An index formula for families of end-periodic Dirac operators

... and anomalies in QFT

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1 Index theory for Dirac operators

2 Index theory and anomalies

3 Index theory for end-periodic Dirac operators

4 A new index formula

5 Remarks on the proof depending on how much time is left

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$E \to M$ vector bundle with connection ∇.
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Consider simple situation where $\dim M = 2$. Gauss-Bonnet theorem tells us that

$$\operatorname{ind}(d + d^*) = \chi(M) = \frac{1}{2\pi} \int_M K \, dA.$$

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Notation: $\operatorname{ind} D^+ = \int_M AS(D(M))$.

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where the *eta invariant* is defined as

$$\eta(D(\partial Z)) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \operatorname{Tr}(D(\partial Z) e^{-tD(\partial Z)^2}) \, dt.$$

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Consider the mapping torus $(M \times S^1)_\pi = (M \times [0,1])/\sim$, where $(x, 1) \sim (\pi(x), 0)$. Suppose $(M \times S^1)_\pi = \partial B$. Then using APS, Witten computes ΔI in terms of the eta invariant $\eta(D(\partial B))$.

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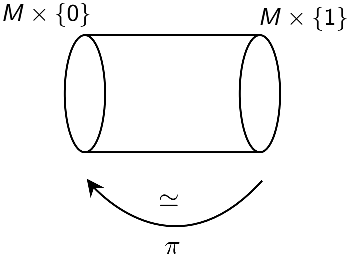

Anomalies in QFT

Global gravitational anomalies, Witten, 1985

where $H = dB + \omega_3^L$. Thus, the non-covariant objects dB and ω combine, as they must, into the covariant field strength H. ($\Delta\tilde{S}$ and ΔI_{Reg} were invariant under coordinate transformations that vanish at $t=0$ and $t=1$, but not otherwise. As far as I know, it does not make sense to identify $t=1$ with $t=0$ by means of a non-trivial diffeomorphism π until these expressions are combined into a covariant form.) Thus we finally get an expression for the change in the action in terms of topological invariants:

$$\begin{aligned}\Delta I_{\mathrm{TOT}} &= \Delta I_{\mathrm{det}} + \Delta\tilde{S} + \Delta I_{\mathrm{Reg}} \\ &= -2\pi i\left[\frac{1}{(2\pi)^6}\int_B\left(\tfrac{1}{1536}(\mathrm{Tr}\,R^2)^3 + \tfrac{1}{384}\mathrm{Tr}\,R^2\,\mathrm{Tr}\,R^4\right)\right. \\ &\qquad \left. - \int_{(M\times S^1)_\pi} H\cdot\left(\tfrac{1}{1536}(\mathrm{Tr}\,R^2)^2 + \tfrac{1}{384}\mathrm{Tr}\,R^4\right)\right] \\ &= -2\pi i\cdot\tfrac{1}{192}\left[-3p_1^3(B) + 4p_1p_2(B)\right], \end{aligned} \tag{57}$$

where p_1^3 and p_1p_2 were defined in Sect. II.

Families of Dirac operators

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$D = (D^z)_{z\in B}$ = family of Dirac operators on family of Clifford modules $E_z \to M_z$.
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Important construction: the **Bismut superconnection**

$$\begin{aligned} &A \circlearrowright C^\infty(M; E \otimes \Lambda T^* B) \\ &A = D + A_{[1]} + A_{[2]}. \end{aligned}$$

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Bismut-Cheeger 1990, Melrose-Piazza 1997: under the appropriate global boundary conditions, the family $(D^+(Z)^z)_{z\in B}$ is Fredholm, and the Chern character of the index bundle $\mathrm{ch}(\mathrm{Ind}\, D)$ is represented in de Rham cohomology by the differential form

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A path of metrics g_t is a loop in the parameter space B and Witten's global anomaly is recovered as the holonomy around this loop of the connection on the determinant line bundle.

