monoidal functor constructed in theorem 2.2.35. Then we have an isomorphism of $\mathfrak{h}$-algebroids*

$$B \otimes_{\mathrm{U}_q(\mathfrak{h})^{\mathrm{gen}} \otimes \mathrm{U}_q(\mathfrak{h})^{\mathrm{gen}}} \otimes (\mathrm{Frac}(\mathrm{U}_q(\mathfrak{h})) \otimes \mathrm{Frac}(\mathrm{U}_q(\mathfrak{h}))) \cong F_q(G).$$

## 2.6. Dynamical Weyl groups

In this section we introduce a Weyl symmetry of the parabolic restriction functors $\mathrm{res}\colon \mathrm{HC}(G) \to \mathrm{HC}(H)$ and $\mathrm{res}\colon \mathrm{HC}_q(G) \to \mathrm{HC}_q(H)$ introduced in section 2.5 and relate it to dynamical Weyl groups.

**2.6.1. Classical Zhelobenko operators.** Fix a reductive group $G$ and its Lie algebra $\mathfrak{g}$ as in section 2.5.1. Recall that the Weyl group is

$$W = \mathrm{N}(H)/H,$$

where $\mathrm{N}(H)$ is the normalizer of $H$ in $G$. Denote by $\hat{W}$ the corresponding braid group generated by simple reflections $s_\alpha \in W$ with the relation (for $\alpha \neq \beta$)

$$\underbrace{s_\alpha s_\beta s_\alpha \dots}_{m_{\alpha\beta}} = \underbrace{s_\beta s_\alpha s_\beta \dots}_{m_{\alpha\beta}},$$

where $m_{\alpha\beta}$ is the Coxeter matrix. There is a canonical map $W \to \hat{W}$ which sends a reduced expression $w = s_1 \cdot \dots \cdot s_n \in W$ to the corresponding element in $\hat{W}$.

We may lift $w \in W$ to elements $T_w \in \mathrm{N}(H)$ satisfying the braid relations. Moreover, for a simple reflection $s_\alpha \in W$ the element $T^2_{s_\alpha} \in H$ has order at most 2 [Tit66]. For concreteness, we assume that the elements $T_w$ act via the $q = 1$ version of Lusztig's operators $T'_{w,1}$ as in [Lus10, Section 5.2.1].

Denote by $\rho \in \mathfrak{h}^*$ the half-sum of positive roots. For an element $w \in W$ and $\lambda \in \mathfrak{h}^*$ denote by $w \cdot \lambda \in \mathfrak{h}^*$ the ***dot action***:

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

The induced action on $h \in \mathfrak{h} \subset \mathrm{U}\mathfrak{h}$ is denoted by

$$w \cdot h = w(h) + \langle h, w(\rho) - \rho \rangle,$$

where the usual $W$ action on $\mathrm{U}\mathfrak{h}$ is simply denoted by $w(d)$ for $d \in \mathrm{U}\mathfrak{h}$.

Recall that for a right $\mathrm{U}\mathfrak{g}$-module $X$, $X \otimes_{\mathrm{U}\mathfrak{g}} M^{\mathrm{univ}} \cong X/X\mathfrak{n}$. In the study of Mickelsson algebras Zhelobenko [Zhe87] has introduced a collection of operators acting on $\mathrm{U}\mathfrak{g}$-bimodules for each element of the Weyl group. We refer to [KO08, Section 6] for the proof of the following results.

Theorem 2.6.1. *Suppose $X \in \mathrm{HC}(G)$. Suppose $\alpha$ is a simple root and denote by $\{e_\alpha, h_\alpha, f_\alpha\}$ the standard generators of the corresponding $\mathfrak{sl}_2$ subalgebra $\mathfrak{g}_\alpha \subset \mathfrak{g}$. Consider the* **Zhelobenko operator** *$\check{q}_\alpha \colon X \to X$ given by an infinite series*

$$\check{q}_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\operatorname{ad} e_\alpha)^n (T_{s_\alpha}(x)) f_\alpha^n g_{n,\alpha}^{-1},$$

*where*

$$g_{n,\alpha} = \prod_{j=1}^{n} (h_\alpha - j + 1)$$

*and $\operatorname{ad} e_\alpha$ refers to the diagonal $\mathfrak{g}$-action on $X$. Then the operators $\check{q}_\alpha$ descend to well-defined linear isomorphisms*

$$\check{q}_\alpha \colon (X \otimes_{\mathrm{U}\mathfrak{g}} M^{\mathrm{univ,gen}})^N \longrightarrow (X \otimes_{\mathrm{U}\mathfrak{g}} M^{\mathrm{univ,gen}})^N$$

*which satisfy the following relations:*

(1) $\check{q}_\alpha((\operatorname{ad} d)(x)) = (\operatorname{ad} s_\alpha(d))(\check{q}_\alpha(x))$ *for every $d \in \mathfrak{h}$ and $x \in X$.*
(2) $\check{q}_\alpha(dx) = (s_\alpha \cdot d)\check{q}_\alpha(x)$ *for every $d \in \mathfrak{h}$ and $x \in X$.*
(3) $\underbrace{\check{q}_\alpha \check{q}_\beta \check{q}_\alpha \dots}_{m_{\alpha\beta}} = \underbrace{\check{q}_\beta \check{q}_\alpha \check{q}_\beta \dots}_{m_{\alpha\beta}}$ *for $\alpha \neq \beta$.*
(4) $\check{q}_\alpha^2(x) = (h_\alpha + 1)^{-1} T_{s_\alpha}^2(x)(h_\alpha + 1)$ *for every $x \in X$.*

For an element $w \in W$ with a reduced decomposition $w = s_{\alpha_1} \dots s_{\alpha_n}$ we define

$$\check{q}_w = \check{q}_{\alpha_{i_1}} \dots \check{q}_{\alpha_{i_n}}.$$

The third relation in theorem 2.6.1 shows that $\check{q}_w$ is independent of the chosen decomposition.

In addition, we have the following important multiplicativity property of the Zhelobenko operators proven in [KO08, Theorem 3].

Theorem 2.6.2. *Let $X, Y \in \mathrm{HC}(G)$ and take $x \in X$ and $y \in Y$, where $\mathfrak{n}y \in Y\mathfrak{n}$. Then we have an equality*

$$\check{q}_w(x \otimes y) = \check{q}_w(x) \otimes \check{q}_w(y)$$

*in $X \otimes_{\mathrm{U}\mathfrak{g}} Y \otimes_{\mathrm{U}\mathfrak{g}} M^{\mathrm{univ,gen}}$.*

**2.6.2. Classical dynamical Weyl group.** Given a group $G$ we may regard it as a discrete (i.e. having no morphisms except for identities) monoidal category $\mathrm{Cat}(G)$. Let us recall the notion of a $G$-action on a monoidal category and the category of $G$-equivariant objects (see e.g. [Eti+15, Section 2.7]).

**Definition 2.6.3.** Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}$ be a monoidal category. A ***G*-*action on*** $\mathcal{C}$ is a monoidal functor

$$\mathrm{Cat}(G) \longrightarrow \mathrm{Fun}^{\mathrm{L},\otimes}(\mathcal{C}, \mathcal{C})$$

to the monoidal category of monoidal colimit-preserving endofunctors on $\mathcal{C}$.

Explicitly, for every element $g \in G$ we have a monoidal functor $S_g \colon \mathcal{C} \to \mathcal{C}$ together with a natural isomorphism $S_e \cong \mathrm{id}$ and natural isomorphisms $S_{gh} \cong S_g \circ S_h$ for a pair of elements $g, h \in G$ satisfying an associativity axiom.

**Definition 2.6.4.** Suppose $\mathcal{C}$ is a monoidal category with a $G$-action. A ***G*-*equivariant object*** is an object $x \in \mathcal{C}$ equipped with isomorphisms $S_g(x) \cong x$ compatible with the isomorphisms $S_{gh} \cong S_g \circ S_h$ and $S_e \cong \mathrm{id}$. We denote by $\mathcal{C}^G$ the category of $G$-equivariant objects.

The category $\mathrm{HC}(H) \cong \mathrm{LMod}_{\mathrm{U}\mathfrak{h}}(\mathrm{Rep}\, H)$ carries a natural action of the Weyl group $W$ defined as follows. Let us regard $X \in \mathrm{HC}(H)$ as a $\mathrm{U}\mathfrak{h}$-bimodule. Then the action of $w \in W$ twists the left and the right $\mathrm{U}\mathfrak{h}$-action by the dot action: $S_w(X) = X$ as a plain vector space with the $\mathrm{U}\mathfrak{h}$-bimodule structure given by

$$d \triangleright^w x = (w \cdot d) \triangleright x, \qquad x \triangleleft^w d = x \triangleleft (w \cdot d)$$

for $x \in X$ and $d \in \mathrm{U}\mathfrak{h}$. The dot action of $W$ on $\mathfrak{h}$ is given by affine transformations, so the corresponding diagonal $\mathfrak{h}$-action on $S_w(X)$ is given by its linear part, i.e. we twist the diagonal $\mathfrak{h}$-action on $X$ by the usual $W$-action. By construction $S_e = \mathrm{id}$ and $S_{w_1 w_2} = S_{w_1} \circ S_{w_2}$. Moreover, the identity map of vector spaces

$$S_w(X) \otimes_{\mathrm{U}\mathfrak{h}} S_w(Y) \longrightarrow S_w(X \otimes_{\mathrm{U}\mathfrak{h}} Y)$$

together with the dot action

$$\mathrm{U}\mathfrak{h} \longrightarrow S_w(\mathrm{U}\mathfrak{h})$$

define a monoidal structure on the collection $\{S_w\}_{w \in W}$.

The functor

$$\mathrm{free}\colon \mathrm{Rep}(H) \longrightarrow \mathrm{HC}(H)$$

is naturally $W$-equivariant, where the maps

$$\mathrm{U}\mathfrak{h} \otimes S_w(V) \to S_w(\mathrm{U}\mathfrak{h} \otimes V) \tag{2.6.1}$$

are given by the dot action on the $\mathrm{U}\mathfrak{h}$ factor.

Restricting the $W$-action on $\mathrm{HC}(H)$ under the quotient map $\hat{W} \to W$ from the braid group we obtain a natural action of $\hat{W}$ on $\mathrm{HC}(H)$.

Recall that by corollary 2.5.18 the parabolic restriction functor

$$\mathrm{res}^{\mathrm{gen}}\colon \mathrm{HC}(G) \longrightarrow \mathrm{HC}(H)^{\mathrm{gen}}$$

given by $X \mapsto (X \otimes_{\mathrm{U}\mathfrak{g}} M^{\mathrm{univ}})^N$ is monoidal. We will now show that it factors through $\hat{W}$-invariants.

Theorem 2.6.5. *The Zhelobenko operators define a factorization*

$$\begin{tikzcd} \mathrm{HC}(G) \arrow[r, dashed] \arrow[dr, "\mathrm{res}^{\mathrm{gen}}"'] & \mathrm{HC}(H)^{\mathrm{gen},\hat{W}} \arrow[d] \\ & \mathrm{HC}(H)^{\mathrm{gen}} \end{tikzcd}$$

*of* $\mathrm{res}^{\mathrm{gen}}\colon \mathrm{HC}(G) \to \mathrm{HC}(H)^{\mathrm{gen}}$ *through a monoidal functor* $\mathrm{res}^{\mathrm{gen}}\colon \mathrm{HC}(G) \to \mathrm{HC}(H)^{\mathrm{gen},\hat{W}}$.

Proof. Let us first construct a factorization of $\mathrm{res}^{\mathrm{gen}}$ through $\mathrm{HC}(H)^{\mathrm{gen},\hat{W}} \to \mathrm{HC}(H)^{\mathrm{gen}}$ as a plain (non-monoidal) functor. Since the braid group $\hat{W}$ is generated by simple reflections $\{s_\alpha\}$, for $X \in \mathrm{HC}(G)$ we have to specify natural isomorphisms

$$\mathrm{res}^{\mathrm{gen}}(X) \xrightarrow{\sim} S_{s_\alpha}(\mathrm{res}^{\mathrm{gen}}(X))$$

satisfying the braid relations. We define them to be the Zhelobenko operators $\check{q}_\alpha$. The compatibility with the $\mathrm{U}\mathfrak{h}$-bimodule action follows from parts (1) and (2) of theorem 2.6.1. The braid relations follow from part (3) of the same theorem.

Next, we have to construct a monoidal structure on $\mathrm{HC}(G) \to \mathrm{HC}(H)^{\mathrm{gen},\hat{W}}$ compatible with the one on $\mathrm{res}^{\mathrm{gen}}\colon \mathrm{HC}(G) \to \mathrm{HC}(H)^{\mathrm{gen}}$ which we recall is given by (2.5.3). The unit map is the natural inclusion $\mathrm{U}\mathfrak{h} \hookrightarrow (M^{\mathrm{univ}})^N$.

We begin by showing compatibility with the tensor products. By proposition 2.5.16 the functor of $N$-invariants $\mathcal{O}^{\mathrm{univ}} \to \mathrm{HC}(H)^{\mathrm{gen}}$ is exact. In particular, we may exchange the order of left $N$-invariants and right $\mathfrak{n}$-coinvariants in the definition of $\mathrm{res}(X) = (X/X\mathfrak{n})^N$. But then the diagram

$$\begin{tikzcd} \mathrm{res}^{\mathrm{gen}}(X) \otimes_{(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}} \mathrm{res}^{\mathrm{gen}}(Y) \arrow[r] \arrow[d, "\check{q}_\alpha \otimes \check{q}_\alpha"] & \mathrm{res}^{\mathrm{gen}}(X \otimes_{\mathrm{U}\mathfrak{g}} Y) \arrow[d, "\check{q}_\alpha"] \\ S_{s_\alpha}(\mathrm{res}^{\mathrm{gen}}(X)) \otimes_{(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}} S_{s_\alpha}(\mathrm{res}^{\mathrm{gen}}(Y)) \arrow[r] & S_{s_\alpha}(\mathrm{res}^{\mathrm{gen}}(X \otimes_{\mathrm{U}\mathfrak{g}} Y)) \end{tikzcd} \tag{2.6.2}$$

is commutative by theorem 2.6.2.

Next, we have to show compatibility with the unit maps. Consider the diagram

$$\begin{tikzcd}
\mathrm{U}\mathfrak{h} \arrow[r] \arrow[d] & (M^{\mathrm{univ}})^N \arrow[d, "\check{q}_\alpha"] \\
S_{s_\alpha}(\mathrm{U}\mathfrak{h}) \arrow[r] & S_{s_\alpha}((M^{\mathrm{univ}})^N)
\end{tikzcd}$$

To show that it is commutative, we have to compute the action of $\check{q}_\alpha$ on $\mathrm{U}\mathfrak{h} \hookrightarrow M^{\mathrm{univ}}$. By part (2) of theorem 2.6.1 $\check{q}_\alpha(d\cdot 1) = (s_\alpha \cdot d)\check{q}_\alpha(1)$, where $d \in \mathrm{U}\mathfrak{h}$ and $1 \in \mathrm{U}\mathfrak{g}$ is the unit. But it is obvious from the explicit formula for $\check{q}_\alpha$ that $\check{q}_\alpha(1) = 1$. ☐

Let us now analyze the composite monoidal functor

$$\mathrm{Rep}(G) \xrightarrow{\mathrm{free}_G} \mathrm{HC}(G) \xrightarrow{\mathrm{res}^{\mathrm{gen}}} \mathrm{HC}(H)^{\mathrm{gen},\hat{W}}.$$

Recall that by theorem 2.5.19 we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}(G) \arrow[r, "\mathrm{free}_G"] \arrow[d] & \mathrm{HC}(G) \arrow[d, "\mathrm{res}^{\mathrm{gen}}"] \\
\mathrm{Rep}(H) \arrow[r, "\mathrm{free}_H"] & \mathrm{HC}(H)^{\mathrm{gen}}
\end{tikzcd}$$

of plain (non-monoidal) categories.

