YANGIANS, MIRABOLIC SUBALGEBRAS, AND WHITTAKER VECTORS

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Abstract. We construct an element, which we call the Kirillov projector, that connects the topics of the title: on the one hand, it is defined using the Yangian of $\mathfrak{gl}_N$, on the other hand, it gives a canonical projection onto the space of Whittaker vectors for any Whittaker module over the mirabolic subalgebra. Using the Kirillov projector, we deduce some categorical properties of Whittaker modules, for instance, we prove a mirabolic analog of Skryabin's theorem. We also show that it quantizes a rational version of the Cremmer-Gervais r-matrix. As application, we construct a universal vertex-IRF transformation from the standard dynamical R-matrix to this constant one in categorical terms.

Introduction

Solutions to the *quantum Yang-Baxter equation* (QYBE shortly)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

the so-called *quantum R-matrices*, have many applications in mathematical physics and theory of quantum groups. The most well-studied example is the *standard solution*, associated to any reductive Lie algebra $\mathfrak{g}$, that leads to Drinfeld-Jimbo quantum groups [Dri87; Jim85].

The starting point for the paper is a particular non-standard solution in type A which is called the *Cremmer-Gervais R-matrix*. Originally, it appeared in [CG90] and was later studied, for instance, in [Hod95; BRT07]. This solution is special among other non-standard solutions in at least two ways (which might be connected to each other).

First, it is related to the so-called *vertex-IRF transformation*. Namely, there is another important equation in mathematical physics, which is a generalization of the QYBE, called *quantum dynamical Yang-Baxter equation*:

$$R_{12}(\lambda - \hbar h_3)R_{13}(\lambda)R_{23}(\lambda - \hbar h_1) = R_{23}(\lambda)R_{13}(\lambda - \hbar h_2)R_{12}(\lambda);$$

where λ is a parameter on the dual space $\mathfrak{h}^*$ to an abelian Lie algebra $\mathfrak{h}$ and h_i are weight operators, see Definition 1.7. As explained by Felder [Fel95], it is closely related to the star-triangle relation for face-type statistical models [Bax89]. Moreover, it naturally appears in the description of the Liouville and Toda field theories [GN84]. As in the non-dynamical case, it leads to the theory of *dynamical quantum groups*, and for any reductive $\mathfrak{g}$, there are canonically defined standard solutions, see [ES01; Eti02]. It turns out that in type A, it is possible to gauge the standard dynamical R-matrix to a constant one which is exactly the Cremmer-Gervais R-matrix (in fact, this is how it was discovered in [CG90]). What is even more interesting is that such kind of gauge transformations (which are called *vertex-IRF* transformations) is unique, that is, it exists only in type A and the only constant R-matrix we can get is the Cremmer-Gervais one, see [BDF90]. So, it is natural to ask: what is a representation-theoretic explanation of this fact? Or, at least, what is a representation-theoretic meaning of *the* vertex-IRF transformation?

This brings to the second special feature of the Cremmer-Gervais R-matrix: its relation to W-algebras. In principle, the exchange algebra of the Toda field theory (which is related to affine W-algebras) is "controlled" by the standard dynamical R-matrix [GN84], but, as we mentioned before, it was shown in [CG90] that is possible to gauge away the dependence on the dynamical parameter. Likewise, it was observed in [BRT12] that the Cremmer-Gervais R-matrix can be constructed using the *Sevostyanov characters* [Sev00] which are used to define a q-analog of the principal finite W-algebra. So, one may ask: what is a direct relation between the Cremmer-Gervais R-matrix and W-algebras?

This paper is a small attempt to answer these questions in a somewhat simpler situation of the rational analog of the Cremmer-Gervais solution (or, rather, a degeneration thereof), see [EH00]. It turns out that, first, it is intrinsically defined not for the whole group GL_N, but for the so-called *mirabolic subgroup* M_N of invertible transformations preserving the last basis vector in $\mathbf{C}^N$ with the Lie algebra $\mathfrak{m}_N$; second, as we show in the paper, it naturally comes in a family over the subspace of the diagonal matrices in $\mathfrak{m}_N$. Explicitly, if $e \in \mathfrak{gl}_N$ is the principal nilpotent element and h is such a diagonal matrix, then a *rational Cremmer-Gervais r-matrix* is the inverse of the trace pairing $x \wedge y \mapsto \mathrm{Tr}((e-h)\cdot[x,y])$ for $x, y \in \mathfrak{m}_N$.

To quantize this family of classical r-matrices, we adopt the so-called *exchange construction* of [EV99] to the setting of Whittaker modules over the mirabolic subalgebra, or, rather, its categorical interpretation as in [KS22]. The main ingredient of the construction is a natural isomorphism

$$\mathrm{Whit}(\mathcal{W}\otimes V) \cong \mathrm{Whit}(\mathcal{W}) \otimes V$$

between the Whittaker vectors in the translated module $\mathcal{W}\otimes V$ and those of $\mathcal{W}$, where $\mathcal{W}$ is an arbitrary Whittaker module over $\mathfrak{m}_N$ and V is an M_N-representation. We do it with the help of a Whittaker analog of the *extremal projector* [AST71] in the dynamical setting that we call the *Kirillov projector* as it bears some resemblance to the Kirillov model from p-adic groups [Kir63]. This is an element $P_{\mathfrak{m}_N}(\vec{u})$ lying in a certain completion of the universal enveloping algebra $\mathrm{U}_\hbar\mathfrak{m}_N$ and depending on a vector of parameters $\vec{u} = (u_1, \ldots, u_{N-1})$. Theorem 4.3 is the main result of the paper.

Theorem. *As an operator acting on right Whittaker modules over $\mathfrak{m}_N$, the Kirillov projector satisfies*

$$\begin{aligned} P_{\mathfrak{m}_N}(\vec{u})^2 &= P_{\mathfrak{m}_N}(\vec{u}),\\ P_{\mathfrak{m}_N}(\vec{u})\cdot(E_{ij}+\delta_{i,j+1}) &= 0,\ 1\le j<i\le N,\\ (E_{ij}+\delta_{ij}u_j)\cdot P_{\mathfrak{m}_N}(\vec{u}) &= 0,\ 1\le i\le j\le N-1. \end{aligned}$$

Here, $\{E_{ij}\}$ is the standard basis of matrix units in $\mathfrak{gl}_N$.

It is constructed inductively via the sequence of subalgebras $\mathfrak{m}_2 \subset \mathfrak{m}_3 \subset \ldots \subset \mathfrak{m}_N$ embedded in the lower right corner. Surprisingly, at each step, the construction involves the *Yangian* $\mathrm{Y}(\mathfrak{gl}_k)$, more precisely, the image of its Gelfand-Tsetlin subalgebra under the evaluation homomorphism to $\mathrm{U}(\mathfrak{gl}_k)$ with evaluation parameter u_{N-k+1}. In particular, the Kirillov projector transforms spectral variables into "dynamical" ones, that is, lying on the Cartan subalgebra.

With the help of the Kirillov projector, we can show some categorical properties of the Whittaker module over the mirabolic subalgebra. For instance, Theorem 5.1 is a mirabolic analog of Skryabin's theorem [Pre02].

Theorem. *The category of Whittaker modules over $\mathfrak{m}_N$ is equivalent to the category of vector spaces.*

Likewise, we reprove Skryabin's theorem for $\mathfrak{gl}_N$ in Theorem 5.12.

Finally, adopting the construction of [KS22] to the Whittaker setting, we obtain a quantization of the rational Cremmer-Gervais r-matrix in Theorem 5.5 and Theorem 5.9.

Theorem. *The Kirillov projector $P_{\mathfrak{m}_N}(\vec{u})$ provides a tensor structure on the forgetful functor $\mathrm{Rep}(\mathrm{M}_N) \to \mathrm{Vect}$ from the category of M_N-representations to the category of vector spaces. The semiclassical limit of the corresponding quantum R-matrix is the rational Cremmer-Gervais r-matrix for $h = u_1E_{11}+\ldots+u_{N-1}E_{N-1,N-1}$.*

As application, we interpret the vertex-IRF transformation via a categorical equivalence between the Kostant-Whittaker reduction functor of [BF08] and the parabolic restriction functor of [KS22].

Structure of the paper. In Section 1, we give a general setup for the paper and review the categorical approach to (standard) dynamical quantum groups via the category of Harish-Chandra bimodules and parabolic restriction functor as in [KS22] . Proofs are mostly omitted unless there is a version that we need later in the paper and it is different from *loc. cit.* In Section 2, we define the mirabolic subgroup and a family of r-matrices that deform the degenerate version of the Cremmer-Gervais r-matrix. In Section 3, we review Yangians and related constructions such as quantum minors. We prove various technical results that we use later in the paper. In Section 4, we introduce the Kirillov projector and prove its defining properties. In Section 5, we recall the Kostant-Whittaker reduction functor, introduce its mirabolic version, and show some properties thereof. We also obtain quantization of the family of rational Cremmer-Gervais solutions. In Section 6, we define a categorical version of the vertex-IRF transformation and show that an equivalence between parabolic restriction and Kostant-Whittaker reduction gives the vertex-IRF transformation in the classical sense.

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1. Background: dynamical quantum groups

In this section, we will give a brief overview of the classical extremal projector and its relation to dynamical quantum groups through the category of Harish-Chandra bimodules. Note: the cited literature mostly deals with the case $\hbar = 1$; however, the proofs of all the statements we use are easily generalizable to the case of an arbitrary $\hbar$.

1.1. **General setup.** For the rest of the paper, we work over an algebraically closed field $\mathbf{C}$ of characteristic zero. All categories and functors we will consider are $\mathbf{C}$-linear. Throughout this paper we work with locally presentable categories (we refer to [AR94] and [BCJ15, Section 2] for more details). We will also use Deligne's tensor product of locally presentable categories introduced in [Del90].

Almost all the constructions involve a parameter $\hbar$: to avoid categorical complications, we treat $\hbar$ as a *non-formal* parameter, i.e. $\hbar \in \mathbf{C}^\times$ (in particular, we can still deal with $\mathbf{C}$-linear categories); the reader may safely assume that $\hbar = 1$. The only purpose of introducing $\hbar$ is to compute classical limits of certain formulas, and it will be clear from the context how to make sense of the corresponding $\hbar$-family over $\mathbb{A}^1$. We hope it will not lead to any confusion.

1.2. **Harish-Chandra bimodules.** For details and proofs in a more general setting, we refer the reader to [KS22, Section 2].

Let G be an affine algebraic group and $\mathfrak{g}$ be its Lie algebra with bracket $[,]$.

Definition 1.1. The ***universal enveloping algebra*** $\mathrm{U}_\hbar(\mathfrak{g})$ of $\mathfrak{g}$ is a tensor algebra over $\mathbf{C}$, generated by the vector space $\mathfrak{g}$, with the relations

$$xy - yx = \hbar[x, y],\ \ x, y \in \mathfrak{g}.$$

Remark 1.2. In principal, such $\hbar$-version is usually called the *asymptotic* universal enveloping algebra; however, it is usually defined over $\mathbf{C}[\hbar]$, but in our case, parameter $\hbar$ is a number, see Section 1.1. Moreover, some formulas will use $\hbar^{-1}$. So, we decided to drop the adjective "asymptotic".

Denote by $\mathrm{Rep}(G)$ the category of G-representations. Naturally, $\mathrm{U}_\hbar(\mathfrak{g})$ is an object in $\mathrm{Rep}(G)$.

Definition 1.3. A ***Harish-Chandra bimodule*** is a left $\mathrm{U}_\hbar(\mathfrak{g})$-module X in the category $\mathrm{Rep}(G)$. In other words, it has a structure of a G-representation and a left $\mathrm{U}_\hbar(\mathfrak{g})$-module such that the action morphism

$$\mathrm{U}_\hbar(\mathfrak{g}) \otimes X \to X$$

is a homomorphism of G-representations. The category of Harish-Chandra bimodules is denoted by $\mathrm{HC}_\hbar(G)$.

There is a natural right $\mathrm{U}_\hbar(\mathfrak{g})$-module structure on any Harish-Chandra bimodule X (justifying the name). Namely, for $\xi \in \mathfrak{g}$, denote by $\mathrm{ad}_\xi \colon X \to X$ the derivative of the G-action on X along ξ. Then we can define

$$x\xi := \xi x - \hbar\,\mathrm{ad}_\xi(x),\ \ x \in X,$$

and extend it to a right $\mathrm{U}_\hbar(\mathfrak{g})$-action. Therefore, the category $\mathrm{HC}_\hbar(G)$ is a subcategory of $\mathrm{U}_\hbar(\mathfrak{g})$-bimodules, hence is equipped with a tensor structure:

$$X \otimes^{\mathrm{HC}_\hbar(G)} Y := X \otimes_{\mathrm{U}_\hbar(\mathfrak{g})} Y.$$

There is a natural functor of the so-called ***free Harish-Chandra bimodules***:

$$\mathrm{free}\colon\ \mathrm{Rep}(G) \to \mathrm{HC}_\hbar(G), \qquad V \mapsto \mathrm{U}_\hbar(\mathfrak{g}) \otimes V.$$

One can check that this functor is monoidal. In fact, all Harish-Chandra bimodules can be "constructed" from the free ones:

Proposition 1.4. *[KS22, Proposition 2.7] The category* $\mathrm{HC}_\hbar(G)$ *is generated by* $\mathrm{free}(V)$ *for* $V \in \mathrm{Rep}(G)$.

1.3. **Constant and dynamical R-matrices.** For introduction to the theory of dynamical quantum groups, see [ES01]; for the interpretation in terms of the category of Harish-Chandra bimodules, see [KS22].

Definition 1.5.
- Let $F_{VW}\colon V\otimes W\to V\otimes W$ be a collection of linear maps natural in $V, W\in \mathrm{Rep}(G)$. The ***twist*** equation is

$$F_{U\otimes V,W}\circ(F_{UV}\otimes \mathrm{id}_W) = F_{U,V\otimes W}\circ(\mathrm{id}_U\otimes F_{VW})\in \mathrm{End}(U\otimes V\otimes W). \tag{1.1}$$

  A solution is called a ***(Drinfeld) twist***.
- Let $R_{VW}\colon V\otimes W\to V\otimes W$ be a collection of linear maps natural in $V, W\in \mathrm{Rep}(G)$. The ***quantum Yang-Baxter equation*** is

$$R_{UV}R_{UW}R_{VW} = R_{VW}R_{UW}R_{UV}\in \mathrm{End}(U\otimes V\otimes W). \tag{1.2}$$

  A solution is called a ***quantum R-matrix***.
- Let $r_{VW}\colon V\otimes W\to V\otimes W$ be a collection of linear map natural in $V, W\in \mathrm{Rep}(G)$. The ***classical Yang-Baxter equation*** is

$$[r_{UV}, r_{UW}] + [r_{UV}, r_{VW}] + [r_{UW}, r_{VW}] = 0\in \mathrm{End}(U\otimes V\otimes W). \tag{1.3}$$

  A solution is called a ***classical r-matrix***.

The following result is standard.

Proposition 1.6.
1. *A tensor structure on the forgetful functor* $\mathrm{Rep}(G)\to \mathrm{Vect}$ *is equivalent to the data of a Drinfeld twist* (1.1).
2. *Using the natural symmetric monoidal structure on* $\mathrm{Rep}(G)$*, one can consider F_{WV} as an element of* $\mathrm{End}(V\otimes W)$*. Define $R_{VW} = F_{WV}^{-1}F_{VW}$. Then R_{VW} is a quantum R-matrix* (1.2).
3. *Assume that there is a family $R^{\hbar}_{VW}$ depending on $\hbar$ such that $R^{\hbar}_{VW} = \mathrm{id}_{V\otimes W} + \hbar r_{VW} + O(\hbar^2)$. Then r_{VW} is a classical r-matrix* (1.3).

There is a version of these equations involving dynamical parameter $\lambda\in\mathfrak{h}^*$. Let $H\subset G$ be a torus and $\mathfrak{h}\subset\mathfrak{g}$ be its Lie algebra. Let U, V, W be weight $\mathfrak{h}$-modules. Define the operator

$$h_V\colon U\otimes V\otimes W\to U\otimes V\otimes W,\qquad h_V\cdot u\otimes v\otimes w = \mathrm{wt}(v)\cdot u\otimes v\otimes w,$$

similarly for h_U, h_W.

Definition 1.7.
- Let $J_{VW}(\lambda)\colon V\otimes W\to V\otimes W$ be a collection of $\mathrm{End}_{\mathfrak{h}}(V\otimes W)$-valued functions on $\mathfrak{h}^*$ natural in $V, W\in\mathrm{Rep}(G)$. The ***dynamical twist*** equation is

$$J_{U\otimes V,W}(\lambda)\circ(J_{UV}(\lambda)\otimes\mathrm{id}_W) = J_{U,V\otimes W}\circ(\mathrm{id}_U\otimes J_{VW}(\lambda-\hbar h_U)). \tag{1.4}$$

  A solution is called a ***dynamical (Drinfeld) twist***.
- Let $R_{VW}(\lambda)\colon V\otimes W\to V\otimes W$ be a collection of $\mathrm{End}_{\mathfrak{h}}(V\otimes W)$-valued functions on $\mathfrak{h}^*$ natural in $V, W\in\mathrm{Rep}(G)$. The ***quantum dynamical Yang-Baxter equation*** is

$$R_{UV}(\lambda-\hbar h_W)R_{UW}(\lambda)R_{VW}(\lambda-\hbar h_U) = R_{VW}(\lambda)R_{UW}(\lambda-\hbar h_V)R_{UV}(\lambda). \tag{1.5}$$

  A solution is called a ***quantum dynamical R-matrix***.
- Let $r_{VW}\colon V\otimes W\to V\otimes W$ be a collection of $\mathrm{End}_{\mathfrak{h}}(V\otimes W)$-valued functions on $\mathfrak{h}^*$ natural in $V, W\in\mathrm{Rep}(G)$. The ***classical dynamical Yang-Baxter equation*** is

$$\begin{aligned}&\sum_i\left((x_i)_U\frac{\partial r_{VW}}{\partial x^i} - (x_i)_V\frac{\partial r_{UW}}{\partial x^i} + (x_i)_W\frac{\partial r_{UV}}{\partial x^i}\right)+\\ &+[r_{UV}(\lambda), r_{UW}(\lambda)] + [r_{UV}(\lambda), r_{VW}(\lambda)] + [r_{UW}(\lambda), r_{VW}(\lambda)] = 0,\end{aligned} \tag{1.6}$$

  where $\{x_i\}\subset\mathfrak{h}$ is a basis in $\mathfrak{h}$, and $\{x^i\}$ is the dual basis in $\mathfrak{h}^*$. A solution is called a ***classical dynamical r-matrix***.

It turns out that there is a Tannakian interpretation of dynamical R-matrices in terms of functors to the category of Harish-Chandra bimodules. Since $\mathfrak{h}$ is commutative, the universal enveloping algebra $\mathrm{U}_\hbar(\mathfrak{h})$ is equal to the space $\mathcal{O}(\mathfrak{h}^*)$ of polynomial functions on $\mathfrak{h}^*$. In particular, for any $X\in\mathrm{Rep}(H)$, a map between free Harish-Chandra bimodules

$$\Phi\colon \mathrm{U}_\hbar(\mathfrak{h})\otimes V\to \mathrm{U}_\hbar(\mathfrak{h})\otimes V$$

can be equivalently given by an $\mathrm{End}_{\mathfrak{h}}(V)$-valued function $\Phi(\lambda)$ on $\mathfrak{h}^*$. We use this identification for the rest of this paper.

Theorem 1.8. *[KS22] Consider the composition*

$$\mathrm{Rep}(G) \xrightarrow{\text{forget}} \mathrm{Rep}(H) \xrightarrow{\text{free}} \mathrm{HC}_\hbar(H),$$

where the first arrow is the forgetful functor.

(1) *A monoidal structure on $\mathrm{Rep}(G) \to \mathrm{HC}_\hbar(H)$ is equivalent to the data of a dynamical twist (1.4).*
(2) *Using the natural symmetric monoidal structures on $\mathrm{Rep}(G)$, one can consider $J_{WV}(\lambda)$ as an element of $\mathrm{End}_{\mathfrak{h}}(V \otimes W)$. Define $R_{VW}(\lambda) = J_{WV}(\lambda)^{-1} J_{VW}(\lambda)$. Then $R_{VW}(\lambda)$ is a quantum dynamical R-matrix (1.5).*
(3) *Assume that there is a family $R^\hbar_{VW}(\lambda)$ depending on a parameter $\hbar$ such that*

$$R^\hbar_{VW}(\lambda) = \mathrm{id}_{V\otimes W} + \hbar r_{VW}(\lambda) + O(\hbar^2).$$

Then $r_{VW}(\lambda)$ is a classical dynamical r-matrix (1.6).

From now on, let G be a reductive group. As in the non-dynamical case, there is a standard solution to classical dynamical Yang-Baxter equation. Namely, choose a Borel subgroup $B \subset G$ and a maximal torus $H \subset B$. It induces a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ with respect to the set of positive roots Δ_+. For $\alpha \in \Delta_+$, denote by $\{f_\alpha, h_\alpha, e_\alpha\}$ the corresponding $\mathfrak{sl}_2$-triple of the Chevalley base. Define (see [EV99])

$$r(\lambda) = \sum_{\alpha\in\Delta_+} \frac{e_\alpha \wedge f_\alpha}{\langle \lambda, h_\alpha\rangle}. \tag{1.7}$$

One can check that it indeed satisfies (1.6). In what follows, we will construct its quantization in Section 1.5.

1.4. **Extremal projector.**

The original construction can be found in [AST71], see also [Zhe90].

Denote by $\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}} = \mathrm{Frac}(\mathrm{U}_\hbar(\mathfrak{h}))$ the ring of fractions of $\mathrm{U}_\hbar(\mathfrak{h})$ and by $\mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}} = \mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}} \otimes_{\mathrm{U}_\hbar(\mathfrak{h})} \mathrm{U}_\hbar(\mathfrak{g})$ the localized version of the universal enveloping algebra. For $\alpha \in \Delta_+$, define the following element lying in a certain completion of $\mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}}$:

$$P_\alpha(t) = \sum_{k\geq 0} \frac{(-1)^k \hbar^{-k}}{k!} \frac{1}{\prod_{j=1}^k (h_\alpha + \hbar(t+j))} f_\alpha^k e_\alpha^k, \tag{1.8}$$

Choose a normal ordering $<$ on Δ_+ (i.e. such that $\alpha+\beta$ lies between α and β). Denote by ρ the half-sum of positive roots. Define

$$P := \prod_{\alpha\in\Delta_+}^{<} P_\alpha(t_\alpha), \tag{1.9}$$

where the product is taken in the normal ordering, and $t_\alpha = h_\alpha(\rho)$.

Theorem 1.9. *[AST71]. The element P satisfies*

$$e_\alpha P = P f_\alpha = 0 \; \forall \alpha \in \Delta_+.$$

The action of P is well-defined on left (resp. right) $\mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}}$-modules with a locally nilpotent $\mathfrak{n}$- (resp. $\mathfrak{n}_-$-) action.

For any left $\mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}}$-module X, denote by $\mathfrak{n}_-\backslash X := (\mathfrak{n}_- \mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}} \cdot X)\backslash X$ the quotient by the right $\mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}}$ ideal generated by $\mathfrak{n}_-$, and by $X^{\mathfrak{n}}$ the space of left $\mathfrak{n}$-invariants. Then the extremal projector defines an inverse of the projection

$$X^{\mathfrak{n}} \to \mathfrak{n}_-\backslash X \tag{1.10}$$

for any left module with a locally nilpotent $\mathfrak{n}$-action.

1.5. **Parabolic restriction functor.** The main reference for the setup of this paper is [KS22], but the history of the subject is certainly much richer, for instance, see [ES01].

Definition 1.10. A ***universal generic category*** $\mathcal{O}$ is the category $\mathcal{O}^{\mathrm{univ,gen}}$ of $(\mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}}, \mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}})$-bimodules such that the diagonal $\mathfrak{b}$-action integrates to a B-action. The ***universal generic Verma module*** is

$$M^{\mathrm{univ}} := \mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}} \otimes_{\mathrm{U}_\hbar(\mathfrak{n})} \mathbf{C}.$$

Similarly to Section 1.2, we can define a generic version $\mathrm{HC}_\hbar(H)^{\mathrm{gen}}$ of the category of Harish-Chandra bimodules. We have two functors:

$$\mathrm{act}_H\colon \mathrm{HC}_\hbar(H)^{\mathrm{gen}} \to \mathcal{O}^{\mathrm{univ,gen}}, \qquad X \mapsto M^{\mathrm{univ}} \otimes_{\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}} X, \tag{1.11}$$

and

$$\mathrm{act}_G\colon \mathrm{HC}_\hbar(G) \to \mathcal{O}^{\mathrm{univ,gen}}, \qquad X \mapsto X \otimes_{\mathrm{U}_\hbar(\mathfrak{g})} M^{\mathrm{univ}}. \tag{1.12}$$

Denote by $X/\mathfrak{n} := X/(X \cdot \mathrm{U}_\hbar(\mathfrak{g})\mathfrak{n})$ the quotient by the left $\mathrm{U}_\hbar(\mathfrak{g})$-ideal generated by $\mathfrak{n}$. Then the latter functor can also be presented as $X \mapsto (X\mathfrak{n}) \otimes_{\mathrm{U}_\hbar(\mathfrak{h})} \mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}$.

