Mirror symmetry for Jacobians of hyperelliptic curves

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Abstract

In this work, we construct a mirror family for the Jacobian of a hyperelliptic curve, examining in details the genus two case. More definitely, we propose a notion of a mirror family as the complex moduli space of a homological mirror together with an additional information, and compute its Picard-Fuchs equation and monodromy representation. Then we compute the (twisted) Givental J-function of the Jacobian by means of the abelian/nonabelian correspondence, and show that in some sense its coefficients are indeed the solutions to the Picard-Fuchs equation.

Introduction

One of the mathematical ways, in which mirror symmetry for Calabi-Yau manifolds (V^o, V) can be formulated, is an isomorphism between certain variations of Hodge structures on the complex moduli space of V (B-model) and the Kähler moduli space of V^o (A-model). One of the consequences of such an isomorphism is that the generating function for genus zero Gromov-Witten invariants can be computed just in terms of the solutions to the Picard-Fuchs equations on the B-side.

However, the main problem is to find a mirror for a given Calabi-Yau manifold V. One of the approaches to this problem is the following: instead of searching for a mirror family, we can find (at least conjecturally) its numerical part, namely the solutions to the Picard-Fuchs equation, and then try to prove that it is equal to a function encoding the information about Gromov-Witten invariants. This is the Givental approach [Giv98]: it states an equality between the so-called (twisted) $I-$ and *J-functions*, responsible for the B- and A-side respectively. It works only for toric complete intersections, since it uses a toric action through the localization principle.

Nonetheless, there is a method, introduced in [CFKBSC08], to compute the (twisted) J-function for a larger class of varieties; it is called *the abelian/nonabelian correspondence*. Namely, we need to consider a variety X with an action of a reductive group G with a maximal torus T; then the J-function of the GIT-quotient $X//T$ (a toric variety) and that of $X//G$ are related in a certain way (Theorem 3.3.1). Moreover, given a representation V of G, the (twisted) J-functions of the corresponding bundles $\mathcal{V}_T$ and $\mathcal{V}_G$ are related in the same way (Theorem 3.3.2).

This work is based on the following observation. Due to M. Reid [Rei72], we can realize the Jacobian J of any hyperelliptic curve as the zero locus of a section of the bundle $(S^2\mathcal{U}^*)^{\oplus 2}$ on the Grassmannian. In this case we can apply the abelian/nonabelian correspondence to obtain its (twisted) J-function. Certainly, all genus zero Gromov-Witten invariants of any abelian variety are trivial, since there are no non-constant maps from a rational curve to a compex torus; however, it turns out that the J-function is non-trivial.

In this work we show that there is an analogue of a mirror family for the Jacobian J. First, we introduce the I-function for this bundle on the Grassmannian. Second, we use the results of the papers [KO03] and [GLO01], which state an equivalence between a mirror isomorphism of two superconformal vertex algebras and an isomorphism between certain linear algebraic data of two tori, to guess what a homological mirror J^o to J should look like. Then we consider the complex moduli space of J^o together with some additional information, related to an immersion of J into the Grassmannian, and compute its Picard-Fuchs equation. It turns out that its solutions coincide with the coefficients of the I-function as in the case of complete intersections in toric varieties.

The work is organized as follows: in Section 1 we discuss homological mirror symmetry and the derived category of coherent sheaves of a generic principally polarized abelian variety; in Section 2 we briefly recall some results of the papers [KO03] and [GLO01]; in Section 3 we expain how the Givental approach can be applied in the case of the Jacobian of a curve, in particular, we briefly discuss M. Reid's construction and the abelian/nonabelian correspondence; in Section 4 we construct a mirror family and expain why it can be actually considered as a mirror family computing the Picard-Fuchs equation and the monodromy group. All the calculations are done for genus one and two cases; in Section 5 we outline possible approaches to higher genera.

1 Homological mirror symmetry

1.1 Generalities

According to Kontsevich ([Kon95]), mirror symmetry between (weak) Calabi-Yau manifolds (X,ω_X) and (Y,ω_Y), both equipped with a complex structure and a compatible symplectic form, can be formulated as an equivalence between two triangulated categories, namely the derived category of coherent sheaves and the derived Fukaya category:

$$\tilde{m} : \mathrm{D}^b\mathrm{Coh}(X) \xrightarrow{\sim} \mathrm{D}\mathcal{F}(Y). \tag{1.1}$$

Remark 1.1.1. The definition of the derived Fukaya category is highly non-trivial (for example, see [Aur13]), and we will not give it here. The only thing we need to know is that it is constructed purely and functorially in symplectic terms, using Lagrangian submanifolds and their Floer homology.

We will also use a cohomological version m of the mirror symmetry map $\tilde{m}$. Namely, consider the Grothendieck K-group of $\mathrm{D}^b(X) := \mathrm{D}^b\mathrm{Coh}(X)$. By definition, the Grothendieck K-group of a triangulated category $\mathcal{A}$ is an abelian group generated by the classes of objects of $\mathcal{A}$ modulo the following relation: for any distinguished triangle $X \to Y \to Z \to X[1]$, the alternating sum $[X] - [Y] + [Z]$ is equivalent to zero. In the case of the derived category of coherent sheaves it is easy to see that its K-group $K(X) := K_0(\mathrm{D}^b(X))$ is generated by the classes of coherent sheaves, and the relations are

$$\sum_{i=0}^{n} (-1)^i [X_i] = 0$$

for every complex

$$\cdots \to 0 \to X_0 \to X_1 \to \cdots \to X_n \to 0 \to \ldots.$$

Definition 1.1.2. A bilinear form on $K(X)$ defined on sheaves by

$$\langle E, F\rangle := \sum_{i=0}^{n} (-1)^i \dim \mathrm{Ext}^i(E,F), \tag{1.2}$$

and extended by linearity to the whole $K(X)$, is called *the Euler bilinear form.*

If the canonical bundle ω_X is trivial, then by the Serre duality

$$\mathrm{Ext}^i(E,F) \cong \mathrm{Ext}^{n-i}(F,E)^\vee,$$

hence the form is (anti)-symmetric depending on the parity of n.

Using Koszul resolution, one can represent any sheaf as a sum of locally free sheaves in $K(X)$. Thus there is a homomorphism

$$\mathrm{Ch} : K(X) \xrightarrow{\mathrm{ch}(\cdot)\sqrt{\mathrm{Td}_X}} H^{even}(X;\mathbb{Q}), \tag{1.3}$$

defined on locally free sheaves by taking the (twisted) Chern character and extended by linearity to the whole $K(X)$. The factor $\sqrt{\mathrm{Td}_X}$ is necessary for this homomorphism to respect bilinear forms, since by the Grothendieck-Riemann-Roch theorem

$$\langle E,F\rangle = \chi(E^\vee \otimes F) = \int\limits_X \mathrm{ch}(E^\vee \otimes F)\mathrm{Td}_X = \int\limits_X \mathrm{Ch}(E^\vee)\mathrm{Ch}(F).$$

On the mirror side, the similiar homomorphism is expected to be just taking cohomology class of Lagrangian submanifolds with a slightly modified cup-product as a bilinear form:

$$\mathcal{Q}(\alpha,\beta) := (-1)^{\frac{n(n-1)}{2}} \int\limits_Y \alpha \cup \beta. \tag{1.4}$$

Denote by $H^n_L(Y)$ a sublattice generated by the classes of Lagrangian submanifolds. Then the cohomological mirror symmetry map is an isomorphism

$$m : \mathbf{Im}(\mathrm{Ch}) \xrightarrow{\sim} H^n_L(Y). \tag{1.5}$$

The group of symplectic automorphisms $\mathrm{Symp}(Y)$ acts on $\mathrm{D}\mathcal{F}(Y)$ by autoequivalences, moreover, it is expected that symplectomorphisms homotopic (in C^∞ topology) to the identity act trivially. By mirror symmetry, an autoequivalence of $\mathrm{D}\mathcal{F}(Y)$ gives rise to an autoequivalence of $\mathrm{D}^b(X)$, thus we have a map

$$\pi_0(\mathrm{Symp}(Y)) \longrightarrow \mathrm{Autoeq}(\mathrm{D}^b(X)).$$

Now consider a family $\pi : \mathcal{Y} \to B$ with the following properties: $\mathcal{Y}, B$ smooth; π flat; $B_0 \subset B$ a (dense) open subset, whose complement is a divisor with simple normal crossings, such that for any $b \in B_0$ the fiber $\mathcal{Y}_b$ is a smooth Calabi-Yau manifold. Moreover, assume that there is a symplectic form Ω on $\mathcal{Y}$ such that its restriction to any smooth fiber is still non-degenerate; then there is a natural connection defined by the orthogonal complements to $T_y(\mathcal{Y}_b)$ in $T_y\mathcal{Y}$, where $b \in B_0$, $y \in \mathcal{Y}$. Given a path $\gamma : [0,1] \to B_0$, parallel transport induces a symplectomorphism $\mathcal{Y}_{\gamma(0)} \to \mathcal{Y}_{\gamma(1)}$; in particular, when γ is a loop with a base point b, it induces a (symplectic) automorphism of $\mathcal{Y}_b$, and its action on $\mathrm{D}\mathcal{F}(\mathcal{Y}_b)$ depends only on the homotopy type of γ, that is on the class $[\gamma] \in \pi_1(B_0,b)$.