Consider $V \in \mathrm{Rep}(G)$. Using the natural isomorphism

$$\mathrm{res}^{\mathrm{gen}}(\mathrm{U}\mathfrak{g} \otimes V) \cong (\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \otimes V$$

in $\mathrm{HC}(H)^{\mathrm{gen}}$ provided by the above diagram we obtain that the $\hat{W}$-invariance of $\mathrm{res}^{\mathrm{gen}}(\mathrm{U}\mathfrak{g} \otimes V)$ boils down to maps $(\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \otimes V \to (\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \otimes S_w(V)$ obtained via the composite

$$(\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \otimes V \xleftarrow{\sim} \mathrm{res}^{\mathrm{gen}}(\mathrm{U}\mathfrak{g} \otimes V) \xrightarrow{\check{q}_w} S_w(\mathrm{res}^{\mathrm{gen}}(\mathrm{U}\mathfrak{g} \otimes V)) \xrightarrow{\sim} S_w((\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \otimes V) \xrightarrow{\sim} (\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \otimes S_w(V).$$

Such maps are uniquely determined by their value on $1 \otimes v$, which gives linear maps

$$A_{w,V}(\lambda)\colon V \longrightarrow V$$

depending rationally on a parameter $\lambda \in \mathfrak{h}^*$.

Let $V, U \in \mathrm{Rep}(G)$ and recall the matrix $J_{V,U}(\lambda)\colon V \otimes U \to V \otimes U$ defined in section 2.5.1 which controls the monoidal structure on the composite $\mathrm{Rep}(G) \to \mathrm{Rep}(H) \xrightarrow{\mathrm{free}_H} \mathrm{HC}(H)^{\mathrm{gen}}$.

**Proposition 2.6.6.** *For any simple reflection $s_\alpha$ and $V, U \in \mathrm{Rep}(G)$ we have an equality*

$$A_{s_\alpha, V\otimes U}(\lambda) J_{V,U}(\lambda) = J_{V,U}(s_\alpha \cdot \lambda) A^{(1)}_{s_\alpha, U}(\lambda) A^{(2)}_{s_\alpha, U}(\lambda - h^{(1)})$$

*of rational functions $\mathfrak{h}^* \to \mathrm{End}(V \otimes U)$, where $A^{(1)}$ denotes $A \otimes 1$ and $A^{(2)}$ denotes $1 \otimes A$.*

Proof. Consider the diagram

$$\begin{tikzcd}
((\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \otimes V) \otimes_{(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}} ((\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \otimes U) \arrow[r] \arrow[d] & (\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \otimes V \otimes U \arrow[d] \\
\mathrm{res}^{\mathrm{gen}}(\mathrm{U}\mathfrak{g} \otimes V) \otimes_{(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}} \mathrm{res}^{\mathrm{gen}}(\mathrm{U}\mathfrak{g} \otimes U) \arrow[r] \arrow[d, "\check{q}_\alpha \otimes \check{q}_\alpha"] & \mathrm{res}^{\mathrm{gen}}((\mathrm{U}\mathfrak{g} \otimes V) \otimes_{\mathrm{U}\mathfrak{g}} (\mathrm{U}\mathfrak{g} \otimes U)) \arrow[d, "\check{q}_\alpha"] \\
S_{s_\alpha}(\mathrm{res}^{\mathrm{gen}}(\mathrm{U}\mathfrak{g} \otimes V)) \otimes_{(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}} S_{s_\alpha}(\mathrm{res}^{\mathrm{gen}}(\mathrm{U}\mathfrak{g} \otimes U)) \arrow[r] \arrow[d] & S_{s_\alpha}(\mathrm{res}^{\mathrm{gen}}((\mathrm{U}\mathfrak{g} \otimes V) \otimes_{\mathrm{U}\mathfrak{g}} (\mathrm{U}\mathfrak{g} \otimes U))) \arrow[d] \\
(\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \otimes V \otimes U \arrow[r] & (\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \otimes V \otimes U
\end{tikzcd}$$

where the middle square is (2.6.2).

The left vertical arrow is $A^{(1)}_{s_\alpha,V}(\lambda)A^{(2)}_{s_\alpha,U}(\lambda-h^{(1)})$ and the right vertical arrow is $A_{s_\alpha,V\otimes U}(\lambda)$. Using the isomorphism (2.6.1) the bottom horizontal arrow is $J_{V,U}(s_\alpha\cdot\lambda)$. $\square$

Let us now compute a particular example of the operators $A_{w,V}(\lambda)$. Consider $G=\mathrm{SL}_2$, $V$ the two-dimensional irreducible representation, $H\subset G$ the subgroup of diagonal matrices and $w$ the unique simple reflection. We can lift it to the matrix $T\in\mathrm{N}(H)$ given by

$$T=\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}.$$

Let $\{e,h,f\}$ be the standard basis of $\mathfrak{sl}_2$. Let $\{v_+,v_-\}$ be the basis of $V$ such that

$$hv_+=v_+,\qquad hv_-=-v_-,\qquad fv_+=v_-.$$

**Proposition 2.6.7.** *The action of $A_{w,V}(\lambda)$ is given as follows:*

$$\begin{aligned} A_{w,V}(\lambda)v_+ &= v_-\\ A_{w,V}(\lambda)v_- &= -\frac{\lambda+2}{\lambda+1}v_+\end{aligned}$$

Proof. The isomorphism $(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}\otimes V\to (V\otimes M^{\mathrm{univ,gen}})^N$ is given by

$$\begin{aligned} 1\otimes v_+ &\mapsto v_+\otimes x^{\mathrm{univ}},\\ 1\otimes v_- &\mapsto v_-\otimes x^{\mathrm{univ}} - v_+\otimes fh^{-1}x^{\mathrm{univ}},\end{aligned}$$

where $x^{\mathrm{univ}}\in M^{\mathrm{univ,gen}}$ is the generator. We have

$$\check{q}_w(v_+\otimes 1)=\sum_n \frac{(-1)^n}{n!}\,\mathrm{ad}_e^n(v_-\otimes 1)f^n g_n^{-1}=v_-\otimes 1-v_+\otimes fh^{-1},$$

hence $A_{w,V}(\lambda)(v_+)=v_-$. To compute $A_{w,V}(\lambda)(v_-)$, we use property (4) from theorem 2.6.1, namely,

$$\check{q}_w(v_-\otimes 1-v_+\otimes fh^{-1})=\check{q}_w^2(v_+\otimes 1)=-(h+1)^{-1}(v_+\otimes 1)(h+1)=-h(h+1)^{-1}(v_+\otimes 1).$$

Under identification $S_w(\mathrm{res}^{\mathrm{gen}}(V\otimes\mathrm{U}\mathfrak{g}))\cong(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}\otimes S_w(V)$, we have

$$\check{q}_w(v_-\otimes 1-v_+\otimes fh^{-1})\mapsto -\frac{w\cdot\lambda}{w\cdot\lambda+1}\otimes v_+=-\frac{\lambda+2}{\lambda+1}\otimes v_+,$$

and the claim follows.

$\square$

We return to the case of arbitrary $G$. Recall that Tarasov and Varchenko [TV00] have introduced the dynamical Weyl group, i.e. a collection of operators $A^{TV}_{w,V}(\lambda)\colon V\to V$ for every finite-dimensional $\mathfrak{g}$-representation $V$ and $w\in W$ depending rationally on the parameter $\lambda\in\mathfrak{h}^*$. We will now prove that the operators $A_{w,V}$ constructed from the Zhelobenko operators coincide with the dynamical Weyl group.

Theorem 2.6.8. *For any $V\in\mathrm{Rep}(G)$ and $w\in W$ we have an equality of rational functions*

$$A^{TV}_{w,V}(\lambda)=A_{w,V}(\lambda).$$

Proof. Both $A^{TV}_{w,V}(\lambda)$ and $A_{w,V}(\lambda)$ are given by products in terms of simple reflections, so it is enough to establish the fact for a simple reflection $w=s_\alpha$ along a simple root $\alpha$.

In turn, both $A^{TV}_{s_\alpha,V}(\lambda)$ and $A_{s_\alpha,V}(\lambda)$ are defined by considering the corresponding $\mathfrak{sl}_2$-subalgebra $\mathfrak{g}_\alpha\subset\mathfrak{g}$ generated by $\{e_\alpha,h_\alpha,f_\alpha\}$. So, it is enough to prove the claim for $G=\mathrm{SL}_2$.

For a tensor product of representations $A_{w,V}(\lambda)$ satisfies a multiplicativity property given by proposition 2.6.6 and so does $A^{TV}_{w,V}(\lambda)$ (see [TV00, Lemma 7], where the relationship between $J_{V,U}(\lambda)$ and $J^{EV}_{V,U}(\lambda)$ is given by proposition 2.5.21). Therefore, it is enough to check the equality on the 2-dimensional irreducible representation of $\mathfrak{sl}_2$, which follows by comparing the expressions given in proposition 2.6.7 with the explicit expressions given in [TV00, Section 2.5] (see also [EV02, Lemma 5] for an explicit description of the dynamical Weyl group in the 2-dimensional representation of quantum $\mathfrak{sl}_2$). $\square$

**2.6.3. Quantum Zhelobenko operators.** We continue to use notations for quantum groups from section 2.5.2. It was shown by Lusztig [Lus10], Soibelman [Soi90] and Kirillov–Reshetikhin [KR90] that one can introduce an action of the braid group $\hat{W}$ on modules in $\mathrm{Rep}_q(G)$. For $V \in \mathrm{Rep}_q(G)$ and $w \in W$ we denote by $T_w \colon V \to V$ the corresponding operator of the quantum Weyl group (for definitiveness, we consider $T'_{w,+1}$ in the notation of [Lus10, Chapter 5]).

**Example 2.6.9.** Consider $\mathrm{U}_q(\mathfrak{sl}_2)$ with generators $E, K, F$ as in example 2.2.21, $V \in \mathrm{Rep}_q(\mathrm{SL}_2)$ and $v \in V$ a vector of weight $n$. Then

$$T_w(v) = \sum_{a,b,c;a-b+c=n} (-1)^b q^{-ac+b} \frac{F^a E^b F^c}{[a]![b]![c]!} v$$

for the unique nontrivial element $w \in W$.

The Weyl group $W$ acts in the standard way on the weight lattice $\Lambda$. We introduce the dot action of $W$ on $\mathrm{U}_q(\mathfrak{h}) = k[\Lambda]$ by

$$w \cdot K_\mu = K_{w(\mu)} q^{(\mu, w(\rho)-\rho)}$$

for every $\mu \in \Lambda$.

Recall that for a root $\alpha$ we denote $q_\alpha = q^{(\alpha,\alpha)/2}$. The quantum integer is

$$[n]_\alpha = \frac{q_\alpha^n - q_\alpha^{-n}}{q_\alpha - q_\alpha^{-1}}$$

and the quantum factorial is defined similarly. The quantum Zhelobenko operators were introduced in [KO08, Section 9]. For the following statement recall lemma 2.5.25 which explains that the infinite sums in the quantum Zhelobenko operators are well-defined.

Theorem 2.6.10. *Suppose $X \in \mathrm{HC}_q(G)$. For a simple root $\alpha$ we denote by $\{E_\alpha, K_\alpha, F_\alpha\}$ the corresponding subset of generators of $\mathrm{U}_q(\mathfrak{g})$. Consider the* **quantum Zhelobenko operator** *on $X$ given by*

$$\check{q}_\alpha(x) = \sum_{n=0}^\infty \frac{(-1)^n}{[n]_\alpha!} (\mathrm{ad}(K_\alpha^{-1} E_\alpha))^n ((\mathrm{ad}\, T_{s_\alpha})(x)) F_\alpha^n g_{n,\alpha}^{-1},$$

*where*

$$g_{n,\alpha} = \prod_{j=1}^n [h_\alpha - j + 1]_\alpha$$

*and $\mathrm{ad}(K_\alpha^{-1} E_\alpha)$ refers to the diagonal $\mathrm{U}_q(\mathfrak{g})$-action. Then the operators $\check{q}_\alpha$ descend to linear isomorphisms*

$$(X \otimes_{\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}} M_q^{\mathrm{univ,gen}})^{\mathrm{U}_q(\mathfrak{n})} \longrightarrow (X \otimes_{\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}} M_q^{\mathrm{univ,gen}})^{\mathrm{U}_q(\mathfrak{n})}$$

*which satisfy the following relations:*

1. *$\check{q}_\alpha((\mathrm{ad}\, d)(x)) = (\mathrm{ad}\, s_\alpha(d))(\check{q}_\alpha(x))$ for every $d \in \mathrm{U}_q(\mathfrak{h})$ and $x \in X$.*
2. *$\check{q}_\alpha(dx) = (s_\alpha \cdot d)\check{q}_\alpha(x)$ for every $d \in \mathrm{U}_q(\mathfrak{h})$ and $x \in X$.*
3. *$\underbrace{\check{q}_\alpha \check{q}_\beta \check{q}_\alpha \dots}_{m_{\alpha\beta}} = \underbrace{\check{q}_\beta \check{q}_\alpha \check{q}_\beta \dots}_{m_{\alpha\beta}}$ for $\alpha \neq \beta$.*

The third property allows us to define $\check{q}_w$ for any element $w \in \hat{W}$. We also have a multiplicativity property.

Theorem 2.6.11. *Let $X, Y \in \mathrm{HC}_q(G)$ and take $x \in X$ and $y \in Y$, where $\mathrm{U}_q^{\geq 0}(\mathfrak{n}) y \in Y \mathrm{U}_q^{\geq 0}(\mathfrak{n})$. Then we have an equality*

$$\check{q}_w(x \otimes y) = \check{q}_w(x) \otimes \check{q}_w(y)$$

*in $X \otimes_{\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}} Y \otimes_{\mathrm{U}_q(\mathfrak{g})^{\mathrm{lf}}} M_q^{\mathrm{univ,gen}}$.*

**2.6.4. Quantum dynamical Weyl group.** As in section 2.6.2, quantum Zhelobenko operators define Weyl symmetry of the parabolic restriction functor $\mathrm{res}^{\mathrm{gen}}\colon \mathrm{HC}_q(G) \to \mathrm{HC}_q(H)^{\mathrm{gen}}$.

The $W$-action on $\mathrm{HC}_q(H)$ is defined similarly to the $W$-action on $\mathrm{HC}(H)$. An element $w \in W$ gives rise to a functor $S_w\colon \mathrm{HC}_q(H) \to \mathrm{HC}_q(H)$ given as follows. For $X \in \mathrm{HC}_q(H)$ we set $S_w(X) = X$ as a vector space with the $\mathrm{U}_q(\mathfrak{h})$-bimodule structure given by

$$d \triangleright^w x = (w \cdot d) \triangleright x, \qquad x \triangleleft^w d = x \triangleleft (w \cdot d),$$

where $d \in \mathrm{U}_q(\mathfrak{h})$ and $x \in X$. The functors $\{S_w\}$ have obvious monoidal structures.