Proposition 1.11. *The functor* act_H *is an equivalence.*

Proof. We repeat an argument from [KS22, Theorem 4.17].

By [KS22, Proposition 4.6], the functor $(-)^{\mathfrak{n}}\colon \mathcal{O}^{\mathrm{univ,gen}} \to \mathrm{HC}_\hbar(H)^{\mathrm{gen}}$ is right adjoint to act_H. The unit of the adjunction is given by

$$X \mapsto (M^{\mathrm{univ}} \otimes_{\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}} X)^{\mathfrak{n}}.$$

The extremal projector gives an isomorphism with the quotient functor as in (1.10):

$$_{\mathfrak{n}_-}(-)\colon \mathcal{O}^{\mathrm{univ,gen}} \to \mathrm{HC}_\hbar(H), \qquad M \mapsto \mathfrak{n}_-\backslash M.$$

So, the functor $(-)^{\mathfrak{n}}$, is exact. We have

$$(M^{\mathrm{univ}} \otimes_{\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}} X)^{\mathfrak{n}} \cong \mathfrak{n}_-\backslash(M^{\mathrm{univ}} \otimes_{\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}} X),$$

and the composition

$$X \mapsto \mathfrak{n}_-\backslash(M^{\mathrm{univ}} \otimes_{\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}} X)$$

is an isomorphism by the PBW theorem. Therefore, act_H is fully faithful.

Since the $\mathfrak{n}$-action on $M \in \mathcal{O}^{\mathrm{univ,gen}}$ is locally nilpotent, $M^{\mathfrak{n}} = 0$ if and only if $M = 0$. But $(-)^{\mathfrak{n}}$ is exact, therefore, it is conservative. Since its left adjoint act_H is fully faithful, it is an equivalence. $\square$

Definition 1.12. The ***parabolic restriction*** functor $\mathrm{res}_{\mathfrak{g}}\colon \mathrm{HC}_\hbar(G) \to \mathrm{HC}_\hbar(H)$ is the composition

$$\mathrm{HC}_\hbar(G) \xrightarrow{\mathrm{act}_G} \mathcal{O}^{\mathrm{univ,gen}} \xrightarrow{(-)^{\mathfrak{n}}} \mathrm{HC}_\hbar(H)^{\mathrm{gen}}.$$

Explicitly, it is given by quantum Hamiltonian reduction and extension of scalars

$$X \mapsto ((X/\mathfrak{n}) \otimes_{\mathrm{U}_\hbar(\mathfrak{h})} \mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}})^{\mathfrak{n}}.$$

Equivalently, we can extend the scalars on the $\mathrm{HC}_\hbar(G)$-side first and then take the quantum Hamiltonian reduction:

$$X \mapsto (\mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}} \otimes_{\mathrm{U}_\hbar(\mathfrak{g})} X/\mathfrak{n})^{\mathfrak{n}}.$$

There is a natural lax monoidal structure on $\mathrm{res}_{\mathfrak{g}}$:

$$(X/\mathfrak{n})^{\mathfrak{n}} \otimes_{\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}} (Y/\mathfrak{n})^{\mathfrak{n}} \to (X \otimes_{\mathrm{U}_\hbar(\mathfrak{g})} Y/\mathfrak{n})^{\mathfrak{n}}, \qquad [x] \otimes [y] \mapsto [x \otimes y]. \tag{1.13}$$

Theorem 1.13. *[KS22, Corollary 4.18] The functor* $\mathrm{res}_{\mathfrak{g}}$ *is colimit-preserving and monoidal.*

Similarly to Section 1.2, one can define a generic version of the functor of free Harish-Chandra bimodules $\mathrm{Rep}(H) \to \mathrm{HC}_\hbar(H)^{\mathrm{gen}}$. Then, on the one hand, we have a monoidal functor $\mathrm{Rep}(G) \to \mathrm{HC}_\hbar(H)^{\mathrm{gen}}$ obtained by composition with $\mathrm{res}_{\mathfrak{g}}$. On the other hand, there is the forgetful functor $\mathrm{Rep}(G) \to \mathrm{Rep}(H)$ which is also monoidal. So, we have a diagram

$$\begin{tikzcd} \mathrm{Rep}(G) \arrow[r] \arrow[d] & \mathrm{HC}_\hbar(G) \arrow[d] \\ \mathrm{Rep}(H) \arrow[r] & \mathrm{HC}_\hbar(H)^{\mathrm{gen}} \end{tikzcd} \tag{1.14}$$

Theorem 1.14. *[KS22, Theorem 4.20] The extremal projector defines a natural isomorphism*

$$P_V \colon \mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}} \otimes V \to (\mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}} \otimes V/\mathfrak{n})^{\mathfrak{n}}.$$

In other words, the diagram (1.14) is commutative.

Proof. We will use a slightly different homomorphism as in the citation. By the PBW theorem, we have a vector space isomorphism

$$\mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}} \otimes V \cong \mathrm{U}_\hbar(\mathfrak{n}_-) \otimes U(\mathfrak{h})^{\mathrm{gen}} \otimes V \otimes \mathrm{U}_\hbar(\mathfrak{n})$$

for any G-representation V. In particular,

$$\mathfrak{n}_- \backslash \mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}} \otimes V/\mathfrak{n} \cong \mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}} \otimes V.$$

Therefore, the extremal projector gives a natural isomorphism as in (1.10):

$$(\mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}} \otimes V/\mathfrak{n})^{\mathfrak{n}} \xrightarrow{\sim} \mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}} \otimes V.$$

Explicitly, the inverse is given by

$$f(\lambda) \otimes v \mapsto f(\lambda) P v \in \mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}} \otimes V/\mathfrak{n}$$

for any $f(\lambda) \in \mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}$ and $v \in V$. □

In particular, one can translate the natural tensor structure on $\mathrm{res}_{\mathfrak{g}}$ to the free module functor $\mathrm{Rep}(G) \to \mathrm{HC}_\hbar(H)^{\mathrm{gen}}$.

Theorem 1.15. (1) *There is a collection of maps $J_{VW}(\lambda)$ natural in $V, W \in \mathrm{Rep}(G)$, such that the diagram*

$$\begin{tikzcd} (\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}} \otimes V) \otimes_{\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}} (\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}} \otimes W) \arrow[r, "J_{VW}(\lambda)"] \arrow[d, "P_V \otimes P_W"] & \mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}} \otimes V \otimes W \arrow[d, "P_{V\otimes W}"] \\ \mathrm{res}_{\mathfrak{g}}(\mathrm{U}_\hbar(\mathfrak{g}) \otimes V) \otimes_{\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}} \mathrm{res}_{\mathfrak{g}}(\mathrm{U}_\hbar(\mathfrak{g}) \otimes W) \arrow[r] & \mathrm{res}_{\mathfrak{g}}(\mathrm{U}_\hbar(\mathfrak{g}) \otimes V \otimes W) \end{tikzcd}$$

is commutative, where the lower arrow is the natural tensor structure on $\mathrm{res}_{\mathfrak{g}}$. In particular, the collection of $J_{VW}(\lambda)$ satisfies the dynamical twist equation Eq. (1.4), and $R^{\mathrm{dyn}}_{VW}(\lambda) := J_{WV}(\lambda)^{-1} J_{VW}(\lambda)$ is a quantum dynamical R-matrix from Definition 1.7.

(2) *The map $J_{VW}(\lambda)$ has the form*

$$J_{VW}(\lambda) \in \mathrm{id}_{V\otimes W} + \hbar \mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}} \otimes \mathrm{U}_\hbar(\mathfrak{b}_-)^{>0} \otimes \mathrm{U}_\hbar(\mathfrak{b}_-)^{>0},$$

where the upper subscript >0 means the augmentation ideal.

(3) *The coefficient $j_{VW}(\lambda)$ of the first $\hbar$-power of $J_{VW}(\lambda)$ is given by*

$$j_{VW}(\lambda) = -\sum_{\alpha \in \Delta_+} \frac{1}{h_\alpha} f_\alpha \otimes e_\alpha = -\frac{f_\alpha \otimes e_\alpha}{\langle \lambda, h_\alpha \rangle}.$$

In particular, the dynamical R-matrix $R^{\mathrm{dyn}}(\lambda)_{VW}$ quantizes the standard classical dynamical r-matrix (1.7):

$$r(\lambda) := j(\lambda) - j^{21}(\lambda) = \sum_{\alpha \in \Delta_+} \frac{e_\alpha \wedge f_\alpha}{\langle \lambda, h_\alpha \rangle}.$$

Proof. We use the argument from [Kho04]. In what follows, we drop the tensor product sign, i.e. for any $x \in \mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}} \otimes V, y \in \mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}} \otimes W$, we denote

$$xy := [x \otimes_{\mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}}} y] \in (\mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}} \otimes V) \otimes_{\mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}}} (\mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}} \otimes W) \cong \mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}} \otimes V \otimes W.$$

Observe that by Theorem 1.14, every element in $(\mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}} \otimes V/\mathfrak{n})^{\mathfrak{n}}$ can written as $f(\lambda)PvP$, where $f(\lambda) \in \mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}$ and the right P multiplier corresponds to taking the quotient by $\mathfrak{n}$. Then the statement can be presented as follows: for every $v \in V, w \in W$, we have

$$PvPwP = PJ_{V,W}(\lambda)(v \otimes w)P. \tag{1.15}$$

Since the collection of $J_{VW}(\lambda)$ defines a tensor structure on $\mathrm{Rep}(G) \to \mathrm{HC}_\hbar(H)^{\mathrm{gen}}$, it automatically satisfies the dynamical twist equation by Theorem 1.8; the same is true for the dynamical R-matrix $R(\lambda)$.

Let us write P in the PBW basis:

$$P = 1 + \sum_i \hbar^{k_i} a_i(\lambda) f_i e_i,$$

where $a_i(\lambda) \in \mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}, f_i \in \mathrm{U}_\hbar(\mathfrak{n}_-), e_i \in \mathrm{U}_\hbar(\mathfrak{n})$, and k_i are some negative numbers. It follows from the construction Eq. (1.8) that

$$P \in 1 + (\mathfrak{n}_- \mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}} \cap \mathrm{U}_\hbar(\mathfrak{g})^{\mathrm{gen}} \mathfrak{n}).$$

Therefore, we have $f_i = 1$ if and only if $e_i = 1$. Then we proceed as follows: consider the middle P in (1.15). Using the fact that P commutes with $\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}$, we can push $a_i(\lambda)$ to the left. Using Theorem 1.9, we can push f_i to the left until it meets P and becomes zero. Likewise, we can push e_i to the right until it meets P and becomes zero as well.

Let us demonstrate how it works when f_i and e_i are Lie algebra elements. Then we are dealing with the term $\hbar^{-1} a_i(\lambda) f_i e_i$. Let us compute $\hbar^{-1} Pva_i(\lambda) f_i e_i wP$. First, we push $a_i(\lambda)$ to the left:

$$Pva_i(\lambda) = Pa_i(\lambda - \hbar \mathrm{wt}(v))v = a_i(\lambda - \hbar \mathrm{wt}(v))Pv.$$

By $Pf_i = 0$, we have

$$Pvf_i = Pf_i v - \hbar P \,\mathrm{ad}_{f_i}(v) = -\hbar P \,\mathrm{ad}_{f_i}(v).$$

Likewise, by $e_i P = 0$, we get

$$e_i wP = \hbar \,\mathrm{ad}_{e_i}(w)P.$$

Therefore,

$$\hbar^{-1} Pva_i(\lambda) f_i e_i wP = -\hbar a_i(\lambda - \hbar \mathrm{wt}(v)) P \,\mathrm{ad}_{f_i}(v) \,\mathrm{ad}_{e_i}(v) P,$$

which has the necessary form.

In general, we see that each term $\hbar^{-l} f_\alpha^l e_\alpha^l$ in (1.8) acts with a minimal power of $\hbar^l$, and the second part of the theorem follows. One can easily check that the first power of $\hbar$ is given by the sum of actions of $k = 1$ terms of (1.8) in the product (1.9), which is

$$j(\lambda) = -\sum_{\alpha \in \Delta_+} \frac{f_\alpha \otimes e_\alpha}{\langle \lambda, h_\alpha \rangle}.$$

□

The last part motivates the following definition, see [EV99].

Definition 1.16. The ***standard dynamical twist*** is the collection $J_{VW}(\lambda)$. The ***standard quantum dynamical R-matrix*** is $R^{\mathrm{dyn}}_{VW}(\lambda)$.

2. Mirabolic subgroup

Denote by GL_N the group of invertible $N \times N$-matrices and by $\mathfrak{gl}_N$ its Lie algebra. We choose a natural basis $\{E_{ij} | 1 \leq i, j \leq N\}$ of matrix units in $\mathfrak{gl}_N$ with the commutator

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}.$$

For the rest of the paper, we will adopt the following notations. For any $k \leq N$, we consider $\mathfrak{gl}_k \subset \mathfrak{gl}_N$ embedded as the *upper left corner*:

$$\mathfrak{gl}_k = \begin{pmatrix} * & \dots & * & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & \dots & * & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}\right\} N-k$$

Likewise, for any $k \leq N$, we denote by ${}_k\mathfrak{gl}_N$ the *left k-th truncation* of $\mathfrak{gl}_N$: it is the subalgebra isomorphic to $\mathfrak{gl}_{N-k}$, but embedded as the *lower right corner*:

$${}_k\mathfrak{gl}_N = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & * & \dots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & * & \dots & * \end{pmatrix} \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}\right\} N-k$$

We will also use combinations of these notations. For instance, here is an example of ${}_1\mathfrak{gl}_4$ inside $\mathfrak{gl}_5$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & 0 \\ 0 & * & * & * & 0 \\ 0 & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

It turns out that almost all the constructions of the paper involve not the whole group GL_N, but its almost parabolic subgroup:

Definition 2.1. The ***mirabolic subgroup*** M_N is the subgroup of GL_N preserving the last basis vector:

$$\mathrm{M}_N = \begin{pmatrix} * & \dots & * & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \dots & * & 0 \\ * & \dots & * & 1 \end{pmatrix}$$

The ***mirabolic subalgebra*** $\mathfrak{m}_N$ is the Lie algebra of M_N identified with the space of matrices whose last column is zero:

$$\mathfrak{m}_N = \begin{pmatrix} * & \dots & * & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \dots & * & 0 \end{pmatrix}$$

We will also adopt notation ${}_i\mathfrak{m}_j \subset \mathfrak{m}_N$ as in the case of $\mathfrak{gl}_N$. Here is an example of ${}_1\mathfrak{m}_4 \subset \mathfrak{m}_5$:

$${}_1\mathfrak{m}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 \\ 0 & * & * & 0 & 0 \\ 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For the rest of the paper, we will use the notations

$$\begin{aligned} \mathfrak{n}_- &= \mathrm{span}(E_{ij} | N \geq i > j \geq 1), \\ \mathfrak{b} &= \mathrm{span}(E_{kl} | 1 \leq k \leq l \leq N-1). \end{aligned} \tag{2.1}$$

unless specified otherwise. In particular, $\mathfrak{b}$ refers to the Borel subalgebra *inside* $\mathfrak{m}_N$, not $\mathfrak{gl}_N$. Likewise, we will denote by ${}_k\mathfrak{n}_-$ and ${}_k\mathfrak{b}$ the corresponding truncated subalgebras in ${}_k\mathfrak{m}_N$.

Let $e = E_{12} + \ldots + E_{N-1,N} \in \mathfrak{gl}_N$ be the so-called ***principal nilpotent element*** and $\vec{u} = (u_1, \ldots, u_{N-1})$ be a vector of parameters. Denote by

$$e_{\vec{u}} := e - u_1 E_{11} - \ldots - u_{N-1} E_{N-1,N-1} \tag{2.2}$$

While $e_{\vec{u}} \notin \mathfrak{m}_N$, we can still define the following 2-form on $\mathfrak{m}_N$:

$$\omega(\vec{u})\colon \mathfrak{m}_N \wedge \mathfrak{m}_N \to k, \qquad x \wedge y \mapsto \mathrm{Tr}(e_{\vec{u}} \cdot [x, y]). \tag{2.3}$$

Proposition 2.2. *The form $\omega(\vec{u})$ is non-degenerate, and its inverse $r^{\mathrm{CG}}(\vec{u})$ is uniquely specified by the condition*

$$r^{\mathrm{CG}}(\vec{u})(E^*_{i+k,i}) = -E_{i,i+k-1} - (u_i - u_{i+k-1}) \cdot r^{\mathrm{CG}}(\vec{u})(E^*_{i+k-1,i}) + \delta_{i>1} r^{\mathrm{CG}}(\vec{u})(E^*_{i+k-1,i-1}). \tag{2.4}$$

for any $1 \leq i < i + k \leq N$, where E^*_{ij} is the dual basis and we consider $r^{\mathrm{CG}}(\vec{u})$ as a map $\mathfrak{m}^*_N \to \mathfrak{m}_N$.

Proof. Note that both $\mathfrak{n}_-$ and $\mathfrak{b}$ are isotropic subspaces of $\mathfrak{m}_N$. Since $\omega(\vec{u})$ is skew-symmetric, it is enough to construct an inverse only of one map, say $\omega(\vec{u})\colon \mathfrak{b} \to \mathfrak{n}_-$. Observe that

$$\omega(\vec{u})(E_{i,i+k-1}) = -E^*_{i+k,i} + \delta_{i>1} E^*_{i+k-1,i-1} - (u_i - u_{i+k-1}) \cdot E^*_{i+k-1,i}.$$

Hence, to satisfy $(r^{\mathrm{CG}}(\vec{u}) \circ \omega(\vec{u}))(E_{i,i+k-1}) = E_{i,i+k-1}$, we get the equation (2.4). It can be solved inductively which proves the existence part. ☐

It follows by general arguments (see [ES02]) that $r^{\mathrm{CG}}(\vec{u})$ satisfies the classical Yang-Baxter equation (1.3) for any $\vec{u}$. In particular, for $\vec{u} = (0, \ldots, 0)$, we get

$$r^{\mathrm{CG}}(0) = -\sum_{N \geq i > j \geq 1} E_{ij} \wedge \sum_{k=1}^{j} E_{k,k+i-1-j}.$$

This is a certain degeneration of the Cremmer-Gervais r-matrix mentioned in the introduction, see [EH00]. This motivates the following definition.

Definition 2.3. A ***rational Cremmer-Gervais r-matrix*** is $r^{\mathrm{CG}}(\vec{u})$ for $\vec{u} \in \mathbf{C}^{N-1}$.

As any classical r-matrix, it defines a Poisson-Lie structure on the mirabolic subgroup M_N (in this case, even symplectic one); moreover, by considering $r^{\mathrm{CG}}(\vec{u})$ as an element of $\mathfrak{gl}_N \wedge \mathfrak{gl}_N$, it also defines a Poisson-Lie structure on the whole group GL_N. We will construct its quantization in Section 5.

3. Yangian

In this section we recall some facts about the Yangian of the Lie algebra $\mathfrak{gl}_N$ and prove some results that we use later in the paper. For details, we refer the reader to [Mol07]; note that we use an $\hbar$-version of constructions in question (see Section 1.1), while in *loc. cit.* it is specialized to $\hbar = 1$. However, all the proofs can be easily generalized to the case of an arbitrary $\hbar$.

3.1. General definition and properties.

The Yangian $\mathrm{Y}_\hbar(\mathfrak{gl}_N)$ is an associative algebra generated over $\mathbf{C}$ by the elements $t^{(r)}_{ij}$, where $1 \leq i, j \leq N$ and $r \geq 1$. In what follows, we also define $t^{(0)}_{ij} := \delta_{ij}$. Introduce the formal Laurent series in u^{-1}:

$$t_{ij}(u) := \sum_{k \geq 0} t^{(r)}_{ij} u^{-r}.$$

We combine them into a generating matrix

$$T(u) := (t_{ij}(u)) \in \mathrm{Y}_\hbar(\mathfrak{gl}_N) \otimes \mathrm{End}(\mathbf{C}^N)[[u^{-1}]].$$

Let $P \in \mathrm{End}(\mathbf{C}^N \otimes \mathbf{C}^N)$ be the permutation matrix. Consider the ***Yang R-matrix***

$$R(u - v) = \mathrm{id} - \frac{\hbar P}{u - v} \in \mathrm{End}(\mathbf{C}^N \otimes \mathbf{C}^N).$$

Then the defining relation of $\mathrm{Y}_\hbar(\mathfrak{gl}_N)$ can be presented in the form

$$R(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R(u - v),$$

where the equality takes place in $\mathrm{Y}_\hbar(\mathfrak{gl}_N) \otimes \mathrm{End}(\mathbf{C}^n \otimes \mathbf{C}^N)$ and $T_1(u) := T(u) \otimes \mathrm{id}$, $T_2(v) = \mathrm{id} \otimes T(v)$. Equivalently, it is given by

$$(u-v)[t_{ij}(u), t_{kl}(v)] = \hbar \frac{t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)}{u-v}. \tag{3.1}$$

Consider the following $\mathrm{U}_\hbar(\mathfrak{gl}_N)$-valued matrix:

$$E = \begin{pmatrix} E_{11} & \dots & E_{1,n-1} & E_{1N} \\ \vdots & \ddots & \vdots & \vdots \\ E_{N-1,1} & \dots & E_{N-1,N-1} & E_{N-1,N} \\ E_{N1} & \dots & E_{N,N-1} & E_{NN} \end{pmatrix},$$

One can define a homomorphism $\mathrm{Y}_\hbar(\mathfrak{gl}_N) \to \mathrm{U}_\hbar(\mathfrak{gl}_N)$, called the ***evaluation homomorphism***, by

$$T(u) \mapsto \mathrm{id} + u^{-1}E. \tag{3.2}$$

In what follows, we will use the same notation $T(u)$ both for the elements of $\mathrm{Y}_\hbar(\mathfrak{gl}_N)$ and their images in $\mathrm{U}_\hbar(\mathfrak{gl}_N)$ under the evaluation homomorphism; all the statements involving $T(u)$ will hold in $\mathrm{Y}_\hbar(\mathfrak{gl}_N)$ unless specified otherwise. We hope it will not lead to any confusion.

Actually, we will be mainly dealing with a slightly different version of the T-matrix. Namely, define

$$L(u) = (L_{ij}(u)) := u \cdot \mathrm{id} + E. \tag{3.3}$$

Obviously, we have $L(u) = uT(u)$. Almost all the properties of $T(u)$ that we use in the paper are satisfied *mutatis mutandis* by $L(u)$; we will explicitly indicate the cases where it is not true or not trivial.

Let $\{a_1, \dots a_m\}, \{b_1, \dots, b_m\}$ be two sets of indices. Define a ***quantum minor*** as the sum over all permutations of $1, 2, \dots, m$, see [Mol07, (1.54), (1.55)]:

$$\begin{aligned} t^{a_1 \dots a_m}_{b_1 \dots b_m}(u) &:= \sum_\sigma \mathrm{sgn}(\sigma) t_{a_{\sigma(1)} b_1}(u) t_{a_{\sigma(2)} b_2}(u-\hbar) \dots t_{a_{\sigma(m)} b_m}(u - \hbar m + \hbar) = \\ &= \sum_\sigma \mathrm{sgn}(\sigma) t_{a_1 b_{\sigma(1)}}(u) t_{a_2 b_{\sigma(2)}}(u-\hbar) \dots t_{a_m b_{\sigma(m)}}(u - \hbar m + \hbar). \end{aligned}$$

Similarly, we can define a quantum minor for $L(u)$:

$$\begin{aligned} L^{a_1 \dots a_m}_{b_1 \dots b_m}(u) &:= \sum_\sigma \mathrm{sgn}(\sigma) L_{a_{\sigma(1)} b_1}(u) L_{a_{\sigma(2)} b_2}(u-\hbar) \dots L_{a_{\sigma(m)} b_m}(u - \hbar m + \hbar) = \\ &= \sum_\sigma \mathrm{sgn}(\sigma) L_{a_1 b_{\sigma(1)}}(u) L_{a_2 b_{\sigma(2)}}(u-\hbar) \dots L_{a_m b_{\sigma(m)}}(u - \hbar m + \hbar). \end{aligned} \tag{3.4}$$

Observe that

$$L^{a_1 \dots a_m}_{b_1 \dots b_m}(u) = u(u-\hbar) \dots (u - \hbar m + \hbar) t^{a_1 \dots a_m}_{b_1 \dots b_m}(u). \tag{3.5}$$

We will need the following properties of quantum minors.