Let Y be a fiber of the family, say $Y = \mathcal{Y}_b$. Composing the maps defined above, we obtain a homomorphism

$$\tilde{\mu} : \pi_1(B_0,b) \longrightarrow \mathrm{Autoeq}(\mathrm{D}^b(X)). \tag{1.6}$$

We can also consider a cohomological variant μ of the homomorphism $\tilde{\mu}$

$$\mu : \pi_1(B_0,b) \longrightarrow O(\mathbf{Im}(\mathrm{Ch})). \tag{1.7}$$

1.2 Derived category of coherent sheaves on an abelian variety

Recall that for X,Y smooth projective varieties and $\mathcal{E}$ an object of the bounded derived category of coherent sheaves $\mathrm{D}^b(X\times Y)$, *a Fourier-Mukai transform with a kernel* $\mathcal{E}$ is a functor $\Phi_{\mathcal{E}}:\mathrm{D}^b(X)\to\mathrm{D}^b(Y)$ such that for any object $A\in\mathrm{D}^b(X)$

$$\Phi_{\mathcal{E}}(A)=p_{2,*}(\mathcal{E}\otimes p_1^*A),$$

where $p_1:X\times Y\to X$, $p_2:X\times Y\to Y$ are the corresponding projections (note that all the operations are understood as derived functors). It is known that for any exact equivalence $F:\mathrm{D}^b(X)\to\mathrm{D}^b(Y)$ there is an object $\mathcal{E}\in\mathrm{D}^b(X\times Y)$ such that F is isomorphic to $\Phi_{\mathcal{E}}$ ([Orl97]).

Definition 1.2.1. A variety Y is called *a Fourier-Mukai partner* of X if there is an exact equivalence $F:\mathrm{D}^b(X)\to\mathrm{D}^b(Y)$.

Let A be an abelian variety, and $\hat{A}$ its dual. Denote by $\mathcal{P}$ the Poincaré bundle on $A\times\hat{A}$, and by $\mathcal{S}:=\Phi_{\mathcal{P}}$ the corresponding Fourier-Mukai transform. It is well-known that $\mathcal{S}$ is an exact equivalence (for example, see [Pol03, §11]).

Consider another abelian variety B and an isomorphism $f:A\times\hat{A}\to B\times\hat{B}$ (as abelian varieties). We can represent f as a matrix

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}, \tag{1.8}$$

where $x:A\to B$, $y:A\to\hat{B}$, $z:\hat{A}\to B$, $w:\hat{A}\to\hat{B}$ are the corresponding homomorphisms.

Definition 1.2.2. The isomorphism f is called *isometric* if

$$f^{-1}=\begin{pmatrix} \hat{w} & -\hat{y} \\ -\hat{z} & \hat{x} \end{pmatrix}.$$

It was proved in [Orl02] that there is a surjective map from the set of all equivalences between $\mathrm{D}^b(A)$ and $\mathrm{D}^b(B)$ to the set of isometric isomorphisms between $A\times\hat{A}$ and $B\times\hat{B}$. In particular, there is a homomorphism from the group of autoequivalences of $\mathrm{D}^b(A)$ to the group $U(A\times\hat{A})$ of isometric automorphisms of $A\times\hat{A}$; the kernel is isomorphic to $\mathbb{Z}\oplus(A\times\hat{A})_{\mathbb{C}}$, where $\mathbb{Z}$ corresponds to the shift in the derived category and $(A\times\hat{A})_{\mathbb{C}}$ to the functors of the form $t_a^*(\cdot)\otimes\mathcal{P}_\alpha$ with $(a,\alpha)\in A\times\hat{A}$. ([Orl02, Theorem 4.14])

Let H be a divisor on an abelian variety A; denote by $\mathcal{H}:=\mathcal{O}_J(-H)$ the corresponding ideal sheaf. There is a map

$$\varphi_{\mathcal{H}}:A\to\hat{A},$$
$$x\mapsto t_x^*\mathcal{H}\otimes\mathcal{H},$$

where $t_x:A\to A$, $t_x(a)=a+x$. An abelian variety A is called *principally polarized with* $\mathcal{H}$, if $\varphi_{\mathcal{H}}$ is an isomorphism. Assume that $\mathrm{End}(A)=\mathbb{Z}$; then there are no Fourier-Mukai partners of A besides A itself, and the group $U(A\times\hat{A})$ is isomorphic to $\mathrm{SL}_2(\mathbb{Z})$ ([Orl02, Example 4.16]). Up to a shift, the generators of the latter are $T:=(\cdot)\otimes\mathcal{H}$ and $S:=\phi_{\mathcal{H}}^*\circ\mathcal{S}$, corresponding to the matrices

$$T=\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ S=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.9}$$

It is known that the Néron-Severi group $\mathrm{NS}(A)$ of a principally polarized abelian variety A is isomorphic to the fixed by the Rosati involution part of the endomorphism ring $\mathrm{End}(A)$ (see [BL04, Proposition 5.2.1]). In particular, in the case of a very general Jacobian J its Néron-Severi group is generated by the theta-divisor H ([BL04, §9.9]).

Now let C be a very general curve of genus 2 (in the sense that the endomorphism ring $\mathrm{End}(J)$ of its Jacobian $J := J(C)$ is isomorphic to $\mathbb{Z}$). By the previous discussion, the image of the Chern character is generated by the classes $[\mathcal{O}_J], [\mathcal{O}_p], [\mathcal{H}]$, where p is a point. Let us compute the bilinear form (1.2).

Proposition 1.2.3. *In a suitable $\mathbb{Z}$-basis, the matrix of the Euler form is*

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Proof. First, let us compute the form in a natural basis $([\mathcal{O}_J], [\mathcal{H}], [\mathcal{O}_p])$:

- $\langle \mathcal{O}_J, \mathcal{O}_J \rangle$: This is just the Euler characteristic $\chi(\mathcal{O}_J)$ of the structure sheaf which is zero by Noether's formula:
$$\chi(\mathcal{O}_J) = \frac{c_1^2 + c_2}{12} = 0.$$
- $\langle \mathcal{O}_J, \mathcal{H} \rangle$: This is the Euler characteristic of the line bundle $\mathcal{H}$; by the Riemann-Roch formula,
$$\chi(H) = \frac{H^2}{2} = 1$$
- $\langle \mathcal{O}_J, \mathcal{O}_p \rangle$: Again, this is the Euler characteristic of a skyscraper sheaf, which is equal to 1.
- $\langle \mathcal{H}, \mathcal{H} \rangle$: Since $\mathcal{H}$ is a locally free sheaf, $\langle \mathcal{H}, \mathcal{H} \rangle = \langle \mathcal{O}_J, \mathcal{H} \otimes \mathcal{H}^{-1} \rangle = \langle \mathcal{O}_J, \mathcal{O}_J \rangle = 0$.
- $\langle \mathcal{H}, \mathcal{O}_p \rangle$: This is equal to $\langle \mathcal{O}_J, \mathcal{O}_p \rangle = 1$, since $\mathcal{O}_p \otimes H \cong \mathcal{O}_p$.
- $\langle \mathcal{O}_p, \mathcal{O}_p \rangle$: Here we can use the Fourier-Mukai transform. Consider $J \times \hat{J}$ and p, q the projections on the first and the second factor correpsondingly. Then $\mathcal{S}(\mathcal{O}_P) = q_*(p^* \mathcal{O}_p \otimes \mathcal{P}) \cong q_*(\mathcal{P}_{p \times \hat{J}}) \cong L$, where $L \in \mathrm{Pic}^0(\hat{J})$ is some line bundle on $\hat{J}$. Since the Fourier-Mukai transform is an exact equivalence, $\langle \mathcal{O}_p, \mathcal{O}_p \rangle = \langle \mathcal{S}(\mathcal{O}_p), \mathcal{S}(\mathcal{O}_p) \rangle = \langle L, L \rangle = \chi(\mathcal{O}_{\hat{J}}) = 0$.

Therefore, in the basis $[\mathcal{O}_J], [\mathcal{H}], [\mathcal{O}_p]$ the matrix of the bilinear form (1.2) is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \tag{1.10}$$

and in the basis $\xi := [\mathcal{O}_J], \eta := [\mathcal{H}] - [\mathcal{O}_J] - [\mathcal{O}_p], \zeta := [\mathcal{O}_p]$ it has the form

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.11}$$

$\square$

Remark 1.2.4. The class η has a geometric interpretation. Namely, consider the embedding $i : C \to J$ defined by $x \mapsto \mathcal{O}_J(x-c)$ for some fixed point $c \in C$. Denote by Θ any theta-characteristic, i.e. a line bundle such that its square is isomorphic to the canonical bundle. Then

$$\mathrm{ch}(\Theta)\cdot \mathrm{td}(C) = (1-\frac{c_1(C)}{2})(1+\frac{c_1(C)}{2}) = 1,$$

and by the Grothendieck-Riemann-Roch theorem

$$\mathrm{ch}(i_!\Theta)\sqrt{\mathrm{td}(J)} = \mathrm{ch}(i_!\Theta) = i_*\mathrm{ch}(\Theta) = c_1(\mathcal{H}).$$

One can easily verify that

$$\mathrm{ch}(\mathcal{O}_J) = 1,\ \ \mathrm{ch}(\mathcal{O}_p) = c_1(\mathcal{H})^2/2.$$

Therefore, we get that $\mathrm{ch}(i_!\Theta) = \mathrm{ch}([\eta])$.

Now let us observe the action of T and S (1.9).

Proposition 1.2.5. *In the basis* (ξ,η,ζ). *defined above, the matrices of* T *and* S *are*

$$\begin{pmatrix} 1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 2 & 1\end{pmatrix},\ \begin{pmatrix} 0 & 0 & 1\\ 0 & -1 & 0\\ 1 & 0 & 0\end{pmatrix}$$

respectively.