Consider the action of the quantum Zhelobenko operators

$$\check{q}_\alpha\colon \mathrm{res}^{\mathrm{gen}}(X) \xrightarrow{\sim} S_{s_\alpha}(\mathrm{res}^{\mathrm{gen}}(X)).$$

Theorem 2.6.12. *The quantum Zhelobenko operators define a factorization*

$$\begin{tikzcd} \mathrm{HC}_q(G) \arrow[r, dashed] \arrow[dr, "\mathrm{res}^{\mathrm{gen}}"'] & \mathrm{HC}_q(H)^{\mathrm{gen},\hat{W}} \arrow[d] \\ & \mathrm{HC}_q(H)^{\mathrm{gen}} \end{tikzcd}$$

*of* $\mathrm{res}_q^{\mathrm{gen}}\colon \mathrm{HC}_q(G) \to \mathrm{HC}_q(H)^{\mathrm{gen}}$ *through a monoidal functor* $\mathrm{res}^{\mathrm{gen}}\colon \mathrm{HC}_q(G) \to \mathrm{HC}_q(H)^{\mathrm{gen},\hat{W}}$.

By theorem 2.5.34 we have a commutative diagram

$$\begin{tikzcd} \mathrm{Rep}_q(G) \arrow[r, "\mathrm{free}_G"] \arrow[d] & \mathrm{HC}_q(G) \arrow[d, "\mathrm{res}^{\mathrm{gen}}"] \\ \mathrm{Rep}_q(H) \arrow[r, "\mathrm{free}_H"] & \mathrm{HC}_q(H)^{\mathrm{gen}} \end{tikzcd}$$

which gives rise to a monoidal structure on the composite

$$\mathrm{Rep}_q(G) \longrightarrow \mathrm{Rep}_q(H) \xrightarrow{\mathrm{free}_H} \mathrm{HC}_q(H)^{\mathrm{gen},\hat{W}}.$$

As in section 2.6.2, we obtain linear maps $A_{w,V}(\lambda)\colon V \to V$ for every $V \in \mathrm{Rep}_q(G)$, which are rational functions on $H$. For $V, U \in \mathrm{Rep}_q(G)$ recall the matrix $J_{V,U}(\lambda)\colon V \otimes U \to V \otimes U$ defined in section 2.5.2.

**Proposition 2.6.13.** *For any simple reflection* $s_\alpha$ *and* $V, U \in \mathrm{Rep}_q(G)$ *we have an equality*

$$A_{s_\alpha, V\otimes U}(\lambda) J_{V,U}(\lambda) = J_{V,U}(s_\alpha \cdot \lambda) A^{(1)}_{s_\alpha,V}(\lambda) A^{(2)}_{s_\alpha,U}(\lambda - h^{(1)})$$

*of rational functions* $H \to \mathrm{End}(V \otimes U)$.

Let us now compute the operators $A_{w,V}$ for $G = \mathrm{SL}_2$. Consider the irreducible two-dimensional representation $V \in \mathrm{Rep}_q(G)$ with the basis $\{v_+, v_-\}$, such that

$$Kv_+ = qv_+, \qquad Kv_- = q^{-1}v_+, \qquad Fv_+ = v_-.$$

**Proposition 2.6.14.** *The action of* $A_{w,V}(\lambda)$ *is given as follows:*

$$A_{w,V}(\lambda)v_+ = v_-$$
$$A_{w,V}(\lambda)v_- = -\frac{[\lambda+2]}{[\lambda+1]}v_+$$

Proof. The isomorphism $\mathrm{U}_q(\mathfrak{h})^{\mathrm{gen}} \otimes V \to (V \otimes M^{\mathrm{univ},\mathrm{gen}})^N$ is given by

$$1 \otimes v_+ \mapsto v_+ \otimes x^{\mathrm{univ}},$$
$$1 \otimes v_- \mapsto v_- \otimes 1 - q^{-1} v_+ \otimes F \cdot [h]^{-1} \cdot x^{\mathrm{univ}}.$$

By [Lus10, Proposition 5.2.2] we have

$$T_w(v_+) = v_-, \qquad T_w(v_-) = -qv_+.$$

Therefore,

$$\check{q}_w(v_+\otimes 1)=\sum_{n=0}^{\infty}\frac{(-1)^n}{[n]!}(\mathrm{ad}(K^{-1}E))^n(v_-\otimes 1)F^n g_n^{-1}=v_-\otimes 1-q^{-1}v_+\otimes F[h]^{-1},$$

which implies that

$$A_{w,V}(\lambda)v_+=v_-.$$

Using the formula for the square of the quantum Zhelobenko operator [KO08, Corollary 9.6] we obtain

$$\check{q}_w(v_-\otimes 1-q^{-1}v_+\otimes F[h]^{-1})=-[h+1]^{-1}(v_+\otimes 1)[h+1]=-\frac{[h]}{[h+1]}(v_+\otimes 1),$$

which implies that

$$A_{w,V}(\lambda)v_-=-\frac{[\lambda+2]}{[\lambda+1]}v_+.$$

□

**Remark 2.6.15.** The formulas (9.10) and (9.11) in [KO08] are missing a sign, see [Lus10, Proposition 5.2.2].

Etingof and Varchenko [EV02] have introduced a quantum analog of the dynamical Weyl group, i.e. a collection of rational functions $A^{EV}_{w,V}(\lambda)\colon V\to V$ for every $V\in\mathrm{Rep}_q(G)$ and $w\in W$. We are now ready to relate $A_{w,V}$ and $A^{EV}_{w,V}$.

Theorem 2.6.16. *For any $V\in\mathrm{Rep}_q(G)$ an $w\in W$ we have an equality of rational functions*

$$A^{EV}_{w,V}(\lambda)=q^{(w(\rho)-\rho,h)}A_{w,V}(\lambda). \tag{2.6.3}$$

Proof. The proof is analogous to the proof of theorem 2.6.8. Both $A^{EV}_{w,V}$ and $A_{w,V}$ are given by a product over simple reflections, so it is enough to establish this equality for a simple reflection $w=s_\alpha$.

We have $s_\alpha(\rho)=\rho-\alpha$, so

$$q^{(s_\alpha(\rho)-\rho,h)}A_{s_\alpha,V}(\lambda)=q^{-(\alpha,h)}A_{s_\alpha,V}(\lambda)=K_\alpha^{-1}A_{s_\alpha,V}(\lambda).$$

In particular, both sides of the equality (2.6.3) are defined in terms of the corresponding $\mathrm{U}_q(\mathfrak{sl}_2)$-subalgebra, so it is enough to restrict our attention to $G=\mathrm{SL}_2$. Using the multiplicativity property of $A^{EV}_{w,V}$ and $A_{w,V}$ given by [EV02, Lemma 4] and proposition 2.6.13, we reduce to the case of the defining representation. The equality on the defining representation of $\mathrm{SL}_2$ follows from comparing the formulas in [EV02, Lemma 5] and proposition 2.6.14. □

## 2.7. Relation to the Kostant-Whittaker reduction

Let $\psi\colon\mathfrak{n}_-\to\mathbf{C}$ be a nondegenerate character. Unless otherwise stated, we assume without loss of generality that $\psi(f_i)=1$ for every simple root vector $f_i\in\mathfrak{n}_-$. Denote by $\mathfrak{n}_-^\psi$ the right ideal in $\mathrm{U}\mathfrak{g}$ generated by $x-\psi(x)$ for $x\in\mathfrak{n}_-$. For a Harish-Chandra bimodule $X$, consider the quantum Hamiltonian reduction $(\mathfrak{n}_-^\psi X\backslash X)^{N_-}$. Observe that it is equipped with a left and right $\mathrm{Z}(\mathrm{U}\mathfrak{g})$-action.

**Definition 2.7.1.** [BF08] The ***Kostant-Whittaker reduction*** is the functor $\mathrm{res}^\psi\colon\mathrm{HC}(G)\to\mathrm{RMod}_{\mathrm{Z}(\mathrm{U}\mathfrak{g})}$ defined by $X\mapsto(\mathfrak{n}_-^\psi X\backslash X)^{N_-}$.

In fact, this functor admits an additional structure:

**Proposition 2.7.2.** *There is a monoidal action of $\mathrm{HC}(G)$ on $\mathrm{RMod}_{\mathrm{Z}(\mathrm{U}\mathfrak{g})}$ defined by*

$$F\boxtimes X\mapsto F\otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})}\mathrm{res}^\psi(X)$$

Proof. The category $\mathrm{RMod}_{\mathrm{Z}(\mathrm{U}\mathfrak{g})}$ is generated by $\mathrm{Z}(\mathrm{U}\mathfrak{g})$, hence it is enough to consider the case $F=\mathrm{Z}(\mathrm{U}\mathfrak{g})$. The data of a monoidal action then is equivalent to the data of natural isomorphisms

$$\mathrm{res}^\psi(X)\otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})}\mathrm{res}^\psi(Y)\to\mathrm{res}^\psi(X\otimes_{\mathrm{U}\mathfrak{g}}Y) \tag{2.7.1}$$

for any Harish-Chandra bimodules $X,Y$. We define them by

$$[x]\otimes[y]\mapsto[x\otimes y]$$

for $[x] \in (\mathfrak{n}_-^\psi X\backslash X)^{N_-}$, $[y] \in (\mathfrak{n}_-^\psi Y\backslash Y)^{N_-}$, which is well-defined by $N_-$-invariance of $[x]$. It was shown by Kostant [Kos78] that the natural right action map $(\mathfrak{n}_-^\psi X\backslash X)^{N_-} \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \mathfrak{n}_-^\psi\backslash \mathrm{U}\mathfrak{g} \to \mathfrak{n}_-^\psi\backslash X$ is an isomorphism. Therefore, we have a sequence of isomorphisms

$$\begin{aligned} \mathrm{res}^\psi(X) \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \mathrm{res}^\psi(Y) \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} (\mathfrak{n}_-^\psi\backslash \mathrm{U}\mathfrak{g}) \to \mathrm{res}^\psi(X) \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \mathfrak{n}_-^\psi Y\backslash Y \to \\ \to \mathrm{res}^\psi(X) \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} (\mathfrak{n}_-^\psi\backslash \mathrm{U}\mathfrak{g} \otimes_{\mathrm{U}\mathfrak{g}} Y) \to (\mathrm{res}^\psi(X) \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} (\mathfrak{n}_-^\psi\backslash \mathrm{U}\mathfrak{g})) \otimes_{\mathrm{U}\mathfrak{g}} Y \to \\ \to \mathfrak{n}_-^\psi(X \otimes_{\mathrm{U}\mathfrak{g}} Y)\backslash X \otimes_{\mathrm{U}\mathfrak{g}} Y. \end{aligned}$$

Observe that this composition is a map of right $\mathrm{U}\mathfrak{g}$-modules and it coincides with the morphism above for $[x] \otimes [y] \otimes [1]$. $\square$

Recall the Harish-Chandra homomorphism $\mathrm{Z}(\mathrm{U}\mathfrak{g}) \to \mathrm{U}\mathfrak{h}$.

**Definition 2.7.3.** The ***extended*** Kostant-Whittaker reduction is the functor $\mathrm{HC}(G) \to \mathrm{RMod}_{\mathrm{U}\mathfrak{h}}$ defined by

$$X \mapsto \mathrm{res}^\psi(X) \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \mathrm{U}\mathfrak{h}.$$

As in the previous section, we can also define its generic version.

Composition of the parabolic reduction $\mathrm{HC}(G) \to \mathrm{HC}(H)$ and the forgetful functor $\mathrm{HC}(H) \to \mathrm{RMod}_{\mathrm{U}\mathfrak{h}}$ provides another functor $\mathrm{HC}(G) \to \mathrm{RMod}_{\mathrm{U}\mathfrak{h}}$ (and also its generic version) which we denote by res in this section. Therefore, we have the diagram

$$\begin{tikzcd} & \mathrm{HC}(H)^{\mathrm{gen}} \arrow[dr] & \\ \mathrm{HC}(G) \arrow[ur] \arrow[dr] & & \mathrm{RMod}_{(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}} \\ & \mathrm{RMod}_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \arrow[ur] & \end{tikzcd}$$

which should be thought of as a quantization of (1.3.11). According to proposition 1.3.24, the classical version of this diagram is commutative (also on the level of Lagrangian structures). We will show a quantum analog of this statement.

Recall that in the case of parabolic reduction, there is a generic isomorphism $(X/X\mathfrak{n})^N \to \mathfrak{n}_- X\backslash X/X\mathfrak{n}$. A similar statement is true in the Kostant-Whittaker case even without genericity assumption:

**Proposition 2.7.4.** *[GK18, Lemma 6.2.1] Let $X \in \mathrm{HC}(G)$, where $G$ is an arbitrary reductive Lie group. Then the natural map*

$$(\mathfrak{n}_-^\psi X\backslash X)^{N_-} \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \mathrm{U}\mathfrak{h} \xrightarrow{p_X \otimes \mathrm{id}} \mathfrak{n}_-^\psi X\backslash X/X\mathfrak{n} \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \mathrm{U}\mathfrak{h} \xrightarrow{\mathrm{act}} \mathfrak{n}_-^\psi X\backslash X/X\mathfrak{n}$$

*is an isomorphism of right* $\mathrm{U}\mathfrak{h}$*-modules. Here* $p_X$ *is the projection and* act *is the right action of* $\mathrm{U}\mathfrak{h}$.

Therefore, it is enough to show that the natural morphism

$$(X/X\mathfrak{n})^N \to \mathfrak{n}_-^\psi X\backslash X/X\mathfrak{n} \tag{2.7.2}$$

is a generic isomorphism.

**Proposition 2.7.5.** *For a free Harish-Chandra bimodule $X = V \otimes \mathrm{U}\mathfrak{g}$, the natural map (2.7.2) is a generic isomorphism.*

Proof. As in remark 2.5.20, let $\{v_\mu\}$ be a weight basis of $V$, then the elements $P(v_\mu \otimes 1)$, where $P$ is the extremal projector, are $(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}$-generators of the free module $(X/\mathfrak{n})^N \otimes_{\mathrm{U}\mathfrak{h}} (\mathrm{U}\mathfrak{h})^{\mathrm{gen}}$. Likewise, by the PBW theorem, we have an isomorphism

$$(\mathfrak{n}_-^\psi\backslash X/\mathfrak{n}) \otimes_{\mathrm{U}\mathfrak{h}} (\mathrm{U}\mathfrak{h})^{\mathrm{gen}} \cong V \otimes (\mathrm{U}\mathfrak{h})^{\mathrm{gen}}$$

with the elements $v_\mu \otimes 1$ as generators.

Observe that property (2) of the extremal projector from theorem 2.5.14 implies that its image $\bar{P}$ in the quotient $\mathfrak{n}_-^\psi\backslash T(\mathfrak{g})$ lies in the completion of $1+(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}\otimes \mathrm{U}^{>0}\mathfrak{n}$, where the superscript means the augmentation ideal of the algebra $\mathrm{U}\mathfrak{n}$. Observe also that for any $x\in X/\mathfrak{n}$, we have ${}_{\mathfrak{n}_-^\psi}[Px] = {}_{\mathfrak{n}_-^\psi}[\bar{P}x]$ in $\mathfrak{n}_-^\psi\backslash X/\mathfrak{n}$. For $x = v_\mu\otimes x^{\mathrm{univ}}$, we obtain

$$\bar{P}(v_\mu\otimes x^{\mathrm{univ}}) = v_\mu\otimes x^{\mathrm{univ}} + \sum_{\lambda>\mu} v_\lambda\otimes f_\lambda\cdot x^{\mathrm{univ}},$$

where $f_\lambda\in(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}$ are some functions. Therefore, under identifications of both sides of (2.7.2) with $V\otimes(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}$, this map restricted to the generators is given by an upper-triangular $(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}$-valued matrix with ones on the diagonal, thus it is invertible. Therefore, the morphism (2.7.2) is a generic isomorphism. □

**Remark 2.7.6.** See similar result [GK18, Theorem 7.1.8].

**Corollary 2.7.7.** *There is an equivalence of functors* $\mathrm{res}^{\mathrm{gen}}$ *and* $\mathrm{res}^\psi\otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})}(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}$.