Proposition 3.1. (1) *For any permutation σ, we have*

$$L^{a_{\sigma(1)} \dots a_{\sigma(m)}}_{b_1 \dots b_m}(u) = \mathrm{sgn}(\sigma) L^{a_1 \dots a_m}_{b_1 \dots b_m} = L^{a_1 \dots a_m}_{b_{\sigma(1)} \dots b_{\sigma(m)}}(u).$$

In particular, if $a_i = a_j$ or $b_i = b_j$ for some $i \neq j$, then $L^{a_1 \dots a_m}_{b_1 \dots b_m}(u) = 0$.

(2) *[Mol07, Proposition 1.6.8] A quantum minor can be decomposed as*

$$\begin{aligned} L^{a_1\dots a_m}_{b_1\dots b_m}(u) &= \sum_{l=1}^m (-1)^{m-l} L^{a_1\dots\hat{a}_l\dots a_m}_{b_1\dots b_{m-1}}(u) L_{a_l b_m}(u-\hbar m+\hbar) = \\ &= \sum_{l=1}^m (-1)^{m-l} L^{a_1\dots a_{m-1}}_{b_1\dots\hat{b}_l\dots b_m}(u-\hbar) L_{a_m b_l}(u) = \\ &= \sum_{l=1}^m (-1)^{l-1} L_{a_l b_1}(u) L^{a_1\dots\hat{a}_l\dots a_m}_{b_2\dots b_m}(u-\hbar) = \\ &= \sum_{l=1}^m (-1)^{l-1} L_{a_1 b_l}(u-\hbar m+\hbar) L^{a_2\dots a_m}_{b_1\dots\hat{b}_l\dots b_m}(u). \end{aligned}$$

Quantum minors enjoy the following commutation properties.

Proposition 3.2. *For $k \neq l$, we have*

$$[E_{kl}, L^{a_1\dots a_m}_{b_1\dots b_m}(u)] = \hbar\left(\sum_{i=1}^m \delta_{a_i,l} L^{a_1\dots k\dots a_m}_{b_1\dots b_m}(u) - \sum_{i=1}^m \delta_{k,b_i} L^{a_1\dots a_m}_{b_1\dots l\dots b_m}(u)\right).$$

where k, resp. l, are on the i-th position.

Proof. It follows from [Mol07, Proposition 1.7.1] that

$$(u-v)[L_{kl}(u), L^{a_1\dots a_m}_{b_1\dots b_m}(v)] = \hbar\left(\sum_{i=1}^m L_{a_i l}(u) L^{a_1\dots k\dots a_m}_{b_1\dots b_m}(v) - \sum_{i=1}^m L^{a_1\dots a_m}_{b_1\dots l\dots b_m}(v) L_{k b_i}(u)\right),$$

where the indices k and l replace the corresponding indices a_i and b_i respectively. Since $L_{ij}(u) = \delta_{ij}\cdot u + E_{ij}$, the statement follows by comparing the coefficients of u on both sides. □

A particular case of a quantum minor will be important.

Definition 3.3. The ***quantum determinant*** of $L(u)$ (respectively $T(u)$) is

$$\operatorname{qdet}(L(u)) = L^{1\dots N}_{1\dots N}(u)$$

(respectively $\operatorname{qdet}(T(u)) = T^{1\dots N}_{1\dots N}(u)$). In what follows, we will also call $\operatorname{qdet}(L(u))$ the ***quantum characteristic polynomial***.

It follows from Proposition 3.2 that $\operatorname{qdet}(L(u))$ is central:

$$[L_{ij}(u), \operatorname{qdet}(L(v))] = 0 \ \forall i,j.$$

Observe the matrix $T(u)$ has the form

$$T(u) \in \mathrm{id} + u^{-1} \mathrm{Y}_\hbar(\mathfrak{gl}_N) \otimes \operatorname{End}(\mathbf{C}^N)[[u^{-1}]],$$

hence is invertible. We can explicitly construct its inverse as follows.

Definition 3.4. [Mol07, Definition 1.9.1] Denote by $\hat{t}_{ij}(u) := (-1)^{i+j} t^{1\dots\hat{j}\dots N}_{1\dots\hat{i}\dots N}(u)$ the complementary quantum minor. The ***quantum comatrix*** is

$$\hat{T}(u) := (\hat{t}_{ij}(u)).$$

Likewise, one can define $\hat{L}_{ij}(u) := L^{1\dots\hat{j}\dots N}_{1\dots\hat{i}\dots N}(u)$.

Proposition 3.5. *[Mol07, Proposition 1.9.2] The quantum comatrix satisfies*

$$\hat{T}(u) T(u-\hbar N+\hbar) = \operatorname{qdet}(T(u))\cdot\mathrm{id}.$$

In particular, the inverse of $T(u)$ is given by

$$T(u)^{-1} = \operatorname{qdet}(T(u+\hbar N-\hbar)^{-1}\cdot \hat{T}(u+\hbar N-\hbar).$$

Under the evaluation homomorphism, the inverse of $T(u)$ can be explicitly presented in terms of E as

$$T(u)^{-1} = (\mathrm{id}+u^{-1}E)^{-1} = \sum_{k\geq 0}(-1)^k u^{-k} E^k. \tag{3.6}$$

3.2. **Technical lemmas.** In what follows, we will prove some technical statements that we will use later in the paper.

Proposition 3.6. *For any $k \leq l$, we have*

$$L^{1\dots k}_{1\dots k}(u) L^{1,\dots,k-1,l}_{1,\dots,k-1,k}(u-\hbar) = L^{1,\dots,k-1,l}_{1,\dots,k-1,k}(u) L^{1\dots k}_{1\dots k}(u-\hbar).$$

Proof. By [Mol07, Theorem 1.12.1], the map

$$\mathrm{Y}_\hbar(\mathfrak{gl}_{l-k+1}) \to \mathrm{Y}_\hbar(\mathfrak{gl}_l), \quad t_{ij}(u) \mapsto t^{1\dots k-1,k-1+i}_{1\dots k-1,k-1+j}(u)$$

is an algebra homomorphism. The defining commutation relation (3.1) gives

$$[t_{11}(u), t_{l-k+1,1}(v)] = \hbar \frac{t_{l-k+1,1}(u)t_{11}(v) - t_{l-k+1,1}(v)t_{11}(u)}{u-v}.$$

Substituting $u - v = \hbar$, we conclude. □

Proposition 3.7. *For any $a > d$, we have*

$$L^{1\dots d-1}_{1\dots d-1}(u) L^{a,2,\dots,d-1}_{1\dots d-1}(u-\hbar) = L^{a,2,\dots d-1}_{1\dots d-1}(u) L^{1\dots d-1}_{1\dots d-1}(u-\hbar).$$

Proof. We can suppose that the statement takes place in $\mathfrak{gl}_a$. Consider the automorphism of $\mathfrak{gl}_a$ cyclically permuting the indices:

$$\phi\colon \{1,2,\dots,a\} \mapsto \{a,1,\dots,a-1\}, \quad E_{ij} \mapsto E_{\phi(i),\phi(j)}.$$

After some permutation of indices in the minors, the statement of the lemma becomes

$$L^{1,\dots,d-2,a}_{1,\dots,d-2,a}(u) L^{1,\dots,d-2,a-1}_{1,\dots,d-2,a}(u-\hbar) = L^{1,\dots,d-2,a-1}_{1,\dots,d-2,a}(u) L^{1,\dots,d-2,a}_{1,\dots,d-2,a}(u-\hbar).$$

Then we can proceed as in the proof of Proposition 3.6 reducing to

$$\begin{aligned}&[t_{a-d+2,a-d+2}(u), t_{a-d+1,a-d+2}(v)] = \\ =&\hbar\frac{t_{a-d+2,a-d+1}(u)t_{a-d+2,a-d+2}(v) - t_{a-d+2,a-d+1}(v)t_{a-d+2,a-d+2}(u)}{u-v},\end{aligned}$$

the defining commutation relation. □

Let $\psi\colon \mathfrak{n}_- \to \mathbf{C}$ be the character such that $\psi(E_{i+1,i}) = 1$ for all $1 \leq i \leq N-1$, and

$$\mathfrak{n}_-^\psi = \operatorname{span}(x - \psi(x) | x \in \mathfrak{n}_-) \subset \mathrm{U}_\hbar(\mathfrak{gl}_N)$$

be the shift of $\mathfrak{n}_-$ as in Section 4.

Proposition 3.8. *For all $c \leq N$,*

$$L^{1\dots\hat{l}\dots c}_{1\dots c-1}(u) \equiv L^{1\dots l-1}_{1\dots l-1}(u) \bmod \mathfrak{n}_-^\psi \mathrm{U}_\hbar(\mathfrak{gl}_N).$$

Proof. For $l = c$ the lemma is obvious. By Proposition 3.1, we can permute the upper indices:

$$L^{1\dots\hat{l}\dots c}_{1\dots c-1}(u) = (-1)^c L^{c,1,\dots,\hat{l},\dots,c-1}_{1\dots c-1}(u).$$

By part (2), the right-hand side is equal to

$$\sum_{k=1}^{c-1} (-1)^{c+k-1} E_{ck} L^{1\dots\hat{l}\dots c-1}_{1\dots\hat{k}\dots c-1}(u).$$

But $E_{lk} \equiv 0$ for $k \neq l-1$ and $E_{l,l-1} \equiv 1$. Hence, the right-hand side is equivalent to $L^{1\dots\hat{l}\dots c-1}_{1\dots c-2}(u)$. The statement follows from the the obvious induction. □

Proposition 3.9. *For any $1 \leq i,j \leq c$, we have*

$$\frac{L^{1\dots\hat{j}\dots c}_{1\dots\hat{i}\dots c}(v) L^{1\dots c}_{1\dots c}(u) - L^{1\dots\hat{j}\dots c}_{1\dots\hat{i}\dots c}(v) L^{1\dots c}_{1\dots c}(u)}{u-v} = \sum_{l=1}^{c} L^{1\dots\hat{l}\dots c}_{1\dots\hat{i}\dots c}(v) L^{1\dots\hat{j}\dots c}_{1\dots\hat{l}\dots c}(u).$$

Proof. We can assume that the statement takes place in $\mathfrak{gl}_c$. Then, for any k, l:

$$L^{1\dots\hat{l}\dots c}_{1\dots\hat{k}\dots c}(u) = \hat{L}_{kl}(u),$$
$$L^{1\dots c}_{1\dots c}(u) = \mathrm{qdet}(L(u)).$$

By (3.5), we have

$$\hat{L}_{ab}(u) = u(u-\hbar)\dots(u-\hbar N+2\hbar)\hat{t}_{ab},$$
$$\mathrm{qdet}(L(u)) = u(u-\hbar)\dots(u-\hbar N+\hbar)\mathrm{qdet}(T(u))$$

for any a, b. Therefore, if we divide both sides by

$$\mathrm{qdet}(T(u))\mathrm{qdet}(T(v))\prod_{i=0}^{N-2}(u-\hbar i)\prod_{i=0}^{N-2}(v-\hbar i),$$

(recall that $\mathrm{qdet}(T(u))$ is central), we get

$$\frac{(u-\hbar N+\hbar)\mathrm{qdet}(T(v))^{-1}\hat{t}_{ij}(v)-(v-\hbar N+\hbar)\mathrm{qdet}(T(u))^{-1}\hat{t}_{ij}(u)}{u-v} =$$
$$= \sum_{l=1}^{N}(\mathrm{qdet}(T(v))^{-1}\hat{t}_{il}(v))(\mathrm{qdet}(T(u))^{-1}\hat{t}_{lj}(u)).$$

Let us shift the variables $u \mapsto u+\hbar N-\hbar, v \mapsto v+\hbar N-\hbar$:

$$\frac{u\cdot\mathrm{qdet}(T(v+\hbar N-\hbar))^{-1}\hat{t}_{ij}(v+\hbar N-\hbar)-v\cdot\mathrm{qdet}(T(u+\hbar N-\hbar))^{-1}\hat{t}_{ij}(u+\hbar N-\hbar)}{u-v} =$$
$$= \sum_{l=1}^{N}\mathrm{qdet}(T(v+\hbar N-\hbar))^{-1}\hat{t}_{il}(v+\hbar N-\hbar)\cdot\mathrm{qdet}(T(u+\hbar N-\hbar))^{-1}\hat{t}_{lj}(u+\hbar N-\hbar).$$

Denote by $\tilde{t}_{ij}(u)$ the matrix entry of $T(u)^{-1}$. Then by Proposition 3.5 we have

$$\tilde{t}_{ij}(u) = \mathrm{qdet}(T(u+\hbar N-\hbar))^{-1}\hat{t}_{ij}(u+\hbar N-\hbar).$$

Therefore, the equality above is equivalent to

$$\frac{u\cdot\tilde{t}_{ij}(v)-v\cdot\tilde{t}_{ij}(u)}{u-v} = \sum_{l=1}^{N}\tilde{t}_{il}(v)\tilde{t}_{lj}(u).$$

In terms of series coefficients

$$\tilde{t}_{ij}(u) = \sum_{r\geq 0}\tilde{t}^{(r)}_{ij}u^{-r},$$

this equality is equivalent to

$$\tilde{t}^{(r)}_{ij} = \sum_{l=1}^{N}\tilde{t}^{(r')}_{il}\tilde{t}^{(s')}_{lj} \tag{3.7}$$

for every $r'+s'=r$. Now let us recall that the inverse is given by the series (3.6) under the evaluation homomorphism[1]. In particular, the element $\tilde{t}^{(r)}_{ij}$ is the ij-entry of the matrix power $(-E)^r$. Therefore, equation (3.7) is just the tautological formula $(-E)^{r'+s'} = (-E)^{r'}\cdot(-E)^{s'}$ in terms of the matrix entries. □

[1] I would like to thank Vasily Krylov for suggesting the argument!

4. Kirillov projector

Recall the notion of the mirabolic subalgebra from Section 2. In this section, we define the Kirillov projector and show its key properties.

Denote by $\mathfrak{n}_- = \mathrm{span}(E_{ji}|j > i)$ the negative nilpotent subalgebra of $\mathfrak{m}_N$. Let $\psi\colon \mathfrak{n}_- \to \mathbf{C}$ be a nondegenerate character of $\mathfrak{n}_-$; we will always assume that $\psi(E_{i+1,i}) = 1$ for any $i \leq N-1$. For any $x \in \mathfrak{n}_-$, denote by $x^\psi := x - \psi(x)$. Define the shift $\mathfrak{n}_-^\psi = \mathrm{span}(x^\psi|x \in \mathfrak{n}_-) \subset \mathrm{U}_\hbar(\mathfrak{gl}_N)$.

Definition 4.1. A right $\mathrm{U}_\hbar(\mathfrak{m}_N)$-module is ***Whittaker*** if the $\mathfrak{n}_-^\psi$-action is locally nilpotent. A ***Whittaker vector*** in a Whittaker module is an $\mathfrak{n}_-^\psi$-invariant element.

Recall the notion of a quantum minor (3.4). For $i > j$, consider the following element in a certain completion of $\mathrm{U}_\hbar(\mathfrak{m}_N)[u]$:

$$P_{ij}^\psi(u) := \sum_{k\geq 0} (-1)^{(i+j)k} \hbar^{-k} (-E_{ij}^\psi)^k \frac{L_{j\dots i-1}^{j\dots i-1}(u)\dots L_{j\dots i-1}^{j\dots i-1}(u-\hbar k+\hbar)}{k!} \tag{4.1}$$

For instance, its action is well-defined on any *right* $\mathrm{U}_\hbar(\mathfrak{m}_N)$-module where $\mathfrak{n}_-^\psi$ acts locally nilpotently.

In what follows, we denote by $\vec{u} = (u_1, \dots, u_{N-1})$ a vector of variables. Let us introduce the main object of the paper.

Definition 4.2. The ***Kirillov projector*** $P_{\mathfrak{m}_N}^\psi(\vec{u})$ is the element

$$P_{\mathfrak{m}_N}^\psi(\vec{u}) = \left(P_{N,N-1}^\psi(u_{N-1})\right)\left(P_{N-1,N-2}^\psi(u_{N-2})P_{N,N-2}^\psi(u_{N-2})\right)\dots\left(P_{21}^\psi(u_1)\dots P_{N1}^\psi(u_1)\right). \tag{4.2}$$

Denote by

$$\mathfrak{b}^{\vec{u}} = \mathrm{span}(E_{ij} + \delta_{ij}\cdot u_i|1 \leq i \leq j \leq N-1). \tag{4.3}$$

This is the main theorem of the paper.

Theorem 4.3. *For any right Whittaker module W over $\mathfrak{m}_N$, the Kirillov projector defines a unique linear operator $P(\vec{u})\colon W \to W$ (acting on the right), satisfying*

$$P(\vec{u})(x - \psi(x)) = 0,\ x \in \mathfrak{n}_-,$$
$$(E_{ij} + \delta_{ij}\cdot u_i)P(\vec{u}) = 0,\ 1 \leq i \leq j \leq N-1,$$

and the normalization condition

$$wP(\vec{u}) = w$$

for any Whittaker vector $w \in W^{\mathfrak{n}_-^\psi}$. In particular, it defines a canonical isomorphism

$$W^{\mathfrak{n}_-^\psi} \to W/(W\cdot \mathrm{U}_\hbar(\mathfrak{m}_N)\mathfrak{b}^{\vec{u}}), \tag{4.4}$$

with the quotient $W/\mathfrak{b}^{\vec{u}} := W/(W\cdot \mathrm{U}_\hbar(\mathfrak{m}_N)\mathfrak{b}^{\vec{u}})$.

Proof. Uniqueness is clear: assume we have another operator $P_2(\vec{u})$ satisfying these assumptions. By Proposition 3.1,

$$L_{j\dots i-1}^{j\dots i-1}(u_j) = (-1)^{i-j} L_{j+1,\dots,i-1,j}^{j,\dots,i-1}(u_j) = \sum_{l=j}^{i-1} (-1)^l L_{j\dots\hat{l}\dots i-1}^{j+1\dots i-1}(u_j - \hbar)L_{jl}(u_j).$$

Observe that $L_{jl}(u_j) = E_{jl} + \delta_{jl}\cdot u_j$, hence $L_{j\dots i-1}^{j\dots i-1}(u_j)P_2(\vec{u}) = 0$ and

$$P_{ij}^\psi(u_j)P_2(\vec{u}) = P_2(\vec{u})$$

for every $i > j$. Therefore,

$$P_{\mathfrak{m}_N}(\vec{u})P_2(\vec{u}) = P_2(\vec{u}).$$

Likewise, for any vector $v \in W$, the vector $vP_{\mathfrak{m}_N}(\vec{u})$ is Whittaker. Therefore, by the normalization condition on $P_2(\vec{u})$, we have

$$vP_{\mathfrak{m}_N}(\vec{u})P_2(\vec{u}) = vP_{\mathfrak{m}_N}(\vec{u}).$$

In particular, as operators on W, we get

$$P_{\mathfrak{m}_N}(\vec{u})P_2(\vec{u}) = P_{\mathfrak{m}_N}(\vec{u}),$$

and so

$$P_2(\vec{u}) = P_{\mathfrak{m}_N}(\vec{u}).$$

The properties for $P_{\mathfrak{m}_N}(\vec{u})$ are proven in Theorem 4.5 and Theorem 4.13. $\square$

Remark 4.4. Classically, these properties can be interpreted as follows. The element $P_{\mathfrak{m}_N}(\vec{u})$ defines a projection

$$\mathfrak{n}_-^{\psi}\backslash \mathrm{U}_\hbar(\mathfrak{m}_N) \to (\mathfrak{n}_-^{\psi}\backslash \mathrm{U}_\hbar(\mathfrak{m}_N))^{\mathfrak{n}_-^{\psi}}.$$

It follows from Definition 1.1 and the explicit formula (4.2) that this map is defined over $k[\hbar]$. By the second property, it induces an isomorphism

$$\mathfrak{n}_-^{\psi}\backslash \mathrm{U}_\hbar(\mathfrak{m}_N)/\mathfrak{b}^{\vec{u}} \xrightarrow{\sim} (\mathfrak{n}_-^{\psi}\backslash \mathrm{U}_\hbar(\mathfrak{m}_N))^{\mathfrak{n}_-^{\psi}}.$$

Composing with a natural projection $\mathrm{U}_\hbar(\mathfrak{m}_N) \to \mathrm{U}_\hbar(\mathfrak{m}_N)/\mathfrak{b}^{\vec{u}}$, we get a map

$$\mathrm{U}_\hbar(\mathfrak{m}_N) \to \mathfrak{n}_-^{\psi}\backslash \mathrm{U}_\hbar(\mathfrak{m}_N)/\mathfrak{b}^{\vec{u}}.$$

Since it is defined over $k[\hbar]$ we can take the limit $\hbar \to 0$ which gives

$$\mathrm{Sym}(\mathfrak{m}_N) \cong \mathcal{O}(\mathfrak{m}_N^*) \to \mathfrak{n}_-^{\psi}\backslash \mathcal{O}(\mathfrak{m}_N^*)/\mathfrak{b}^{\vec{u}}.$$

Observe that the target is the space of functions on the closed subvariety of $\mathfrak{m}_N^*$ specified by the equations $E_{ij} = \psi(E_{ij})$ for $E_{ij} \in \mathfrak{n}_-$ and $E_{kl} = -\delta_{kl} \cdot u_k$ for $1 \le k \le l$, which is just the point $e_{\vec{u}} \in \mathfrak{m}_N^*$ of (2.2). Moreover, by the tensor property from Theorem 5.4, this is an algebra map when $\hbar = 0$. Therefore, the classical limit of the Kirillov projector gives a map of varieties

$$\mathrm{pt} \to \mathfrak{m}_N^*, \qquad \mathrm{pt} \mapsto e_{\vec{u}} \in \mathfrak{m}_N^*.$$

4.1. **Kirillov projector: right $\mathfrak{n}_-^{\psi}$-invariance.** In this subsection, we will prove the first property of Theorem 4.3.

Theorem 4.5. *The Kirillov projector satisfies*

$$P^{\psi}_{\mathfrak{m}_N}(\vec{u})(x - \psi(x)) = 0 \ \forall x \in \mathfrak{n}_-.$$

In particular, for any Whittaker module X, it defines a canonical projection

$$P^{\psi}_{\mathfrak{m}_N}(\vec{u})\colon X \mapsto X^{\mathfrak{n}_-^{\psi}}$$

to the space of Whittaker vectors.

We will prove it by two-fold induction. In what follows, we will always identify $u = u_1$.

4.1.1. *Global induction.* The first one is the induction on the dimension N of $\mathfrak{m}_N$. The base case is $N = 2$.

Proposition 4.6. *We have $P^{\psi}_{21}(u)(E_{21} - 1) = 0$.*

Proof. Recall that $L_1^1(u) = E_{11} + u$. Then

$$L_1^1(u)E_{21} = E_{21}L_1^1(u - \hbar).$$

Therefore,

$$
\begin{aligned}
P^{\psi}_{21}(u)E_{21} = &\sum_{k\geq 0} \hbar^{-k}(E_{21}-1)^k \frac{L^1_1(u)\dots L^1_1(u-\hbar k+\hbar)}{k!}E_{21} = \\
=&\sum_{k\geq 0} \hbar^{-k}(E_{21}-1)^k E_{21}\frac{L^1_1(u-\hbar)\dots L^1_1(u-\hbar k)}{k!} = \\
=&\sum_{k\geq 0} \hbar^{-k}(E_{21}-1)^{k+1}\frac{L^1_1(u-\hbar)\dots L^1_1(u-\hbar k)}{k!} + \\
+&\sum_{k\geq 0} \hbar^{-k}(E_{21}-1)^{k}\frac{L^1_1(u-\hbar)\dots L^1_1(u-\hbar k)}{k!}
\end{aligned}
\tag{4.5}
$$

The constant term (i.e. with no powers of $E_{21}-1$) is 1. Combining the coefficients of $(E_{21}-1)^{k+1}$, we get

$$
\begin{aligned}
&\hbar^{-k}\frac{L^1_1(u-\hbar)\dots L^1_1(u-\hbar k)}{k!}\cdot\left(1+\frac{L^1_1(u-\hbar k-\hbar))}{\hbar(k+1)}\right) = \\
=&\hbar^{-k}\frac{L^1_1(u-\hbar)\dots L^1_1(u-\hbar k)}{k!}\cdot\frac{E_{11}+u}{\hbar(k+1)} = \\
=&\hbar^{-k-1}\frac{L^1_1(u)\dots L^1_1(u-\hbar k)}{(k+1)!}
\end{aligned}
$$

Therefore, equation (4.5) becomes

$$P^{\psi}_{21}(u)E_{21} = 1+\sum_{k\geq 0}\hbar^{-k-1}(E_{21}-1)^{k+1}\frac{L^1_1(u)\dots L^1_1(u-\hbar k)}{(k+1)!} = P^{\psi}_{21}(u).$$

□

Then we proceed to the induction step. Recall the notation ${}_1\mathfrak{m}_N$ from Section 2:

$$
{}_1\mathfrak{m}_N = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & * & \dots & * & 0 \\ \vdots & \ddots & \vdots & \vdots & \\ 0 & * & \dots & * & 0 \end{pmatrix}
\tag{4.6}
$$

and ${}_1\mathfrak{n}^{\psi}_{-} = \mathrm{span}(E_{ij}|i>j\geq 2)$. Denote by ${}_1\vec{u}=(u_2,\dots,u_{N-1})$ the truncated vector of parameters. Assume we proved that the element

$$P_{{}_1\mathfrak{m}_N}({}_1\vec{u}) = \left(P^{\psi}_{N,N-1}(u_{N-1})\right)\dots\left(P^{\psi}_{32}(u_2)\dots P^{\psi}_{N2}(u_2)\right)$$

is invariant under ${}_1\mathfrak{n}^{\psi}_{-}$. Denote by

$$\mathcal{P}_b(\vec{u}) = P_{{}_1\mathfrak{m}_N}({}_1\vec{u})P^{\psi}_{21}\dots P^{\psi}_{b1}(u)$$

the b-truncation of the Kirillov projector for $\mathfrak{m}_N$. Then it would be enough to prove that

$$\mathcal{P}_N(\vec{u})E^{\psi}_{ij} = 0.$$

for all $i>j$.