Proof. Again, let us first find the action of T and S on the classes $([\mathcal{O}_J],[\mathcal{H}],[\mathcal{O}_p])$:

- T: Obviously, $T([\mathcal{O}_J]) = [\mathcal{H}]$, $T([\mathcal{O}_p]) = [\mathcal{O}_p]$. To find $T([\mathcal{H}])$, we use an exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{O}_J \longrightarrow \mathcal{O}_H \longrightarrow 0. \tag{1.12}$$

  We know that H is isomorphic to the curve C, thus by the Riemann-Roch formula $H^2 = 2$, and we obtain another exact sequence:

$$0 \longrightarrow \mathcal{O}_H(\mathcal{H}) \longrightarrow \mathcal{O}_H \longrightarrow \mathcal{O}_x \oplus \mathcal{O}_y \longrightarrow 0, \tag{1.13}$$

  where x, y are points on H corresponding to the line bundle $\mathcal{H}$.

  From (1.13) we get

$$[\mathcal{O}_H(\mathcal{H})] = [\mathcal{O}_H] - 2[\mathcal{O}_p] = [\mathcal{O}_J] - [\mathcal{H}] - 2[\mathcal{O}_p],$$

  and (1.12) gives

$$T([\mathcal{H}]) = [\mathcal{H}\otimes\mathcal{H}] = [\mathcal{H}] - [\mathcal{O}_H\otimes\mathcal{H}] = 2[\mathcal{H}] - [\mathcal{O}_J] + 2[\mathcal{O}_p].$$

  Therefore, the matrix is

$$T = \begin{pmatrix} 0 & -1 & 0\\ 1 & 2 & 0\\ 0 & 2 & 1\end{pmatrix},$$

  and in the basis (ξ,η,ζ)

$$T = \begin{pmatrix} 1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 2 & 1\end{pmatrix}. \tag{1.14}$$

- S: We have already seen that $\mathcal{S}(\mathcal{O}_p) \cong L$, where L is a degree zero line bundle on $\hat{J}$. Since $\varphi_{\mathcal{H}}$ is an isomorphism, $\varphi_{\mathcal{H}}^* L$ is also of degree zero, thus its cohomology class is equal to $[\mathcal{O}_J]$. Therefore, $S([\mathcal{O}_p]) = [\mathcal{O}_J]$.

  From [Pol03, Proposition 11.9] we see that $S(\mathcal{H}) \cong \pi^* R\pi_* \mathcal{H} \otimes \mathcal{H}^{-1}$, where $\pi : J \to \mathrm{Spec}(\mathbb{C})$ is projection to the point. On the cohomology level

$$S([\mathcal{H}]) = [\pi^* R\pi_* \mathcal{H} \otimes \mathcal{H}^{-1}] = \chi(H)[\mathcal{O}_J \otimes \mathcal{H}^{-1}] = [\mathcal{H}^{-1}].$$

  To compute the latter, we need to tensor (1.12) and (1.13) by the sheaf $\mathcal{H}^{-1}$ to get

$$[\mathcal{H}^{-1}] = [\mathcal{O}_J] + [\mathcal{O}_H(\mathcal{H}^{-1})] = 2[\mathcal{O}_J] - [\mathcal{H}] + 2[\mathcal{O}_p].$$

  Finally, we need to compute $S([\mathcal{O}_J])$. Again, consider $J \times \hat{J}$ and p, q the corresponding projections. Then $q_*(\mathcal{P} \otimes p^* \mathcal{O}_J) = q_*(\mathcal{P} \otimes \mathcal{O}_{J \times \hat{J}}) = q_* \mathcal{P}$. By [Bot14, Proposition 2.2] $q_* \mathcal{P} \cong \mathcal{O}_{e^\vee}[2]$, where $e^\vee \in \hat{J}$ is the group identity. Since $\varphi_{\mathcal{H}}$ is an isomorphism, $\varphi_{\mathcal{H}}^* \mathcal{O}_{e^\vee} \cong \mathcal{O}_e$, thus $S([\mathcal{O}_J]) = [\mathcal{O}_p]$.

  Therefore, the matrix is

$$S = \begin{pmatrix} 0 & 2 & 1 \\ 0 & -1 & 0 \\ 1 & 2 & 0 \end{pmatrix},$$

  and in the basis (ξ, η, ζ)

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.15}$$

□

2 Mirror symmetry for abelian varieties

In this section, we briefly recall the mirror symmetry theorems concerning a mirror isomorphism between $N = 2$ superconformal vertex algebras (SCVA) arising from complex tori with a flat Kähler metric. For the proofs and all the details we refer the reader to [GLO01] and [KO03].

Let T be a $2d$-dimensional torus U/Γ for $U \simeq \mathbb{R}^{2d}$ and Γ a lattice in U. Let I be a complex structure, G be a flat Kähler metric with the corresponding form $\omega = GI$, and b be a class in the second cohomology group $H^2(T; \mathbb{R})$ represented by a constant 2-form B. To this data we associate a superconformal vertex algebra, denoted by $\mathrm{Vert}(\Gamma, I, G, B)$ (see [KO03, Section 4]). Denote by Γ^* the dual lattice, and by $T^* = U^*/\Gamma^*$ the dual torus. There is a natural pairing $l : \Gamma \oplus \Gamma^* \to \mathbb{Z}$ and a natural bilinear form q on $\Gamma \oplus \Gamma^*$ defined by

$$q((v_1, \xi_1), (v_2, \xi_2)) := l(v_1, \xi_2) + l(v_2, \xi_1)$$

Define two complex structures on $T \times T^*$:

$$\mathcal{I}(I, B) := \begin{pmatrix} I & 0 \\ BI + I^t B & -I^t \end{pmatrix},$$

$$\mathcal{J}(G,I,B) := \begin{pmatrix} -IG^{-1}B & IG^{-1} \\ GI - BIG^{-1}B & BIG^{-1} \end{pmatrix}.$$

Then the following theorem holds:

Theorem 2.0.1. [KO03, Theorem 2.2] $\mathrm{Vert}(\Gamma, I, G, B)$ *is mirror to* $\mathrm{Vert}(\Gamma', I', G', B')$ *if and only if there is an isomorphism of lattices* $\Gamma \oplus \Gamma^*$ *and* $\Gamma' \oplus \Gamma'^*$ *which takes* q *to* q'*,* $\mathcal{I}$ *to* $\mathcal{J}'$*, and* $\mathcal{J}$ *to* $\mathcal{I}'$*.*

Now let $T = A$ be an abelian variety. Denote by $C_A^a \subset \mathrm{NS}_A(\mathbb{R})$ the ample cone of A, and put

$$C_A^{\pm} := \mathrm{NS}_A(\mathbb{R}) \pm iC_A^a,$$

$$C_A := C_A^+ \sqcup C_A^-.$$

Let us call the pair (A, ω_A) *algebraic* if $\omega_A \in C_A$. Assume that NS_A is generated by one element, say $\mathrm{NS}_A \simeq \langle \varphi \rangle$, then the following theorem holds:

Theorem 2.0.2. [GLO01, Proposition 9.6.1, 9.6.3] *For any* $(a+ib)\varphi =: \omega_A \in C_A$ *there exist isogenous elliptic curves* $E_1, \ldots, E_d$ *and a form* $\omega \in C_E$ *on the product* $E := E_1 \times \ldots \times E_d$ *such that the pair* (A, ω_A) *is a mirror to the algebraic pair* (E, ω)*. Moreover, if* A *is principally polarized then all the curves are pairwise isomorphic.*

Remark 2.0.3. Note that here the B-field is represented by purely real class $a \cdot \varphi$, and a 'usual' symplectic form by purely imaginary class $ib \cdot \varphi$. Throughout the work, the B-field will be zero and a symplectic form will lie in real cohomology. We hope that would not lead to any confusion.

3 Givental mirror symmetry

3.1 Linear subspaces on the intersection of two quadrics

Let Q_0, $Q_1 \subset \mathbb{P}^{2n+3}$ be two non-singular quadrics in a projective space such that their intersection $X := Q_0 \cap Q_1$ is nonsingular of dimension $2n+1$; it follows from [Rei72, Proposition 2.1] that in this case there are precisely $2n+4$ singular quadrics Q_{λ_i}, $\lambda_i \in \mathbb{P}^1$, in the pencil generated by Q_0, Q_1, and each Q_{λ_i} is a simple cone. Let C be the double cover of $\mathbb{P}^1$ ramified at the points λ_i, then C is a hyperelliptic curve of genus $n+1$. Thus to every non-singular complete intersection of two quadrics we can associate a hyperelliptic curve.

Vice versa, let C be a hyperelliptic curve of genus $n+1$ with a morphism $C \to \mathbb{P}^1$ ramified at the points $\{\mu_i\}$. Consider quadrics in $\mathbb{P}^{2n+3}$ defined by the equations

$$\sum x_i^2 = 0, \ \sum \mu_i x_i^2 = 0.$$

Then their intersection X is non-singular of codimesion two ([Rei72, Proposition 2.1]).

Let $F(X)$ be the variety of n-subspaces on X. It can be realized as the zero locus of a section of the bundle $(S^2\mathcal{U}^*)^{\oplus 2}$, where $\mathcal{U}$ is the tautological bundle on the Grassmannian $\mathrm{Gr}(n+1, 2n+4)$. Therefore, one can easily compute the first Chern class of $F(X)$ which turns out to be zero ([Rei72, Theorem 4.8]). Also $F(X)$ is birationally equivalent to the symmetric power $S^{(n+1)}C$ of the curve C ([Rei72, Corollary 4.5]), and hence to the Jacobian $J(C)$. So we have a birational equivalence $F(X) \to J(C)$ which can be extended to an isomorphism $F(X) \xrightarrow{\sim} J(C)$ ([Rei72, Lemma 4.7]).

Thus the Jacobian J_g of a hyperelliptic curve of genus g is isomorphic to the zero locus of a section of the bundle $(S^2\mathcal{U}^*)^{\oplus 2}$ on the Grassmannian $\mathrm{Gr}(g, 2g+2)$.