Proof. By corollary 2.5.18, functor $\mathrm{res}^{\mathrm{gen}}$ is exact. Likewise, by [BF08, Lemma 4], functor $\mathrm{res}^\psi$ is exact and $(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}$ is a flat $\mathrm{Z}(\mathrm{U}\mathfrak{g})$-module, so that the composition $\mathrm{res}^\psi\otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})}(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}$ is exact. Since $\mathrm{HC}(G)$ is generated by free Harish-Chandra bimodules, we conclude. □

In the case of $\mathfrak{g}=\mathfrak{sl}_2$, there is another construction that uses a version of the extremal projector.

**Definition 2.7.8.** The ***semi-Whittaker extremal projector*** is the sum (lying in some completion of $\mathrm{U}\mathfrak{sl}_2$)

$$P^\psi = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}P_k^\psi e^k, \tag{2.7.3}$$

where $P_k^\psi := \prod_{i=0}^{k-1} f^\psi(h-i)^{-1}$ with $P_0^\psi := 1$.

Observe that it is defined for any character $\psi$, and if $\psi=0$, then $P^\psi$ becomes the usual extremal projector of $\mathfrak{sl}_2$.

**Proposition 2.7.9.** *The element* $P^\psi$ *satisfies*

$$eP^\psi = P^\psi f^\psi = 0.$$

*In particular,* $(P^\psi)^2 = P^\psi$.

The proof is technical and given in Appendix, section 3.2. Such element then defines a generic map $\mathfrak{n}_-^\psi\backslash X/\mathfrak{n}\to(X/\mathfrak{n})^N$ which is the inverse of (2.7.2).

Recall that the parabolic reduction functor admits a natural extension $\mathrm{HC}(G)\to\mathrm{HC}(H)^{\mathrm{gen},\hat{W}}$ to the $\hat{W}$-equivariant category.

**Corollary 2.7.10.** *There is a natural extension of the functor* $\mathrm{res}^{\mathrm{gen}}$ *to the* $W$*-equivariant category* $\mathrm{RMod}^W_{(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}}$.

Proof. Recall from the proof of [GK18, Lemma 6.2.1] that the morphism $\mathrm{Z}(\mathrm{U}\mathfrak{g})\to\mathrm{U}\mathfrak{h}$ is given by $\mathrm{Z}(\mathrm{U}\mathfrak{g})\to\mathrm{U}\mathfrak{g}\to\mathfrak{n}_-^\psi\backslash\mathrm{U}\mathfrak{g}/\mathfrak{n}_+$. A priori, it is not the Harish-Chandra homomorphism. However, observe that for any character $\lambda$ of $\mathfrak{h}$, we can $\mathrm{Z}(\mathrm{U}\mathfrak{g})$-equivariantly identify $\mathfrak{n}_-^\psi\backslash\mathrm{U}\mathfrak{g}/\mathfrak{n}_+\otimes_{\mathrm{U}\mathfrak{h}}\lambda$ with the one-dimensional space spanned by the highest-weight vector. Therefore, by construction, this map coincides with the Harish-Chandra homomorphism, see [Hum08, Section 1.7].

It follows then that there is well-defined $W$-action on $\mathrm{res}^\psi(X)\otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})}\mathrm{U}\mathfrak{h}$ defined by the one on $\mathrm{U}\mathfrak{h}$. We conclude by Corollary 2.7.7. □

**Remark 2.7.11.** Observe that this $W$-action does *not* coincide with the one provided by the Zhelobenko operators: for instance, the former satisfies $s_\alpha^2=\mathrm{id}$. They seem to be related as we will show in section 2.7.1, but the precise connection is unclear.

Recall that the categrory $\mathrm{RMod}_{(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}}$ is naturally a $\mathrm{HC}(H)^{\mathrm{gen}}$-module category, so, the monoidal functor $\mathrm{HC}(G) \to \mathrm{HC}(H)^{\mathrm{gen}}$ of parabolic restriction provides a monoidal action of $\mathrm{HC}(G)$ on it. Likewise, we have a monoidal action of $\mathrm{HC}(G)$ on $\mathrm{RMod}_{\mathrm{Z}(\mathrm{U}\mathfrak{g})}$ by proposition 2.7.2.

**Proposition 2.7.12.** *The functor* $\mathrm{RMod}_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \to \mathrm{RMod}_{(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}}$, $F \mapsto F \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} (\mathrm{U}\mathfrak{h})^{\mathrm{gen}}$ *is a functor of* $HC(G)$*-module categories.*

Proof. As in the proof of proposition 2.4.3, it is enough to check compatibility of actions on $\mathrm{Z}(\mathrm{U}\mathfrak{g})$. By corollary 2.7.7, we have an isomorphism

$$\mathrm{res}^{\psi}(X) \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \mathrm{U}\mathfrak{h} \to \mathrm{res}(X)$$

(for brevity, we omit the index "gen"). Therefore, we only need to check the pentagon axiom:

$$\begin{tikzcd}
 & \mathrm{res}^{\psi}(X) \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \mathrm{res}^{\psi}(Y) \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \mathrm{U}\mathfrak{h} \arrow[dl] \arrow[dr] & \\
\mathrm{res}^{\psi}(X \otimes_{\mathrm{U}\mathfrak{g}} Y) \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \mathrm{U}\mathfrak{h} \arrow[dr] & & (X/X\mathfrak{n})^N \otimes_{\mathrm{U}\mathfrak{h}} (Y/Y\mathfrak{n})^N \arrow[dl] \\
 & (X \otimes_{\mathrm{U}\mathfrak{g}} Y/(X \otimes_{\mathrm{U}\mathfrak{g}} Y)\mathfrak{n})^N &
\end{tikzcd}$$

In terms of elements, let ${}_{\mathfrak{n}_-^{\psi}}[y] \in (\mathfrak{n}_-^{\psi}\backslash Y)^{N_-}$ and ${}_{\mathfrak{n}_-^{\psi}}[x] \in (\mathfrak{n}_-^{\psi}\backslash X)^{N_-}$. We denote their image in the double quotient $\mathfrak{n}_-^{\psi}\backslash Y/\mathfrak{n}$ by ${}_{\mathfrak{n}_-^{\psi}}[y]_{\mathfrak{n}}$ (same for $X$). Let $F$ be the inverse of (2.7.2). Then we need to show that

$$F({}_{\mathfrak{n}_-^{\psi}}[x]_{\mathfrak{n}}) \otimes_{\mathrm{U}\mathfrak{g}} F({}_{\mathfrak{n}_-^{\psi}}[y]_{\mathfrak{n}}) = F({}_{\mathfrak{n}_-^{\psi}}[x \otimes_{\mathrm{U}\mathfrak{g}} y]_{\mathfrak{n}}).$$

We show that the projections of both sides to $\mathfrak{n}_-^{\psi}\backslash(X \otimes_{\mathrm{U}\mathfrak{g}} Y)/\mathfrak{n}$ coincide. Indeed: since $F$ is an isomorphism, we have

$$F({}_{\mathfrak{n}_-^{\psi}}[y]_{\mathfrak{n}}) \in [y]_{\mathfrak{n}} + \mathfrak{n}_-^{\psi} Y/\mathfrak{n}.$$

The element ${}_{\mathfrak{n}_-^{\psi}}[x]$ is $N_-$-invariant, i.e. $x \cdot \mathfrak{n}_-^{\psi} \in \mathfrak{n}_-^{\psi} X$, therefore, the left-hand side belongs to

$$[x \otimes_{\mathrm{U}\mathfrak{g}} y]_{\mathfrak{n}} + \mathfrak{n}_-^{\psi}(X \otimes_{\mathrm{U}\mathfrak{g}} Y)/\mathfrak{n},$$

and so the projections to both sides coincide. □

**2.7.1. Case of $\mathfrak{sl}_2$.** In this subsection, we explicitly describe the equivalence of the Kostant-Whittaker reduction and the parabolic restriction in the case of $\mathfrak{g} = \mathfrak{sl}_2$. In particular, we argue that it provides a categorical vertex-IRF transformation in the sense of section 1.4.

First, we give an explicit presentation of the Whittaker vectors in free Harish-Chandra bimodules.

**Proposition 2.7.13.** *Let* $[x] \in \mathfrak{n}_-^{\psi} X\backslash X$. *Then the element* $[Q^{\psi}(x)] \in \mathfrak{n}_-^{\psi} X\backslash X$, *where*

$$Q^{\psi}(x) := \sum_{k=0}^{\infty} (-1)^k \mathrm{ad}_f^k(x) \frac{h(h-2)\cdot \ldots \cdot (h-2k+2)}{k! 2^k} \tag{2.7.4}$$

*is a Whittaker vector.*

Proof. The action of $Q^{\psi}$ is well-defined: indeed,

$$\mathrm{ad}_f^k(fx) = \sum_{l=0}^{k} \mathrm{ad}_f^l(f)\, \mathrm{ad}_f^{k-l}(x) = f\, \mathrm{ad}_f^k(x),$$

hence $[\mathrm{ad}_f^k(fx)] = [\mathrm{ad}_f^k(x)]$. The property $[Q^{\psi}(x) f] = [Q^{\psi}(x)]$ is verified by direct computations. □

**Corollary 2.7.14.** *For a free Harish-Chandra bimodule* $X = V \otimes \mathrm{U}\mathfrak{g}$, *there is a natural isomorphism of right* $\mathrm{Z}(\mathrm{U}\mathfrak{g})$*-modules* $V \otimes \mathrm{Z}(\mathrm{U}\mathfrak{g}) \cong \mathrm{res}^{\psi}(X)$.

Proof. We can assume that $V$ is irreducible with a weight basis $\{v_\lambda\}$. Observe that

$$Q^\psi(v_\lambda \otimes 1) = v_\lambda \otimes 1 + \sum_\mu v_\mu \otimes x_\mu,$$

where $x_\mu \in \mathrm{U}\mathfrak{g}$ and $\mu < \lambda$. In particular, the vectors $[Q^\psi(v_\lambda \otimes 1)]$ are $\mathrm{Z}(\mathrm{U}\mathfrak{g})$-independent and generate a free module $V \otimes \mathrm{Z}(\mathrm{U}\mathfrak{g})$ inside $\mathrm{res}^\psi(X)$. By the same reasoning, these vectors are linearly independent after specializing to any character of $\mathrm{Z}(\mathrm{U}\mathfrak{g})$; by [Kos78, Theorem 4.6], the dimension of the space of Whittaker vectors in $\mathrm{res}^\psi(X)$ is $\dim(V)$ after specialization to any character of $\mathrm{Z}(\mathrm{U}\mathfrak{g})$, therefore, $\mathrm{res}^\psi(X)$ coincides with $V \otimes \mathrm{Z}(\mathrm{U}\mathfrak{g})$; moreover, such identification is independent of the choice of basis of $V$. □

**Remark 2.7.15.** We can induce a *left* $\mathrm{Z}(\mathrm{U}\mathfrak{g})$-module structure on $V \otimes \mathrm{Z}(\mathrm{U}\mathfrak{g})$ via the isomorphism with $\mathrm{res}^\psi(V \otimes \mathrm{U}\mathfrak{g})$. In fact, we have $V \otimes \mathrm{Z}(\mathrm{U}\mathfrak{g}) \cong \mathrm{Z}(\mathrm{U}\mathfrak{g}) \otimes V$ as left $\mathrm{Z}(\mathrm{U}\mathfrak{g})$-modules. Indeed: using the same operator $Q^\psi$, one can show that there is a natural isomorphism $\mathrm{res}^\psi(\mathrm{U}\mathfrak{g} \otimes V) \cong \mathrm{Z}(\mathrm{U}\mathfrak{g}) \otimes V$ as left $\mathrm{Z}(\mathrm{U}\mathfrak{g})$-modules, then we use the composition

$$V \otimes \mathrm{Z}(\mathrm{U}\mathfrak{g}) \to \mathrm{res}^\psi(V \otimes \mathrm{U}\mathfrak{g}) \xrightarrow{\tau} \mathrm{res}^\psi(\mathrm{U}\mathfrak{g} \otimes V) \to \mathrm{Z}(\mathrm{U}\mathfrak{g}) \otimes V.$$

In particular, for any $\mathrm{Z}(\mathrm{U}\mathfrak{g})$-module $F$, we have a vector space isomorphism $F \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} (V \otimes \mathrm{Z}(\mathrm{U}\mathfrak{g})) \cong F \otimes V$. See [Goo11, Theorem 5.1] for a general version of the latter isomorphism for an arbitrary $W$-algebra, also [BK08, Theorem 8.1] for $\mathfrak{g} = \mathfrak{gl}_n$.

Let $X = V \otimes \mathrm{U}\mathfrak{g}$, where $V = \mathbf{C}^2$ is the defining representation of $\mathfrak{sl}_2$ with the highest weight vector $v_+$ and $v_- = f v_+$. The elements

$$\begin{aligned} v_-^\psi &= Q^\psi(v_- \otimes 1) = {}_{\mathfrak{n}_-^\psi}[v_- \otimes 1], \\ v_+^\psi &= Q^\psi(v_+ \otimes 1) = {}_{\mathfrak{n}_-^\psi}[v_+ \otimes 1 - v_- \otimes \frac{h}{2}] \end{aligned} \tag{2.7.5}$$

are generators of $(\mathfrak{n}_-^\psi X \backslash X)^{N_-}$. Likewise, the elements

$$\begin{aligned} Q^\psi(v_- \otimes v_- \otimes 1) &= {}_{\mathfrak{n}_-^\psi}[v_- \otimes v_- \otimes 1], \\ Q^\psi(v_- \otimes v_+ \otimes 1) &= {}_{\mathfrak{n}_-^\psi}[v_- \otimes v_+ 1 - v_- \otimes v_- \otimes \frac{h}{2}], \\ Q^\psi(v_+ \otimes v_- \otimes 1) &= {}_{\mathfrak{n}_-^\psi}[v_+ \otimes v_- \otimes 1 - v_- \otimes v_- \otimes \frac{h}{2}], \\ Q^\psi(v_+ \otimes v_+ \otimes 1) &= {}_{\mathfrak{n}_-^\psi}[v_+ \otimes v_+ \otimes 1 - (v_+ \otimes v_- + v_- \otimes v_+) \otimes \frac{h}{2} + v_- \otimes v_- \otimes \frac{h(h-2)}{4}] \end{aligned}$$

are generators of $\mathrm{res}^\psi(V \otimes V \otimes \mathrm{U}\mathfrak{g})$. Then the tensor structure morphism $V \otimes V \otimes \mathrm{Z}(\mathrm{U}\mathfrak{g}) \to V \otimes V \otimes \mathrm{Z}(\mathrm{U}\mathfrak{g})$ of (2.7.1) is given by the *constant* matrix

$$J = \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In particular, the matrix $(J^{21})^{-1}J$ is a solution to the quantum Yang-Baxter equation. To find the corresponding classical $r$-matrix, we need to consider the asymptotic version of this construction involving the deformation parameter $\hbar$ (as in [BF08], for instance). The resulting asymptotic $R$-matrix is then

$$R_\hbar = \begin{pmatrix} 1 & -\frac{\hbar}{2} & \frac{\hbar}{2} & -\frac{\hbar^2}{4} \\ 0 & 1 & 0 & \frac{\hbar}{2} \\ 0 & 0 & 1 & -\frac{\hbar}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and one can see that it is indeed a quantization of the rational version of the Cremmer-Gervais $r$-matrix $b_{CG}$, see (1.4.5).