4.1.2. *Local induction.* The second induction is on the truncation b.

Proposition 4.7. *For any $b\geq i>j$, we have*

$$\mathcal{P}_b(\vec{u})E^{\psi}_{ij} = 0.$$

Again, the base case $b=2$ is Proposition 4.6. For the rest of the subsection, we assume that we proved for it for b. Obviously, $[E_{b+1,1}, E_{ij}] = 0$ for any $b \geq i > j$. By centrality of $L^{1\dots b}_{1\dots b}(u)$, we also have $[L^{1\dots b}_{1\dots b}(u), E_{ij}] = 0$ for the same i, j. Therefore,

$$\mathcal{P}_{b+1}(\vec{u}) E^{\psi}_{ij} = \mathcal{P}_b(\vec{u}) P^{\psi}_{b+1,1}(u) E^{\psi}_{ij} = \mathcal{P}_b(\vec{u}) E^{\psi}_{ij} P^{\psi}_{b+1,1}(u) = 0$$

by the local induction assumption. Hence, it is enough to prove for $i = b+1$ and $j = b$, since other elements are generated by the corresponding commutators.

For the proof, we need the following lemma.

Lemma 4.8. *For any $c \leq d \leq b < a$, we have*

$$\mathcal{P}_d(\vec{u}) E^{\psi}_{ac} = (-1)^{c+1} \mathcal{P}_d(\vec{u}) E_{a1} L^{1\dots c-1}_{1\dots c-1}(u).$$

We will prove it by induction on d. Base is $d = 2$: the case $c = 1$ is trivial, and for $c = 2$, we use

$$[(E^{\psi}_{21})^k, E^{\psi}_{a2}] = -k\hbar E_{a1} (E^{\psi}_{21})^{k-1},$$

so that

$$\begin{aligned}
&P_{1\mathfrak{m}_N}({}_1\vec{u}) P^{\psi}_{21}(u_1) E_{a2} = \\
=&P_{1\mathfrak{m}_N}({}_1\vec{u}) \sum_{k\geq 0} (E^{\psi}_{21})^k \hbar^{-k} \frac{\prod_{i=0}^{k-1} L^1_1(u_1 - i\hbar)}{k!} E_{a2} = \\
=&P_{1\mathfrak{m}_N}({}_1\vec{u}) E^{\psi}_{a2} P^{\psi}_{21}(u_1) - \\
-&P_{1\mathfrak{m}_N}({}_1\vec{u}) \sum_{k\geq 0} (E^{\psi}_{21})^k E_{a1} \hbar^{-k} \frac{\prod_{i=0}^{k} L^1_1(u_1 - i\hbar + \hbar)}{k!} = \\
=&-\mathcal{P}_2(\vec{u}) E_{a1} L^1_1(u),
\end{aligned}$$

where we used $P_{1\mathfrak{m}_N}({}_1\vec{u}) E^{\psi}_{a2} = 0$ by global induction assumption.

Lemma 4.9. *For $c \leq d \leq b < a$ and any $k \geq 0$, we have*

$$\mathcal{P}_d(\vec{u})(L^{a,2,\dots,c}_{1,2,\dots,c}(v) + (-1)^c \psi(E_{ac})) = \mathcal{P}_d(\vec{u}) E_{a1} \frac{L^{1\dots c}_{1\dots c}(v) - L^{1\dots c}_{1\dots c}(u)}{v-u}.$$

In particular,

$$\mathcal{P}_d(\vec{u}) \partial^k_u L^{a,2,\dots,c}_{1\dots c}(u) + (-1)^c \delta_{k,0} \psi(E_{ac}) \mathcal{P}_d(\vec{u}) = \frac{1}{k+1} \mathcal{P}_d(\vec{u}) E_{a1} (\partial^{k+1}_u L^{1\dots c}_{1\dots c}(u)),$$

where by ∂_u we mean the derivative with respect to u.

Proof. By Proposition 3.1,

$$L^{a,2,\dots,c}_{1\dots c}(v) = \sum_{l=1}^{c} (-1)^{l-1} E_{al} L^{2\dots c}_{1\dots\hat{l}\dots c}(v).$$

Since $E_{al} = E^{\psi}_{al} + \psi(E_{al})$ and $\psi(E_{al}) = 0$ except for, possibly, $l = c = a-1$, we can rewrite the sum as

$$\sum_{l=1}^{c-1} (-1)^{l-1} E^{\psi}_{al} L^{2\dots c}_{1\dots\hat{l}\dots c}(v) + (-1)^{c-1} \psi(E_{ac}) L^{2\dots c}_{1\dots c-1}(v).$$

Therefore, by Lemma 4.8,

$$\mathcal{P}_d(\vec{u}) L^{a,2,\dots,c}_{1\dots c}(v) = \mathcal{P}_d(\vec{u}) E_{a1} \sum_{l=1}^{c} L^{1\dots l-1}_{1\dots l-1}(u) L^{2\dots c}_{1\dots\hat{l}\dots c}(v) + (-1)^{c-1} \mathcal{P}_d(\vec{u}) \psi(E_{ac}) L^{2\dots c}_{1\dots c-1}(v).$$

Observe that $[E_{a1}, E_{ij}] = 0$ for any $c \geq i > j$. By Proposition 3.8, the classes of $L^{1\dots l-1}_{1\dots l-1}(u)$ and $L^{1\dots\hat{l}\dots c}_{1\dots c-1}(u)$ in the left quotient by the shift $\mathrm{span}(E_{ij} - \psi(E_{ij}) | c \geq i > j \geq 1)$ are equal for every l; therefore, by Proposition 4.7,

$$\mathcal{P}_d(\vec{u}) E_{a1} L^{1\dots l-1}_{1\dots l-1}(u) = \mathcal{P}_d(\vec{u}) E_{a1} L^{1\dots\hat{l}\dots c}_{1\dots c-1}(u)$$

and the right-hand side is equal to

$$\mathcal{P}_d(\vec{u})E_{a1}\sum_{l=1}^{c} L^{1\ldots\hat{l}\ldots c}_{1\ldots c-1}(u)L^{2\ldots c}_{1\ldots\hat{l}\ldots c}(v)+(-1)^{c-1}\psi(E_{ac})\mathcal{P}_d(\vec{u}).$$

Recall Proposition 3.9

$$\sum_{l=1}^{c} L^{1\ldots\hat{l}\ldots c}_{1\ldots c-1}(u)L^{2\ldots c}_{1\ldots\hat{l}\ldots c}(v)=\frac{L^{2\ldots c}_{1\ldots c-1}(v)L^{1\ldots c}_{1\ldots c}(u)-L^{2\ldots c}_{1\ldots c-1}(u)L^{1\ldots c}_{1\ldots c}(v)}{u-v}$$

Again, by Proposition 3.8, the class $L^{2\cdots c}_{1\ldots c-1}(u)$ in the quotient is just 1, so,

$$\mathcal{P}_d(\vec{u})(L^{a,2,\ldots,c}_{1\ldots c}(v)+(-1)^c\psi(E_{ac}))=\mathcal{P}_d(\vec{u})E_{a1}\frac{L^{1\ldots c}_{1\ldots c}(v)-L^{1\ldots c}_{1\ldots c}(u)}{v-u}.$$

The second part of the lemma follows by setting $v=u+t$ and comparing the Taylor series in t on both sides. $\square$

Proof of Lemma 4.8. First, let us show that it will be enough to prove it for $c=d$. Indeed: let $c<d$, then

$$\begin{aligned}\mathcal{P}_d(\vec{u})E^{\psi}_{ac}&=\hbar^{-1}(\mathcal{P}_d(\vec{u})E^{\psi}_{ad}E^{\psi}_{dc}-\mathcal{P}_d(\vec{u})E^{\psi}_{dc}E^{\psi}_{ad})=\\ =&\hbar^{-1}\mathcal{P}_d(\vec{u})E^{\psi}_{ad}E^{\psi}_{dc}=(-1)^{d+1}\hbar^{-1}\mathcal{P}_d(\vec{u})E_{a1}L^{1\ldots d-1}_{1\ldots d-1}(u)E^{\psi}_{dc}\end{aligned}$$

where we used Proposition 4.7 in the second equality. By Proposition 3.2

$$[L^{1\ldots d-1}_{1\ldots d-1}(u),E^{\psi}_{dc}]=-\hbar L^{1\ldots d\ldots d-1}_{1\ldots d-1}(u)=(-1)^{d-c}\hbar L^{1\ldots\hat{c}\ldots d}_{1\ldots d-1}(u).$$

so,

$$(-1)^{d+1}\hbar^{-1}\mathcal{P}_d(\vec{u})E_{a1}L^{1\ldots d-1}_{1\ldots d-1}(u)E^{\psi}_{dc}=(-1)^{c+1}\mathcal{P}_d(\vec{u})E_{a1}L^{1\ldots\hat{c}\ldots d}_{1\ldots d-1}(u).$$

The statement then follows from Proposition 3.8 as in the proof of Lemma 4.9.

So, let us assume that $c=d$. Then

$$\begin{aligned}&\mathcal{P}_{d-1}(\vec{u})P^{\psi}_{d1}(u)E^{\psi}_{ad}=\\ =&\mathcal{P}_{d-1}(\vec{u})\sum_{k\geq 0}\hbar^{-k}(-1)^{(d+1)k}(-E^{\psi}_{d1})^k\frac{\prod_{i=0}^{k-1}L^{1\ldots d-1}_{1\ldots d-1}(u-i\hbar)}{k!}E^{\psi}_{ad}=\\ =&\mathcal{P}_{d-1}(\vec{u})\left(E^{\psi}_{ad}P^{\psi}_{d1}(u)+E_{a1}\sum_{k\geq 1}\hbar^{-k+1}(-1)^{(d+1)k}(-E^{\psi}_{d1})^{k-1}\frac{\prod_{i=0}^{k-1}L^{1\ldots d-1}_{1\ldots d-1}(u-i\hbar)}{(k-1)!}\right)=\\ =&\mathcal{P}_{d-1}(\vec{u})\left(E^{\psi}_{ad}P^{\psi}_{d1}(u)+(-1)^{d+1}E_{a1}P^{\psi}_{d1}(u-\hbar)L^{1\ldots d-1}_{1\ldots d-1}(u)\right),\end{aligned}$$

where we used

$$[(-E^{\psi}_{d1})^k,E^{\psi}_{ad}]=k\hbar(-E^{\psi}_{d1})^{k-1}E_{a1}.$$

It follows from the definition that $[\mathcal{P}_{d-1}(\vec{u}),E^{\psi}_{ad}]=0$, therefore, the first summand is zero. Hence,

$$\mathcal{P}_d(\vec{u})E^{\psi}_{ad}=(-1)^{d+1}\mathcal{P}_{d-1}(\vec{u})E_{a1}P^{\psi}_{d1}(u-\hbar)L^{1\ldots d-1}_{1\ldots d-1}(u).$$

Therefore, the statement of the lemma would follow from

$$\mathcal{P}_{d-1}(\vec{u})P^{\psi}_{d1}(u)E_{a1}=\mathcal{P}_{d-1}(\vec{u})E_{a1}P^{\psi}_{d1}(u-\hbar). \tag{4.7}$$

Let us commute E_{a1} past $P^{\psi}_{d1}(u)$. We need three formulas:

- It follows from Proposition 3.2 that

$$[L^{1\ldots d-1}_{1\ldots d-1}(u),E_{a1}]=-\hbar L^{a,2,\ldots,d-1}_{1\ldots d-1}(u).$$

- From Proposition 3.7 we get

$$L^{1\ldots d-1}_{1\ldots d-1}(v)L^{a,2,\ldots,d-1}_{1\ldots d-1}(v-\hbar)=L^{a,2,\ldots d-1}_{1\ldots d-1}(v)L^{1\ldots d-1}_{1\ldots d-1}(v-\hbar).$$

- By Proposition 3.2
$$[L^{a,2,\dots d-1}_{1\dots d-1}(u), E_{a1}] = 0.$$

Using them, one can show that

$$[\prod_{i=0}^{k-1} L^{1\dots d-1}_{1\dots d-1}(u-\hbar i), E_{a1}] = -k\hbar L^{a,2,\dots,d-1}_{1\dots d-1}(u) \prod_{i=1}^{k-1} L^{1\dots d-1}_{1\dots d-1}(u-\hbar i)$$

Then one uses the induction on d. By Lemma 4.9, we have

$$\begin{aligned}
&\mathcal{P}_{d-1}(\vec{u}) P^{\psi}_{d1}(u) E_{a1} = \\
=&\mathcal{P}_{d-1}(\vec{u}) \sum_{k\geq 0} \hbar^{-k} (-1)^{(d+1)k} (-E^{\psi}_{d1})^k \frac{\prod L^{1\dots d-1}_{1\dots d-1}(u-i\hbar)}{k!} E_{a1} = \\
=&\mathcal{P}_{d-1}(\vec{u}) E_{a1} \sum_{k\geq 0} (-1)^{(d+1)k} \hbar^{-k} (-E^{\psi}_{d1})^k \frac{\prod_{i=0}^{k-1} L^{1\dots d-1}_{1\dots d-1}(u-i\hbar)}{k!} - \\
-&\mathcal{P}_{d-1}(\vec{u}) L^{a,2,\dots,d-1}_{1\dots d-1}(u) \sum_{k\geq 1} \hbar^{-k+1} (-1)^{(d+1)k} (-E^{\psi}_{d1})^k \frac{\prod_{i=1}^{k-1} L^{1\dots d-1}_{1\dots d-1}(u-i\hbar)}{(k-1)!} = \\
=&\mathcal{P}_{d-1}(\vec{u}) E_{a1} \sum_{k\geq 0} (-1)^{(d+1)k} \hbar^{-k} (-E^{\psi}_{d1})^k \frac{\prod_{i=0}^{k-1} L^{1\dots d-1}_{1\dots d-1}(u-i\hbar)}{k!} - \\
-&\mathcal{P}_{d-1}(\vec{u}) E_{a1} \partial_u L^{1\dots d-1}_{1\dots d-1}(u) \sum_{k\geq 1} \hbar^{-k+1} (-1)^{(d+1)k} (-E^{\psi}_{d1})^k \frac{\prod_{i=1}^{k-1} L^{1\dots d-1}_{1\dots d-1}(u-i\hbar)}{(k-1)!}.
\end{aligned}$$

Recall (4.7). By term-by-term comparison, the lemma would follow from

$$\begin{aligned}
&\mathcal{P}_{d-1}(\vec{u}) \left[E_{a1} (-E^{\psi}_{d1})^k \frac{L^{1\dots d-1}_{1\dots d-1}(u)}{k} - \hbar E_{a1} \partial_u L^{1\dots d-1}_{1\dots d-1}(u) (-E^{\psi}_{d1})^k \right] = \\
=&\mathcal{P}_{d-1}(\vec{u}) E_{a1} (-E^{\psi}_{d1})^k \frac{L^{1\dots d-1}_{1\dots d-1}(u-k\hbar)}{k}.
\end{aligned} \tag{4.8}$$

For $d=2$, it is obviously satisfied. Assume $d>2$. Then we can identify $E^{\psi}_{d1} = E_{d1}$. Let us study the term $\hbar E_{a1} \partial_u L^{1\dots d-1}_{1\dots d-1}(u) (-E^{\psi}_{d1})^k$. Taking derivative of Proposition 3.2, we obtain

$$[\partial_u L^{1\dots d-1}_{1\dots d-1}(u), (-E^{\psi}_{d1})^k] = \hbar k \partial_u L^{d,2,\dots,d-1}_{1,2,\dots,d-1}(u) (-E^{\psi}_{d1})^{k-1}.$$

Hence, by Lemma 4.9, we get

$$\begin{aligned}
&\hbar \mathcal{P}_{d-1}(\vec{u}) E_{a1} \partial_u L^{1\dots d-1}_{1\dots d-1}(u) (-E^{\psi}_{d1})^k = \\
=&\hbar \mathcal{P}_{d-1}(\vec{u}) \left(E_{a1} (-E^{\psi}_{d1})^k \partial_u L^{1\dots d-1}_{1\dots d-1}(u) + k\hbar^2 E_{a1} \partial_u L^{d,2,\dots,d-1}_{1,2,\dots,d-1}(u) (-E^{\psi}_{d1})^{k-1} \right) = \\
=&\hbar \mathcal{P}_{d-1}(\vec{u}) \left(E_{a1} (-E^{\psi}_{d1})^k \partial_u L^{1\dots d-1}_{1\dots d-1}(u) + k\hbar^2 \partial_u L^{d,2,\dots,d-1}_{1,2,\dots,d-1}(u) E_{a1} (-E^{\psi}_{d1})^{k-1} \right) = \\
=&\hbar \mathcal{P}_{d-1}(\vec{u}) \left(E_{a1} (-E^{\psi}_{d1})^k \partial_u L^{1\dots d-1}_{1\dots d-1}(u) - \frac{k\hbar^2}{2} (-E^{\psi}_{d1}) \partial_u^2 L^{1\dots d-1}_{1\dots d-1}(u) E_{a1} (-E^{\psi}_{d1})^{k-1} \right)
\end{aligned} \tag{4.9}$$

We will need the following technical statement.

Lemma 4.10. *For any l, we have*

$$\begin{aligned}
&\mathcal{P}_{d-1}(\vec{u}) (-E^{\psi}_{d1}) \partial_u^l L^{1\dots d-1}_{1\dots d-1}(u) (-E_{a1}) (-E^{\psi}_{d1})^{k-1} = \\
=&\mathcal{P}_{d-1}(\vec{u}) \left((-E_{a1}) (-E^{\psi}_{d1})^k \partial_u^l L^{1\dots d-1}_{1\dots d-1}(u) - k\hbar (-E^{\psi}_{d1}) \frac{\partial_u^{l+1} L^{1\dots d-1}_{1\dots d-1}(u)}{l+1} (-E_{a1}) (-E^{\psi}_{d1})^{k-1} \right).
\end{aligned}$$

Proof. Here is a series of equalities where we repeatedly use Lemma 4.9:

$$\begin{aligned}
&\mathcal{P}_{d-1}(\vec{u})(-E^{\psi}_{d1})\partial^l_u L^{1\dots d-1}_{1\dots d-1}(u)(-E_{a1})(-E^{\psi}_{d1})^{k-1} = \\
=&\mathcal{P}_{d-1}(\vec{u})(-E^{\psi}_{d1})(-E_{a1})\partial^l_u L^{1\dots d-1}_{1\dots d-1}(u)(-E^{\psi}_{d1})^{k-1} + \hbar\partial^l_u L^{a,2,\dots,d-1}_{1\dots,d-1}(u)(-E^{\psi}_{d1})^{k} = \\
=&\mathcal{P}_{d-1}(\vec{u})(-E_{a1})(-E^{\psi}_{d1})^{k}\partial^l_u L^{1\dots d-1}_{1\dots d-1}(u) + (k-1)\hbar\partial^l_u L^{d,2,\dots,d-1}_{1,2\dots,d-1}(u)(-E_{a1})(-E^{\psi}_{d1})^{k-1} - \\
&-\mathcal{P}_{d-1}(\vec{u})\hbar(-E_{a1})\frac{\partial^{l+1}_u L^{1\dots d-1}_{1\dots d-1}(u)}{l+1}(-E^{\psi}_{d1})^{k} = \\
=&\mathcal{P}_{d-1}(\vec{u})(-E_{a1})(-E^{\psi}_{d1})^{k}\partial^l_u L^{1\dots d-1}_{1\dots d-1}(u) - (k-1)\hbar(-E^{\psi}_{d1})\frac{\partial^{l+1}_u L^{1\dots d-1}_{1\dots d-1}(u)}{l+1}(-E_{a1})(-E^{\psi}_{d1})^{k-1} - \\
&-\mathcal{P}_{d-1}(\vec{u})\hbar(-E_{a1})\frac{\partial^{l+1}_u L^{1\dots d-1}_{1\dots d-1}(u)}{l+1}(-E^{\psi}_{d1})^{k}.
\end{aligned}$$

Therefore, the claim would follow from

$$\mathcal{P}_{d-1}(\vec{u})(-E_{a1})\partial^{l+1}_u L^{1\dots d-1}_{1\dots d-1}(u)(-E^{\psi}_{d1}) = \mathcal{P}_{d-1}(\vec{u})(-E^{\psi}_{d1})\partial^{l+1}_u L^{1\dots d-1}_{1\dots d-1}(u)(-E_{a1}).$$

It can be proven be descending induction on l. Indeed, for l large, the corresponding derivative is just zero and the equation is trivial. Assume it holds for the k-th derivative. Then, on the one hand,

$$\begin{aligned}
&\mathcal{P}_{d-1}(\vec{u})(-E_{a1})\partial^{k-1}_u L^{1\dots d-1}_{1\dots d-1}(u)(-E^{\psi}_{d1}) = \\
=&\mathcal{P}_{d-1}(\vec{u})\left((-E_{a1})(-E^{\psi}_{d1})\partial^{k-1}_u L^{1\dots d-1}_{1\dots d-1}(u) + \hbar\partial^{k-1}_u L^{d,2,\dots,d-1}_{1,2,\dots d-1}(u)(-E_{a1})\right) = \\
=&\mathcal{P}_{d-1}(\vec{u})\left((-E_{a1})(-E^{\psi}_{d1})\partial^{k-1}_u L^{1\dots d-1}_{1\dots d-1}(u) + \hbar(-E^{\psi}_{d1})\frac{\partial^{k}_u L^{1\dots d-1}_{1\dots d-1}(u)}{k}(-E_{a1})\right),
\end{aligned}$$

on the other hand,

$$\begin{aligned}
&\mathcal{P}_{d-1}(\vec{u})(-E^{\psi}_{d1})\partial^{k-1}_u L^{1\dots d-1}_{1\dots d-1}(u)(-E_{a1}) = \\
=&\mathcal{P}_{d-1}(\vec{u})\left((-E^{\psi}_{d1})(-E_{a1})\partial^{k-1}_u L^{1\dots d-1}_{1\dots d-1}(u) + \hbar\partial^{k-1}_u L^{a,2,\dots,d-1}_{1,2,\dots d-1}(u)(-E^{\psi}_{d1})\right) = \\
=&\mathcal{P}_{d-1}(\vec{u})\left((-E_{a1})(-E^{\psi}_{d1})\partial^{k-1}_u L^{1\dots d-1}_{1\dots d-1}(u) + \hbar(-E_{a1})\frac{\partial^{k}_u L^{1\dots d-1}_{1\dots d-1}(u)}{k}(-E^{\psi}_{d1})\right),
\end{aligned}$$

and we can apply induction. □

By repeated application of the lemma, one can see that (4.9) is equal to

$$-\frac{1}{k}\mathcal{P}_{d-1}(\vec{u})E_{a1}(-E^{\psi}_{d1})^{k}\sum_{l=1}^{+\infty}\frac{(-\hbar k)^l}{l!}\partial^l_u L^{1\dots k-1}_{1\dots k-1}(u).$$

However, this is nothing but

$$\frac{1}{k}\mathcal{P}_{d-1}(\vec{u})E_{a1}(-E^{\psi}_{d1})^{k}(L^{1\dots d-1}_{1\dots d-1}(u) - L^{1\dots d-1}_{1\dots d-1}(u-k\hbar))$$

Substituting it back into (4.8), we conclude.

□

Finally, we are ready to prove Proposition 4.7.