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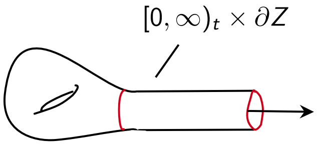
Cylindrical end = boundary



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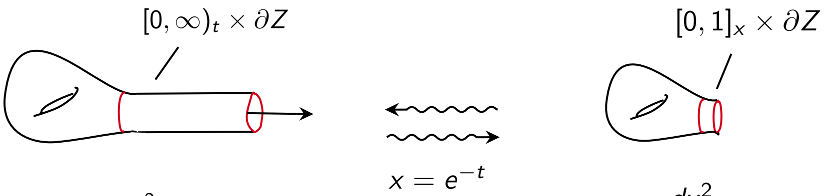

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$$g = \frac{dx^2}{x^2} + g_{\partial Z}$$

End-periodic manifolds

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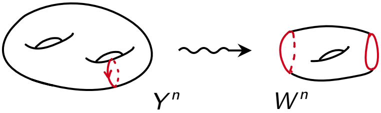
(Taubes 1987)



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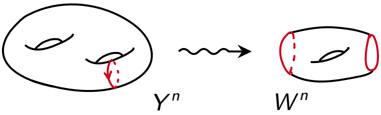

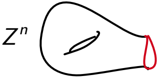

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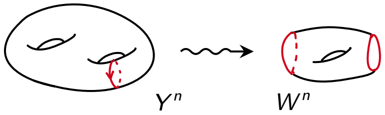

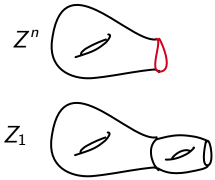

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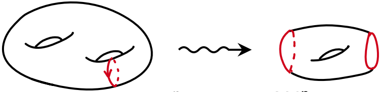
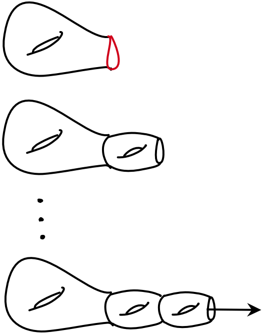


Y^n W^n



Z^n

Z_1

$M^n = Z_\infty$

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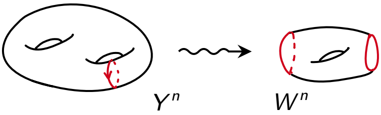

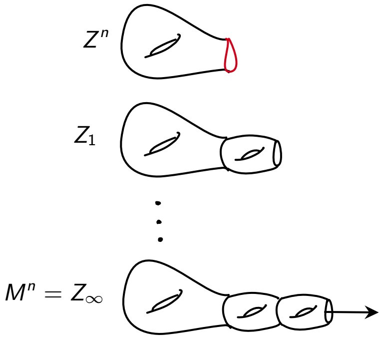

$$
\begin{tikzcd}
\widetilde{Y} \arrow[r, "f"] \arrow[d, "p"] & \mathbb{R} \arrow[d, "\exp"] \\
Y \arrow[r, "\overline{f}"'] & S^1
\end{tikzcd}
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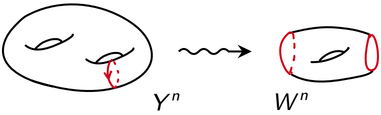

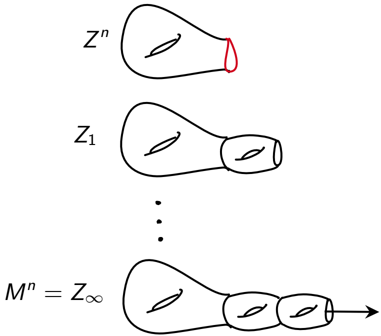

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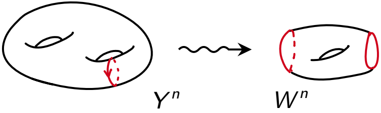

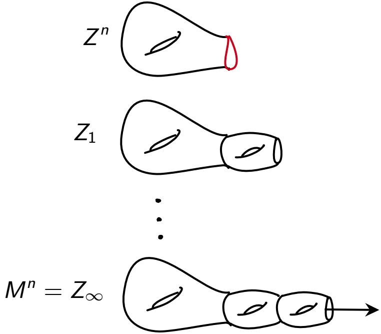

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 & E|_{\mathrm{end}(M)} \arrow[rr] \arrow[d, hook] & & \mathrm{end}(M) \arrow[d, hook] \arrow[urr, dash, dashed] & & \\
 & \widetilde{E} \arrow[rr] \arrow[d] & & \widetilde{Y} \arrow[r, "f"] \arrow[d, "p"] & \mathbb{R} \arrow[d, "\exp"] & \\
 & E_Y \arrow[rr] \arrow[dl, dash, dashed] & & Y \arrow[r, "\overline{f}"'] \arrow[dl, dash, dashed] & S^1 & \\
D(Y) & & g_Y & & &
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$$