In general, one can easily show that the tensor structure is constant for any two $\mathfrak{sl}_2$-modules (enough to show for $V_m \otimes \mathbf{C}^2$, where $V_m$ is the weight $m$ irreducible representation). In particular, the monoidal action of $\mathrm{Rep}(G)$ on $\mathrm{RMod}_{\mathrm{Z}(\mathrm{U}\mathfrak{g})}$ factorizes through the one on Vect given by the quantization of the rational Cremmer-Gervais $r$-matrix. Therefore, the isomorphism of monoidal actions $\mathrm{res}^\psi \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} (\mathrm{U}\mathfrak{h})^{\mathrm{gen}}$ and $\mathrm{res}^{\mathrm{gen}}$ provides a vertex-IRF transformation. Let us compute it for the defining representation $V$.

For the parabolic restriction $(X/X\mathfrak{n})^N$, we consider the basis of remark 2.5.20 instead of the one from theorem 2.5.19. Namely, it is given by the elements of the form $[Pv]$, where $v \in V$ and $P$ is the (standard) extremal projector:

$$\begin{aligned}\tilde{v}_- &= v_- \otimes \frac{h}{h+1} x^{\mathrm{univ}} - v_+ \otimes f \frac{1}{h+1} x^{\mathrm{univ}} \\ \tilde{v}_+ &= v_+ \otimes x^{\mathrm{univ}},\end{aligned}$$

Then we compute:

$$\begin{aligned}P^\psi v_-^\psi &= \tilde{v_-} + \tilde{v}_+ (h+1)^{-1}, \\ P^\psi v_+^\psi &= -\tilde{v}_- \frac{h}{2} + \tilde{v}_+ \frac{h+2}{2(h+1)}.\end{aligned}$$

In general, the asymptotic version of the isomorphism of Corollary 2.7.7 can be presented by the matrix

$$\Omega_\hbar(h) = \begin{pmatrix} \frac{1}{h+\hbar} & \frac{h+2\hbar}{2(h+\hbar)} \\ 1 & -h/2 \end{pmatrix}$$

(under identification with right free modules). For $\hbar = 0$, it becomes the matrix of the form $A(\lambda)S(\lambda)$, where $S(\lambda)$ is the Vandermonde matrix (1.4.4) and $A(\lambda)$ is a diagonal matrix. One can check directly that it provides a semi-classical vertex-IRF transformation.

Finally, let us compute the $W$-action. Unraveling the construction of Corollary 2.7.10, we see that it is given by the matrix $s_\alpha(h) := \Omega(h)\Omega^{-1}(-h-2)$; the latter does not depend on the choice of generators of $\mathrm{res}^\psi(X)$, since, by construction, it amounts to the transformation $\Omega(h) \mapsto \Omega(h)G(h)$ for some invertible matrix $G(h)$ satisfying $G(h) = G(-h-2)$. We have

$$s_\alpha(h) = \begin{pmatrix} 0 & -1-h \\ \frac{1}{h+1} & 0 \end{pmatrix}$$

**2.7.2. Case of $\mathfrak{gl}_n$.** In this section, we construct, as in the case of $\mathfrak{sl}_2$, a quantization of the rational Cremmer-Gervais $r$-matrix for $\mathfrak{g} = \mathfrak{gl}_n$ in terms of the Kostant-Whittaker reduction.

Kostant's theorem states that there is an isomorphism of algebras $(\mathfrak{n}_-^\psi \backslash \mathrm{U}\mathfrak{g})^{N_-} \cong \mathrm{Z}(\mathrm{U}\mathfrak{g})$. In type A, there is also a presentation in terms of the so-called shifted Yangians, see [BK06b]. We will construct another one (quite possible, related to the previous one) via the Gelfand-Tsetlin subalgebra.

Recall the set $S$ of companion matrices (1.4.6) which we write in the form

$$\begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix}$$

Up to transposition, coming from the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ via the trace form, we have $\mathcal{O}(S) = \mathfrak{n}_-^\psi \backslash \mathrm{Sym}(\mathfrak{g})/\mathfrak{b}_\mathfrak{l}$, where $\mathfrak{l} = \mathfrak{gl}_{n-1}$ embedded in left upper corner and $\mathfrak{b}_\mathfrak{l} = \mathfrak{b} \cap \mathfrak{l}$ is its Borel subalgebra. As we mentioned in remark 1.4.13, this is a transversal slice introduced in [MV03]. In particular, we have an isomorphism $\mathcal{O}(S) \cong \mathrm{Sym}(\mathfrak{g})^G$. The latter is rather standard fact: the algebra $\mathrm{Sym}(\mathfrak{g}^*)^G$ is generated by the coefficients of the characteristic polynomial and the latter for $S$ is $\sum_i a_i \lambda^i$.

In fact, this statement can be upgraded. Consider the embeddings $\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \ldots \subset \mathfrak{gl}_n$ of Lie algebras via the left upper corners and let $\mathrm{GT}_n^{clas}$ be the subalgebra generated by the centers $\mathrm{Sym}(\mathfrak{gl}_k)^{GL_k}$ for

$1 \leq k \leq n$. Let $\tilde{S}$ the set of matrices

$$\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1,n-1} & b_{1n} \\ 1 & b_{22} & \dots & b_{2,n-1} & b_{2n} \\ 0 & 1 & \dots & b_{3,n-1} & b_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & b_{nn} \end{pmatrix}$$

whose ring of functions is $\mathfrak{n}_-^\psi \backslash \mathrm{Sym}(\mathfrak{g})$. It is shown in [KW06] that the map $\mathrm{GT}_n^{clas} \to \mathfrak{n}_-^\psi \backslash \mathrm{Sym}(\mathfrak{g})$ is an isomorphism.

Denote by $\mathrm{GT}_n$ the so-called *Gelfand-Tsetlin* subalgebra formed by the centers $\mathrm{Z}_k := \mathrm{Z}(\mathrm{U}\mathfrak{gl}_k)$ for $1 \leq k \leq n$. We will show quantum analogs of this statements.

The center $\mathrm{Z}(\mathrm{U}\mathfrak{gl}_n)$ has an explicit presentation. Namely, consider the matrix

$$E = \begin{pmatrix} E_{11}+t & E_{12} & \dots & E_{1n} \\ E_{21} & E_{22}+t-1 & \dots & E_{2n} \\ \vdots & \vdots & & \vdots \\ E_{n1} & E_{n2} & \dots & E_{nn}+t-n+1 \end{pmatrix}$$

with values in $\mathrm{U}\mathfrak{gl}_n[t]$. For a noncommutative matrix $A = (a_{ij})$, its "determinant" is defined as the sum

$$\det(A) := \sum_\sigma (-1)^\sigma a_{\sigma(1),1} \dots a_{\sigma(n),n}. \tag{2.7.6}$$

It follows (for instance, see [Mol07, Theorem 7.1.1] that the coefficients of the so-called *Capelli determinant* $\det(E)$ lie in the center of $\mathrm{U}\mathfrak{g}$ and, moreover, they generate $\mathrm{Z}(\mathrm{U}\mathfrak{g})$.

In what follows, we will use a slightly different generators. Namely, consider the antiautomorphism $\varpi$ of $\mathrm{U}\mathfrak{gl}_n$ defined by $\varpi(E_{ij}) = E_{ji}$. It maps the center to itself isomorphically; we define

$$\tilde{Q}_{\mathfrak{gl}_n}(t) := \varpi(\det(E)) = \sum_\sigma (-1)^\sigma a_{n,\sigma(n)} \dots a_{1,\sigma(1)}. \tag{2.7.7}$$

Recall that the map

$$\mathrm{Z}(\mathrm{U}\mathfrak{g}) \to \mathrm{U}\mathfrak{g} \to \mathfrak{n}_-^\psi \backslash \mathrm{U}\mathfrak{g} \cong \mathrm{U}\mathfrak{b}$$

is an algebra homomorphism by [Kos78]. We define $Q_{\mathfrak{gl}_n}(t)$ to be the image of $\tilde{Q}_{\mathfrak{gl}_n}(t)$ under the map to $\mathrm{U}\mathfrak{b}[t]$.

**Lemma 2.7.16.** *The polynomial $Q_{\mathfrak{gl}_n}(t)$ can be expressed as*

$$Q_{\mathfrak{gl}_n}(t) = (t-n+1) Q_{\mathfrak{gl}_{n-1}}(t) + \sum_{k=n}^{1} (-1)^{n-k} E_{kn} Q_{\mathfrak{gl}_{k-1}}(t). \tag{2.7.8}$$

Proof. We decompose

$$\tilde{Q}_{\mathfrak{gl}_n}(t) = (E_{nn}+t-n+1) \cdot \tilde{Q}_{\mathfrak{gl}_{n-1}}(t) - E_{n,n-1} \cdot \mathrm{Aux}_1(t) + \dots,$$

where $\mathrm{Aux}_1(t)$ is some polynomial and the first terms in $\cdots$ are $E_{n,i}$ for $i < n-1$. Therefore, after factorization by $\mathfrak{n}_-^\psi$, we have

$$Q_{\mathfrak{gl}_n}(t) = (E_{nn}+t-n+1) \cdot Q_{\mathfrak{gl}_{n-1}}(t) - \mathrm{Aux}_1(t).$$

In its turn, polynomial $\mathrm{Aux}_1(t)$ can be presented as

$$\mathrm{Aux}_1(t) = E_{n-1,n} \cdot Q_{\mathfrak{gl}_{n-2}}(t) - E_{n-1,n-2} \cdot \mathrm{Aux}_2(t) + \dots$$

with the same meaning of the symbols. Continuing this process, we obtain the formula (2.7.8). □

Define the elements $Z_n^i \in \mathrm{U}\mathfrak{b}$ by

$$Q_{\mathfrak{gl}_n}(t) = t^n + \sum_{i=1}^n Z_n^i t^{n-i} \tag{2.7.9}$$

**Proposition 2.7.17.** *The map* $\mathrm{GT}_n \to \mathrm{U}\mathfrak{g} \to \mathfrak{n}_-^{\psi}\backslash \mathrm{U}\mathfrak{g} \cong \mathrm{U}\mathfrak{b}$ *is an isomorphism of left* $\mathrm{Z}(\mathrm{U}\mathfrak{g})$*-modules.*

Proof. By arguments similar to [Kos78, Lemma 2.3], this map is a homomorphism of left $\mathrm{Z}(\mathrm{U}\mathfrak{gl}_n)$-modules. We will show that it is an isomorphism of vector spaces.

Observe that $\mathrm{GT}_k \subset \mathrm{GT}_n$ and the image of $\mathrm{GT}_k$ under the map to $\mathrm{U}\mathfrak{b}$ lands in $\mathrm{U}\mathfrak{b}_k \subset \mathrm{U}\mathfrak{b}$, where $\mathfrak{b}_k = \mathfrak{b} \cap \mathfrak{gl}_k$. Therefore, we can proceed by induction.

The base $n = 1$ is trivial. Assume the statement for $n-1$. We have a left $\mathrm{Z}(\mathrm{U}\mathfrak{gl}_n)$-module isomorphism $\mathrm{GT}_n = \mathrm{Z}(\mathrm{U}\mathfrak{gl}_n) \otimes \mathrm{GT}_{n-1}$. Likewise, the PBW isomorphism gives the splitting $\mathrm{U}\mathfrak{b} = \mathrm{U}\mathfrak{u} \otimes \mathrm{U}\mathfrak{b}_{n-1}$, where $\mathfrak{u} \subset \mathfrak{gl}_n$ is the subalgebra whose only nonzero column is the last one.

First, let us show that the subspace $\mathrm{U}\mathfrak{u}^{\leq 1}\mathrm{U}\mathfrak{b}_{n-1}$, where the subscript means degree less or equal to 1, is covered. We show for the subspaces $E_{kn}\mathrm{U}\mathfrak{b}_{n-1}$ by induction on $k$ starting from $k = n$. Consider the formula (2.7.8). By global induction assumption, the polynomial $Q_{\mathfrak{gl}_{n-1}}(t)$ has a preimage in $\mathrm{GT}_{n-1}$. The coefficient in front of $t^{n-1}$ of the difference $Q_{\mathfrak{gl}_n}(t) - (t-n+1)Q_{\mathfrak{gl}_{n-1}}(t)$ is equal to the one of $E_{nn}Q_{\mathfrak{gl}_{n-1}}(t)$ (the others have strictly lower degree); since we can write $Q_{\mathfrak{gl}_{n-1}}(t) = t^{n-1} + l.d.$, where $l.d.$ means "lower degrees", we have $Z_n^1 \in E_{nn} + \mathrm{U}\mathfrak{b}_{n-1}$. It follows that the element $E_{nn}$ has a preimage under this map and so does the subspace $E_{nn}\mathrm{U}\mathfrak{b}_{n-1}$. This proves the base; the induction step is performed similarly using the polynomial

$$Q_{\mathfrak{gl}_n}(t) - (t-n+1)Q_{\mathfrak{gl}_{n-1}}(t) - \sum_{l=n}^{k+1} (-1)^{n-l} E_{ln} Q_{\mathfrak{gl}_{l-1}}(t).$$

By normality of $\mathrm{U}\mathfrak{u}$ inside $\mathrm{U}\mathfrak{b}_{n-1}$, the space $\mathrm{U}\mathfrak{u}^{\leq k}\mathrm{U}\mathfrak{b}_{n-1}$ also has a preimage. Indeed: consider, for instance, $k = 2$, then

$$\begin{aligned} Z_n^{n-i+1}Z_n^{n-j+1} \mapsto (E_{in} + \mathrm{U}\mathfrak{b}_{n-1})(E_{jn} + \mathrm{U}\mathfrak{b}_{n-1}) &= E_{in}E_{jn} + E_{in}\mathrm{U}\mathfrak{b}_{n-1} + \mathrm{U}\mathfrak{b}_{n-1}E_{jn} + \mathrm{U}\mathfrak{b}_{n-1} = \\ &= E_{in}E_{jn} + E_{in}\mathrm{U}\mathfrak{b}_{n-1} + x\mathrm{U}\mathfrak{b}_{n-1} + \mathrm{U}\mathfrak{b}_{n-1}, \end{aligned}$$

where $x$ is some element coming from commuting $E_{jn}$ past $\mathrm{U}\mathfrak{b}_{n-1}$. Therefore, we conclude that the map $\mathrm{GT}_n \to \mathrm{U}\mathfrak{b}_n$ is surjective. It also clear that it is injective: we have a filtration on $\mathrm{GT}_n$ by $\mathrm{Z}(\mathrm{U}\mathfrak{g})^{\leq k}\mathrm{GT}_{n-1}$ by degree of polynomials in generators $Z_n^i$ and the one on $\mathrm{U}\mathfrak{b}$ by $\mathrm{U}\mathfrak{u}^{\leq k}\mathrm{U}\mathfrak{b}_{n-1}$. This map preserves filtrations and, as computation above shows, induces an isomorphism on the associated graded spaces. □

In fact, the following statement follows from the proof:

**Corollary 2.7.18.** *There is an isomorphism of* $\mathrm{Z}(\mathrm{U}\mathfrak{g})$*-bimodules* $\mathrm{Z}(\mathrm{U}\mathfrak{g}) \cong \mathfrak{n}_-^{\psi}\backslash \mathrm{U}\mathfrak{g}/\mathfrak{b}_{n-1}$.