3.2 Mirror theorems for toric varieties

Let X be a toric variety. Denote by $\mathcal{M}_{g,n}(X,\beta)$ the moduli space of degree β stable maps from genus g curves with n marked points to X, and by $[\mathcal{M}_{g,n}(X,\beta)]^{vir}$ its virtual fundamental class. Let $\mathrm{ev}: \mathcal{M}_{g,n}(X,\beta) \to X$ be the evaluation map at the first marked point, and $\mathcal{L}$ be the universal cotangent line, that is a line bundle on $\mathcal{M}_{g,n}(X,\beta)$ with a fibre over a stable map $(f: C \to X,\ p_1,\ldots,p_n)$ equal to $T^*_{p_1}C$; set $\psi = c_1(\mathcal{L})$.

Definition 3.2.1. The Givental J-function is

$$J_X(\sigma+\tau) = e^{\sigma/z}e^{\tau/z}\left(1+\sum_{\beta\in H_2(X,\mathbb{Z})} Q^\beta e^{\langle\beta,\tau\rangle} ev_*\left([\mathcal{M}_{0,1}(X,\beta)]^{vir}\cap\frac{1}{z(z-\psi)}\right)\right). \tag{3.1}$$

Here $\sigma\in H^0(X;\mathbb{Q})$, $\tau\in H^2(X,\mathbb{Q})$, Q^β is the representative of β in the group ring $\mathbb{Q}[H_2(X,\mathbb{Z})]$. Let ω be a Kähler class of X. Denote by Λ_X the completion of $\mathbb{Q}[H_2(X;\mathbb{Z})]$ with respect to a valuation $v(Q^\beta) = \langle\beta,\omega\rangle$, then we regard $J_X(\sigma+\tau)$ as a function on $H^0(X;\mathbb{Z})\oplus H^2(X;\mathbb{Z})$ to $H^*(X;\Lambda_X)[[z^{-1}]]$. We have

$$J_X(\sigma+\tau) = 1+(\sigma+\tau)z^{-1}+O(z^{-2})$$

Denote by D_i, $i=1..N$, the cohomology classes of codimension one toric orbits on X.

Definition 3.2.2. The Givental I-function is

$$I_X(\tau) = e^{\tau/z}\sum_{\beta\in H_2(X,\mathbb{Z})} Q^\beta e^{\langle\beta,\tau\rangle}\frac{\prod_{i=1}^N\prod_{m\le 0}(D_i+mz)}{\prod_{i=1}^N\prod_{m=-\infty}^{\langle D_i,\beta\rangle}(D_i+mz)}.$$

If X is Fano, then the Givental mirror theorem ([Giv98]) states that the J-function is equal to the I-function up to an exponential factor, that is

$$J_X(\tau) = e^{c\tau}I_X(\tau), \tag{3.2}$$

where c is a rational number; moreover, $c=0$ if the Fano index of X is greater than 2.

Let Y be the zero locus of a section of a vector bundle $\mathcal{E} = L_1\oplus\ldots\oplus L_s$ on X, where each L_i is a line bundle; denote by $\rho_i := c_1(L_i)$ the corresponding first Chern classes. Suppose $\mathbb{C}^*$ acts on the fibers by rescaling and trivially on the base. According to Coates and Givental ([CG07]), there is a $\mathbb{C}^*$-equivariant bundle $\mathcal{E}_{0,1,\beta}$ on $\mathcal{M}_{0,1}(X,\beta)$ with a natural map $\mathcal{E}_{0,1,\beta}\to \mathrm{ev}^*\mathcal{E}$; let $\mathcal{E}'_{0,1,\beta}$ be its kernel. Denote by $\mathbf{e}(\cdot)$ the $\mathbb{C}^*$-equivariant Euler class, and by λ the multiplicative generator of $H^*_{\mathbb{C}^*}(\mathrm{pt})\simeq\mathbb{C}[\lambda]$.

Definition 3.2.3. The twisted J-function is

$$J_{e,\mathcal{E}}(\sigma+\tau) = e^{\sigma/z}e^{\tau/z}\left(1+\sum_{\beta\in H_2(X,\mathbb{Z})} Q^\beta e^{\langle\beta,\tau\rangle} ev_*\left([\mathcal{M}_{0,1}(X,\beta)]^{vir}\cap\mathbf{e}(\mathcal{E}'_{0,1,\beta})\cap\frac{1}{z(z-\psi)}\right)\right). \tag{3.3}$$

Denote by $j: Y\to X$ the inclusion. The twisted J-function admits a non-equivariant limit $J_{X,Y}$ satisfying

$$j_*J_Y(j^*(\sigma+\tau)) = J_{X,Y}\cup\prod_{i=1}^s\rho_i\ .$$

Consider the Taylor expansion of the function $J_X(\tau)$ in a variable Q

$$J_X(\tau) = \sum_{\beta \in H_2(X,\mathbb{Z})} Q^\beta J_\beta(\tau). \tag{3.4}$$

Definition 3.2.4. The twisted I-function is

$$I_{\mathbf{e},\mathcal{E}} = \sum_{\beta \in H_2(X,\mathbb{Z})} Q^\beta J_\beta(\tau) \prod_{i=1}^{s} \prod_{m=1}^{\langle \beta, \rho_i \rangle} (\lambda + \rho_i + mz). \tag{3.5}$$

Expand the function $I_{\mathbf{e},\mathcal{E}}$ in powers of z

$$I_{\mathbf{e},\mathcal{E}} = A(\tau) + B(\tau) z^{-1} + O(z^{-2}). \tag{3.6}$$

Here $A(\tau)$, $B(\tau)$ are functions on $H^2(X;\mathbb{Q})$ to $H^0(X;\Lambda_X)$ and $H^0(X;\Lambda_X[\lambda]) \oplus H^2(X;\Lambda_X[\lambda])$ respectively.

Theorem 3.2.5. [CG07, Corollary 7] *Suppose $c_1(\mathcal{E}) \le c_1(X)$, then*

$$J_{\mathbf{e},\mathcal{E}}(\theta(\tau)) = \tfrac{I_{\mathbf{e},\mathcal{E}}(\tau)}{A(\tau)},$$

where $\theta(\tau) = \frac{B(\tau)}{A(\tau)}$.

Let $I_{X,Y}$ be the non-equivariant limit of $I_{\mathbf{e},\mathcal{E}}$.

Corollary 3.2.6. *Under assumptions of Theorem 3.2.5*

$$J_{X,Y}(\tilde{\theta}(\tau)) = \frac{I_{X,Y}(\tau)}{A(\tau)},$$

where $\tilde{\theta}(\tau)$ is the non-equivariant limit of $\theta(\tau)$.

3.3 Abelian/non-abelian correspondence

Recall that the Jacobian of a genus two curve can be obtained as the zero locus of a section of the bundle $(S^2\mathcal{U}^*)^{\oplus 2}$ on the Grassmannian $\mathrm{Gr}(2,6)$, where $\mathcal{U}$ is the tautological bundle. Unfortunately, nor the Grassmannian is a toric variety, neither $(S^2\mathcal{U}^*)^{\oplus 2}$ is a direct sum of line bundles. Nevertheless, we are still able to determine the J-function due to the abelian/nonabelian correspondence.

3.3.1 Generalities

Here we briefly recall the abelian/non-abelian correspondence. For the proofs and technical details we refer the reader to [CFKBSC08].

Let X be a complex smooth projective variety with a (linearized) action of a complex reductive Lie group G with a maximal torus $T \subset G$. Assume that the GIT-quotients $X//T$ and $X//G$ are smooth projective varieties and the G-unstable locus is of codimension at least two. Denote by $X^s(T)$ ($X^s(G)$) the corresponding stable loci. We have the following diagram:

$$X//T \xleftarrow{j} X^s(G)/T \xrightarrow{\pi} X//G$$

with j an open immersion and π a G/T-bundle.

Let S be another complex torus (possibly trivial) acting on X such that this action commutes with those of G and it preserves G-unstable locus. Let $W := \mathrm{N}(T)/T$ be the Weyl group, $\Delta = \Delta_+ \cup \Delta_-$ be the root system with a chosen decomposition into positive and negative roots, and $\mathbb{C}_\alpha$ be the one-dimensional representation corresponding to a root $\alpha \in \Delta$. The Weyl group W acts on $X//T$, hence on the equivariant cohomology ring $H_S^*(X//T;\mathbb{C})$. The representations $\mathbb{C}_\alpha$ define S-equivariant line bundles

$$L_\alpha := X^s(T) \times_T \mathbb{C}_\alpha$$

on $X//T$ with equivariant first Chern classes $c_1(L_\alpha)$. Consider the following class:

$$\Omega := \sqrt{\frac{(-1)^{|\Delta_+|}}{|W|}} \prod_{\alpha\in\Delta_+} c_1(L_\alpha) \tag{3.7}$$

It is W-anti-invariant, and any other W-anti-invariant class can be written (non-uniquely) as $\gamma \cup \Omega$ with $\gamma \in H_S^*(X//T;\mathbb{C})^W$.

We will need the following properties:

- π^* induces an isomorphism $H_S^*(X//G) \cong H_S^*(X^s(G)/T)^W$;
- There is an exact sequence
$$0 \longrightarrow \ker(\cup\Omega) \longrightarrow H_S^*(X//T)^W \overset{(\pi^*)^{-1}\circ j^*}{\longrightarrow} H_S^*(X//G) \longrightarrow 0;$$
- For any $\sigma \in H_S^*(X//G)$, $\tilde{\sigma} \in H_S^*(X//T)$ with $j^*\tilde{\sigma} = \pi^*\sigma$
$$\int_{X//T} \Omega^2 \tilde{\sigma} = \int_{X//G} \sigma.$$
Such $\tilde{\sigma}$ are called *lifts* of σ.