Proof of Proposition 4.7. As it was mentioned in the discussion after the proposition statement, it is enough to show

$$\mathcal{P}_{b+1}(\vec{u})E^{\psi}_{b+1,b} = 0.$$

As usual, let us commute $E^{\psi}_{b+1,b}$ past $P^{\psi}_{b+1,1}(u)$. We need three formulas:

- It follows from Proposition 3.2 that

$$[L^{1\dots b}_{1\dots b}(u), E^{\psi}_{b+1,b}] = -\hbar L^{1,\dots,b-1,b+1}_{1,\dots,b-1,b}(u).$$

- From Proposition 3.6 we get

$$L^{1\ldots b}_{1\ldots b}(v)L^{1,\ldots,b-1,b+1}_{1,\ldots,b-1,b}(v-\hbar) = L^{1,\ldots,b-1,b+1}_{1,\ldots,b-1,b}(v)L^{1\ldots b}_{1\ldots b}(v-\hbar).$$

- By Proposition 3.2

$$[L^{1,\ldots,b-1,b+1}_{1,\ldots,b-1,b}(u), E^{\psi}_{b+1,b}] = 0.$$

It implies that

$$\left[\prod_{i=0}^{k-1} L^{1\ldots b}_{1\ldots b}(u-i\hbar), E^{\psi}_{b+1,b}\right] = -k\hbar L^{1,\ldots,b-1,b+1}_{1,\ldots,b-1,b}(u)\prod_{i=1}^{k-1} L^{1\ldots b}_{1\ldots b}(u-i\hbar).$$

Hence,

$$\begin{aligned}
&\mathcal{P}_b(\vec{u})P^{\psi}_{b+1,1}(u)E^{\psi}_{b+1,b} = \\
=&\mathcal{P}_b(\vec{u})\sum_{k\geq 0}(-E^{\psi}_{b+1,1})^k(-1)^{(b+2)k}\hbar^{-k}\frac{\prod_{i=0}^{k-1} L^{1\ldots b}_{1\ldots b}(u-i\hbar)}{k!}E^{\psi}_{b+1,b} = \\
=&\mathcal{P}_b(\vec{u})E^{\psi}_{b+1,b}P^{\psi}_{b+1,1}(u) - \\
-&\mathcal{P}_b(\vec{u})\sum_{k\geq 1}(-E^{\psi}_{b+1,1})^k(-1)^{(b+2)k}\hbar^{-k+1}L^{1,\ldots,b-1,b+1}_{1,\ldots,b-1,b}(u)\frac{\prod_{i=1}^{k-1} L^{1\ldots b}_{1\ldots b}(u-i\hbar)}{(k-1)!}.
\end{aligned}$$

Denote

$$\begin{aligned}
&A := \mathcal{P}_b(\vec{u})E^{\psi}_{b+1,b}P^{\psi}_{b+1,1}(u), \\
&B := -\mathcal{P}_b(\vec{u})\sum_{k\geq 1}(-E^{\psi}_{b+1,1})^k(-1)^{(b+2)k}\hbar^{-k+1}L^{1,\ldots,b-1,b+1}_{1,\ldots,b-1,b}(u)\frac{\prod_{i=1}^{k-1} L^{1\ldots b}_{1\ldots b}(u-i\hbar)}{(k-1)!}
\end{aligned} \tag{4.10}$$

Then, naturally, we need to prove that

$$A+B=0.$$

Let us study the A-term first. By Lemma 4.8

$$\mathcal{P}_b(\vec{u})E^{\psi}_{b+1,b} = (-1)^{b+1}\mathcal{P}_b(\vec{u})E_{b+1,1}L^{1\ldots b-1}_{1\ldots b-1}(u).$$

Since we assume $b>2$, we can identify $E^{\psi}_{b+1,1} = E_{b+1,1}$. Therefore,

$$\begin{aligned}
&\mathcal{P}_b(\vec{u})E^{\psi}_{b+1,b}P^{\psi}_{b+1,1}(u) = \mathcal{P}_b(\vec{u})E_{b+1,1}L^{1\ldots b-1}_{1\ldots b-1}(u) = \\
=&\mathcal{P}_b(\vec{u})(-1)^{b+1}E_{b+1,1}\sum_{k\geq 0}L^{1\ldots b-1}_{1\ldots b-1}(u)(-E_{b+1,1})^k(-1)^{(b+2)k}\hbar^{-k}\frac{\prod_{i=0}^{k-1} L^{1\ldots b}_{1\ldots b}(u-i\hbar)}{k!}.
\end{aligned}$$

Let us look at the term

$$\mathcal{P}_b(\vec{u})(-E_{b+1,1})L^{1\ldots b-1}_{1\ldots b-1}(u)(-E_{b+1,1})^k.$$

Similarly to the proof of Lemma 4.8, we need the following technical statement.

Lemma 4.11. *For any $l \leq b-1$, we have*

$$\mathcal{P}_b(\vec{u})(-E_{b+1,1})L^{1\ldots l}_{1\ldots l}(u)(-E_{b+1,1})^k = \mathcal{P}_b(\vec{u})(-E_{b+1,1})^{k+1}L^{1\ldots l}_{1\ldots l}(u-k\hbar).$$

Proof. Indeed: by Lemma 4.8,

$$\begin{aligned}
&\mathcal{P}_b(\vec{u})(-E_{b+1,1})L^{1\ldots l}_{1\ldots l}(u)(-E_{b+1,1})^k = \\
=&\mathcal{P}_b(\vec{u})(-E_{b+1,1})^{k+1}L^{1\ldots l}_{1\ldots l}(u) + k\hbar L^{b+1,2,\ldots,l}_{1,2,\ldots,l}(u)(-E_{b+1,1})^k = \\
=&\mathcal{P}_b(\vec{u})(-E_{b+1,1})^{k+1}L^{1\ldots l}_{1\ldots l}(u) - k\hbar(-E_{b+1,1})\partial_u L^{1\ldots l}_{1\ldots l}(u)(-E_{b+1,1})^k.
\end{aligned}$$

Then we continue the process by pushing the derivative to the right. For instance:

$$\begin{aligned}
&\mathcal{P}_b(\vec{u})(-E_{b+1,1})^{k+1}L^{1\ldots l}_{1\ldots l}(u) - k\hbar(-E_{b+1,1})\partial_u L^{1\ldots l}_{1\ldots l}(u)(-E_{b+1,1})^k = \\
=&\mathcal{P}_b(\vec{u})(-E_{b+1,1})^{k+1}L^{1\ldots l}_{1\ldots l}(u) - k\hbar(-E_{b+1,1})^{k+1}\partial_u L^{1\ldots l}_{1\ldots l}(u) - \\
-&\mathcal{P}_b(\vec{u})(k\hbar)^2\partial_u L^{b+1,2,\ldots,l}_{1,2,\ldots,l}(u)(-E_{b+1,1})^k = \\
=&\mathcal{P}_b(\vec{u})(-E_{b+1,1})^{k+1}L^{1\ldots l}_{1\ldots l}(u) - (-E_{b+1,1})^{k+1}k\hbar\partial_u L^{1\ldots l}_{1\ldots l}(u) + \\
+&\mathcal{P}_b(\vec{u})(k\hbar)^2(-E_{b+1,1})\frac{\partial_u^2 L^{1\ldots l}_{1\ldots l}(u)}{2}(-E_{b+1,1})^k
\end{aligned}$$

In the end, we arrive at

$$\begin{aligned}
&\mathcal{P}_b(\vec{u})(-E_{b+1,1})L^{1\ldots l}_{1\ldots l}(u)(-E_{b+1,1})^k = \\
=&\mathcal{P}_b(\vec{u})(-E_{b+1,1})^{k+1}\sum_{j=0}^{+\infty}\frac{(-k\hbar)^j}{j!}\partial_u^j L^{1\ldots l}_{1\ldots l}(u) = \\
=&\mathcal{P}_b(\vec{u})(-E_{b+1,1})^{k+1}L^{1\ldots l}_{1\ldots l}(u-k\hbar),
\end{aligned}$$

where the sum is finite. □

Therefore,

$$\begin{aligned}
A =&\mathcal{P}_b(\vec{u})E^{\psi}_{b+1,b}P^{\psi}_{b+1,1}(u) = \\
=&\mathcal{P}_b(\vec{u})\sum_{k\geq 0}(-E_{b+1,1})^{k+1}(-1)^{(b+2)(k+1)}\hbar^{-k}L^{1\ldots b-1}_{1\ldots b-1}(u-k\hbar)\frac{\prod_{i=0}^{k-1}L^{1\ldots b}_{1\ldots b}(u-i\hbar)}{k!}.
\end{aligned} \tag{4.11}$$

Now let us study B of (4.10). Consider the term

$$\mathcal{P}_b(\vec{u})(-E_{b+1,1})^k L^{1,\ldots,b-1,b+1}_{1,\ldots,b-1,b}(u).$$

By Proposition 3.1, we have

$$L^{1,\ldots,b-1,b+1}_{1,\ldots,b-1,b}(u) = (-1)^{b+1}\sum_{l=1}^{b}(-1)^{l-1}E_{b+1,l}L^{1\ldots b-1}_{1\ldots\hat{l}\ldots b}(u).$$

Recall that $E_{b+1,,b} = E^{\psi}_{b+1,b}+1$. By Lemma 4.8 and Lemma 4.11, we get

$$\begin{aligned}
&\mathcal{P}_b(\vec{u})(-E_{b+1,1})^k L^{1,\ldots,b-1,b+1}_{1,\ldots,b-1,b}(u) = \\
=&\mathcal{P}_b(\vec{u})\left(\sum_{l=1}^{b}E^{\psi}_{b+1,l}(-1)^{b+l}(-E_{b+1,1})^k L^{1\ldots b-1}_{1\ldots\hat{l}\ldots b}(u)\right) + (-E_{b+1,1})^k L^{1\ldots b-1}_{1\ldots b-1}(u) = \\
=&\mathcal{P}_b(\vec{u})(-1)^{b+2}(-E_{b+1,1})\left(\sum_{l=1}^{b}L^{1\ldots l-1}_{1\ldots l-1}(u)(-E_{b+1,1})^k L^{1\ldots b-1}_{1\ldots\hat{l}\ldots b}(u)\right) + (-E_{b+1,1})^k L^{1\ldots b-1}_{1\ldots b-1}(u) = \\
=&\mathcal{P}_b(\vec{u})(-1)^{b+2}(-E_{b+1,1})^{k+1}\left(\sum_{l=1}^{b}L^{1\ldots l-1}_{1\ldots l-1}(u-k\hbar)L^{1\ldots b-1}_{1\ldots\hat{l}\ldots b}(u)\right) + (-E_{b+1,1})^k L^{1\ldots b-1}_{1\ldots b-1}(u).
\end{aligned}$$

By Proposition 3.8 and Proposition 3.9, we have

$$\begin{aligned}
&\mathcal{P}_b(\vec{u})(-E_{b+1,1})^{k+1}\sum_{l=1}^{b}L^{1\ldots l-1}_{1\ldots l-1}(u-k\hbar)L^{1\ldots b-1}_{1\ldots\hat{l}\ldots b}(u) = \\
=&\mathcal{P}_b(\vec{u})(-E_{b+1,1})^{k+1}\frac{L^{1\ldots b-1}_{1\ldots b-1}(u-k\hbar)L^{1\ldots b}_{1\ldots b}(u) - L^{1\ldots b}_{1\ldots b}(u-k\hbar)L^{1\ldots b-1}_{1\ldots b-1}(u)}{k\hbar}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
B = &-\mathcal{P}_b(\vec{u})\sum_{k\geq 1}(-E_{b+1,1})^k(-1)^{(b+2)k}\hbar^{-k+1}L^{1,\ldots,b-1,b+1}_{1,\ldots,b-1,b}(u)\frac{\prod_{i=1}^{k-1}L^{1\ldots b}_{1\ldots b}(u-i\hbar)}{(k-1)!} = \\
&= -\mathcal{P}_b(\vec{u})\sum_{k\geq 1}(-E_{b+1,1})^{k+1}(-1)^{(b+2)(k+1)}\frac{\hbar^{-k}}{k!}L^{1\ldots b-1}_{1\ldots b-1}(u-k\hbar)\prod_{i=0}^{k-1}L^{1\ldots b}_{1\ldots b}(u-i\hbar)- \\
&+\mathcal{P}_b(\vec{u})\sum_{k\geq 1}(-E_{b+1,1})^{k+1}(-1)^{(b+2)(k+1)}\frac{\hbar^{-k}}{k!}L^{1\ldots b-1}_{1\ldots b-1}(u)\prod_{i=1}^{k}L^{1\ldots b}_{1\ldots b}(u-i\hbar)+ \\
&-\mathcal{P}_b(\vec{u})\sum_{k\geq 1}(-E_{b+1,1})^{k}(-1)^{(b+2)k}\frac{\hbar^{-(k-1)}}{(k-1)!}L^{1\ldots b-1}_{1\ldots b-1}(u)\prod_{i=1}^{k-1}L^{1\ldots b}_{1\ldots b}(u-i\hbar).
\end{aligned}
$$

By comparing the second and the third rows, we see that their sum is $\mathcal{P}_b(\vec{u})(-E_{b+1,1})$. Likewise, recall (4.11) that

$$A = -\mathcal{P}_b(\vec{u})\sum_{k\geq 0}(-E_{b+1,1})^{k+1}(-1)^{(b+2)(k+1)}\hbar^{-k}L^{1\ldots b-1}_{1\ldots b-1}(u-k\hbar)\frac{\prod_{i=0}^{k-1}L^{1\ldots b}_{1\ldots b}(u-i\hbar)}{k!}.$$

In particular, the sum of A and the first row of B is equal to $-\mathcal{P}_b(\vec{u})(-E_{b+1,1})$. The proposition follows. ☐

Remark 4.12. It follows from the proof that instead of $P^{\psi}_{ij}(u_j)$, one can consider its image under the quotient map. Namely, denote by

$$[L^{j\ldots i-1}_{j\ldots i-1}(u_j)] \in \mathfrak{n}^{\psi}_{-}\backslash \mathrm{U}_{\hbar}(\mathfrak{m}_j) \cong \mathrm{U}_{\hbar}(\mathfrak{b}_j)$$

where $\mathfrak{b}_j \subset \mathfrak{m}_j$ is the positive Borel subalgebra. The image of $L^{j\ldots i-1}_{j\ldots i-1}(u_j)$ in $\mathrm{U}_{\hbar}(\mathfrak{b}_j)$. Consider an analog of (4.1):

$$[P^{\psi}_{ij}(u)] := \sum_{k\geq 0}(-1)^{(i+j)k}\hbar^{-k}(-E^{\psi}_{ij})^k\frac{[L^{j\ldots i-1}_{j\ldots i-1}(u)]\ldots[L^{j\ldots i-1}_{j\ldots i-1}(u-\hbar k+\hbar)]}{k!}$$

and $[P_{\mathfrak{m}_N}(\vec{u})]$ given by a similar formula as (4.2). It follows that it defines the same element as $P_{\mathfrak{m}_N}(\vec{u})$. Indeed: as usual, one can show it by two-fold induction. Consider the element

$$P_{{}_1\mathfrak{m}_N}({}_1\vec{u})P^{\psi}_{21}(u_1)\ldots P^{\psi}_{b1}(u_1).$$

We know that it is invariant under the right action of E^{ψ}_{cd} for $1 \leq d < c \leq b$. Assume we proved it for b. To show the statement for $b+1$, observe that

$$P^{\psi}_{b+1,1}(u) := \sum_{k\geq 0}(-1)^{(b+2)k}\hbar^{-k}(-E^{\psi}_{b+1,1})^k\frac{L^{1\ldots b}_{1\ldots b}(u_1)\ldots L^{1\ldots b}_{1\ldots b}(u_1-\hbar(k-1))}{k!}$$

Since $L^{1\ldots b}_{1\ldots b}(v)$ is central in $\mathfrak{gl}_b$ and $E^{\psi}_{b+1,1}$ commutes with all E^{ψ}_{cd} as above, the statement follows.

In particular, it should be possible to express the Kirillov projector using the generators from [BK06].

4.2. **Kirillov projector: left $\mathfrak{b}^{\vec{u}}$-invariance.** In this subsection, we will show the second property from Theorem 4.3.

Theorem 4.13. *The Kirillov projector satisfies*

$$
\begin{aligned}
(E_{ii}+u_i)P^{\psi}_{\mathfrak{m}_N}(\vec{u}) = 0,\ 1\leq i\leq N-1, \\
E_{ij}P^{\psi}_{\mathfrak{m}_N}(\vec{u}) = 0,\ 1\leq i<j\leq N-1.
\end{aligned}
\tag{4.12}
$$

We will prove this statement by global induction. The base case is $N=2$.

Proposition 4.14. *The Kirillov projector $P^{\psi}_{\mathfrak{m}_2}(u_1)$ satisfies*

$$(E_{11}+u)P^{\psi}_{21}(u) = 0.$$

Proof. First, observe that

$$E_{21} P^{\psi}_{21}(u-\hbar) = P_{21}(u). \tag{4.13}$$

Indeed, we have

$$\begin{aligned} P^{\psi}_{21}(u) E_{21} =& \sum_{k\geq 0} (E^{\psi}_{21})^k \hbar^{-k} \frac{L^1_1(u)\dots L^1_1(u-\hbar k+\hbar)}{k!} E_{21} = \\ =& E_{21} \sum_{k\geq 0} (E^{\psi}_{21})^k \hbar^{-k} \frac{L^1_1(u-\hbar)\dots L^1_1(u-\hbar k)}{k!} = \\ =& E_{21} P^{\psi}_{21}(u-\hbar). \end{aligned}$$

But by Theorem 4.13, we have $P^{\psi}_{21}(u) E_{21} = P^{\psi}_{21}(u)$.

Then, we will use the following commutation relation:

$$[E_{11}, (E^{\psi}_{21})^k] = -k\hbar E_{21} (E^{\psi}_{21})^{k-1}.$$

Therefore,

$$\begin{aligned} (E_{11}+u) P^{\psi}_{21} =& E_{11} \sum (E^{\psi}_{21})^k \hbar^{-k} \frac{L^1_1(u)\dots L^1_1(u-\hbar k+\hbar)}{k!} = \\ =& \sum (E^{\psi}_{21})^k \hbar^{-k} (E_{11}+u) \frac{L^1_1(u)\dots L^1_1(u-\hbar k+\hbar)}{k!} - \\ -& E_{21} \sum (E^{\psi}_{21})^{k-1} \hbar^{1-k} \frac{L^1_1(u)\dots L^1_1(u-\hbar k+\hbar)}{(k-1)!} = \\ =& P^{\psi}_{21}(u) L^1_1(u) - E_{21} P^{\psi}_{21}(u-\hbar) L^1_1(u) = 0 \end{aligned}$$

by (4.13). □

Now we make an induction step. Recall the notation ${}_1\mathfrak{m}_N$ from (4.6). Assume we proved Theorem 4.13 for $P^{\psi}_{{}_1\mathfrak{m}_N}({}_1\vec{u})$. Observe that

$$P^{\psi}_{\mathfrak{m}_N}(\vec{u}) = P^{\psi}_{{}_1\mathfrak{m}_N}({}_1\vec{u}) P^{\psi}_{21}(u_1)\dots P^{\psi}_{N1}(u_1).$$

Then the statement is reduced to

$$\begin{aligned} (E_{11}+u_1) P^{\psi}_{\mathfrak{m}_N}(\vec{u}) = 0, \\ E_{1b} P^{\psi}_{\mathfrak{m}_N}(\vec{u}) = 0,\ 1<k\leq N-1. \end{aligned}$$

The first property follows from Proposition 4.14, as E_{11} commutes with $P^{\psi}_{{}_1\mathfrak{m}_N}({}_1\vec{u})$. For the second property, let us show first an a priori weaker statement.

Proposition 4.15. *We have*

$$P^{\psi}_{{}_1\mathfrak{m}_N}({}_1\vec{u}) E_{1b} P^{\psi}_{\mathfrak{m}_N}(\vec{u}) = 0.$$

In other words, the equality $E_{1b} P^{\psi}_{\mathfrak{m}_N}(\vec{u}) = 0$ holds in the left quotient by ${}_1\mathfrak{n}^{\psi}_{-}$.

Proof. By global induction and by Corollary 4.19, we have

$$E_{1b} P_{\mathfrak{m}_b}(u_1, u_2, \dots, u_{b-1}) = P_{\mathfrak{m}_b}(u_1-\hbar, u_2, \dots, u_{b-1}) L^{1\dots b}_{1\dots b}(u_1).$$

We claim that nothing changes in the quotient when we pass to a larger algebra, namely,

$$\begin{aligned} & P^{\psi}_{{}_1\mathfrak{m}_N}({}_1\vec{u}) E_{1b} P^{\psi}_{{}_1\mathfrak{m}_N}({}_1\vec{u}) P^{\psi}_{21}(u_1)\dots P^{\psi}_{b1}(u_1) = \\ =& P^{\psi}_{{}_1\mathfrak{m}_N}({}_1\vec{u}) P^{\psi}_{21}(u_1-\hbar)\dots P^{\psi}_{b1}(u_1-\hbar) L^{1\dots b}_{1\dots b}(u_1). \end{aligned}$$

It follows from the following technical, but in fact simple lemma.

Lemma 4.16. *Assume that $x = \sum_{i=0}^{l-1} E_{1,b-i}\alpha_i$ with $\alpha_i \in \mathrm{U}_\hbar({}_{b-1-i}\mathfrak{gl}_{b-1})$. Then*

$$P^{\psi}_{{}_1\mathfrak{m}_N}({}_1\vec{u})xP^{\psi}_{b-l+1,b-l}(u_{b-l})\dots P_{b,b-l}(u_{b-l}) \in \sum_{i=0}^{l} E_{1,b-i}\cdot({}_{b-1-i}\mathfrak{gl}_{b-1})$$

and

$$P^{\psi}_{{}_1\mathfrak{m}_N}({}_1\vec{u})xP^{\psi}_{b+1,b-l+1}(u_{b-l+1})\dots P^{\psi}_{N,b-l+1}(u_{b-l+1}) = P^{\psi}_{{}_1\mathfrak{m}_N}({}_1\vec{u})x.$$

Proof. Let us only show the second property. The condition $\alpha_i \in ({}_{b-1-i}\mathfrak{gl}_{b-1})$ means that α_i is the sum of products of elements of the form $E_{\alpha\beta}$ with $b-l+1 \le \alpha,\beta \le b-1$. It is clear that for any such α,β, the commutator $[E_{j,b-l+1}, E_{\alpha\beta}]$ for $j > b$ will be a nilpotent element of the form E_{jc} with $b-l+1 \le c \le b-1$, in particular, the value of ψ on it is zero. Therefore, we have

$$P^{\psi}_{{}_1\mathfrak{m}_N}({}_1\vec{u})\mathrm{ad}_{E_{j,b-l+1}}(x) = 0.$$

Since

$$P^{\psi}_{{}_1\mathfrak{m}_N}({}_1\vec{u})x\xi^{\psi} = P^{\psi}_{{}_1\mathfrak{m}_N}({}_1\vec{u})\mathrm{ad}_{\xi}(x)$$

for any ξ in the negative nilpotent subalgebra of ${}_1\mathfrak{m}_N$, we conclude that every $P^{\psi}_{j,b-l+1}(u_{b-l+1})$ for $j > b$ acts on $P^{\psi}_{{}_1\mathfrak{m}_N}({}_1\vec{u})x$ by identity. Hence,

$$P^{\psi}_{{}_1\mathfrak{m}_N}({}_1\vec{u})xP^{\psi}_{b+1,b-l+1}(u_{b-l+1})\dots P^{\psi}_{N,b-l+1}(u_{b-l+1})(u_{b-l+1}) = P^{\psi}_{{}_1\mathfrak{m}_N}({}_1\vec{u})x.$$

The first property follows by similar analysis of commutators and is left to the reader. $\square$

Therefore, the claim of the proposition would follow from

$$P^{\psi}_{{}_1\mathfrak{m}_N}({}_1\vec{u})P^{\psi}_{21}(u_1-\hbar)\dots P^{\psi}_{b1}(u_1-\hbar)L^{1\dots b}_{1\dots b}(u_1)P^{\psi}_{b+1,1}(u_1) = 0.$$

For brevity, denote by $\vec{u}-\hbar := (u_1-\hbar, u_2, \dots, u_{N-1})$ and $u = u_1$. Adopting notations of Section 4.1, it can be reformulated as

$$\mathcal{P}_b(\vec{u}-\hbar)L^{1\dots b}_{1\dots b}(u)P^{\psi}_{b+1,1}(u) = 0.$$

Observe that $b \ge 2$, in particular, $E_{b+1,1} = E^{\psi}_{b+1,1}$. We need two facts. The first one: by Lemma 4.9, we have

$$\begin{aligned}
&\mathcal{P}_b(\vec{u}-\hbar)L^{1\dots b}_{1\dots b}(u_1)(-E_{b+1,1}) = \\
=&\mathcal{P}_b(\vec{u}-\hbar)(-E_{b+1,1})L^{1\dots b}_{1\dots b}(u) + \hbar\mathcal{P}_b(\vec{u}-\hbar)L^{b+1,2,\dots,b}_{1,2,\dots,b}(u) = \\
=&\mathcal{P}_b(\vec{u}-\hbar)(-E_{b+1,1})L^{1\dots b}_{1\dots b}(u) + \mathcal{P}_b(\vec{u}-\hbar)\hbar E_{b+1,1}\frac{L^{1\dots b}_{1\dots b}(u) - L^{1\dots b}_{1\dots b}(u-\hbar)}{\hbar} - \\
-&(-1)^{b+2}\hbar\mathcal{P}_b(\vec{u}-\hbar) = \\
=&\mathcal{P}_b(\vec{u}-\hbar)(-E_{b+1,1})L^{1\dots b}_{1\dots b}(u-\hbar) - (-1)^{b+2}\hbar\mathcal{P}_b(\vec{u}-\hbar).
\end{aligned}$$