In particular, it implies the following result:

Theorem 2.7.19. *The functor* $\mathrm{HC}(G) \to {}_{\mathrm{Z}(\mathrm{U}\mathfrak{g})}\mathrm{BMod}_{\mathrm{Z}(\mathrm{U}\mathfrak{g})}$ *given by*

$$X \mapsto \mathfrak{n}_-^{\psi}\backslash X/\mathfrak{b}_{n-1}, \tag{2.7.10}$$

*is naturally isomorphic to the Kostant-Whittaker reduction.*

Proof. Recall that the natural action map

$$(\mathfrak{n}_-^{\psi}\backslash X)^{N_-} \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \mathfrak{n}_-^{\psi}\backslash \mathrm{U}\mathfrak{g} \to \mathfrak{n}_-^{\psi}\backslash X$$

is an isomorphism of right $\mathrm{U}\mathfrak{g}$-modules. Therefore, be corollary 2.7.18, we have

$$\mathfrak{n}_-^{\psi}\backslash X/\mathfrak{b}_{n-1} \cong (\mathfrak{n}_-^{\psi}\backslash X)^{N_-} \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \mathfrak{n}_-^{\psi}\backslash \mathrm{U}\mathfrak{g}/\mathfrak{b}_{n-1} \cong (\mathfrak{n}_-^{\psi}\backslash X)^{N_-} \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \mathrm{Z}(\mathrm{U}\mathfrak{g}) \cong (\mathfrak{n}_-^{\psi}\backslash X)^{N_-}.$$

□

Let $V$ be the defining representation of $\mathfrak{g}$. We can identify $\mathfrak{n}_-^{\psi}\backslash \mathrm{U}\mathfrak{g} \otimes V \cong \mathrm{U}\mathfrak{b} \otimes V$; likewise, using the isomorphism $\tau\colon \mathrm{U}\mathfrak{g} \otimes V \to V \otimes \mathrm{U}\mathfrak{g}$ as in (2.2.5), we have an isomorphism $\mathfrak{n}_-^{\psi}\backslash V \otimes \mathrm{U}\mathfrak{g} \cong V \otimes \mathrm{U}\mathfrak{b}$.

**Proposition 2.7.20.** *Let $\{v_i\}$ be the natural basis of $V$, where $v_n$ is the lowest weight vector. Then the elements*

$$v_{n-k}^{\psi} = \sum_{i=0}^{k} (-1)^i v_{n-i} \otimes Z_{n-i-1}^{k-i} \tag{2.7.11}$$

*in $\mathfrak{n}_-^{\psi}\backslash V \otimes \mathrm{U}\mathfrak{g}$ for $1 \leq k \leq n$ freely generate the $\mathrm{Z}(\mathrm{U}\mathfrak{g})$-module $\mathrm{res}^{\psi}(V \otimes \mathrm{U}\mathfrak{g})$. Here, we set $Z_j^0 = 1$ for any $j$.*

Proof. Similarly to arguments in corollary 2.7.14, we only need to show that this elements are $\mathfrak{n}_-^{\psi}$-invariant. Observe the following “triangle” pattern: invariance under the right action of $E_{k,k-1} - 1$ for $k < n$ reduces to the statement for $\mathfrak{gl}_k$, since the coefficients $Z_j^i$ are the images of central elements. Therefore, by induction, it is enough to prove invariance under $E_{n,n-1} - 1$. Since $Z_j^i$ commutes with $E_{n,n-1}$ for $j \leq n-2$, we only need to show that

$$Z_{n-1}^i E_{n,n-1} + Z_{n-2}^{i-1} - Z_{n-1}^i \in \mathfrak{n}_-^{\psi} \mathrm{U}\mathfrak{gl}_n.$$

Again, consider the polynomial (2.7.7). Then

$$Q_{\mathfrak{gl}_{n-1}}(t) E_{n,n-1} = (t+n-2) Q_{\mathfrak{gl}_{n-2}}(t) E_{n,n-1} + \sum_{k=n-1}^{1} (-1)^{n-1-k} E_{kn} Q_{\mathfrak{gl}_{k-1}}(t) E_{n,n-1}.$$

For $k \leq n-1$, the polynomials $Q_{\mathfrak{gl}_{k-1}}(t)$ commute with $E_{n,n-1}$. The only term in the sum having non-trivial commutator with $E_{n,n-1}$ is $E_{k,n-1}$ for $k = n-1$. Therefore,

$$Q_{\mathfrak{gl}_{n-1}}(t) E_{n,n-1} = E_{n,n-1} Q_{\mathfrak{gl}_{n-1}}(t) - E_{n,n-1} Q_{\mathfrak{gl}_{n-2}}(t),$$

which implies the claim for coefficients. □

**Remark 2.7.21.** As we mentioned, there is a certain “triangle” pattern: the generators of $\mathfrak{n}_-^{\psi}\backslash V \otimes \mathrm{U}\mathfrak{g}$ can be presented in the matrix form

$$\begin{pmatrix} Z_{n-1}^0 & -Z_{n-1}^1 & \dots & (-1)^{n-2} Z_{n-1}^{n-2} & (-1)^{n-1} Z_{n-1}^{n-1} \\ 0 & Z_{n-2}^0 & \dots & (-1)^{n-3} Z_{n-2}^{n-3} & (-1)^{n-2} Z_{n-2}^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & Z_1^0 & -Z_1^1 \\ 0 & 0 & \dots & 0 & Z_0^0. \end{pmatrix}$$

in the basis $\{v_n, \dots, v_1\}$, so that the lower right corners define the generators of similar Kostant-Whittaker reductions for $\mathfrak{gl}_k$ with $k \leq n$.

Recall that for any finite-dimensional $\mathfrak{g}$-module $W$, there exists a natural $\mathfrak{b}_-$-stable filtration $W^{\leq \mu}$, where $\mu \in \Lambda$ such that the quotient $W^{\leq \mu}/W^{< \mu}$ is isomorphic to the $\mu$-weight space of $W$.

**Corollary 2.7.22.** *Let $W$ be arbitrary representation of $GL_n$. Then for any $w \in W$ of weight $\mu$ there exists a unique element $w^{\psi} \in \mathrm{res}^{\psi}(W \otimes \mathrm{U}\mathfrak{g})$ such that under identification $\mathfrak{n}_-^{\psi}\backslash W \otimes \mathrm{U}\mathfrak{g} \cong W \otimes \mathrm{U}\mathfrak{b}$, it has the form*

$$w^{\psi} \in w \otimes 1 + W^{< \mu} \otimes \mathrm{U}\mathfrak{b}_{n-1}^{>0}.$$

*In particular, the collection $\{w^{\psi} |\ w \in W\}$ freely generates $\mathrm{res}^{\psi}(W \otimes \mathrm{U}\mathfrak{g})$ as a right $\mathrm{Z}(\mathrm{U}\mathfrak{g})$-module, i.e. we have a natural isomorphism*

$$\mathrm{res}^{\psi}(W \otimes \mathrm{U}\mathfrak{g}) \cong W \otimes \mathrm{Z}(\mathrm{U}\mathfrak{g}). \tag{2.7.12}$$

Proof. Uniqueness can be easily shown by induction on the filtration of $W$ starting from the lowest-weight vectors. The arguments similar to corollary 2.7.14 also show that such a collection freely generates $\mathrm{res}^{\psi}(W \otimes \mathrm{U}\mathfrak{g})$. Therefore, we only need to show existence.

For $W = V$ the defining representation, consider the generators of proposition 2.7.20. They do not quite satisfy the assumptions because of the constant terms in the coefficients $Z_j^i$, however, we can correct by applying a suitable constant upper-triangular matrix to the basis $\{v_i^{\psi}\}$.

Consider $W = V \otimes V$. By proposition 2.7.2 we have an isomorphism

$$(\mathfrak{n}_-^\psi \backslash V \otimes \mathrm{U}\mathfrak{g})^{N_-} \otimes_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} (\mathfrak{n}_-^\psi \backslash V \otimes \mathrm{U}\mathfrak{g})^{N_-} \xrightarrow{\sim} (\mathfrak{n}_-^\psi \backslash V \otimes V \otimes \mathrm{U}\mathfrak{g})^{N_-}.$$

Restrict it to the generators $\{v_i^\psi\}$ of $\mathrm{res}^\psi(V \otimes \mathrm{U}\mathfrak{g})$. Careful analysis of the coproduct of the formula (2.7.8) shows that

$$(Z_j^i)_{(1)} \cdot v_k \otimes (Z_j^i)_{(2)} \in \delta_{k \leq j} \cdot \mathrm{span}(v_k, v_{k-1}, \dots, v_{k-i+1}) \otimes \mathrm{U}\mathfrak{b}_{n-1}.$$

Since

$$v_{n-k}^\psi \otimes v_{n-l}^\psi = \sum_{\substack{i=0,\dots,k \\ j=0,\dots,l}} (-1)^{i+j} v_{n-i} \otimes (Z_{n-i-1}^{k-i})_{(1)} \cdot v_{n-j} \otimes (Z_{n-i-1}^{k-i})_{(2)} Z_{n-j-1}^{l-j} \in V \otimes V \otimes \mathrm{U}\mathfrak{b}_{n-1},$$

we see that the generators $\{v_{n-k}^\psi \otimes v_{n-l}^\psi\}$ satisfy the condition of the claim except for the constant term (i.e., although we apply some elements from $\mathrm{U}\mathfrak{b}_{n-1}$, the weights do not become "too high", so that the filtration assumption is not violated). The constant term can also corrected by applying a certain constant upper-triangular (in the weight filtration) matrix to the generators $v_{n-k}^\psi \otimes v_{n-l}^\psi$.

Similarly, we can find such generators for any tensor power $V^{\otimes n}$. Since any representation can be presented as a direct summand of a suitable tensor power of $V$, we conclude. □

In other terms, for every representation $W$, we have a natural (due to uniqueness) map

$$W \to \mathrm{res}^\psi(W \otimes \mathrm{U}\mathfrak{g}),$$

such its composition with isomorphism

$$\mathrm{res}^\psi(W \otimes \mathrm{U}\mathfrak{g}) \cong \mathfrak{n}_-^\psi \backslash W \otimes \mathrm{U}\mathfrak{g}/\mathfrak{b}_{n-1}$$

of theorem 2.7.19 factorizes through the identity map $w \mapsto [w \otimes 1]$. As corollary, we have the following result:

**Proposition 2.7.23.** *For any $V, W \in \mathrm{Rep}(GL_n)$, there is a commutative diagram*

$$\begin{tikzcd} V \otimes W \otimes \mathrm{Z}(\mathrm{U}\mathfrak{g}) \arrow[r] & V \otimes W \otimes \mathrm{Z}(\mathrm{U}\mathfrak{g}) \\ V \otimes W \arrow[u, "\mathrm{id}_{V\otimes W} \otimes 1"] \arrow[r, "J_{V,W}"] & V \otimes W, \arrow[u, "\mathrm{id}_{V\otimes W} \otimes 1"'] \end{tikzcd} \tag{2.7.13}$$

*where the upper horizontal arrow is the monoidal structure (2.7.1) combined with the natural isomorphisms (2.7.12) and $J_{V,W}$ defines a monoidal structure on the forgetful functor $\mathrm{Rep}(GL_n) \to \mathrm{Vect}$.*

As a corollary, we obtain a vertex-IRF transformation:

THEOREM 2.7.24. *For $\mathfrak{g} = \mathfrak{gl}_n$, there is a monoidal functor $\mathrm{Rep}(G) \to \mathrm{Vect}$ (i.e. a there is a structure of a $\mathrm{Rep}(G)$-module category on $\mathrm{Vect}$) such that the functor $\mathrm{Vect} \to \mathrm{RMod}_{\mathrm{Z}(\mathrm{U}\mathfrak{g})}$ given by $V \mapsto V \otimes \mathrm{Z}(\mathrm{U}\mathfrak{g})$ is a functor of $\mathrm{Rep}(G)$-module categories. In particular, its composition with the functor $\mathrm{RMod}_{\mathrm{Z}(\mathrm{U}\mathfrak{g})} \to \mathrm{RMod}_{(\mathrm{U}\mathfrak{h})^{\mathrm{gen}}}$ of $\mathrm{Rep}(G)$-module categories as in proposition 2.7.12 provides a categorical vertex-IRF transformation in the sense of proposition 2.4.3.*

As in the case of $\mathfrak{sl}_2$, one can consider the asymptotic version of this construction involving the deformation parameter $\hbar$, so that the semi-classical limit provides a classical vertex-IRF transformation in the sense of section 1.4. In fact, we computed in low ranks that the constant tensor structure quantizes the rational Cremmer-Gervais $r$-matrix. If this is true in general, then an equivalence between the Kostant-Whittaker reduction and the parabolic restriction provides a quantization of *the* classical vertex-IRF transformation of remark 1.4.13.

CHAPTER 3

# Appendix

### 3.1. Proof of theorem 1.4.7.

The proof is base on [BRT07, Section 9.1] and the notations are taken from there.

Consider an arbitrary generalized Belavin-Drinfeld triple $(\Gamma_1, \Gamma_2, \tau)$. Let $\Gamma_3 \subset \Gamma_1 \cap \Gamma_2$ be the largest subset stable under $\tau$. Note that a BD-triple is nilpotent (hence gauge-equivalent to a constant $r$-matrix) iff $\Gamma_3 = \emptyset$. Therefore, assume that $\Gamma_3 \neq \emptyset$.

Let $\alpha_1 \in \Gamma_3$ be a root such that $\tau(\alpha_1) \neq \alpha_1$. Let $n$ be the minimal natural number such that $\tau^n(\alpha_1) = \alpha_1$. Define $\alpha_i := \tau^{i-1}(\alpha_0),\ i = 1, \ldots, n$. Then

$$\begin{aligned} K(\lambda)(e_{\alpha_1}) &= \frac{1}{2} e_{\alpha_1} + e^{-(\alpha_1,\lambda)} e_{\alpha_2} + \ldots + e^{(n-1)(\alpha_1,\lambda)} e_{\alpha_n} + \\ &+ e^{-n(\alpha_1,\lambda)} e_{\alpha_1} + e^{-(n+1)(\alpha_1,\lambda)} e_{\alpha_2} + \ldots = \\ &= \left( -\frac{1}{2} + \frac{1}{1 - e^{-n(\alpha_1,\lambda)}} \right) e_{\alpha_1} + \frac{e^{-(\alpha_1,\lambda)}}{1 - e^{-n(\alpha_1,\lambda)}} e_{\alpha_2} + \ldots + \frac{e^{-(n-1)(\alpha_1,\lambda)}}{1 - e^{-n(\alpha_1,\lambda)}} e_{\alpha_n} = \\ &= \frac{1}{2} \left( \frac{x_1^n + 1}{x_1^n - 1} e_{\alpha_1} + \frac{2x_1^{n-1}}{x_1^n - 1} e_{\alpha_2} + \ldots + \frac{2x_1}{x_1^n - 1} e_{\alpha_n} \right) \end{aligned}$$

Likewise, $K(\lambda)(e_{\alpha_k})$ are cyclic permutations of the expression above and with $x_1$ changed to $x_k$. For instance,

$$K(\lambda)(e_{\alpha_2}) = \frac{1}{2} \left( \frac{x_2^n + 1}{x_2^n - 1} e_{\alpha_2} + \frac{2x_2^{n-1}}{x_2^n - 1} e_{\alpha_3} + \ldots + \frac{2x_2^2}{x_2^n - 1} e_{\alpha_n} + \frac{2x_2}{x_2^n - 1} e_{\alpha_1} \right).$$

Let $M(x)$ be a vertex-IRF transform and $A_i := x_i (\partial_i M) M^{-1}$. Then we have equations

$$D_A(A) = 0, \tag{3.1.1}$$

$$x_i \partial_i (r(x) + \sum_j h_{\alpha_j} \wedge A_j) + [r(x) + \sum h_{\alpha_j} \wedge A_j, A_i \otimes 1 + 1 \otimes A_i] = 0, \tag{3.1.2}$$

where $D_A(\bullet) = x_i \partial_i + [\bullet, A_i \otimes 1 + 1 \otimes A_i]$. Let us compare these equations to the ones from [BRT07]. Beforehand, observe that the $r$-matrix is of the form $r(x) = r_{\mathfrak{h}_0,\mathfrak{h}_0} + \sum_{\alpha \in \Delta_+} g_{\alpha\beta}(x) e_\alpha \wedge e_{-\beta}$, that is, its $\mathfrak{h} \wedge \mathfrak{h}$-part is constant and there are no mixed terms (i.e. of the form $h \wedge e_\alpha$ for $h \in \mathfrak{h}$).