Denote by $\mathrm{NE}(X//G)$ the cone of curves on $X//G$. Define a function

$$\epsilon : \mathrm{NE}(X//G) \longrightarrow \mathbb{Z}/2\mathbb{Z},$$
$$\epsilon(\beta) = \left(\int_{\tilde{\beta}} \sum_{\alpha\in\Delta_+} c_1^S(L_\alpha) \right) \pmod 2,$$

where $\tilde{\beta}$ is a lift of β ($\epsilon(\beta)$ does not depend on it; see [CFKBSC08, §3.2]).

Choose a homogeneous basis $\{\sigma_0 = 1, \sigma_1, \dots, \sigma_r, \dots, \sigma_m\}$ of $H_S^*(X//G)$ such that $\{\sigma_1, \dots, \sigma_r\}$ is a basis of $H_S^2(X//G)$. If

$$J_{X//G}(t,z) = \sum_{i=0}^{m} J_{i,X//G}(t_0,\dots,t_m,z)\sigma_i$$

is the J-function of $X//G$, put

$$\tilde{J}_{X//G} = \sum_{i=0}^{r} J_{i,X//G}(t_0,\dots,t_m,z)\gamma_i, \tag{3.8}$$

where γ_i are chosen W-invariant lifts of σ_i.

Theorem 3.3.1. [CFKBSC08, Lemma 5.3.1] *Up to a change of coordinate $t \mapsto \theta(t)$*

$$\tilde{J}_{X//G}(\theta(t), z) \cup \Omega = (z\partial_\Omega J_{X//T})|_{Q^{\tilde{\beta}} = (-1)^{\epsilon(\beta)} Q^{\beta}}(t, z).$$

The function $\theta(t)$ is uniquely determined by the condition $J_{X//G}(t, z) = 1 + \frac{t}{z} + O(z^{-2})$.

Now consider a representation space $\mathcal{V}$ of G. There is $S \times G \times \mathbb{C}^*$ action on X and $\mathcal{V}$ ($\mathbb{C}^*$ acts trivially on X and homothetically on $\mathcal{V}$), hence we have equivariant bundles

$$\mathcal{V}_T := X^s(T) \times_T \mathcal{V}, \ \ \mathcal{V}_G := X^s(G) \times_G \mathcal{V}$$

on $X//T$, $X//G$ respectively.

Let $J_{\mathcal{V}_G}(t, z)$ be the twisted J-function of a pair $(X//G, \mathcal{V}_G)$, and $\tilde{J}_{\mathcal{V}_G}(t, z)$ be its lift defined in a way similiar to (3.11).

Theorem 3.3.2. [CFKBSC08, Theorem 6.1.2] *Up to a change of coordinate $t \mapsto \theta(t)$*

$$\tilde{J}_{\mathcal{V}_G}(\theta(t), z) \cup \Omega = (z\partial_\Omega J_{\mathcal{V}_T})|_{Q^{\tilde{\beta}} = (-1)^{\epsilon(\beta)} Q^{\beta}}(t, z).$$

3.3.2 Our case

We fix the following data:

- $X = \mathbb{C}^{rn}$ regarded as the space of $r \times n$ matrices;
- $G = \mathrm{GL}(r)$ acting on the left;
- T is the diagonal torus in G;
- $\mathcal{V} = (S^2 V_{st})^{\oplus 2}$, where V_{st} is the standard representation of $\mathrm{GL}(r)$;
- S is a trivial torus.

Then

- $X//G = \mathrm{Gr}(r, n)$;
- $X//T = (\mathbb{P}^{n-1})^r$;
- $\mathcal{V}_G = (S^2 \mathcal{U}^*)^{\oplus 2}$;
- $\mathcal{V}_T = (\bigoplus_{i=1}^{r} \bigoplus_{j=i}^{r} p_i^* \mathcal{O}(1) \otimes p_j^* \mathcal{O}(1))^{\oplus 2}$, where $p_i : (\mathbb{P}^{n-1})^r \to \mathbb{P}^{n-1}$ is the projection on the i-th factor.

Up to the constant, the class Ω (3.7) is the top Chern class of the bundle

$$\prod_{i=1}^{r-1} \prod_{j=i+1}^{r} p_i^* \mathcal{O}(1) \otimes p_j^* \mathcal{O}(-1).$$

Let $Y \subset \mathrm{Gr}(r, n)$ be a subvariety determined by the bundle $\mathcal{V}_G$. We see that $X//T$ is a toric variety and $\mathcal{V}_T$ is a direct sum of line bundles, hence we may apply Theorem 3.2.5 and Theorem 3.3.2 to obtain the (twisted) J-function of Y.

Now put $r = 2$, $n = 6$. In this case $X//T = (\mathbb{P}^5)^2$, $\mathcal{V}_T = (\mathcal{O}(2,0) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(0,2))^{\oplus 2}$. Denote by $P_i := p_i^* H$ the pull-back of the hyperplane class of the i-th factor, then the twisted I-function is

$$I_{\mathcal{V}_T}(\tau) = e^{\tau/z} \sum_{k,l=0}^{\infty} Q_1^k Q_2^l e^{kt_1+lt_2} \frac{\prod_{i=1}^{2k}(2P_1+iz+\lambda)^2 \prod_{i=1}^{k+l}(P_1+P_2+iz+\lambda)^2 \prod_{i=1}^{2l}(2P_2+iz+\lambda)^2}{\prod_{i=1}^{k}(P_1+iz)^6 \prod_{i=1}^{l}(P_2+iz)^6} =$$
$$= e^{\tau/z} \sum_{k,l=0}^{\infty} Q_1^k Q_2^l e^{kt_1+lt_2} \Gamma_{k,l}, \quad (3.9)$$

where

$$\Gamma_{k,l} := \frac{\prod_{i=1}^{2k}(2P_1+iz+\lambda)^2 \prod_{i=1}^{k+l}(P_1+P_2+iz+\lambda)^2 \prod_{i=1}^{2l}(2P_2+iz+\lambda)^2}{\prod_{i=1}^{k}(P_1+iz)^6 \prod_{i=1}^{l}(P_2+iz)^6}.$$

Here we idetified $\mathbb{Q}[H_2((\mathbb{P}^5)^2;\mathbb{Z})]$ with the polynomial ring of two variables $\mathbb{Q}[Q_1,Q_2]$ via $Q^\beta \mapsto Q_1^{\langle \beta, P_1\rangle} Q_2^{\langle \beta, P_2\rangle}$, and $\tau = t_1P_1 + t_2P_2$. We may rewrite (3.9) as

$$I_{\mathcal{V}_T}(\tau) = f_0(Q_1,Q_2,t_1,t_2) + \frac{f_1(Q_1,Q_2,t_1,t_2)}{z}P_1 + \frac{f_1(Q_2,Q_1,t_2,t_1)}{z}P_2 + \cdots, \quad (3.10)$$

with $f_0(Q_1,Q_2,t_1,t_2)$ symmetric w.r.t. a change of pairs $(Q_1,t_1) \mapsto (Q_2,t_2)$, since $\mathcal{V}_T$ is invariant under a change of product factors. Let us introduce new variables

$$\tilde{t}_1 = \frac{f_1(Q_1,Q_2,t_1,t_2)}{f_0(Q_1,Q_2,t_1,t_2)},\ \tilde{t}_2 = \frac{f_1(Q_2,Q_1,t_2,t_1)}{f_0(Q_1,Q_2,t_1,t_2)}. \quad (3.11)$$

By Theorem 3.2.5

$$J_{\mathcal{V}_T}(\tilde{t}_1,\tilde{t}_2) = \frac{1}{f_0(Q_1,Q_2,t_1,t_2)} I_{\mathcal{V}_T}(t_1,t_2).$$

In this case $\Omega = P_1 - P_2$ (up to a constant), and

$$\partial_\Omega = \frac{\partial}{\partial\tilde{t}_1} - \frac{\partial}{\partial\tilde{t}_2} = \left(\frac{\partial t_1}{\partial\tilde{t}_1} + \frac{\partial t_1}{\partial\tilde{t}_2}\right)\frac{\partial}{\partial t_1} - \left(\frac{\partial t_2}{\partial\tilde{t}_1} + \frac{\partial t_2}{\partial\tilde{t}_2}\right)\frac{\partial}{\partial t_2}.$$

The change of variables (3.11) is highly non-trivial; instead, we would like to operate with the variables t_1,t_2 in Theorem 3.3.2. Here we need to restrict to $\tilde{t}_1 = \tilde{t}_2 = \tilde{t}$; one can easily see that it is equivalent to $t_1 = t_2 = t$. Moreover, we have

$$\begin{pmatrix} \frac{\partial t_1}{\partial\tilde{t}_1} & \frac{\partial t_1}{\partial\tilde{t}_2} \\ \frac{\partial t_2}{\partial\tilde{t}_1} & \frac{\partial t_2}{\partial\tilde{t}_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial \tilde{t}_1}{\partial t_1} & \frac{\partial \tilde{t}_1}{\partial t_2} \\ \frac{\partial \tilde{t}_2}{\partial t_1} & \frac{\partial \tilde{t}_2}{\partial t_2} \end{pmatrix}^{-1},$$

thus $(\partial_\Omega t_1)|_{Q_1=Q_2=Q,\tilde{t}_1=\tilde{t}_2=\tilde{t}} = (\partial_\Omega t_2)|_{Q_1=Q_2=Q,\tilde{t}_1=\tilde{t}_2=\tilde{t}} =: g(Q,t)$. Therefore, we can apply an operator

$$D := \frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2}$$

instead of ∂_Ω.