The second one: by the same lemma,

$$\begin{aligned}
&\mathcal{P}_b(\vec{u}-\hbar)(-E_{b+1,1})L^{1\dots b}_{1\dots b}(u-\hbar)(-E_{b+1,1})^k = \\
=&\mathcal{P}_b(\vec{u}-\hbar)(-E_{b+1,1})^{k+1}L^{1\dots b}_{1\dots b}(u-\hbar) + k\hbar\mathcal{P}_b(\vec{u}-\hbar)L^{b+1,2,\dots,b}_{1,2,\dots,b}(u-\hbar)(-E_{b1})^k = \\
=&\mathcal{P}_b(\vec{u}-\hbar)(-E_{b+1,1})^{k+1}L^{1\dots b}_{1\dots b}(u-\hbar) - k\hbar(-E_{b+1,1})\partial_u L^{1\dots b}_{1\dots b}(u-\hbar)(-E_{b+1,1})^k - \\
-&(-1)^{b+2}k\hbar\mathcal{P}_b(\vec{u}-\hbar)(-E_{b+1,1})^k.
\end{aligned}$$

Proceeding as in the proof of Lemma 4.11, we conclude that

$$\begin{aligned}
&\mathcal{P}_b(\vec{u}-\hbar)(-E_{b+1,1})L^{1\dots b}_{1\dots b}(u-\hbar)(-E_{b+1,1})^k = \\
=&\mathcal{P}_b(\vec{u}-\hbar)(-E_{b+1,1})^{k+1}L^{1\dots b}_{1\dots b}(u-k\hbar-\hbar) - (-1)^{b+2}k\hbar\mathcal{P}_b(\vec{u}-\hbar)(-E_{b+1,1})^k
\end{aligned}$$

Combining the two facts, we conclude that

$$\begin{aligned}
&\mathcal{P}_b(\vec{u}-\hbar)L^{1\dots b}_{1\dots b}(u)(-E_{b+1,1})^{k+1} = \\
=&\mathcal{P}_b(\vec{u}-\hbar)(-E_{b1})^{k+1}L^{1\dots b}_{1\dots b}(u-k\hbar-\hbar) - (-1)^{b+2}(k+1)\hbar\mathcal{P}_b(\vec{u}-\hbar)(-E_{b+1,1})^k.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\mathcal{P}_b(\vec{u}-\hbar)L^{1\dots b}_{1\dots b}(u)P^{\psi}_{b+1,1}(u)=\\
=&\mathcal{P}_b(\vec{u}-\hbar)\left(1+\sum_{k\geq 0}L^{1\dots b}_{1\dots b}(u)(-E_{b+1,1})^{k+1}(-1)^{(b+2)(k+1)}\hbar^{-k-1}\frac{\prod_{i=0}^{k}L^{1\dots b}_{1\dots b}(u-i\hbar)}{k!}\right)=\\
=&\mathcal{P}_b(\vec{u}-\hbar)L^{1\dots b}_{1\dots b}(u)+\\
+&\sum_{k\geq 0}(-E_{b+1,1})^{k+1}L^{1\dots b}_{1\dots b}(u-k\hbar-\hbar)(-1)^{(b+2)(k+1)}\hbar^{-k-1}\frac{\prod_{i=0}^{k}L^{1\dots b}_{1\dots b}(u-i\hbar)}{(k+1)!}-\\
-&\sum_{k\geq 0}(-E_{b+1,1})^{k}(-1)^{(b+2)k}\hbar^{-k}\frac{\prod_{i=0}^{k}L^{1\dots b}_{1\dots b}(u-i\hbar)}{k!}.
\end{aligned}
$$

Term-by-term comparison gives that this sum is indeed zero, i.e.

$$\mathcal{P}_b(\vec{u}-\hbar)L^{1\dots b}_{1\dots b}(u)P^{\psi}_{b+1,1}(u)=0,$$

and the proposition follows. ∎

It turns out that the weaker version implies the stronger one.

Proposition 4.17. *For $2\leq b\leq N-1$, we have*

$$E_{1b}P_{\mathfrak{m}_N}(\vec{u})=0,$$

where the equality is understood as of linear operators on any Whittaker module.

Proof. Let W be a Whittaker module and $w\in W$. Observe that $\mathrm{span}(E_{1i}|2\leq i\leq N-1)$ can be naturally identified with the dual of the vector representation of ${}_1\mathfrak{m}_N$. Therefore, we can apply Remark 5.6 and conclude that

$$wE_{1b}P_{{}_1\mathfrak{m}_N}({}_1\vec{u})=\sum_{i=2}^{N-1}w_iP_{{}_1\mathfrak{m}_N}({}_1\vec{u})E_{1i}P_{{}_1\mathfrak{m}_N}({}_1\vec{u})$$

for some $w_i\in W$. Hence,

$$
\begin{aligned}
&wE_{1b}P_{\mathfrak{m}_N}(\vec{u})=\\
=&wE_{1b}P_{{}_1\mathfrak{m}_N}({}_1\vec{u})P^{\psi}_{21}(u_1)\dots P^{\psi}_{N1}(u_1)=\\
=&\sum_{i=2}^{N-1}w_iP_{{}_1\mathfrak{m}_N}({}_1\vec{u})E_{1i}P_{{}_1\mathfrak{m}_N}({}_1\vec{u})P^{\psi}_{21}(u_1)\dots P^{\psi}_{N1}(u_1)=\\
=&\sum_{i=2}^{N-1}w_iP_{{}_1\mathfrak{m}_N}({}_1\vec{u})E_{1i}P_{\mathfrak{m}_N}(\vec{u})
\end{aligned}
$$

for some $\{w_i\}\subset W$. Therefore, to prove that it is zero, it is enough to show that

$$P_{{}_1\mathfrak{m}_N}({}_1\vec{u})E_{1i}P_{\mathfrak{m}_N}(\vec{u})=0$$

for every $2\leq i\leq N-1$. But this is exactly Proposition 4.15. ∎

4.3. **Kirillov projector: other properties.** In what follows, we denote

$$\vec{u}-\hbar e_i:=(u_1,\dots,u_{i-1},u_i-\hbar,u_{i+1},\dots,u_{N-1}).$$

Proposition 4.18. *The Kirillov projector satisfies*

$$L^{i+1\dots N}_{i\dots N-1}(u_i-\hbar)P_{\mathfrak{m}_N}(\vec{u}-\hbar e_i)=P_{\mathfrak{m}_N}(\vec{u}).$$

for every $1\leq i\leq N-1$.

Proof. By inductive construction of the Kirillov projector, it is enough to prove the statement for $i=1$. Recall Theorem 4.3: the operator $P_{\mathfrak{m}_N}(\vec{u})$, satisfying

$$P_{\mathfrak{m}_N}(\vec{u})(x-\psi(x))=0,\ x\in\mathfrak{n}_-,$$
$$(E_{ij}+\delta_{ij}\cdot u_j)P_{\mathfrak{m}_N}(\vec{u})=0,\ 1\leq i\leq j\leq N-1$$

together with the normalization condition $P_{\mathfrak{m}_N}(\vec{u})\big|_{W^{\mathfrak{n}_-^\psi}}=\mathrm{id}$, is unique. Therefore, it would be enough to prove that the left-hand side satisfies these properties. The first property is clear, and the normalization condition follows from Proposition 3.8. For $2\leq i\leq j\leq N-1$, the second property follows from

$$[E_{ij},L^{2\ldots N}_{1\ldots N-1}(u_1-\hbar)]=0.$$

Therefore, it is enough to consider the case $i=1$. Let $j=1$. Then

$$[E_{11},L^{2\ldots N}_{1\ldots N-1}(u_1-\hbar)]=-\hbar L^{2\ldots N}_{1\ldots N-1}(u_1-\hbar),$$

hence

$$(E_{11}+u_1)L^{2\ldots N}_{1\ldots N-1}(u_1-\hbar)P_{\mathfrak{m}_N}(\vec{u}-\hbar e_1)=$$
$$=L^{2\ldots N}_{1\ldots N-1}(u_1-\hbar)(E_{11}+u_1-\hbar)P_{\mathfrak{m}_N}(\vec{u}-\hbar e_1)=0.$$

Let $j>1$, then

$$[E_{1j},L^{2\ldots N}_{1\ldots N-1}(u_1-\hbar)]=(-1)^j\hbar L^{1\ldots\hat{j}\ldots N}_{1\ldots N-1}(u_1-\hbar).$$

Since

$$L^{1\ldots\hat{j}\ldots N}_{1\ldots N-1}(u_1-\hbar)=\sum_{l=1}^{N-1}(-1)^{l-1}L^{2\ldots\hat{j}\ldots N}_{1\ldots\hat{l}\ldots N-1}(u_1-2\hbar)L_{1l}(u_1-\hbar)$$

and each term $L_{1l}(u_1-\hbar)$ acts by zero on $P_{\mathfrak{m}_N}(\vec{u}-\hbar)$, we conclude that

$$E_{ij}L^{2\ldots N}_{1\ldots N-1}(u_1-\hbar)P_{\mathfrak{m}_N}(\vec{u}-\hbar)=0.$$

Therefore, the element

$$L^{2\ldots N}_{1\ldots N-1}(u_1-\hbar)P_{\mathfrak{m}_N}(\vec{u}-\hbar)$$

satisfies the conditions of Theorem 4.3, thus coincides with $P_{\mathfrak{m}_N}(\vec{u})$. ◻

Corollary 4.19. *We have*

$$E_{1N}P_{\mathfrak{m}_N}(\vec{u})=P_{\mathfrak{m}_N}(\vec{u}-\hbar e_1)L^{1\ldots N}_{1\ldots N}(u_1).$$

Proof. Since $L^{1\ldots N}_{1\ldots N}(u_1)$ is central, we have

$$P_{\mathfrak{m}_N}(\vec{u}-\hbar e_1)L^{1\ldots N}_{1\ldots N}(u_1)=L^{1\ldots N}_{1\ldots N}(u_1)P_{\mathfrak{m}_N}(\vec{u}-\hbar e_1).$$

By Proposition 3.1, we have

$$L^{1\ldots N}_{1\ldots N}(u_1)=\sum_{l=1}^{N}(-1)^{l-1}L^{2\ldots N}_{1\ldots\hat{l}\ldots N}(u_1-\hbar)L_{1l}(u_1).$$

By Theorem 4.13, the only terms that act non-trivially are for $i=1$ and $i=N$. The former gives

$$L^{2\ldots N}_{2\ldots N}(u_1-\hbar)L_{11}(u_1)P_{\mathfrak{m}_N}(\vec{u}-\hbar e_1)=$$
$$=L^{2\ldots N}_{2\ldots N}(u_1-\hbar)(L_{11}(u_1-\hbar)+\hbar)P_{\mathfrak{m}_N}(\vec{u}-\hbar e_1)=$$
$$=\hbar L^{2\ldots N}_{2\ldots N}(u_1-\hbar)P_{\mathfrak{m}_N}(\vec{u}-\hbar e_1),$$

while the latter

$$(-1)^{N-1}L^{2\ldots N}_{1\ldots N-1}(u_1-\hbar)E_{1N}P_{\mathfrak{m}_N}(\vec{u}-\hbar e_1).$$

We also have

$$L^{2\ldots N}_{1\ldots N-1}(u_1-\hbar)E_{1N}=E_{1N}L^{2\ldots N}_{1\ldots N-1}(u_1-\hbar)+\hbar(-1)^{N-1}L^{1\ldots N-1}_{1\ldots N-1}(u_1-\hbar)-\hbar(-1)^{N-1}L^{2\ldots N}_{2\ldots N}(u_1-\hbar).$$

Since

$$L^{1\ldots N-1}_{1\ldots N-1}(u_1-\hbar)P_{\mathfrak{m}_N}(\vec{u}-\hbar e_1)=0,$$

we have

$$\begin{aligned}
&L^{1\ldots N}_{1\ldots N}(u_1)P_{\mathfrak{m}_N}(\vec{u}-\hbar e_1)=\\
=&\big[\hbar L^{2\ldots N}_{2\ldots N}(u_1-\hbar)+(-1)^{N-1}E_{1N}L^{2\ldots N}_{1\ldots N-1}(u_1-\hbar)+\\
+&\hbar L^{1\ldots N-1}_{1\ldots N-1}(u_1-\hbar)-\hbar L^{2\ldots N}_{2\ldots N}(u_1-\hbar)\big]P_{\mathfrak{m}_N}(\vec{u}-\hbar e_1)=\\
=&(-1)^{N-1}E_{1N}L^{2\ldots N}_{1\ldots N-1}(u_1-\hbar)P_{\mathfrak{m}_N}(\vec{u}-\hbar e_1).
\end{aligned}$$

Therefore, the claim of the proposition follows from

$$L^{2\ldots N}_{1\ldots N-1}(u_1-\hbar)P_{\mathfrak{m}_N}(\vec{u}-\hbar)=P_{\mathfrak{m}_N}(\vec{u}),$$

which is Proposition 4.18. □

It seems that there is no closed expression for the action of $P_{\mathfrak{m}_N}(\vec{u})$ on the last column $\mathfrak{u}_N$ of $\mathfrak{gl}_N$

$$\mathfrak{u}_N=\begin{pmatrix}0&\ldots&0&*\\ \vdots&\ddots&\vdots&\vdots\\ 0&\ldots&0&*\end{pmatrix}$$

except for E_{1N}. However, it turns out that in the quotient by the ideal $\mathfrak{n}_-^{\psi}\mathrm{U}_\hbar(\mathfrak{gl}_N)$, its image is essentially given by the coefficients of the quantum characteristic polynomial.

Proposition 4.20. *Let*

$$L^{1\ldots N}_{1\ldots N}(v)=v^N+\sum_{i=0}^{N-1}(-1)^{N-i}A_iv^i. \tag{4.14}$$

Then

$$E_{kN}P_{\mathfrak{m}_N}(\vec{u})\equiv(A_k-\beta_{kk})+\sum_{j>k}\alpha_{kj}(A_j-\beta_{kj})\ (\mathrm{mod}\ \mathfrak{n}_-^{\psi}\mathrm{U}_\hbar(\mathfrak{gl}_N)),$$

for some functions $\alpha_{kj}=\alpha_{kj}(\vec{u})$ and $\beta_{kj}=\beta_{kj}(\vec{u})$. In particular, the map

$$\mathfrak{n}_-^{\psi}\backslash\mathrm{U}_\hbar(\mathfrak{gl}_N)/\mathfrak{b}^{\vec{u}}\to\mathbf{A}_{\mathfrak{gl}_N}. \tag{4.15}$$

induced by the Kirillov projector, is an isomorphism of vector spaces. Here $\mathbf{A}_{\mathfrak{gl}_N}$ is the center of the universal enveloping algebra $\mathrm{U}_\hbar(\mathfrak{gl}_N)$.

Proof. In what follows, we use "≡" to denote equivalence modulo the ideal $\mathfrak{n}_-^{\psi}\mathrm{U}_\hbar(\mathfrak{gl}_N)$.

Since $L^{1\ldots N}_{1\ldots N}(v)$ is central,

$$L^{1\ldots N}_{1\ldots N}(v)\equiv P_{\mathfrak{m}_N}(\vec{u})L^{1\ldots N}_{1\ldots N}(v)=L^{1\ldots N}_{1\ldots N}(v)P_{\mathfrak{m}_N}(\vec{u}).$$

By Proposition 3.1, we have

$$L^{1\ldots N}_{1\ldots N}(v)=\sum_{l=1}^{N}(-1)^{N-l}L^{1\ldots\hat{l}\ldots N}_{1\ldots N-1}(v)L_{lN}(v-\hbar N+\hbar).$$

By Proposition 3.8, we get

$$\begin{aligned}
L^{1\ldots N}_{1\ldots N}(v)\equiv&\sum_{l=1}^{N}(-1)^{N-l}L^{1\ldots l-1}_{1\ldots l-1}(v)L_{lN}(v-\hbar N+\hbar)=\\
=&\sum_{l=1}^{N}(-1)^{N-l}L_{lN}(v-\hbar N+\hbar)L^{1\ldots l-1}_{1\ldots l-1}(v).
\end{aligned}$$

Using Proposition 3.1, one can show that

$$L^{1\ldots l-1}_{1\ldots l-1}(v)P_{\mathfrak{m}_N}(\vec{u})=(v-u_1)(v-u_2-\hbar)\ldots(v-u_{l-1}-\hbar l+2\hbar)P_{\mathfrak{m}_N}(\vec{u}),$$

therefore,

$$L^{1\dots N}_{1\dots N}(v)P_{\mathfrak{m}_N}(\vec{u}) \equiv \sum_{l=1}^{N}\left((-1)^{N-l}\prod_{i=1}^{l-1}(v-u_i-\hbar i+\hbar)\right)E_{lN}P_{\mathfrak{m}_N}(\vec{u})+$$
$$+(v-N\hbar+\hbar)\prod_{i=1}^{N-1}(v-u_i-\hbar i+\hbar)P_{\mathfrak{m}_N}(\vec{u})$$

The first part of the proposition then follows by comparing the coefficients of v on both sides. As for the second part, it follows from Remark 5.6 that for any $x, y \in \mathrm{U}_\hbar(\mathfrak{u}_N)$,

$$xyP_{\mathfrak{m}_N}(\vec{u}) = xP_{\mathfrak{m}_N}(\vec{u})yP_{\mathfrak{m}_N}(\vec{u}) + (K_{(1)}\cdot x)P_{\mathfrak{m}_N}(\vec{u})(K_{(2)}\cdot y)P_{\mathfrak{m}_N}(\vec{u}),$$

where $K = K_{(1)}\otimes K_{(2)}$ is some nilpotent matrix acting on $\mathrm{U}_\hbar(\mathfrak{u}_N)\otimes \mathrm{U}_\hbar(\mathfrak{u}_N)$, where we identify $\mathrm{U}_\hbar(\mathfrak{u}_N)$ with the double quotient $\mathfrak{n}_-^{\psi}\backslash \mathrm{U}_\hbar(\mathfrak{gl}_N)/\mathfrak{b}^{\vec{u}}$. In particular, we can invert $1+K$. It follows from Theorem 5.11 that $\mathbf{A}_{\mathfrak{gl}_N}$ is a free commutative algebra on generators $\{A_i\}$. Since we know that the transformation induced by the Kirillov projector is invertible on generators, the second part follows by induction on the degree of generators. ☐

Remark 4.21. It follows from the proof that for a special choice of parameters $u_i = -\hbar i + \hbar$, the images $(E_{kN} - \delta_{k,N}(N-1)\hbar)P_{\mathfrak{m}_N}(0, -\hbar, \dots, -\hbar N + 2\hbar)$ in the quotient $\mathfrak{n}_-^{\psi}\backslash \mathrm{U}_\hbar(\mathfrak{gl}_N)$ are *literally* given by the coefficients A_k of the quantum characteristic polynomial (4.14). Classically, it can be interpreted as follows.

Denote $P_\hbar := P_{\mathfrak{m}_N}(0, -\hbar, \dots, -\hbar N + 2\hbar)$. The Kirillov projector gives a linear map

$$P_\hbar\colon \mathfrak{n}_-^{\psi}\backslash \mathrm{U}_\hbar(\mathfrak{gl}_N) \to (\mathfrak{n}_-^{\psi}\backslash \mathrm{U}_\hbar(\mathfrak{gl}_N))^{\mathfrak{n}_-^{\psi}}.$$

Recall that by Kostant's theorem [Kos78], the target is isomorphic to the center $\mathbf{A}_{\mathfrak{gl}_N}$. Composing with the quotient map $\mathrm{U}_\hbar(\mathfrak{gl}_N) \to \mathfrak{n}_-^{\psi}\backslash \mathrm{U}_\hbar(\mathfrak{gl}_N)$, we get a homomorphism

$$\mathrm{U}_\hbar(\mathfrak{gl}_N) \to \mathbf{A}_{\mathfrak{gl}_N}.$$

Taking limit $\hbar \to 0$ as in Remark 4.4 and identifying $\mathfrak{g} \cong \mathfrak{g}^*$, we obtain a linear map

$$P_0\colon \mathcal{O}(\mathfrak{g}) \to \mathcal{O}(\mathbb{A}^N),$$

where we identified $\mathcal{O}(\mathbb{A}^N)$ with the classical limit of $\mathbf{A}_{\mathfrak{gl}_N}$ using the coefficients A_k as the generators. Moreover, it follows from Remark 5.6 that this is an algebra map, since

$$xyP_\hbar = xP_\hbar yP_\hbar + O(\hbar).$$

In particular, it corresponds to a map between varieties $\mathbb{A}^N \to \mathfrak{g}$. It is not hard to determine which one: since P_0 acts as

$$\begin{aligned}(E_{ij} - \psi(E_{ij}))P_0 &= 0,\ i > j,\\ E_{kl}P_0 &= 0,\ k \le l,\\ E_{kN}P_0 &= A_k, 1 \le k \le N,\end{aligned}$$

this map is given by

$$(A_1, \dots, A_N) \mapsto \begin{pmatrix} 0 & \dots & 0 & A_N \\ 1 & \dots & 0 & A_{N-1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & A_1 \end{pmatrix}$$

(up to transposition induced by the isomorphism $\mathfrak{gl}_N \cong \mathfrak{gl}_N^*$). Therefore, the Kirillov projector quantizes the companion matrix which is a form of a Kostant slice.

5. Kostant-Whittaker reduction and quantization

In this section, we consider the Whittaker analog of the parabolic reduction functor from Section 1.5 that we call (following [BF08]) the *Kostant-Whittaker reduction*. This is a finite-dimensional analog of the Drinfeld-Sokolov reduction, see [Ara17].