(1) Equation (3.1.1) is equivalent to [BRT07, (161), (162)]:

$$x_i \partial_i A_j^m - x_j \partial_j A_i^m - \frac{2}{(\alpha,\alpha)} \sum_{\alpha \in \Phi} A_i^\alpha A_j^{-\alpha} \alpha(\zeta^{\alpha_m}) = 0, \tag{3.1.3}$$

$$x_i \partial_i A_j^\alpha - x_j \partial_j A_i^\alpha - \sum_{\beta+\gamma=\alpha} N_{\beta\gamma}^\alpha A_i^\beta A_j^\gamma - A_j^\alpha \sum_n A_i^n \alpha(h_{\alpha_n}) + A_i^\alpha \sum_n A_j^n \alpha(h_{\alpha_n}) = 0, \tag{3.1.4}$$

where $[e_\beta, e_\gamma] = N_{\beta\gamma}^\alpha e_\alpha$.

(2) Equation (3.1.2) gives equations of three types according to the type of the tensor: $h_{\alpha_n} \wedge h_{\alpha_m}$, $h_{\alpha_i} \wedge e_\alpha$, $e_\alpha \wedge e_\beta$. We will be interested only in the first two.

(a) $h_{\alpha_n} \wedge h_{\alpha_m}$: one can easily see that these equations are independent of the $r$-matrix. Indeed, the only terms which could contribute to this type of equations are either derivatives of $\mathfrak{h} \wedge \mathfrak{h}$-part or commutators with the mixed terms, but both are zero. Therefore, the equations are the same as [BRT07, (163)]:

$$x_i\partial_i(A_n^m - A_m^n) + \sum_{\alpha\in\Delta_+}\frac{2}{(\alpha,\alpha)}[A_n^\alpha A_i^{-\alpha}\alpha(\zeta^{\alpha_m}A_m^\alpha A_i^{-\alpha}\alpha(\zeta^{\alpha_n})] = 0. \tag{3.1.5}$$

Combining it with (3.1.3), we get

$$x_m\partial_m A_i^n - x_n\partial_n A_i^m = 0,$$

so that the general solution is $A_i^n = x_n\partial_n\phi_i$ for arbitrary functions $\phi_i$.

(b) $h_{\alpha_i}\wedge e_\xi$: let us study the commutator part of (3.1.2) in detail.

- $[r_{\mathfrak{h}_0,\mathfrak{h}_0}, A_j\otimes 1 + 1\otimes A_j]$: write $r_{\mathfrak{h}_0,\mathfrak{h}_0} = \sum\frac{B_{mn}}{2}h_{\alpha_m}\wedge h_{\alpha_n}$, $B_{mn} = -B_{nm}$. Then

$$[\sum B_{mn}h_{\alpha_m}\wedge h_{\alpha_n}, A_j^l(h_{\alpha_l}\otimes 1 + 1\otimes h_{\alpha_l})] = 0,$$
$$[\sum B_{mn}h_{\alpha_m}\wedge h_{\alpha_n}, A_j^\alpha(e_\alpha\otimes 1 + 1\otimes e_\alpha)] =$$
$$= \sum_{m,n} -B_{mn}A_j^\alpha\alpha(h_{\alpha_m})h_{\alpha_n}\wedge e_\alpha + A_j^\alpha B_{mn}\alpha(h_{\alpha_n})h_{\alpha_m}\wedge e_\alpha.$$

  Therefore, the coefficient in front of $h_{\alpha_i}\wedge e_\xi$ is given by

$$A_j^\xi\sum_m B_{im}\xi(h_{\alpha_m}).$$

- $[g_{\alpha\beta}(x)e_\alpha\wedge e_{-\beta}, A_j\otimes 1 + 1\otimes A_j]$: Let $\gamma\neq -\alpha,\beta$. Then

$$[g_{\alpha\beta}(x)e_\alpha\wedge e_{-\beta}, A_j^l(h_{\alpha_l}\otimes 1 + 1\otimes h_{\alpha_l})] = g_{\alpha\beta}(x)((\beta-\alpha)(h_{\alpha_l}))e_\alpha\wedge e_{-\beta},$$
$$[g_{\alpha\beta}(x)e_\alpha\wedge e_{-\beta}, A_j^\gamma(e_\gamma\otimes 1 + 1\otimes e_\gamma)] = g_{\alpha\beta}(x)A_j^\gamma(N_{\alpha,\gamma}^{\alpha+\gamma}e_{\alpha+\gamma}\wedge e_{-\beta} + N_{-\beta,\gamma}^{\gamma-\beta}e_\alpha\wedge e_{\gamma-\beta}),$$
$$[g_{\alpha\beta}(x)e_\alpha\wedge e_{-\beta}, A_j^{-\alpha}(e_{-\alpha}\otimes 1 + 1\otimes e_{-\alpha})] = g_{\alpha\beta}(x)A_j^{-\alpha}(h_\alpha\wedge e_{-\beta} + N_{-\alpha,-\beta}^{-\alpha-\beta}e_\alpha\wedge e_{-\alpha-\beta}),$$
$$[g_{\alpha\beta}(x)e_\alpha\wedge e_{-\beta}, A_j^\beta(e_\beta\otimes 1 + 1\otimes e_\beta)] = g_{\alpha\beta}(x)A_j^\beta(N_{\alpha,\beta}^{\alpha+\beta}e_{\alpha+\beta}\wedge e_{-\beta} + h_\beta\wedge e_\alpha).$$

  Therefore, if $\xi$ is positive, then the necessary coefficient is given by

$$\sum_{\beta\in\Delta_+}\frac{2\beta(\zeta^{\alpha_i})}{(\beta,\beta)}g_{\xi,\beta}(x)A_j^\beta,$$

  and if negative:

$$\sum_{\alpha\in\Delta_+}\frac{2\alpha(\zeta^{\alpha_i})}{(\alpha,\alpha)}g_{\alpha,-\xi}(x)A_j^{-\alpha}$$

- $[\sum h_{\alpha_i}\wedge A_i, A_j\otimes 1 + 1\otimes A_j]$: The coefficient of interest will be the same as in [BRT07, (164)], namely

$$\sum_{\beta+\gamma=\xi}N_{\beta,\gamma}^\xi A_i^\beta A_j^\gamma + \sum_m(A_i^m A_j^\xi - A_i^\xi A_j^m - A_m^i A_j^\xi)\xi(h_{\alpha_m}).$$

The derivative part gives only $x_j\partial_j A_i^\xi$, since there are no mixed terms in $r(x)$. Combining altogether, for positive $\xi$ we get the equations of the form

$$x_j\partial_j A_i^\xi + \sum_{\beta\in\Delta_+}\frac{2\beta(\zeta^{\alpha_i})}{(\beta,\beta)}g_{\xi,\beta}(x)A_j^\beta +$$
$$+ \sum_{\beta+\gamma=\xi}N_{\beta,\gamma}^\xi A_i^\beta A_j^\gamma + \sum_m(A_i^m A_j^\xi - A_i^\xi A_j^m - A_m^i A_j^\xi)\xi(h_{\alpha_m}) + A_j^\xi\sum_m B_{im}\xi(h_{\alpha_m}) = 0,$$

and if $\xi$ is negative:

$$x_j\partial_j A_i^{\xi} + \sum_{\alpha\in\Delta_+}\frac{2\alpha(\zeta^{\alpha_i})}{(\alpha,\alpha)}g_{\alpha,-\xi}(x)A_j^{-\alpha}+$$
$$+\sum_{\beta+\gamma=\xi}N^{\xi}_{\beta,\gamma}A_i^{\beta}A_j^{\gamma}+\sum_m(A_i^mA_j^{\xi}-A_i^{\xi}A_j^m-A_m^iA_j^{\xi})\xi(h_{\alpha_m})+A_j^{\xi}\sum_m B_{im}\xi(h_{\alpha_m})=0.$$

Summing it up with (3.1.4) and changing $\xi$ to $\alpha$, we get the following equations:

$$x_i\partial_i A_j^{\alpha}+\sum_{\beta\in\Delta_+}\frac{2\beta(\zeta^{\alpha_i})}{(\beta,\beta)}g_{\alpha,\beta}(x)A_j^{\beta}-A_j^{\alpha}\sum_m(A_m^i+B_{mi})\alpha(h_{\alpha_m})=0, \tag{3.1.6}$$

$$x_i\partial_i A_j^{-\alpha}+\sum_{\beta\in\Delta_+}\frac{2\beta(\zeta^{\alpha_i})}{(\beta,\beta)}g_{\beta,\alpha}(x)A_j^{-\beta}+A_j^{-\alpha}\sum_m(A_m^i+B_{mi})\alpha(h_{\alpha_m})=0 \tag{3.1.7}$$

Let $\alpha$ be one of the simple roots $\alpha_1,\dots,\alpha_n$. Then we can assume in the equations above that $\beta$ is also one of these simple roots. Indeed: $g_{\alpha\beta}(x)\neq 0 \Leftrightarrow \tau^i(\alpha)=\beta$ for some $i$, therefore, $\beta=\tau^i(\alpha_k)=\alpha_{k+i}$, if $\alpha=\alpha_k$ (we take $i+k$ modulo $n$ in an appropriate sense). Likewise, if $g_{\beta\alpha}(x)\neq 0$, then $\tau^i(\beta)=\alpha_k \Leftrightarrow \beta=\tau^{-i}(\alpha_k)=\alpha_{k-i}$.

Denote by $g_{ij}(x):=g_{\alpha_i,\alpha_j}(x)$. Observe that $g_{ij}(x)$ depends only on $x_j$, so we may write $g_{ij}(x_j)$. Simple case-by-case analysis of Dynkin diagrams shows that if $\tau$ is nontrivial, then the lengths of $\alpha_i$ is always 2. Therefore, the equations above become

$$x_i\partial_i A_j^{\alpha_k}+g_{ki}(x_i)A_j^{\alpha_i}-A_j^{\alpha_k}\sum_m(A_m^i+B_{mi})\alpha(h_{\alpha_m})=0, \tag{3.1.8}$$

$$x_i\partial_i A_j^{-\alpha_k}+g_{ik}(x_k)A_j^{-\alpha_i}+A_j^{-\alpha_k}\sum_m(A_m^i+B_{mi})\alpha(h_{\alpha_m})=0. \tag{3.1.9}$$

Recall that $g_{ii}(x)=\frac{x_i^n+1}{x_i^n-1}$. Then a general solution to the equations (3.1.8) and (3.1.9) can be written as

$$A_j^{\pm\alpha_k}=C_j^{\pm\alpha_k}(\hat{x}_k)\cdot\frac{x_k}{(x_k^n-1)^{2/n}}\cdot\exp(\pm\sum_m\tilde{\phi}_m\alpha_k(h_{\alpha_m})), \tag{3.1.10}$$

where $\tilde{\phi}$ are arbitrary functions such that $A_i^m+B_{im}=x_m\partial_m\tilde{\phi}_i$, and $\hat{x}_k$ means that the function does not depend on that variable. Substituting it into (3.1.8), we get

$$x_i\partial_i C_j^{\alpha_k}(\hat{x}_k)=-g_{ki}(x_i)\cdot\frac{F(x_i)}{F(x_k)}\cdot C_j^{\alpha_i}(\hat{x}_i)\cdot\exp(\sum_m\tilde{\phi}_m(\alpha_i-\alpha_k)(h_{\alpha_m})), \tag{3.1.11}$$

where $F(x)=\frac{x}{(x^n-1)^{2/n}}$.

Observe that the left-hand side does not depend on $x_k$. Therefore, on the right-hand side, this dependence should be absorbed by the exponential factor, i.e.

$$\frac{C_j^{\alpha_i}(\hat{x}_i)}{F(x_k)}\cdot\exp(\sum_m\tilde{\phi}_m(\alpha_i-\alpha_k)(h_{\alpha_m}))=G_j^{ki}(\hat{x}_k),$$

or

$$\exp(\sum_m\tilde{\phi}_m(\alpha_i-\alpha_k)(h_{\alpha_m}))=\frac{F(x_k)\cdot G_j^{ki}(\hat{x}_k)}{C_j^{\alpha_i}(\hat{x}_i)}.$$

Likewise,

$$\exp(\sum_m\tilde{\phi}_m(\alpha_k-\alpha_i)(h_{\alpha_m}))=\frac{F(x_i)\cdot G_j^{ik}(\hat{x}_i)}{C_j^{\alpha_k}(\hat{x}_k)}=$$
$$=\exp(-\sum_m\tilde{\phi}_m(\alpha_i-\alpha_k)(h_{\alpha_m})=\frac{C_j^{\alpha_i}(\hat{x}_i)}{F(x_k)\cdot G_j^{ki}(\hat{x}_k)},$$

Denote by

$$Q_j^{ki}(\hat{x}_k) := \frac{G_j^{ki}(\hat{x}_k)}{C_j^{\alpha_k}(\hat{x}_k)}, \tag{3.1.12}$$

then the equality above can be rewritten as

$$F(x_k)F(x_i) = \frac{1}{Q_j^{ki}(\hat{x}_k)\cdot Q_j^{ik}(\hat{x}_i)}.$$

Denote by $\partial_i^{log}$ the partial logarithmic derivative, i.e. $\partial_i^{log} f(x) = \partial_i(\log(f(x))) = \frac{\partial_i f(x)}{f(x)}$. Then we have

$$\partial_i^{log} F(x_i) = \partial_i^{log}(LHS) = \partial_i^{log}(RHS) = -\partial_i^{log} Q_j^{ki}(\hat{x}_k),$$

hence

$$Q_j^{ki}(\hat{x}_k) = \frac{\tilde{Q}^{ki}(\hat{x}_i, \hat{x}_k)}{F(x_i)} \tag{3.1.13}$$

for any $i \neq k$ and some functions $\tilde{Q}_j^{ki}(\hat{x}_i, \hat{x}_k)$ not depending on $x_i, x_k$. Therefore,

$$\exp(\sum_m \tilde{\phi}_m(\alpha_i - \alpha_k)(h_{\alpha_m})) = \frac{F(x_k)}{F(x_i)}\cdot\frac{C_j^{\alpha_k}(\hat{x}_k)}{C_j^{\alpha_i}(\hat{x}_i)}\cdot\tilde{Q}_j^{ki}(\hat{x}_i, \hat{x}_k).$$