The operator D is linear, thus

$$D\left(\frac{1}{f_0}I_{\mathcal{V}_T}\right) = \frac{1}{f_0^2}\left(DI_{\mathcal{V}_T}\cdot f_0 - I_{\mathcal{V}_T}\cdot Df_0\right).$$

Since f_0 is symmetric, $(Df_0)|_{t_1=t_2=t} = 0$. Computing $DI_{\mathcal{V}_T}$, we get

$$zDI_{\mathcal{V}_T}(\tau) = e^{\tau/z}\sum_{k,l=0}^{\infty} Q_1^k Q_2^l e^{kt_1+lt_2}\Gamma_{k,l}\cdot(P_1-P_2+k-l). \tag{3.12}$$

Therefore, under some change of variables $t\mapsto\phi(t)$

$$\tilde{J}_{\mathcal{V}_G}(\phi(t)) = G(Q,t)e^{\tau/z}\left(\sum_{k,l=0}^{\infty}\frac{Q^{k+l}e^{(k+l)t}\Gamma_{k,l}\cdot(P_1-P_2+k-l)}{P_1-P_2}\right) =: G(Q,t)e^{\tau/z}\cdot I(Q,t), \tag{3.13}$$

where $G(Q,t) = g(Q,t)/f_0(Q,t)$.

4 Mirror family

4.1 Numerical computations

Conjecturally, the Givental I-function encodes an information about a mirror family. More definitely, let Y be a Calabi-Yau complete intersection in a toric variety X defined by a bundle $\mathcal{E}$, $\dim Y = d$. Assume for simplicity that X is a projective space, and denote by H the hyperplane class. Putting $z=1,\lambda=0,t=0$, and $Q^\beta = s^{\langle\beta,H\rangle}$, we can rewrite I-function in the form

$$I_{\mathcal{E}} = y_0(s)+y_1(s)H+\ldots+y_d(s)H^d.$$

Then it is expected that the solutions of the Picard-Fuchs equation of a mirror family are $y_0(s), \log(s)y_0(s)+y_1(s), \frac{\log^2(s)}{2!}y_0(s)+\log(s)y_1(s)+y_2(s)$ and so on (up to constant factors).

Consider the function $I(Q,t)$ from (3.13). As usual, put $t=0, z=1, Q=s$, and rewrite I in the form

$$I(s) = y_0(s)+y_1(s)L+y_2(s)Q,$$

where $L\in H^2((\mathbb{P}^5)^2)$ and $Q\in H^4((\mathbb{P}^5)^2)$.

Numerical computation gives

$$\begin{aligned}
y_0(s) &= 1+8s+88s^2+1088s^3+14296s^4+195008s^5+2728384s^6+38879744s^7+\ldots,\\
y_1(s) &= 8s+116s^2+\frac{4832}{3}s^3+\frac{67822}{3}s^4+\frac{4831024}{15}s^5+\frac{69732112}{15}s^6+\frac{7122740864}{105}s^7+\ldots,\\
y_2(s) &= 64s^2+1344s^3+\frac{68528}{3}s^4+\frac{1094240}{3}s^5+\frac{255910352}{45}s^6+\frac{3950391424}{45}s^7+\ldots.
\end{aligned} \tag{4.1}$$

Conjecturally, the Taylor coefficients a_n of $y_0(s)$ are

$$a_n = \sum_{i=0}^{n}\binom{2i}{i}^2\binom{2(n-i)}{n-i}^2$$

Thus the period $y_0(s)$ is the square of a hypergeometric series

$$g_0(s) := F(\frac{1}{2}, \frac{1}{2}, 1; 16s) = 1 + 4s + 36s^2 + 400s^3 + 4900s^4 + \dots \tag{4.2}$$

It is known that a hypergeometric series satisfies certain differential equation:

$$\left\{ s(1-s)\frac{d^2}{ds^2} + [c - (a+b+1)s]\frac{d}{ds} - ab \right\} F(a,b,c;s) = 0$$

Hence the function $g_0(s)$ satisfies the differential equation

$$\left\{ s(1-16s)\frac{d^2}{ds^2} + (1-32s)\frac{d}{ds} - 4 \right\} g_0(s) = 0 \tag{4.3}$$

The second solution is given by $\log(s)g_0(s) + g_1(s)$, where

$$g_1(s) = 8s + 84s^2 + \frac{2960}{3}s^3 + \frac{37310}{3}s^4 + \frac{820008}{5}s^5 + \frac{11153912}{5}s^6 + \frac{1086209696}{35}s^7 + \dots \tag{4.4}$$

Remark 4.1.1. Unless otherwise stated, we will deal with differential equations such that $s = 0$ is a regular singular point and a basis of solutions is $y_0(s), \frac{\log(s)}{1!}y_0(s) + y_1(s), \dots, \frac{\log(s)}{r!}y_0(s) + .. + y_r(s)$, where $y_i(s)$ are holomorphic. In this case by *solutions* we will understand the holomorphic parts $y_i(s)$. We hope it would not lead to any confusion.

Computing $g_0^2(s), g_0(s)g_1(s), g_1^2(s)$ gives

$$\begin{aligned} g_0^2(s) &= 1 + 8s + 88s^2 + 1088s^3 + 14296s^4 + 195008s^5 + 2728384s^6 + 38879744s^7 + \dots, \\ g_0(s)g_1(s) &= 8s + 116s^2 + \frac{4832}{3}s^3 + \frac{67822}{3}s^4 + \frac{4831024}{15}s^5 + \frac{69732112}{15}s^6 + \frac{7122740864}{105}s^7 + \dots, \\ g_1^2(s) &= 64s^2 + 1344s^3 + \frac{68528}{3}s^4 + \frac{1094240}{3}s^5 + \frac{255910352}{45}s^6 + \frac{3950391424}{45}s^7 + \dots, \end{aligned} \tag{4.5}$$

and we see that $y_0(s) = g_0^2(s), y_1(s) = g_0(s)g_1(s), y_2(s) = g_1^2(s)$. Therefore, the functions $y_i(s)$ are the solutions to the symmetric square of the differential equation (4.3):

$$\left\{ s^2(16s-1)^2\frac{d^3}{ds^3} + 3s(32s-1)(16s-1)\frac{d^2}{ds^2} + (1792s^2 - 112s + 1)\frac{d}{ds} + 8(32s-1) \right\} y(s) = 0. \tag{4.6}$$

We can also rewrite it using the differential operator $\delta := s\frac{d}{ds}$:

$$\{(1-16s)^2\delta^3 + 48s(16s-1)\delta^2 + 32s(24s-1)\delta + 8(32s-1)\}y(s) = 0. \tag{4.7}$$

The mirror map $q(s) := s \cdot \exp(\frac{y_1(s)}{y_0(s)})$ is given by

$$q(s) = s + 8s^2 + 84s^3 + 992s^4 + 12514s^5 + 164688s^6 + 2232200s^7 + \cdots, \tag{4.8}$$

Expressing s in terms of q and taking reciprocal gives

$$s(q)^{-1} = q^{-1} + 8 + 20q - 62q^3 + 216q^5 - 641q^7 + \cdots \tag{4.9}$$

which is nothing but the Hauptomodul for a congruence subgroup $\Gamma_0(4)$ (see [Gro87]). Expressing $y_0(s)$ in coordinate q gives

$$f(q) := y_0(s(q)) = 1 + 8q + 24q^2 + 32q^3 + 24q^4 + 48q^5 + 96q^6 + 64q^7 + \cdots, \tag{4.10}$$

and this is just the square of the theta function, a modular form for $\Gamma_0(4)$.

4.2 Predicting a mirror family

4.2.1 What is a mirror family?

We need to make some remarks about what we actually understand by a mirror family. In fact, this is a difficult question. The way we obtained numerical information about a mirror family is closely related to the so-called *Hodge mirror symmetry* ([CK99]), which is formulated as an isomorphism of two variations of Hodge structure, one related to the complex moduli (B-side) and the other to the Kähler moduli via counting genus zero instantons (A-side). In principle, the J-function should reflect the latter and the I-function should reflect the former. In our case, there is a problem with both of them.

First, the honest J-function of an abelian variety is trivial since there are no rational curves on it; however, one can see from (3.13) that there is a non-trivial dependence on $1/z$, which can not be eliminated by any change of variable $t \mapsto \phi(t)$. Second, there is no clear definition of the I-function. Note that there were *no* a priori reason to throw off the factor $G(Q,T)$ from $I(Q,t)$ in the equation (3.13).

Remark 4.2.1. In fact, this was inspired by the work [CCGK16], where the authors computed quantum periods (the coefficient of degree zero part of the J-function) for all Fano manifolds. In the Fano case, the I-function (in the sense above) and the J-function differ by some exponential factor, therefore, it is natural to define the I-function precisely in the way we did, similiar to the case of complete intersections in toric varieties.

However, the functions we obtained are reasonable from the mirror symmetry point of view. More definitely, let us define a mirror family for J as the complex deformations of a homological mirror A together with some decorations (we will explain further what this means). Then the functions (4.1) are indeed the solutions to the Picard-Fuchs equation of this family.

Certainly, we expect more than that. For example, a sudden appearence of the theta-functions (4.10) signalizes that there should be hidden some numerical data like the number of holomorphic disks with boundary on a Lagrangian submanifold as we know from the work of K. Fukaya ([Fuk02]).

4.2.2 Why $\Gamma_0(4)$?

From the previous discussion we see that a mirror family should be somehow connected to the group $\Gamma_0(4)$.