5.1. **Mirabolic setting.** Consider the category $\mathrm{HC}_\hbar(\mathrm{M}_N)$ of Harish-Chandra bimodules over the mirabolic subalgebra $\mathfrak{m}_N$ as in Section 1.2. Denote by $\mathrm{Wh}_\hbar(\mathfrak{m}_N)$ the category of right Whittaker modules over $\mathfrak{m}_N$ from Definition 4.1. It has a distinguished object $Q_{\mathfrak{m}_N} = \mathfrak{n}_-^\psi \mathrm{U}_\hbar(\mathfrak{m}_N)\backslash \mathrm{U}_\hbar(\mathfrak{m}_N)$. Similarly to (1.11), we can define a functor:

$$\mathrm{act}_k^\psi \colon \mathrm{Vect} \to \mathrm{Wh}_\hbar(\mathfrak{m}_N), \qquad A \mapsto A \otimes Q_{\mathfrak{m}_N},$$

Then we have an analog of ***Skryabin's equivalence*** [Pre02] (originally proved in [Kos78] for principal W-algebras):

Theorem 5.1 (Mirabolic Skryabin's equivalence)**.** *Functor* act_k^ψ *is an equivalence.*

Proof. We proceed as in Proposition 1.11. The functor $(-)^{\mathfrak{n}_-^\psi} \colon \mathrm{Wh}_\hbar(\mathfrak{m}_N) \to \mathrm{Vect}$ is right adjoint to act_k^ψ:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Wh}_\hbar(\mathfrak{m}_N)}(\mathrm{act}_k^\psi(A), M) &= \mathrm{Hom}_{\mathrm{Wh}_\hbar(\mathfrak{m}_N)}(A \otimes Q_{\mathfrak{m}_N}, M) \cong \\ &\cong \mathrm{Hom}_{\mathrm{U}_\hbar(\mathfrak{m}_N)}(A \otimes Q_{\mathfrak{m}_N}, M) \cong \\ &\cong \mathrm{Hom}(A, M^{\mathfrak{n}_-^\psi}).\end{aligned}$$

The unit of the adjunction is given by

$$A \mapsto (A \otimes Q_{\mathfrak{m}_N})^{\mathfrak{n}_-^\psi}.$$

The Kirillov projector gives an isomorphism with the quotient functor as in (4.4):

$$(-)_{\mathfrak{b}^{\vec{u}}} \colon \mathrm{Wh}_\hbar(\mathfrak{m}_N) \to \mathrm{Vect}, \qquad M \mapsto M/\mathfrak{b}^{\vec{u}}.$$

So, the functor $(-)^{\mathfrak{n}_-^\psi}$ is exact. We have

$$(A \otimes Q_{\mathfrak{m}_N})^{\mathfrak{n}_-^\psi} \cong (A \otimes Q_{\mathfrak{m}_N})/\mathfrak{b}^{\vec{u}}$$

and the composition

$$A \mapsto (A \otimes Q_{\mathfrak{m}_N})^{\mathfrak{n}_-^\psi} \cong (A \otimes Q_{\mathfrak{m}_N})/\mathfrak{b}^{\vec{u}}$$

is an isomorphism by the PBW theorem. The rest follows by the same arguments as in Proposition 1.11. □

Similarly to (1.12), one can define the action functor

$$\mathrm{act}_{\mathfrak{m}_N}^\psi \colon \mathrm{HC}_\hbar(\mathrm{M}_N) \to \mathrm{Wh}_\hbar(\mathfrak{m}_N), \qquad X \mapsto Q \otimes_{\mathrm{U}_\hbar(\mathfrak{m}_N)} X.$$

The following definition is an analog of Definition 1.12:

Definition 5.2. The ***mirabolic Kostant-Whittaker reduction*** functor $\mathrm{res}_{\mathfrak{m}_N}^\psi \colon \mathrm{HC}_\hbar(\mathrm{M}_N) \to \mathrm{Vect}$ is the composition

$$\mathrm{HC}_\hbar(\mathrm{M}_N) \xrightarrow{\mathrm{act}_{\mathfrak{m}_N}} \mathrm{Wh}_\hbar(\mathfrak{m}_N) \xrightarrow{(-)^{\mathfrak{n}_-^\psi}} \mathrm{Vect}.$$

Explicitly, it is given by a quantum Hamiltonian reduction $X \mapsto (\mathfrak{n}_-^\psi \backslash X)^{\mathfrak{n}_-^\psi}$. There is a natural lax monoidal structure on $\mathrm{res}_{\mathfrak{m}_N}^\psi$:

$$(\mathfrak{n}_-^\psi \backslash X)^{\mathfrak{n}_-^\psi} \otimes (\mathfrak{n}_-^\psi \backslash Y)^{\mathfrak{n}_-^\psi} \to (\mathfrak{n}_-^\psi \backslash X \otimes_{\mathrm{U}_\hbar(\mathfrak{m}_N)} Y)^{\mathfrak{n}_-^\psi}.$$

As in Theorem 1.13, we have

Theorem 5.3. *The functor* $\mathrm{res}_{\mathfrak{m}_N}^\psi$ *is colimit-preserving and monoidal.*

Proof. Follows from exactly the same arguments as [KS22, Corollary 4.18]. □

Recall the monoidal functor free: $\mathrm{Rep}(\mathrm{M}_N) \to \mathrm{HC}_\hbar(\mathrm{M}_N)$ from Section 1.2. On the one hand, we get a monoidal functor $\mathrm{Rep}(\mathrm{M}_N) \to \mathrm{Vect}$ by composing with $\mathrm{res}_{\mathfrak{m}_N}^\psi$. On the other hand, there is a forgetful functor forget: $\mathrm{Rep}(\mathrm{M}_N) \to \mathrm{Vect}$. So, we have a diagram

$$\begin{tikzcd} \mathrm{Rep}(\mathrm{M}_N) \arrow[r, "\mathrm{free}"] \arrow[dr, "\mathrm{forget}"'] & \mathrm{HC}_\hbar(\mathrm{M}_N) \arrow[d, "\mathrm{res}_{\mathfrak{m}_N}^\psi"] \\ & \mathrm{Vect} \end{tikzcd} \tag{5.1}$$

The following result is an analog of Theorem 1.14:

Theorem 5.4. *The Kirillov projector defines a natural isomorphism*

$$P_{\mathfrak{m}_N}(\vec{u})_V \colon V \to (\mathfrak{n}_-^\psi \backslash \mathrm{U}_\hbar(\mathfrak{m}_N) \otimes V)^{\mathfrak{n}_-^\psi}.$$

In other words, the diagram (5.1) *is commutative.*

Proof. By (4.4), we have

$$(\mathfrak{n}_-^\psi \backslash \mathrm{U}_\hbar(\mathfrak{m}_N) \otimes V)^{\mathfrak{n}_-^\psi} \cong \mathfrak{n}_-^\psi \backslash \mathrm{U}_\hbar(\mathfrak{m}_N) \otimes V / \mathfrak{b}^{\vec{u}}.$$

By the PBW theorem, the composition

$$V \to (\mathfrak{n}_-^\psi \backslash \mathrm{U}_\hbar(\mathfrak{m}_N) \otimes V)^{\mathfrak{n}_-^\psi} \cong \mathfrak{n}_-^\psi \backslash \mathrm{U}_\hbar(\mathfrak{m}_N) \otimes V / \mathfrak{b}^{\vec{u}}$$

is an isomorphism. ∎

In particular, there is a monoidal structure on forget induced from $\mathrm{res}^\psi_{\mathfrak{m}_N}$:

Theorem 5.5. (1) *There is a collection of maps $F_{VW}(\vec{u})$ natural in $V, W \in \mathrm{Rep}(\mathrm{M}_N)$, such that the diagram*

$$\begin{tikzcd} V \otimes W \arrow[r, "F_{VW}(\vec{u})"] \arrow[d, "(P_{\mathfrak{m}_N}(\vec{u}))_V \otimes (P_{\mathfrak{m}_N}(\vec{u}))_W"'] & V \otimes W \arrow[d, "(P_{\mathfrak{m}_N}(\vec{u}))_{V\otimes W}"] \\ V \otimes W \arrow[r] & V \otimes W \end{tikzcd}$$

is commutative, where the lower right arrow is the natural tensor structure on $\mathrm{res}^\psi_{\mathfrak{m}_N}$. In particular, the collection of $F_{VW}(\vec{u})$ satisfies the twist equation (1.1)*, and $R_{VW}(\vec{u}) := F_{WV}(\vec{u})^{-1} F_{VW}(\vec{u})$ is a solution to the quantum Yang-Baxter equation.*

(2) *The map $F_{VW}(\vec{u})$ has the form*

$$F_{VW}(\vec{u}) \in \mathrm{id}_{V\otimes W} + \hbar \mathrm{U}_\hbar(\mathfrak{n}_-)^{>0} \otimes \mathrm{U}_\hbar(\mathfrak{b})^{>0},$$

where the upper subscript >0 means the augmentation ideal.

(3) *The inverse $F_{VW}(\vec{u})^{-1}$ also satisfies*

$$F_{VW}(\vec{u})^{-1} \in \mathrm{id}_{V\otimes W} + \hbar \mathrm{U}_\hbar(\mathfrak{n}_-)^{>0} \otimes \mathrm{U}_\hbar(\mathfrak{b})^{>0},$$

Proof. We proceed exactly as in Theorem 1.15. For brevity, denote $P = P_{\mathfrak{m}_N}(\vec{u})$. Every element in $(\mathfrak{n}_-^\psi \backslash \mathrm{U}_\hbar(\mathfrak{m}_N) \otimes V)^{\mathfrak{n}_-^\psi}$ can be presented as PvP for $v \in V$ by Theorem 5.4. Therefore, we need to show that

$$PvPwP = PF(v \otimes w)P. \tag{5.2}$$

First, let us choose the following PBW basis: $\{E_{ij}^\psi\}$ for $\mathrm{U}_\hbar(\mathfrak{n}_-)$ and

$$(E_{kl}, E_{kk} + u_k | 1 \le k < l \le N-1)$$

for $\mathrm{U}_\hbar(\mathfrak{b})$ (exactly in this order for the latter). It is clear from the construction (4.2) and Remark 4.12 that

$$P \in 1 + \mathfrak{n}_-^\psi \mathrm{U}_\hbar(\mathfrak{m}_N) \cap \mathrm{U}_\hbar(\mathfrak{m}_N)\mathfrak{b}^{\vec{u}} \tag{5.3}$$

(recall the notation $\mathfrak{b}^{\vec{u}}$ from Eq. (4.3)). Let us write it in this PBW basis:

$$P = 1 + \hbar^{k_i} \sum_i f_i^\psi e_i^u,$$

where $f_i^\psi \in \mathrm{U}_\hbar(\mathfrak{n}_-), e_i^u \in \mathrm{U}_\hbar(\mathfrak{b})$, and k_i is some negative number. This is well-defined, since a part of P that acts non-trivially is actually finite. It is clear from (5.3) that $e_i^u = 1$ if and only if $f_i^\psi = 1$.

Now consider the middle P in (5.2). As in Theorem 1.15, we push the f_i^ψ-term to the left until it meets P and becomes zero. Likewise, we push the e_i^u-term to right until it meets P and becomes zero as well. Since every term

$$\hbar^{-k}(-E_{ij}^\psi)(-1)^{(i+j)k} \frac{\prod_{i=0}^{k-1} L^{j\dots i-1}_{j\dots i-1}(u - \hbar i + \hbar)}{k!}$$

in (4.1) acts with the power at least $\hbar^k$, the second part of the theorem follows. The third part follows from the fact that $F_{VW}(\vec{u})$ is upper-triangular in the natural filtration on $V \otimes W$ induced from a $\mathfrak{n}_-$-filtration on V and $\mathfrak{b}$-filtration on W.

$\square$

Remark 5.6. In general, let X be any Whittaker module and V any representation of M_N. Then for any $x \in X$ and $v \in V$,

$$PxvP = \sum_i Px_iPv_iP$$

for some $x_i \in X, v_i \in V$. Indeed: let us define an adjoint action on X by

$$\mathrm{ad}_\xi(x) := x\xi^\psi,\ \xi \in \mathfrak{n}_-.$$

Since ψ is a character, it is indeed an action, moreover, it is locally nilpotent by the Whittaker assumption on X. Therefore, we can proceed exactly as in the proof of Theorem 5.5 and use the fact that the analog of $F_{VW}(\vec{u})$ in this case is invertible as well.

Example 5.7. Let $N = 2$. The Kirillov projector is given by

$$\sum_{k\geq 0} (E_{21}^\psi)^k \hbar^{-k} \frac{L_1^1(u)\dots L_1^1(u-\hbar k+\hbar)}{k!}.$$

Then the procedure described in the proof of Theorem 5.5 gives the following formula for the twist:

$$F(u) = \sum_{k\geq 0} \frac{(-\hbar)^k}{k!} E_{21}^k \otimes \prod_{i=0}^{k-1}(E_{11}-i). \tag{5.4}$$

Observe that it does not depend on u. A version of it appeared in [GGS92], see also [KST98]. An additional parameter in the formulas from *loc. cit.* (t in the former and λ in the latter) corresponds to a choice of a non-degenerate character $\psi\colon \mathfrak{n}_- \to k$ that we assumed to be $\psi(E_{21}) = 1$ for simplicity.

Definition 5.8. A ***rational Cremmer-Gervais twist*** $F^{\mathrm{CG}}(\vec{u})$ is the collection of isomorphisms $F_{VW}(\vec{u})$. A ***rational Cremmer-Gervais R-matrix*** is $R^{\mathrm{CG}}(\vec{u}) := (F^{\mathrm{CG}}_{WV}(\vec{u}))^{-1}F^{\mathrm{CG}}_{VW}(\vec{u})$.

To justify the name, let us compute the semiclassical limit of $F^{\mathrm{CG}}_{VW}(\vec{u})$:

Theorem 5.9. *The tensor structure of Theorem 5.5 quantizes a family of rational Cremmer-Gervais r-matrices $r^{\mathrm{CG}}(\vec{u})$ from Definition 2.3.*

Proof. We use notations and conventions from the proof of Theorem 5.5. Denote by

$$f_{VW} = \sum_{N\geq i>j\geq 1} E_{ij} \otimes \alpha_{ij},$$

where $\alpha_{ij} \in \mathfrak{b} \subset \mathfrak{m}_N$. It will follow from the proof that the semiclassical limit depends only on the zeroth $\hbar$-power of $\vec{u}$, therefore, we can assume without loss of generality that $\vec{u}$ does not depend on $\hbar$.

Recall that by Proposition 2.2, it is enough to prove that these elements satisfy the relation

$$\alpha_{i+k,i} = -E_{i,i+k-1} - (u_i - u_{i+k-1})\alpha_{i+k-1,i} + \delta_{i>1}\alpha_{i+k-1,i-1}.$$

It will follow from the following

Lemma 5.10. *Let*

$$PvPwP = Pv + P\left(\sum_{N\geq i>j\geq 1} \mathrm{ad}_{E_{ij}}(v)\beta_{ij} + O(\hbar^2)\right)wP$$

Then, up to higher order terms,

(1) *β_{ij} are sums of diagonal minor $L^{a\dots b}_{a\dots b}(u_a)$ with some coefficients;*
(2) *β_{ij} satisfy the relation*

$$\begin{aligned}&\beta_{i+k,i} = \\ &=(-1)^{k+1}(L^{i\dots i+k-1}_{i\dots i+k-1}(u_i) - (u_i-u_{i+k-1})L^{i\dots i+k-2}_{i\dots i+k-2}(u_i)) - (u_i-u_{i+k-1})\beta_{i+k-1,i} + \beta_{i+k-1,i-1}.\end{aligned} \tag{5.5}$$

Proof. Before proving the statement, let us show that

$$PvL^{k\dots l}_{k\dots l}(u_k)xP \in \hbar\sum_{i=k}^{l}(-1)^{i-k}\prod_{m=i+1}^{l}(u_k-u_m)\cdot \mathrm{ad}_{E_{ki}}(x)P + O(\hbar^2).$$

for any $x \in \mathrm{U}_\hbar(\mathfrak{m}_N)\otimes W$ (in particular, nonzero $\hbar$-degrees in $\vec{u}$ do not contribute to the semiclassical limit). Indeed, by Proposition 3.1, we have

$$L^{k\dots l}_{k\dots l}(u_k) = \sum_{i=k}^{l}(-1)^{i-k}L^{k+1\dots l}_{k\dots\hat{i}\dots l}(u_k-\hbar)L_{ki}(u_k).$$

Observe that for every i,

$$L_{ki}(u_k)wP = \hbar\,\mathrm{ad}_{E_{ki}}(w)P + wL_{ki}(u_k)P = \hbar\,\mathrm{ad}_{E_{ki}}(w)$$

by the left invariance of the Kirillov projector from Theorem 4.13. Also observe that for every $\xi \in \mathfrak{n}_-$,

$$Pv\xi^\psi = -\hbar P\,\mathrm{ad}_\xi(v)$$

by the right invariance Theorem 4.5. Therefore, it is enough to consider the terms $L^{k+1\dots l}_{k\dots\hat{i}\dots l}(u_k-\hbar)$ in the quotient $\mathfrak{n}_-^\psi\backslash \mathrm{U}_\hbar(\mathfrak{m}_N)$. Again, by Proposition 3.1,

$$L^{k+1\dots l}_{k\dots\hat{i}\dots l}(u_k-\hbar) = \sum_{j=k+1}^{l}(-1)^{j-k-1}L_{jk}(u-\hbar l+k\hbar-\hbar)L^{k+1\dots\hat{j}\dots l}_{k+1\dots\hat{i}\dots l}(u_k-\hbar).$$

In particular, all terms except for $j=k+1$ will be zero in the quotient. Therefore, the class of $L^{k+1\dots l}_{k\dots\hat{i}\dots l}(u_k)$ is equal to that of $L^{k+2\dots l}_{k+1\dots\hat{i}\dots l}(u_k-\hbar)$. By obvious induction, we can conclude that its class is equal to

$$L^{k+1\dots l}_{k\dots\hat{i}\dots l}(u_k-\hbar) \equiv L^{i+1\dots l}_{i+1\dots l}(u_k-\hbar) \mod (\mathfrak{n}_-^\psi\mathrm{U}_\hbar(\mathfrak{m}_N)). \tag{5.6}$$

Moreover,

$$L^{i+1\dots l}_{i+1\dots l}(u_k-\hbar) = \prod_{m=i+1}^{l}(u_k-u_m)+O(\hbar).$$

Therefore,

$$PvL^{k\dots l}_{k\dots l}(u_k)xP \in \hbar\sum_{i=k}^{l}(-1)^{i-k}\prod_{m=i+1}^{l}(u_k-u_m)\cdot \mathrm{ad}_{E_{ki}}(x)P + O(\hbar^2). \tag{5.7}$$

We will prove the statement of the lemma by induction on N in $\mathfrak{m}_N$. The base $N=2$ follows from (5.4). Assume we proved it for $N-1$, in particular, we can apply it to ${}_1\mathfrak{m}_N$. Let

$$P' := P_{{}_1\mathfrak{m}_N}(u_2,\dots,u_{N-1}).$$

Then we consider

$$PvPwP = PvwP + \sum_{N\geq i>j\geq 2}\mathrm{ad}_{E_{ij}}(v)({}_1\beta_{ij})P^\psi_{21}(u_1)\dots P^\psi_{N1}(u_1)wP,$$

where $({}_1\beta_{ij})$ are the coefficients of the lemma for P'. Let us study the action of $P^\psi_{k1}(u_1)$. It is clear that it is enough to consider only degree one truncations

$$\tilde{P}^\psi_{k1}(u_1) := 1 + (-1)^{k+1}(-E^\psi_{k1})L^{1\dots k-1}_{1\dots k-1}(u_1)$$

of $P^\psi_{k1}(u_1)$, since

$$\prod_{i=0}^{k-1}L^{1\dots k-1}_{1\dots k-1}(u_1-i\hbar)wP \in O(\hbar^k).$$

By our assumption, each ${}_1\beta_{ij}$ is a sum of diagonal minors of the form $L^{a\dots b}_{a\dots b}(u_a)$. So, let $2 \le a < b$ be arbitrary. We have

$$\begin{aligned} &P\,\mathrm{ad}_{E_{ij}}(v)L^{a\dots b}_{a\dots b}(u_a)\cdot(-1)^{k+1}(-E^{\psi}_{k1})L^{1\dots k-1}_{1\dots k-1}(u_1)wP = \\ =&(-1)^{k+1}P\,\mathrm{ad}_{E_{k1}}\,\mathrm{ad}_{E_{ij}}(v)L^{a\dots b}_{a\dots b}(u_a)L^{1\dots k-1}_{1\dots k-1}(u_1)wP+ \\ +&(-1)^{k+1}P\,\mathrm{ad}_{E_{ij}}(v)\,\mathrm{ad}_{E_{k1}}(L^{a\dots b}_{a\dots b}(u_a))L^{1\dots k-1}_{1\dots k-1}(u_1)wP. \end{aligned}$$

Observe that

$$L^{a\dots b}_{a\dots b}(u_a)L^{1\dots k-1}_{1\dots k-1}(u_1)wP \in O(\hbar^2).$$

Since $L^{1\dots k-1}_{1\dots k-1}(u_1)w \in O(\hbar)$, it is enough to consider the image of $\mathrm{ad}_{E_{k1}}(L^{a\dots b}_{a\dots b}(u_a))$ in the quotient

$$\mathfrak{n}_-\backslash \mathrm{U}_\hbar(\mathfrak{m}_N) \cong \mathrm{U}_\hbar(\mathfrak{b}).$$

We have

$$\mathrm{ad}_{E_{k1}}(L^{a\dots b}_{a\dots b}(u_a)) = (-1)^{k-a-1}\delta_{a\le k\le b}L^{a,\dots,b}_{1,a,\dots,\hat{k},\dots,b}(u_a).$$

By Proposition 3.1,

$$L^{a,\dots,b}_{1,a,\dots,\hat{k},\dots,b}(u_a) = \sum_{i=a}^{b}(-1)^{i-a}E_{i1}L^{a\dots\hat{i}\dots b}_{a\dots\hat{k}\dots b}(u_a-\hbar).$$

Therefore, its image in the quotient is zero unless $i=a=2$, hence for $a\neq 2$,

$$P\,\mathrm{ad}_{E_{ij}}(v)L^{a\dots b}_{a\dots b}(u_a)\cdot(-1)^{k+1}(-E^{\psi}_{k1})L^{1\dots k-1}_{1\dots k-1}(u_1)wP \in O(\hbar^2) \tag{5.8}$$

So, we need to determine the image of $L^{3\dots b}_{2\dots\hat{k}\dots b}(u_2-\hbar)$ in the quotient, which is $L^{k+1\dots b}_{k+1\dots b}(u_2-(k-1)\hbar)$ by (5.6). Therefore, up to higher order terms, we have for $a=2$

$$\begin{aligned} &(-1)^{k+1}P\,\mathrm{ad}_{E_{ij}}(v)\,\mathrm{ad}_{E_{k1}}(L^{2\dots b}_{2\dots b}(u_2))L^{1\dots k-1}_{1\dots k-1}(u_1)wP = \\ =&P\,\mathrm{ad}_{E_{ij}}(v)L^{k+1\dots b}_{k+1\dots b}(u_2-(k-1)\hbar)L^{1\dots k-1}_{1\dots k-1}(u_1)wP. \end{aligned}$$

Next, let $b \ge 1$ be arbitrary and $k > b+1$. Let us study the term

$$P\,\mathrm{ad}_{E_{ij}}(v)L^{1\dots b}_{1\dots b}(u_1)\cdot(-1)^{k+1}(-E^{\psi}_{k1})L^{1\dots k-1}_{1\dots k-1}(u_1)wP.$$

By exactly the same argument, it is enough to look at the image of

$$\mathrm{ad}_{E_{k1}}(L^{1\dots b}_{1\dots b}(u_1)) = L^{k,2,\dots,b}_{1,2,\dots,b}(u_1) = \sum_{i=1}^{b}(-1)^{i-1}E_{ki}L^{2\dots b}_{1\dots\hat{i}\dots b}(u_1).$$

in the quotient (again, we use Proposition 3.1). However, by our assumption on k and b, they are just zero. Therefore, the terms of this kind do not contribute to degree one coefficient, and the first part of the lemma follows.

As for the second part, let assume first that $i>2$. It follows from (5.8) that the coefficient $\beta_{i+k,i}$ does not depend on N. Therefore, it is enough to prove the statement for β_{Ni}.

By induction assumption, we have

$$\begin{aligned} &{}_1\beta_{Ni} = \\ =&(-1)^{N-i+1}(L^{i\dots N-1}_{i\dots N-1}(u_i)-(u_i-u_{N-1})L^{i\dots N-1}_{i\dots N-1}(u_i)) - (u_i-u_{N-1})\cdot{}_1\beta_{N-1,i}+{}_1\beta_{N-1,i-1}. \end{aligned}$$

It follows from arguments before that

$$\beta_{Ni} = {}_1\beta_{Ni} + \sum_{k=1}^{N}(-1)^{k+1}\,\mathrm{ad}_{E_{k1}}({}_1\beta_{Ni})L^{1\dots k-1}_{1\dots k-1}(u_1).$$

But by (5.8), $\mathrm{ad}_{E_{k1}}(L^{i\dots N-1}_{i\dots N-1}(u_i))$ does not contribute to the first power. In particular,

$$\mathrm{ad}_{E_{k1}}({}_1\beta_{Ni}) = -(u_i-u_{N-1})\cdot\mathrm{ad}_{E_{k1}}({}_1\beta_{N-1,i}) + \mathrm{ad}_{E_{k1}}({}_1\beta_{N-1,i-1}) + O(\hbar^2),$$

and the result follows.

For $i = 2$, observe that

$$
\begin{aligned}
{}_1\beta_{N2} &= (-1)^N L^{2\ldots N-1}_{2\ldots N-1}(u_2),\\
{}_1\beta_{N-1,2} &= (-1)^{N-1} L^{2\ldots N-2}_{2\ldots N-2}(u_2).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\beta_{N2} &= (-1)^N L^{2\ldots N-1}_{2\ldots N-1}(u_2) + (-1)^N \sum_{k=1}^{N-1} \prod_{l=k+1}^{N-1} (u_2 - u_l) L^{1\ldots k-1}_{1\ldots k-1}(u_1),\\
\beta_{N-1,2} &= (-1)^{N-1} L^{2\ldots N-2}_{2\ldots N-2}(u_2) + (-1)^{N-1} \sum_{k=1}^{N-2} \prod_{l=k+1}^{N-2} (u_2 - u_l) L^{1\ldots k-1}_{1\ldots k-1}(u_1),\\
\beta_{N-1,1} &= (-1)^N L^{1\ldots N-2}_{1\ldots N-2}(u_1).
\end{aligned}
$$

The statement follows by direct calculation of (5.5).

Finally, for $i = 1$, the statement follows from $\beta_{k1} = (-1)^{k+1} L^{1\ldots k-1}_{1\ldots k-1}(u_1)$. ☐

Now observe that due to (5.7), the semiclassical limit of (5.5) is

$$
\beta_{i+k,i} = -E_{i,i+k-1} - (u_i - u_{i+k-1})\beta_{i+k-1,i} + \beta_{i+k-1,i-1}.
$$

But this is exactly (2.4). The theorem follows. ☐

5.2. **General setting.** In this subsection, we formulate and show some properties of the Kostant-Whittaker reduction in the case of the full algebra $\mathfrak{gl}_N$. The proof of almost all the statements can be directly translated from the mirabolic setting using the Kirillov projector and will be mostly omitted.

Denote by $\mathbf{A}_{\mathfrak{gl}_N} := \mathbf{A}(\mathrm{U}_\hbar(\mathfrak{gl}_N))$ the center of the universal enveloping algebra of $\mathfrak{gl}_N$. In this case, it has a very explicit presentation (for instance, see [MNO96]):

Theorem 5.11. *Let*

$$
L^{1\ldots N}_{1\ldots N}(v) = v^N + \sum_{i=0}^{N-1} (-1)^{N-i} A_i v^i.
$$

Then $\mathbf{A}_{\mathfrak{gl}_N}$ is a commutative polynomial algebra freely generated by $\{A_i\}$.