Take $l \neq i, k$. Then

$$\begin{aligned}
&\exp(\sum_m \tilde{\phi}_m(\alpha_i - \alpha_l)(h_{\alpha_m}))\cdot\exp(\sum_m \tilde{\phi}_m(\alpha_l - \alpha_k)(h_{\alpha_m})) = \\
&= \frac{F(x_l)}{F(x_i)}\cdot\frac{C_j^{\alpha_l}(\hat{x}_l)}{C_j^{\alpha_i}(\hat{x}_i)}\cdot\tilde{Q}_j^{li}(\hat{x}_i, \hat{x}_l)\cdot\frac{F(x_k)}{F(x_l)}\cdot\frac{C_j^{\alpha_k}(\hat{x}_k)}{C_j^{\alpha_l}(\hat{x}_l)}\cdot\tilde{Q}_j^{kl}(\hat{x}_l, \hat{x}_k) = \\
&= \frac{F(x_k)}{F(x_i)}\cdot\frac{C_j^{\alpha_k}(\hat{x}_k)}{C_j^{\alpha_i}(\hat{x}_i)}\cdot\tilde{Q}_j^{li}(\hat{x}_i, \hat{x}_l)\cdot\tilde{Q}_j^{kl}(\hat{x}_l, \hat{x}_k) = \\
&= \exp(\sum_m \tilde{\phi}_m(\alpha_i - \alpha_k)(h_{\alpha_m})) = \\
&= \frac{F(x_k)}{F(x_i)}\cdot\frac{C_j^{\alpha_k}(\hat{x}_k)}{C_j^{\alpha_i}(\hat{x}_i)}\cdot\tilde{Q}_j^{ki}(\hat{x}_i, \hat{x}_k),
\end{aligned}$$

hence

$$\tilde{Q}_j^{ki}(\hat{x}_i, \hat{x}_k) = \tilde{Q}_j^{li}(\hat{x}_i, \hat{x}_l)\cdot\tilde{Q}_j^{kl}(\hat{x}_l, \hat{x}_k).$$

Observe that the right-hand side does not depend on $x_l$ and so does the left-hand side; moreover, this is true for any $l$. Therefore, $\tilde{Q}_j^{ki}(\hat{x}_k, \hat{x}_i) = K_j^{ki}$ is constant and

$$\exp(\sum_m \tilde{\phi}_m(\alpha_i - \alpha_k)(h_{\alpha_m})) = K_j^{ki}\cdot\frac{F(x_k)}{F(x_i)}\cdot\frac{C_j^{\alpha_k}(\hat{x}_k)}{C_j^{\alpha_i}(\hat{x}_i)}. \tag{3.1.14}$$

Substituting it into (3.1.11), we get

$$x_i\partial_i \log(C_j^{\alpha_k}(\hat{x}_k)) = -K_j^{ki} g_{ki}(x_i).$$

Now, consider the equations for negative roots using (3.1.10) and (3.1.14):

$$\begin{aligned}
x_i\partial_i C_j^{-\alpha_k}(\hat{x}_k) &= -g_{ik}(x_k)\cdot C_j^{-\alpha_i}(\hat{x}_i)\cdot\frac{F(x_i)}{F(x_k)}\cdot\exp(\sum_m \tilde{\phi}_m(\alpha_k - \alpha_i)(h_{\alpha_m})) = \\
&= -g_{ik}(x_k)\cdot C_j^{-\alpha_i}(\hat{x}_i)\cdot\frac{F(x_i)}{F(x_k)}\cdot K_j^{ik}\cdot\frac{F(x_i)}{F(x_k)}\cdot\frac{C_j^{\alpha_i}(\hat{x}_i)}{C_j^{\alpha_k}(\hat{x}_k)} = \\
&= -K_j^{ik}\frac{g_{ik}(x_k)\cdot C_j^{-\alpha_i}(\hat{x}_i)\cdot C_j^{\alpha_i}(\hat{x}_i)}{F(x_k)^2}\cdot\frac{F(x_i)^2}{C_j^{\alpha_k}(\hat{x}_k)} \qquad (3.1.15)
\end{aligned}$$

Observe that the left-hand side does not depend on $x_k$, hence

$$C_j^{-\alpha_i}(\hat{x}_i) \cdot C_j^{\alpha_i}(\hat{x}_i) = Aux_k(\hat{x}_k) \cdot \frac{F(x_k)^2}{g_{ik}(x_k)}$$

for some function $Aux_k(\hat{x}_k)$.

**Lemma 3.1.1.** *$Aux_k(\hat{x}_k) = C_j^{ik} \prod_{l \neq i,k} \frac{F(x_l)^2}{g_{il}(x_l)}$ for some constant $C_j^i$.*

PROOF. The equality above is true for any $k$. Therefore, for any $k_1, k_1$:

$$Aux_{k_1}(\hat{x}_{k_1}) \cdot \frac{F(x_{k_1})^2}{g_{ik_1}(x_{k_1})} = Aux_{k_2}(\hat{x}_{k_2}) \cdot \frac{F(x_{k_2})^2}{g_{ik_2}(x_{k_2})} \tag{3.1.16}$$

Applying $\partial_{k_1}^{log}$ to both sides and solving the equation, we get

$$Aux_{k_2}(\hat{x}_{k_2}) = \frac{F(x_{k_1})^2}{g_{ik_1}(x_{k_1})} \cdot Aux_{k_2 k_1}(\hat{x}_{k_2}, \hat{x}_{k_1}),$$

for some function $Aux_{k_2 k_1}(\hat{x}_{k_2}, \hat{x}_{k_1})$, and applying $\partial_{k_2}^{log}$:

$$Aux_{k_1}(\hat{x}_{k_1}) = \frac{F(x_{k_2})^2}{g_{ik_2}(x_{k_2})} \cdot Aux_{k_1 k_2}(\hat{x}_{k_1}, \hat{x}_{k_2}).$$

Substituting it into (3.1.16), we see that

$$Aux_{k_2 k_1}(\hat{x}_{k_2}, \hat{x}_{k_1}) = Aux_{k_1 k_2}(\hat{x}_{k_1}, \hat{x}_{k_2}).$$

Observe that $k_2$ is arbitrary. Then

$$Aux_{k_1}(\hat{x}_{k_1}) = \frac{F(x_{k_2})^2}{g_{ik_2}(x_{k_2})} \cdot Aux_{k_1 k_2}(\hat{x}_{k_1}, \hat{x}_{k_2}) = \frac{F(x_{k_3})^2}{g_{ik_3}(x_{k_3})} \cdot Aux_{k_1 k_3}(\hat{x}_{k_1}, \hat{x}_{k_3}).$$

Taking $\partial_{k_3}^{log}$ gives

$$Aux_{k_1 k_2}(\hat{x}_{k_1}, \hat{x}_{k_2}) = \frac{F(x_{k_3})^2}{g_{ik_3}(x_{k_3})} \cdot Aux_{k_1 k_2 k_3}(\hat{x}_{k_1}, \hat{x}_{k_2}, \hat{x}_{k_3})$$

for some function $Aux_{k_1 k_2 k_3}(\hat{x}_{k_1}, \hat{x}_{k_2}, \hat{x}_{k_3})$. Continuing this process, we see that

$$Aux_k(\hat{x}_k) = C_j^i \prod_{l \neq i,k} \frac{F(x_l)^2}{g_{il}(x_l)}$$

for some constant $C_j^i$. □

Therefore,

$$C_j^{-\alpha_i}(\hat{x}_i) \cdot C_j^{\alpha_i}(\hat{x}_i) = C_j^i \prod_{l \neq i} \frac{F(x_l)^2}{g_{il}(x_l)}. \tag{3.1.17}$$

Substitute it into (3.1.15):

$$x_i \partial_i C_j^{-\alpha_k}(\hat{x}_k) = -K_j^{ik} \frac{g_{ik}(x_k) \cdot C_j^{-\alpha_i}(\hat{x}_i) \cdot C_j^{\alpha_i}(\hat{x}_i)}{F(x_k)^2} \cdot \frac{F(x_i)^2}{C_j^{\alpha_k}(\hat{x}_k)} = \tag{3.1.18}$$

$$= -K_j^{ik} \cdot C_j^i \cdot \frac{F(x_i)^2}{C_j^{\alpha_k}(\hat{x}_k)} \cdot \prod_{l \neq i,k} \frac{F(x_l)^2}{g_{il}(x_l)}, \tag{3.1.19}$$

Divide both sides by $C_j^{-\alpha_k}(\hat{x}_k)$ and use (3.1.17):

$$x_i \partial_i^{log} C_j^{-\alpha_k}(\hat{x}_k) = -K_j^{ik} \cdot \frac{C_j^i}{C_j^k} \cdot F(x_i)^2 \cdot \frac{\prod_{m \neq i} \frac{F(x_m)^2}{g_{im}(x_m)}}{\prod_{l \neq k} \frac{F(x_l)^2}{g_{kl}(x_l)}} =$$

$$= -K_j^{ik} \cdot \frac{C_j^i}{C_j^k} \cdot F(x_k)^2 \cdot \frac{\prod_{l \neq k} g_{kl}(x_l)}{\prod_{m \neq i} g_{im}(x_m)}.$$

On the other hand, we know from (3.1.17) that

$$C_j^{-\alpha_k}(\hat{x}_k) = C_j^k \prod_{l \neq k} \frac{F(x_l)^2}{g_{kl}(x_l)} \cdot \frac{1}{C_j^{\alpha_k}(\hat{x}_k)}$$

and we can take $x_i \partial_i^{log}$ of both sides; combining with the equation above, we get

$$C_j^k[2x_i\partial_i^{log} F(x_i) - x_i\partial_i^{log} g_{ki}(x_i) - x_i\partial_i^{log} C_j^{\alpha_k}(\hat{x}_k).] =$$
$$= K_j^{ik} \cdot \frac{C_j^i}{C_j^k} \cdot F(x_k)^2 \cdot \frac{\prod_{l \neq k} g_{kl}(x_l)}{\prod_{m \neq i} g_{im}(x_m)}.$$

Observe that the left-hand side does not depend on $x_k$, hence we should have $\frac{F(x_k)^2}{g_{ik}(x_k)} = const$. However, this is not true.

## 3.2. Proof of proposition 2.7.9

In this section, we show that semi-Whittaker extremal projector $P^\psi$ from definition 2.7.8 satisfies $eP^\psi = P^\psi f^\psi = 0$.

Key properties that we need:

- $ef^\psi = f^\psi e + h$;
- $eP(h) = P(h-2)e$ for any rational function $P(h)$.

Let us start with the second property $P^\psi f^\psi = 0$. Recall the standard fact:

$$e^k f = f e^k + k(h - k + 1)e^{k-1},$$

which one can easily prove by induction, using, in fact, only commutation relations above. In particular, the proof also works if we substitute $f$ with $f^\psi$, hence

$$e^k f^\psi = f^\psi e^k + k(h - k + 1)e^{k-1}.$$

Therefore,

$$P_k^\psi e^k f^\psi = P_k^\psi f^\psi e^k + kP_k^\psi (h - k + 1)e^{k-1},$$

so the equality $P^\psi f^\psi = 0$ is equivalent to

$$\frac{1}{(k-1)!} P_{k-1}^\psi f^\psi = \frac{1}{k!} k P_k^\psi (h - k + 1),$$

for every $k \geq 1$, which is obvious.

Let us prove the first property $eP^\psi = 0$. To stress the dependence of $h$, denote by

$$Q_k(h-a) := \prod_{i=0}^{k-1} f^\psi (h - a - i)^{-1}.$$

for $a \in \mathbf{C}$. The following property is tautological:

$$Q_i(h-a)Q_j(h-a-i) = Q_{i+j}(h-a), \tag{3.2.1}$$

but turns out to be quite useful.

Claim:

$$e \cdot Q_k(h-a) = Q_k(h-a-2) \cdot e + \sum_{i=0}^{k-1} Q_i(h-a-2) \frac{h}{h-a-i} Q_{k-1-i}(h-a-i-1).$$

We can prove it by induction: for $k = 1$,

$$e \cdot f^\psi (h-a)^{-1} = f^\psi e(h-a)^{-1} + \frac{h}{h-a} = Q_1(h-a-2) + \frac{h}{h-a},$$

and the induction step for $k$:

$$\begin{aligned}
e\cdot Q_{k+1}(h-a) &= ef^{\psi}(h-a)^{-1}Q_k(h-a-1) = (f^{\psi}e+h)(h-a)^{-1}Q_k(h-a-1) = \\
&= f^{\psi}(h-a-2)^{-1}eQ_k(h-a-1) + \frac{h}{h-a}Q_k(h-a-1) = \\
&= Q_1(h-a-2)\left[Q_k(h-a-3)e + \sum_{i=0}^{k-1} Q_i(h-a-3)\frac{h}{h-a-(i+1)}Q_{k-(i+1)}(h-a-(i+1)-1)\right] + \\
&+ \frac{h}{h-a}Q_k(h-a-1) = \\
&= Q_{k+1}(h-a-2)\cdot e + \sum_{i=0}^{k-1} Q_{i+1}(h-a-2)\frac{h}{h-a-(i+1)}Q_{k-(i+1)}(h-a-(i+1)-1) + \\
&+ \frac{h}{h-a}Q_k(h-a-1) = \\
&= Q_{k+1}(h-a-2)\cdot e + \sum_{i=0}^{k} Q_i(h-a-2)\frac{h}{h-a-i}Q_{k-i}(h-a-i-1),
\end{aligned}$$

where we have used (3.2.1) for $i=1$.

Therefore, the property $eP^{\psi}=0$ is equivalent to

$$Q_k(h-2) = \frac{1}{k+1}\sum_{i=0}^{k} Q_i(h-2)\frac{h}{h-i}Q_{k-i}(h-i-1),$$

which we prove by induction. Consider the sum of the right-hand side for $k+1$ multiplied by overall factor $k+2$:

$$\begin{aligned}
&\sum_{i=0}^{k+1} Q_i(h-2)\frac{h}{h-i}Q_{k+1-i}(h-i-1) = \\
&= Q_{k+1}(h-2)\frac{h}{h-k-1} + \sum_{i=0}^{k} Q_i(h-2)\frac{h}{h-i}Q_{k-i}(h-i-1)Q_1(h-k-1) = \\
&= Q_{k+1}(h-2)\frac{h}{h-k-1} + (k+1)Q_k(h-2)Q_1(h-k-1) = \\
&= Q_k(h-2)f^{\psi}\frac{h}{(h-k-2)(h-k-1)} + (k+1)Q_k(h-2)f^{\psi}\frac{1}{h-k-1}.
\end{aligned}$$

Here, we used $Q_{k+1-i}(h-i-1) = Q_{k-i}(h-i-1)Q_1(h-k-1)$ in the second line and the induction assumption in the third one. Since

$$\frac{h}{(h-k-2)(h-k-1)} + \frac{k+1}{h-k-1} = \frac{k+2}{h-(k+2)},$$

the right-hand side (now divided by $k+2$ again) is equal to

$$Q_k(h-2)f^{\psi}(h-k-2)^{-1} = Q_k(h-2)Q_1(h-2-k) = Q_{k+1}(h-2),$$

which proves the claim.

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