Consider genus one case, that is an elliptic curve E and its Jacobian isomorphic to the curve itself. It is known that the mirror to E is an elliptic curve E' with symplectic and complex parametres switched ([PZ98]). The complex deformations of an elliptic curve are well-known – they are parametrized by the modular curve $X(1) \coloneqq \overline{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}$. Here, however, we need to keep track of an additional information. Namely, we have obtained this Jacobian as a subvariety in the Grassmannian. In the case of $g = 1$ the Grassmannian is just the projective space $\mathbb{P}^3$, and the Jacobian E is the intersection of two quadrics. As we can see from the definition of J-function, it remembers only about cohomology classes coming from the ambient space. In our case it means that we need to decorate the derived category $\mathrm{D}^b(E)$ with a subgroup in $H^2(E;\mathbb{Z})$ generated by the cohomology class $[P] = 4[\mathrm{pt}]$ (since E is of degree 4); and the autoequivalences of $\mathrm{D}^b(E)$, we are interested in, should preserve this decoration.

As we know, in the case of generic elliptic curve E the group of autoequivalences of $\mathrm{D}^b(E)$ has a continuous part, shifts, and a discrete part isomorphic to $\mathrm{SL}_2(\mathbb{Z})$ (see Section 1). The only one acting non-trivially on cohomology (up to ± 1) is the latter, and one of its generators (tensor product with the theta-divisor) does not preserve the necessary subgroup. Instead, we need to take a congruence subgroup $\Gamma_0(4)$. Indeed, we may pretend that the curve E is polarized not with a divisor [pt], but with 4[pt]; in this case the only homomorphisms $E \to \hat{E}$ allowed are $4n\varphi_{[\mathrm{pt}]}$, where $n \in \mathbb{Z}$ and $\varphi_{[\mathrm{pt}]}$ defines a principal polarization. Therefore, identifying $U(E)$ with $\mathrm{SL}_2(\mathbb{Z})$, we should take those matrices (1.8), for which $y \equiv 0 \pmod 4$. This is precisely the group $\Gamma_0(4)$.

In fact, the same happens in genus two case. Namely, simple computation with Chern classes shows that the restriction of the hyperplane class $[P] \in H^2(\mathrm{Gr}(2,6);\mathbb{Z})$ to a Jacobian J is of degree 32. If the Jacobian was generic then this class is a positive multiple of the theta-divisor $[H]$; since $[H]^2 = 2$, we obtain $[P] = 4[H]$. Again, the necessary group is $\Gamma_0(4)$.

We can compute generators of $\Gamma_0(4)$ (using SAGE software, for example):

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

4.2.3 Constructing a mirror family: genus one case

From the previous discussion we expect that a mirror family of an elliptic curve should be an elliptic modular surface related to $\Gamma_0(4)$. So let us take a subgroup $\Gamma_1(4)$. Its generators are:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix}$$

So it differs from $\Gamma_0(4)$ just by -1. Since $-1 \notin \Gamma_1(4)$, we can construct an elliptic modular surface $\pi : \mathcal{X} \to B$ associated to this congruence subgroup ([Shi72, Part II]). Let me briefly recall the construction; all the notations are taken from [Shi72].

For $\Gamma = \Gamma_1(4)$, the index μ of $\Gamma_1(4)\cdot(\pm 1) = \Gamma_0(4)$ is equal to 6, there are no elliptic points, and there are $t' = 3$ cusps $\{0, 1/2, \infty\}$; hence, the genus of $B := \overline{\Gamma\backslash\mathbb{H}}$ is equal to $(\mu/6 + 2 - t')/2 = 0$ and $B \simeq \mathbb{P}^1$. Denote by B_0 the curve B without cusps. Since there are no elliptic points, the universal cover of B_0 is the upper half-plane $\mathbb{H}$; denote by $p : \mathbb{H} \to B_0$ the projection. Define a functional invariant J_Γ by

$$J_\Gamma(p(u)) := j(\omega(u)),\ u \in \mathbb{H},$$

where j is j-invariant, $\omega(u)$ is the period of an elliptic curve defined by a lattice $\langle u, 1\rangle$.

There is a unique representation $\varphi_\Gamma : \pi_1(B_0, t) \to \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ such that

$$\omega(\gamma \cdot u) = \varphi(\gamma) \cdot \omega(u),\ u \in \mathbb{H},\ \gamma \in \pi_1(B_0). \tag{4.11}$$

This representation defines a locally constant sheaf G_Γ over B_0. By Kodaira ([Kod63]), we can construct an elliptic surface $\pi : \mathcal{X} \to B$ with such invariants. Note that the sheaf G_Γ is isomorphic to $R^1\pi_*\mathbb{Z} =: \mathcal{H}_\mathbb{Z}$. For $\mathcal{H} := R^1\pi_*\mathbb{C} \otimes \mathcal{O}_{B_0}$, ∇ the Gauss-Manin connection, and $\mathcal{F}^\bullet$ the Hodge filtration, the quadruple $(\mathcal{H}, \mathcal{H}_\mathbb{Z}, \nabla, \mathcal{F}^\bullet)$ defines a variation of Hodge structure.

The stabilizers of $0, \infty$ are

$$\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

correspondingly. Therefore, according to ([Shi72]), the elliptic fibration $\pi : \mathcal{X} \to B$ has three singular fibres, two of each are of the form I_1 and I_4 and the third is either I_a or I_a^* for some a.

Classification of elliptic surfaces ([MP86]) gives that $\mathcal{X}$ is isomorphic to an elliptic surface of type $I_4 I_1 I_1^*$.

To compute the corresponding Picard-Fuchs equation, we can use an explicit formula for this fibration:

$$V^2W + UVW + tVW^2 = U^3 + tU^2W$$

One can easily compute that the singular fibers are of type I_1, I_4, I_1^* over $t = 0, 1/16, \infty$ respectively. We need to do the following:

- Put the equation above into the Weiertrass form $y^2 = x^3 + A(t)x + B(t)$;
- Setting $\omega(t) := \frac{\mathrm{d}x}{y(t)}$, general theory tells us that the periods $\pi(t) := \int_{\gamma_t} \omega(t)$ satisfy a second order differential equation

  $$\frac{\mathrm{d}^2\pi(t)}{\mathrm{d}t^2} + C(t)\frac{\mathrm{d}\pi(t)}{\mathrm{d}t} + D(t) = 0$$

  for some rational funtions $C(t), D(t)$, which we need to find. We can differentiate under an integral sign, thus

  $$\int_{\gamma_t} \frac{\mathrm{d}^2\omega(t)}{\mathrm{d}t^2} + C(t)\frac{\mathrm{d}\omega(t)}{\mathrm{d}t} + D(t) = 0,$$

  and the form $\eta(t) := \frac{\mathrm{d}^2\omega(t)}{\mathrm{d}t^2} + C(t)\frac{\mathrm{d}\omega(t)}{\mathrm{d}t} + D(t)$ is exact.
- The form $\eta(t)$ is equal to $\frac{P(x)}{y^5}\mathrm{d}x$ for some polynomial $P(x)$ of degree 6 with coefficients in $\mathbb{C}[t]$. Subtracting certain multiples of exact forms $\frac{x^k}{y^3}\mathrm{d}x$, $k = 4..0$, we can lower degree of $P(x)$ to one. The functions $C(t), D(t)$ are determined by setting the remaining coefficients equal to zero.

These computations are best done with some mathematical software, for example MAPLE. It gives

$$C(t) = \frac{32t - 1}{t(16t - 1)}, \;\; D(t) = \frac{4}{t(16t - 1)},$$

and the Picard-Fuchs equation is

$$\left\{ t(1 - 16t)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (1 - 32t)\frac{\mathrm{d}}{\mathrm{d}t} - 4 \right\} \pi(t) = 0.$$

This is the same equation as (4.3). Now put $s = t - \frac{1}{16}$, and change the variable in the ODE above. This gives the Picard-Fuchs equation near the point $t = 1/16$:

$$\left\{ s(1 - 16s)\frac{\mathrm{d}^2}{\mathrm{d}s^2} + (1 - 32s)\frac{\mathrm{d}}{\mathrm{d}s} - 4 \right\} \pi(t) = 0.$$

This is *precisely* the equation (4.3).

This observation fits well into mirror symmetry picture. Namely, let $E' = \mathcal{X}_t$ be a (homological) mirror to E. *A priori*, a mirror family is defined in a small neigbourhood of the degeneration point. Here there are two of them, that is t equal to 0 or $1/16$, and we may regard E' lying in a neighbourhood of either 0 or $1/16$. Therefore, there are two mirrors to E', one of them is E and another is some elliptic curve $\tilde{E}$; however, they should be derived equivalent, that is

$\mathrm{D}^b(E) \simeq \mathrm{D}^b(\tilde{E})$. In the case of principally polarized and non CM curve it implies that they are isomorphic (see Section 1). At the same time we expect that the essential information about a mirror is contained in the Picard-Fuchs equation. Hence, it is natural that in two different points the Picard-Fuchs equation has the same form.

Remark 4.2.2. We said nothing about the point $t = \infty$. From mirror symmetry point of view, this is not suitable type of degeneration, since there is a multiple component. In fact, one can check that the solutions to the Picard-Fuchs equation in this point are $\sqrt{s}g_0(s), \sqrt{s}g_1(s)$ for $g_0(s), g_1(s)$ the solutions to (4.3).

The monodromy group is easily determined from the modular representation of the surface $\mathcal{X}$. More definitely, the homomorphism φ_Γ is just a conjugation in $\mathrm{GL}_2(\mathbb{R})$ corresponding to a change of basis

$$\langle \tau, 1\rangle \mapsto \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle =: \langle a, b\rangle$$

for $\tau \in \mathbb{H}$ the modular parameter of the curve $E' = \mathcal{X}_t$. Here $\langle a, b\rangle$ is a hyperbolic basis for which $a, 4b$ are the classes of the vanishing cycles with respect to the points $0, 1/16$ correspondingly. Under a (cohomological) mirror symmetry map m^{-1} (1.5) they should correspond to the K-group classes $[\mathcal{O}_E], [\mathcal{O}_p]$ for the elliptic curve E; in particular, we see that under a representation μ (1.7) the monodromy group generates $\Gamma_1(4)$-subgroup in the group of all discrete autoequivalences. This is not $\Gamma_0(4)$ as we predicted; however, they differ by the element -1, and it actually comes from the automorphism $[-1] : E \to E$. So, in some sense, the monodromy group generates all the 'outer' autoequivalences of the derived category.