Denote by $\mathrm{Wh}_\hbar(\mathfrak{gl}_N)$ the category of $(\mathbf{A}_{\mathfrak{gl}_N}, \mathrm{U}_\hbar(\mathfrak{gl}_N))$-bimodules which are Whittaker with respect to the right $\mathrm{U}_\hbar(\mathfrak{gl}_N)$-action (the definition is the same as in the mirabolic setting of Definition 4.1). It has a distinguished object $Q_{\mathfrak{gl}_N} = \mathfrak{n}_-^\psi \backslash \mathrm{U}_\hbar(\mathfrak{gl}_N)$. In particular, we have a functor

$$
\mathrm{act}^\psi_{\mathbf{A}_{\mathfrak{gl}_N}} \colon {}_{\mathbf{A}_{\mathfrak{gl}_N}}\mathrm{BiMod}_{\mathbf{A}_{\mathfrak{gl}_N}} \to \mathrm{Wh}_\hbar(\mathfrak{gl}_N), \qquad X \mapsto X \otimes_{\mathbf{A}_{\mathfrak{gl}_N}} Q_{\mathfrak{gl}_N},
$$

where ${}_{\mathbf{A}_{\mathfrak{gl}_N}}\mathrm{BiMod}_{\mathbf{A}_{\mathfrak{gl}_N}}$ is the category of $\mathbf{A}_{\mathfrak{gl}_N}$-bimodules. The functor of Whittaker invariants is right adjoint:

$$
(-)^{\mathfrak{n}_-^\psi} \colon \mathrm{Wh}_\hbar(\mathfrak{gl}_N) \to {}_{\mathbf{A}_{\mathfrak{gl}_N}}\mathrm{BiMod}_{\mathbf{A}_{\mathfrak{gl}_N}}.
$$

Theorem 5.12 (Skryabin's equivalence)**.** *The functor $(-)^{\mathfrak{n}_-^\psi}$ is an equivalence.*

Proof. The main difference with the mirabolic setting is the following fact: we need to show that for any $X \in {}_{\mathbf{A}_{\mathfrak{gl}_N}}\mathrm{BiMod}_{\mathbf{A}_{\mathfrak{gl}_N}}$, the unit of adjunction

$$
X \mapsto (X \otimes_{\mathbf{A}_{\mathfrak{gl}_N}} Q_{\mathfrak{gl}_N})^{\mathfrak{n}_-^\psi}
$$

is an isomorphism. As before, the Kirillov projector gives an isomorphism

$$
(X \otimes_{\mathbf{A}_{\mathfrak{gl}_N}} Q_{\mathfrak{gl}_N})^{\mathfrak{n}_-^\psi} \cong (X \otimes_{\mathbf{A}_{\mathfrak{gl}_N}} Q_{\mathfrak{gl}_N})_{\mathfrak{b}^{\vec{u}}}.
$$

By the PBW theorem, we have

$$
(X \otimes_{\mathbf{A}_{\mathfrak{gl}_N}} Q_{\mathfrak{gl}_N})/\mathfrak{b}^{\vec{u}} \cong X \otimes_{\mathbf{A}_{\mathfrak{gl}_N}} \mathrm{U}_\hbar(\mathfrak{u}_N)
$$

with a suitable $\mathbf{A}_{\mathfrak{gl}_N}$-action on $\mathrm{U}_\hbar(\mathfrak{u}_N)$, see Section 4.3. But it follows from Proposition 4.20 that the map $\mathrm{U}_\hbar(\mathfrak{u}_N) \to \mathbf{A}_{\mathfrak{gl}_N}$ induced by the Kirillov projector is an isomorphism. Therefore, the unit of adjunction

$$X \mapsto (X \otimes_{\mathbf{A}_{\mathfrak{gl}_N}} Q_{\mathfrak{gl}_N})^{\mathfrak{n}_-^\psi} \cong (X \otimes_{\mathbf{A}_{\mathfrak{gl}_N}} Q_{\mathfrak{gl}_N})/\mathfrak{b}^{\vec{u}}$$

is an isomorphism as well. The rest follows by the same arguments as Theorem 5.1. □

Recall the category $\mathrm{HC}_\hbar(\mathrm{GL}_N)$ of Harish-Chandra bimodules over $\mathfrak{gl}_N$. We have a functor

$$\mathrm{act}^\psi_{\mathfrak{gl}_N} \colon \mathrm{HC}_\hbar(G) \to \mathrm{Wh}_\hbar(\mathfrak{gl}_N), \qquad X \mapsto Q_{\mathfrak{gl}_N} \otimes_{\mathrm{U}_\hbar(\mathfrak{gl}_N)} X.$$

The following functor was studied in [BF08]:

Definition 5.13. The ***Kostant-Whittaker reduction*** functor $\mathrm{res}^\psi_{\mathfrak{gl}_N} \colon \mathrm{HC}_\hbar(G) \to \mathbf{A}_{\mathfrak{gl}_N}$ is the composition

$$\mathrm{HC}_\hbar(G) \xrightarrow{\mathrm{act}_{\mathfrak{gl}_N}} \mathrm{Wh}_\hbar(\mathfrak{gl}_N) \xrightarrow{(-)^{\mathfrak{n}_-^\psi}} {}_{\mathbf{A}_{\mathfrak{gl}_N}}\mathrm{BiMod}_{\mathbf{A}_{\mathfrak{gl}_N}}.$$

Explicitly, it is given by the quantum Hamiltonian reduction $X \mapsto (\mathfrak{n}_-^\psi \backslash X)^{\mathfrak{n}_-^\psi}$. There is a natural lax monoidal structure on $\mathrm{res}^\psi_{\mathfrak{gl}_N}$:

$$(\mathfrak{n}_-^\psi \backslash X)^{\mathfrak{n}_-^\psi} \otimes (\mathfrak{n}_-^\psi \backslash Y)^{\mathfrak{n}_-^\psi} \to (\mathfrak{n}_-^\psi \backslash X \otimes_{\mathrm{U}_\hbar(\mathfrak{m}_N)} Y)^{\mathfrak{n}_-^\psi}.$$

Theorem 5.14. *The functor* $\mathrm{res}^\psi_{\mathfrak{gl}_N}$ *is colimit-preserving and monoidal.*

Recall the monoidal functor free: $\mathrm{Rep}(G) \to \mathrm{HC}_\hbar(G)$.

Theorem 5.15. *The Kirillov projector defines a natural isomorphism*

$$(P_{\mathfrak{m}_N}(\vec{u}))_V \colon V \otimes \mathbf{A}_{\mathfrak{gl}_N} \to \mathrm{res}^\psi_{\mathfrak{gl}_N}(\mathrm{U}_\hbar(\mathfrak{gl}_N) \otimes V).$$

of right $\mathbf{A}_{\mathfrak{gl}_N}$*-modules. Similarly, there is a natural isomorphism of* left *modules*

$$\mathbf{A}_{\mathfrak{gl}_N} \otimes V \to \mathrm{res}^\psi_{\mathfrak{gl}_N}(\mathrm{U}_\hbar(\mathfrak{gl}_N) \otimes V)$$

Remark 5.16. For a description of the bimodule structure on $\mathrm{res}^\psi_{\mathfrak{gl}_N}(\mathrm{U}_\hbar(\mathfrak{gl}_N) \otimes V)$, see [BF08]. We will only need the fact that it is free on both sides.

Theorem 5.17. *There is a collection of* constant *maps* $F_{VW}(\vec{u})$ *natural in* $V, W \in \mathrm{Rep}(G)$ *such that the diagram*

$$\begin{tikzcd}
(V \otimes \mathbf{A}_{\mathfrak{gl}_N}) \otimes_{\mathbf{A}_{\mathfrak{gl}_N}} (W \otimes \mathbf{A}_{\mathfrak{gl}_N}) \arrow[r, "F_{VW}(\vec{u})"] \arrow[d] & V \otimes W \otimes \mathbf{A}_{\mathfrak{gl}_N} \arrow[d] \\
\mathrm{res}^\psi_{\mathfrak{gl}_N}(\mathrm{U}_\hbar(\mathfrak{gl}_N) \otimes V) \otimes_{\mathbf{A}_{\mathfrak{gl}_N}} \mathrm{res}^\psi_{\mathfrak{gl}_N}(\mathrm{U}_\hbar(\mathfrak{gl}_N) \otimes W) \arrow[r] & \mathrm{res}^\psi_{\mathfrak{gl}_N}(\mathrm{U}_\hbar(\mathfrak{gl}_N) \otimes V \otimes W)
\end{tikzcd}$$

is commutative. The maps $F_{VW}(\vec{u})$ *are the same as in Theorem 5.5, in particular, they quantize the family of rational Cremmer-Gervais solutions* (2.4)*.*

6. Application: vertex-IRF transformation

The main reference for this section is [Kal21]. For reader's convenience, we repeat the main steps of the construction and refer the reader to *loc. cit.* for details.

6.1. *A* vertex-IRF transformation.

Let $R_{VW}(\lambda)$ be a quantum dynamical R-matrix as in Section 1.3.

Definition 6.1. Let $S_V(\lambda) \in \mathrm{End}(V)$ be a collection of invertible $\mathrm{End}(V)$-valued functions on $\mathfrak{h}^*$ natural in $\mathrm{Rep}(G)$. A ***generalized gauge transformation*** is

$$R_{VW}(\lambda) \mapsto R^S_{VW}(\lambda) = (S_V(\lambda - h_W) \otimes S_W(\lambda))R_{VW}(\lambda)(S_V(\lambda)^{-1} \otimes S_W(\lambda - h_U)^{-1}).$$

A generalized gauge transformation is called a ***gauge transformation*** if $S_V(\lambda)$ respects the $\mathfrak{h}$-module structure on V. A generalized gauge transformation is called a ***vertex-IRF transformation*** if $R^S_{VW}(\lambda)$ is constant.

One can similarly define a gauge transformation for dynamical twists, see [ES01].

Proposition 6.2. *Gauge transformations preserve the set of quantum dynamical R-matrices.*

In the setting of Section 1.3, it has the following categorical interpretation:

Proposition 6.3. *[KS22, Proposition 3.8] The data of a gauge transformation between dynamical twists $J_1(\lambda)$ and $J_2(\lambda)$ is equivalent to the data of a natural monoidal isomorphism:*

$$\begin{tikzcd} \mathrm{Rep}(G) \arrow[r, bend left, "J_1(\lambda)"] \arrow[r, bend right, "J_2(\lambda)"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{HC}_\hbar(H). \end{tikzcd}$$

It is also possible to interpret generalized gauge transformations in this language. There is a forgetful functor $\mathrm{HC}_\hbar(H) \to \mathrm{RMod}_{\mathrm{U}_\hbar(\mathfrak{h})}$. While it is not monoidal, it can be upgrade to a *monoidal action* of $\mathrm{HC}_\hbar(H)$ on $\mathrm{RMod}_{\mathrm{U}_\hbar(\mathfrak{h})}$:

Definition 6.4. Let $\mathcal{C}$ be a monoidal category. A $\mathcal{C}$***-module category*** $\mathcal{M}$ is a category $\mathcal{M}$ with a functor $\mathcal{M} \otimes \mathcal{C} \to \mathcal{M}$ which we denote by $(M, X) \mapsto X \otimes M$ with a natural isomorphisms

$$\Psi_{M,X,Y} \colon (M \otimes X) \otimes Y \to M \otimes (X \otimes Y),$$

satisfying a pentagon axiom (see [Eti+15]). A ***functor*** between $\mathcal{C}$-module categories $F\colon \mathcal{M}_1 \to \mathcal{M}_2$ is a functor of plain categories with natural isomorphisms

$$\alpha_{V,X} \colon F(V) \otimes X \to F(V \otimes X),$$

satisfying the unit and the pentagon axioms.

The Harish-Chandra category $\mathrm{HC}_\hbar(H)$ naturally acts on the category $\mathrm{RMod}_{\mathrm{U}_\hbar(\mathfrak{h})}$ of *right* $\mathrm{U}_\hbar(\mathfrak{h})$-modules by

$$(M, X) \mapsto M \otimes_{\mathrm{U}_\hbar(\mathfrak{h})} X.$$

In particular, if we have a monoidal functor $\mathrm{Rep}(G) \to \mathrm{HC}_\hbar(H)$, then there is an action of $\mathrm{Rep}(G)$ on $\mathrm{RMod}_{\mathrm{U}_\hbar(\mathfrak{h})}$. One can easily see that the structure morphisms of Definition 6.4 are given by the dynamical twist maps $J_{VW}(\lambda)$ as in Section 1.3.

Let A be an algebra with a map $A \to \mathrm{U}_\hbar(\mathfrak{h})$. Assume that there is an action of $\mathrm{Rep}(G)$ on RMod_A such that $(A, V) \mapsto A \otimes V$ from Definition 6.4 is a free right A-module. Moreover, assume that the structure maps $\Psi_{A,V,W} \colon (A \otimes V) \otimes Y \to A \otimes (V \otimes W)$ are given by $\mathrm{id}_A \otimes F_{VW}$ for some *constant* F_{VW}. One example is $A = k$; another is the setup of Theorem 5.17.

There is an extension of scalars functor $- \otimes_A \mathrm{U}_\hbar(\mathfrak{h}) \colon \mathrm{RMod}_A \to \mathrm{RMod}_{\mathrm{U}_\hbar(\mathfrak{h})}$.

Theorem 6.5. *[Kal21, Proposition 2.4.3] A generalized gauge transformation between $J_{VW}(\lambda)$ and F_{VW} is equivalent to the data of a $\mathrm{Rep}(G)$-module structure on the functor $- \otimes_A \mathrm{U}_\hbar(\mathfrak{h})$.*

6.2. *The* vertex-IRF transformation.

Recall a monoidal functor $\mathrm{Rep}(\mathrm{GL}_N) \to \mathrm{HC}_\hbar(H)^{\mathrm{gen}}$ from Section 1.5 obtained from parabolic restriction whose monoidal structure morphisms are given by the standard dynamical twist of Definition 1.16. In particular, there is an action of $\mathrm{Rep}(\mathrm{GL}_N)$ on the category $\mathrm{RMod}_{\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}}$ of right $\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}$-modules, whose structure morphisms are given by the standard dynamical twist. Likewise, there is an action of $\mathrm{Rep}(\mathrm{GL}_N)$ on the category $\mathrm{RMod}_{\mathbf{A}_{\mathfrak{gl}_N}}$ of right modules over the center $\mathbf{A}_{\mathfrak{gl}_N}$ whose structure morphisms are given by the rational Cremmer-Gervais twist of Definition 5.8. Also recall the Harish-Chandra homomorphism $\mathbf{A}_{\mathfrak{gl}_n} \to \mathrm{U}_\hbar(\mathfrak{h})$. To show that there is a vertex-IRF transformation between them, it is enough to prove by Theorem 6.5 (or, rather, generic version thereof) that the natural functor $-\otimes_{\mathbf{A}_{\mathfrak{gl}_N}} \mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}\colon \mathrm{RMod}_{\mathbf{A}_{\mathfrak{gl}_N}} \to \mathrm{RMod}_{\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}}$ is a functor of module categories.

Recall that the parabolic restriction $\mathrm{res}_{\mathfrak{gl}_N}$ is a functor $\mathrm{HC}_\hbar(\mathrm{GL}_N) \to \mathrm{HC}_\hbar(H)^{\mathrm{gen}}$. The Harish-Chandra category $\mathrm{HC}_\hbar(H)^{\mathrm{gen}}$ is a subcategory of ${}_{\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}}\mathrm{BiMod}_{\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}}$, and there is restriction functor from the latter to ${}_{\mathbf{A}_{\mathfrak{gl}_N}}\mathrm{BiMod}_{\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}}$. In what follows, we consider $\mathrm{res}_{\mathfrak{gl}_N}$ to take values in ${}_{\mathbf{A}_{\mathfrak{gl}_N}}\mathrm{BiMod}_{\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}}$.

Proposition 6.6. *[Kal21, Corollary 2.7.7] There is a natural isomorphism*

$$\mathrm{res}_{\mathfrak{gl}_N} \to \mathrm{res}^{\psi}_{\mathfrak{gl}_N} \otimes_{\mathbf{A}_{\mathfrak{gl}_N}} \mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}$$

of functors $\mathrm{HC}_\hbar(G) \to {}_{\mathbf{A}_{\mathfrak{gl}_N}}\mathrm{BiMod}_{\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}}$.

Proof. Let $X \in \mathrm{HC}_\hbar(\mathrm{GL}_N)$. Consider a sequence of maps

$$(\mathfrak{n}^{\psi}_{-}\backslash X)^{\mathfrak{n}^{\psi}_{-}} \otimes_{\mathbf{A}_{\mathfrak{gl}_N}} \mathrm{U}_\hbar(\mathfrak{h}) \xrightarrow{p_X \otimes \mathrm{id}_{\mathrm{U}_\hbar(\mathfrak{h})}} (\mathfrak{n}^{\psi}_{-}\backslash X\mathfrak{n}) \otimes_{\mathbf{A}_{\mathfrak{gl}_N}} \mathrm{U}_\hbar(\mathfrak{h}) \xrightarrow{\mathrm{act}} \mathfrak{n}^{\psi}_{-}\backslash X/\mathfrak{n}, \tag{6.1}$$

where p_X is the projection and act is the right action of $\mathrm{U}_\hbar(\mathfrak{h})$ on $\mathfrak{n}^{\psi}_{-}\backslash X/\mathfrak{n}$. By [GK22, Lemma 6.2.1], it is an isomorphism. On the other hand, there is a map

$$(X/\mathfrak{n})^{\mathfrak{n}} \to \mathfrak{n}^{\psi}_{-}\backslash X/\mathfrak{n}. \tag{6.2}$$

We claim that after extension of scalars to $\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}$, it is an isomorphism. Indeed: since $\mathrm{HC}_\hbar(\mathrm{GL}_N)$ is generated by free Harish-Chandra bimodules by Proposition 1.4 and both functors in question are colimit-preserving by Theorem 1.13 and Theorem 5.14, it is enough to check the statement for free Harish-Chandra bimodules $X = \mathrm{U}_\hbar(\mathfrak{g}) \otimes V$. There is an $\mathfrak{n}$-stable filtration on V with one-dimensional quotients; in particular, we can choose a (weight) basis $\{v_\mu\}$ and a partial order on it such that $\mathrm{ad}_{\mathfrak{n}}(v_\mu) > v_\mu$.

By the PBW theorem, we have a right $\mathrm{U}_\hbar(\mathfrak{h})$-module isomorphism

$$\mathfrak{n}^{\psi}_{-}\backslash X/\mathfrak{n} \cong \mathrm{U}_\hbar(\mathfrak{h}) \otimes V,$$

given by the generators $\{1 \otimes v_\mu\}$. Likewise, the extremal projector gives an isomorphism

$$P\colon \mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}} \otimes V \to (X/\mathfrak{n})^{\mathfrak{n}}$$

as in Theorem 1.14. It follows from definition of P that

$$P(1 \otimes v_\mu) = 1 \otimes v_\mu + \sum_{\lambda > \mu} f_\lambda \otimes v_\lambda \in (X/\mathfrak{n}) \otimes_{\mathrm{U}_\hbar(\mathfrak{h})} \mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}} \cong \mathrm{U}_\hbar(\mathfrak{b})^{\mathrm{gen}} \otimes V,$$

where $f_\lambda \in \mathrm{U}_\hbar(\mathfrak{b})$ are some elements. In particular, its class in the left quotient by $\mathfrak{n}^{\psi}_{-}$ is given by some strictly upper-triangular transformation in the basis $\{1 \otimes v_\mu\}$ with coefficients in $\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}$, in particular, it is invertible.

Therefore, composing (6.1) with the inverse of (6.2), we conclude. ∎

Remark 6.7. In particular, (6.1) shows why it was natural to consider right $\mathrm{U}_\hbar(\mathfrak{gl}_N)$-modules in the Whittaker setting instead of left ones.

Proposition 6.8. *The functor* $\mathrm{RMod}_{\mathbf{A}_{\mathfrak{gl}_N}} \to \mathrm{RMod}_{\mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}}$, $M \mapsto M \otimes_{\mathbf{A}_{\mathfrak{gl}_N}} \mathrm{U}_\hbar(\mathfrak{h})^{\mathrm{gen}}$ *is a functor of* $\mathrm{HC}_\hbar(\mathrm{GL}_N)$*-module categories, in particular of* $\mathrm{Rep}(\mathrm{GL}_N)$*-module categories.*

Proof. Since the category $\mathrm{RMod}_{\mathbf{A}_{\mathfrak{gl}_N}}$ is generated by free modules, it is enough to check compatibility of actions on $\mathbf{A}_{\mathfrak{gl}_N}$. By Proposition 6.6, we have an isomorphism

$$\mathrm{res}^{\psi}_{\mathfrak{gl}_N}(X) \otimes_{\mathbf{A}_{\mathfrak{gl}_N}} \mathrm{U}_\hbar(\mathfrak{h}) \to \mathrm{res}_{\mathfrak{gl}_N}(X)$$

(for brevity, we omit the index "gen"). Therefore, we only need to check the pentagon axiom:

$$\begin{tikzcd}[column sep=tiny]
& \operatorname{res}^{\psi}_{\mathfrak{gl}_N}(X) \otimes_{\mathbf{A}_{\mathfrak{gl}_N}} \operatorname{res}^{\psi}_{\mathfrak{gl}_N}(Y) \otimes_{\mathbf{A}_{\mathfrak{gl}_N}} \mathrm{U}_{\hbar}(\mathfrak{h}) \arrow[dl] \arrow[dr] & \\
\operatorname{res}^{\psi}_{\mathfrak{gl}_N}(X \otimes_{\mathrm{U}_{\hbar}(\mathfrak{g})} Y) \otimes_{\mathbf{A}_{\mathfrak{gl}_N}} \mathrm{U}_{\hbar}(\mathfrak{h}) \arrow[dr] & & (X/\mathfrak{n})^{\mathfrak{n}} \otimes_{\mathrm{U}_{\hbar}\mathfrak{h}} (Y/\mathfrak{n})^{\mathfrak{n}} \arrow[dl] \\
& (X \otimes_{\mathrm{U}_{\hbar}\mathfrak{g}} Y/\mathfrak{n})^{\mathfrak{n}} &
\end{tikzcd}$$

In terms of elements, let ${}_{\mathfrak{n}_-^{\psi}}[y] \in (\mathfrak{n}_-^{\psi}\backslash Y)^{\mathfrak{n}_-^{\psi}}$ and ${}_{\mathfrak{n}_-^{\psi}}[x] \in (\mathfrak{n}_-^{\psi}\backslash X)^{\mathfrak{n}_-^{\psi}}$. We denote their image in the double quotient $\mathfrak{n}_-^{\psi}\backslash Y/\mathfrak{n}$ by ${}_{\mathfrak{n}_-^{\psi}}[y]_{\mathfrak{n}}$ (same for X). Let F be the inverse of (6.2). Then we need to show that

$$F({}_{\mathfrak{n}_-^{\psi}}[x]_{\mathfrak{n}}) \otimes_{\mathrm{U}_{\hbar}\mathfrak{g}} F({}_{\mathfrak{n}_-^{\psi}}[y]_{\mathfrak{n}}) = F({}_{\mathfrak{n}_-^{\psi}}[x \otimes_{\mathrm{U}_{\hbar}(\mathfrak{g})} y]_{\mathfrak{n}}).$$

We show that the projections of both sides to $\mathfrak{n}_-^{\psi}\backslash(X \otimes_{\mathrm{U}_{\hbar}(\mathfrak{g})} Y)/\mathfrak{n}$ coincide. Indeed: since F is an isomorphism, we have

$$F({}_{\mathfrak{n}_-^{\psi}}[y]_{\mathfrak{n}}) \in [y]_{\mathfrak{n}} + \mathfrak{n}_-^{\psi} Y/\mathfrak{n}.$$

The element ${}_{\mathfrak{n}_-^{\psi}}[x]$ is $\mathfrak{n}_-^{\psi}$-invariant, i.e. $x \cdot \mathfrak{n}_-^{\psi} \in \mathfrak{n}_-^{\psi} X$, therefore, the left-hand side belongs to

$$[x \otimes_{\mathrm{U}_{\hbar}(\mathfrak{g})} y]_{\mathfrak{n}} + \mathfrak{n}_-^{\psi}(X \otimes_{\mathrm{U}_{\hbar}\mathfrak{g}} Y)/\mathfrak{n},$$

and so the projections to both sides coincide. □

In particular, it implies

Theorem 6.9. *There is a vertex-IRF transformation between the standard quantum dynamical R-matrix $R^{\mathrm{dyn}}(\lambda)$ from Definition 1.16 and the constant quantum rational Cremmer-Gervais R-matrix $R^{\mathrm{CG}}(\vec{u})$ from Definition 5.8.*

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