4.2.4 Constructing a mirror family: genus two case

Now let us return to the main case. We claim that a mirror family should be the resolution of singularities of the fibered product $\mathcal{X} \times_B \mathcal{X} \to B$.

First, it follows from Theorem 2.0.2 that a homological mirror J^o to a generic Jacobian J should be the product of two isomorphic elliptic curves $E \times E$ (in this case a symplectic structure on J is provided by the restriction of the Fubini-Study form on the Grassmannian). The Néron-Severi group of $E \times E$ is generated by the classes of the fibers $[E] \times [\mathrm{pt}], [\mathrm{pt}] \times [E]$, and the class of the diagonal $[\Delta]$. In this basis the intersection matrix is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

and in the basis $\xi = [E] \times [\mathrm{pt}], \eta = [\Delta] - [E] \times [\mathrm{pt}] - [\mathrm{pt}] \times [E], \zeta = [\mathrm{pt}] \times [E]$ it reads

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

For any abelian variety A, the index of $H^2(A; \mathbb{Z})$ is equal to $(c_1^2(A) - 2c_2(A))/2 = 0$ by the Thom-Hirzebruch theorem; thus, the second cohomology lattice is unimodular and has signature (3,3). Due to the general classification of lattices (see [Ser73]), it is isomorphic to $U \oplus U \oplus U$, where U is the hyperbolic lattice. Since in our case the lattice of algebraic cycles is isometric to

$U \oplus \langle -2 \rangle$, the transcendental lattice T is isometric to $U \oplus \langle 2 \rangle$. Therefore, the bilinear form $\mathcal{Q}$ (1.4) is (isometric to)

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In this case the transcendental lattice T has a nice description. More definitely, by the Künneth theorem $H^2(E \times E; \mathbb{Z}) \simeq H^2(E; \mathbb{Z}) \oplus H^1(E; \mathbb{Z}) \otimes H^1(E; \mathbb{Z}) \oplus H^2(E; \mathbb{Z})$, and T is isometric to the symmetric square of $H^1(E; \mathbb{Z})$ understood in a suitable way.

Proposition 4.2.3. *Let $\langle a, b \rangle$ be a hyperbolic basis of $H^1(E; \mathbb{Z})$. Denote by T' the lattice spanned by $a \times a, a \times b + b \times a, b \times b$ under the Künneth isomorphism. Then $T \simeq T'$.*

Proof. The Künneth isomorphism is compatible with the lattice structure, thus $T \subset H^1(E; \mathbb{Z}) \otimes H^1(E; \mathbb{Z})$. Therefore, we need to find the orthogonal to the class of the diagonal $[\Gamma]$ inside $H^1(E; \mathbb{Z}) \otimes H^1(E; \mathbb{Z})$. From [MS74, Theorem 11.11] it follows that

$$[\Gamma] = [\mathrm{pt}] \times 1 + 1 \times [\mathrm{pt}] + a \times b - b \times a.$$

It easy to see that $T' \subset T$, and the intersection matrix restricted to T' has the form

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Therefore, $T' = T$, as required. □

Corollary 4.2.4. *After tensoring by $\mathbb{Q}$, the transcendental subspace $T_{\mathbb{Q}}$ is canonically isomorphic to the symmetric square of $H^1(E; \mathbb{Q})$.*

Picard-Fuchs equation. Consider the fibered product $\mathcal{X}_0 \times_{B_0} \mathcal{X}_0 =: \mathcal{Y}_0 \to B_0$. Its fiber over a point $t \in B_0$ is the product $\mathcal{X}_t \times \mathcal{X}_t$. Denote by $\mathcal{T}$ a locally constant sheaf of transcendental cycles.

Since taking symmetric power is a functorial operation, Corollary 4.2.4 gives that point-wise subspaces of transcendental cycles form a locally constant sheaf $\mathcal{T}_{\mathbb{Q}} := S^2\mathcal{H}_{\mathbb{Q}}$; moreover, we can continue the Gauss-Manin connection ∇ to $\mathcal{T}_{\mathbb{Q}}$. In particular, the corresponding Picard-Fuchs equation is just the symmetric square $S^2 D = 0$ of the original Picard-Fuchs equation $D = 0$. Indeed, for $\omega_t \in H^1(E_t; \mathbb{Q})$ a global section of $\mathcal{H}_{\mathbb{Q}}$, the form $\omega_t \otimes \omega_t$ is a global section of $S^2\mathcal{H}$. Choosing a basis of solutions $y_i(t) = \int_{\gamma_{i,t}} \omega_t$ for some (locally constant) cycles $\gamma_{i,t}$, we see that

$$g_{ij}(t) := \int\limits_{\gamma_{i,t} \times \gamma_{j,t}} \omega_t \otimes \omega_t = y_i(t) y_j(t)$$

are by definition the solutions to the symmetric square of D.

Therefore, the Picard-Fuchs equation corresponding the pair $(\mathcal{T}, \nabla)$ coincides with (4.6).

Monodromy. In genus one case, the monodromy representation $\pi_1(B_0, t) \to H^1(\mathcal{X}_t; \mathbb{Z})$ is (conjugated to) the tautological representation $V =: \mathrm{Span}(a, b)$ of $\Gamma_1(4) \subset \mathrm{SL}_2(\mathbb{Z})$. The generators $\langle a, b \rangle$ can be chosen canonically, since they are the classes of the vanishing cycles. It follow from Proposition 4.2.3 that the monodromy representation $\pi_1(B_0, t) \to \mathcal{T}_t$ is the symmetric square W (in the sense of Proposition 4.2.3) of V, that is $W = \mathrm{Span}(a \times a, a \times b + b \times a, b \times b)$. The $\mathrm{SL}_2(\mathbb{Z})$-generators are

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and they are seen to coincide with (1.14) and (1.15). The corresponding $\Gamma_1(4)$-generators are $T^4, S^{-1}TS$; since the kernel of representation W is -1, we see again that the monodromy generates all the outer autoequivalences of $\mathrm{D}^b(J)$.

Compactification. To compactify $\mathcal{Y}_0$, we may just take $\mathcal{X} \times_B \mathcal{X}$; however, it is singular. More definitely, let the family $\mathcal{X}$ be defined locally by an equation $f(x, y) = t$ for some function $f(x, y)$, then the fibered product is defined by $f(x, y) = f(\tilde{x}, \tilde{y}) = t$. Now consider a fiber of type I_b correspoding to $t = 0$ in a local coordinate t. Locally it is defined by $x = 0$ or $xy = 0$ depending on whether the point is singular or not, hence the local equation of the fibered product is one of three types:

$$x = \tilde{x}, \ \ xy = \tilde{x}, \ \ xy = \tilde{x}\tilde{y}.$$

Globally the only singular type is the third, and it is a cone singularity being resolved just by one blow-up. After the resolution we obtain a fiber with simple normal crossings. The second type is also with simple normal crossings. Therefore, resolving all the singular points, we obtain a good degeneration from the mirror symmetry point of view.

The fiber over $t = \infty$ of type I_1^* is not that good, however, since it has multiple components. Nevertheless, there still exists a resolution $\mathcal{Y} := \widetilde{\mathcal{X} \times_B \mathcal{X}}$ called *a Kuga-Sato variety*; see [Šok81]. In fact, one can check that the solutions to the Picard-Fuchs equation near this point are of the form $zy_0(z), zy_1(z), zy_2(z)$, for $y_i(s)$ the solutions near other points, hence $t = \infty$ is a point of maximally unipotent monodromy, unlike in the genus one case.

5 Some remarks on higher genera case

Unfortunately, in higher genera case it becomes difficult to prove or compute things explicitly; nevertheless, it is rather easy to guess how they should look like.

On the algebraic side, consider the embedding $i : C \hookrightarrow J$ and the images of the induced maps

$$W_k := \mathrm{im}(C^{(k)} \to J),$$

where $C^{(k)}$ is the k-th symmetric power of the curve; in particular, W_{g-1} is the theta-divisor. Consider the cohomology classes $[W_k]$. It is known [AECMGPHJ85] that they are non-divisible and in fact equal to $\theta^{g-k}/(g-k)!$ for $\theta = c_1(W_{g-1})$. Moreover, for a generic Jacobian they generate the whole algebraic subspace in $H^{2*}(J; \mathbb{Q})$ (see Chapter 17 of [BL04]). In this basis the intersection pairing is

$$[W_i] \cdot [W_j] = \delta_{i,g-j} \binom{g}{i}.$$

On the symplectic side, let us again take the product E^n. Let $a, b \in H^1(E;\mathbb{Z})$ be a symplectic basis. Under the Künneth isomorphism consider the sublattice in $H^*(E^n;\mathbb{Z})$ generated by the elements

$$\xi_0 = a^{\times n}, \xi_1 = \sum_i a^{\times(n-i)} \times b \times a^{\times i}, \dots, \xi_n = b^{\times n}$$

(that is, we consider the sum of all the possible permutations of words in k copies of a and $n-k$ copies in b modulo those permutting a (or b) among each other). It is easy to see that the pairing is

$$(\xi_i, \xi_j) = \delta_{i,n-j}\binom{n}{i}.$$

Though everything looks so nice, we are not able either to interpret the classes $[W_k]$ as the Chern classes of particular coherent sheaves over J or to prove why the introduced classes ξ_j are *all* the classes of Lagrangian submanifolds. Nevertheless, one can convince oneself (at least we hope) that the whole mirror symmetry machinery also works in higher genera cases.

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