End-periodic Dirac operators

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D is an end periodic Dirac operator on M, i.e.,

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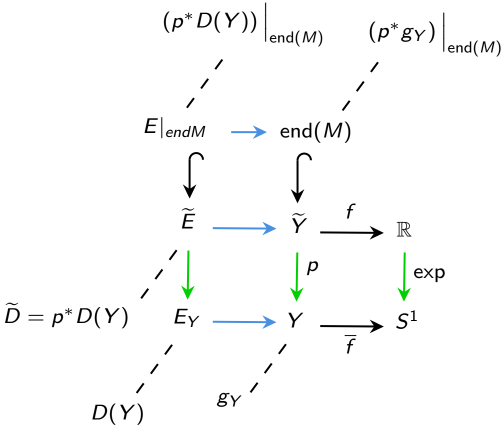
End-periodic Dirac operators



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Family context :
$M \to B,\ Y \to B,\ \widetilde{Y} \to B$
$\mathbb{E} = E \otimes \pi^*\Lambda T^*B,\ etc.$

Replace :
$E_Y \rightsquigarrow \mathbb{E},\ \widetilde{E} \rightsquigarrow \widetilde{\mathbb{E}}$
$E \rightsquigarrow \mathbb{E},\ D \rightsquigarrow A,\ etc.$

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- *Metrics on end-periodic manifolds as models for dark matter*, Duston, 2021.

Index formula for end-periodic Dirac operators

Index formula for end-periodic Dirac operators

Mrowka, Ruberman, Saveliev, 2014: under the appropriate Fredholm condition, the index of the end-periodic Dirac operator D is given by

$$\operatorname{ind} D^+ = \int_Z AS(D(Z)) - \int_Y f\, AS(D(Y)) - \frac{1}{2}\eta_{\mathrm{ep}}(D(Y))$$

where the end-periodic eta invariant is defined by

$$\widehat{\eta}_{\mathrm{ep}}(D(Y)) = \frac{1}{\pi i}\int_0^\infty \oint_{S^1} \operatorname{Tr}\left(c\ell(df) D_\xi^+ e^{-tD_\xi^- D_\xi^+}\right) \frac{d\xi}{\xi}\, dt$$

and $D_\xi = D(Y) - \ln(\xi)\, c\ell(df)$, for $\xi \in S^1$, is the *indicial family* of D.

1 Index theory for Dirac operators

2 Index theory and anomalies

3 Index theory for end-periodic Dirac operators

4 A new index formula

5 Remarks on the proof depending on how much time is left

Index for families of end-periodic Dirac operators

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Theorem (T.) *The Chern character of the index bundle,* $\mathrm{ch}(\mathrm{Ind}\, D)$, *is represented in de Rham cohomology by the differential form*

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where $\mathcal{H}_t(\xi) = \int_0^1 e^{-uA_t^2(\xi)}\,du$.

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where $\mathcal{H}_t(\xi) = \int_0^1 e^{-uA_t^2(\xi)}\,du$. Moreover, the degree zero part of the end-periodic eta form $\widehat{\eta}_{\mathrm{ep}}$ is the fiberwise end-periodic eta invariant: $(\widehat{\eta}_{\mathrm{ep}})_{[0]} \in C^\infty(B)$ given by $z \mapsto \eta_{\mathrm{ep}}(D^z(Y_z))$.

Thank you

Extra slide

Some other important operators:

$$\begin{aligned}
&m_0 : T^*M \to \operatorname{End}\mathbb{E}\\
&\delta_t m_0(df) = t^{1/2}\,\mathrm{c}\ell(d_{Y/B}f) + \pi^* d_B f\\
&A_t(Y)(\xi) = A_t(Y) - \log(\xi)\delta_t m_0(df)
\end{aligned}$$

Extra slide

The Bismut superconnection on Y is

$$A(Y) = D(Y) + A_{[1]}(Y) + A_{[2]}(Y)$$

where

$$A_{[1]}(Y) = \sum_{\alpha} e^{\alpha} \wedge \left(\nabla^{E_Y}_{e_\alpha} + \frac{1}{2} k_Y(e_\alpha) \right)$$

$$A_{[2]}(Y) = -\frac{1}{4} \sum_{\alpha<\beta} \sum_{j} e^{\alpha} \wedge e^{\beta}\, c\ell(e^j)(\Omega_Y(e_\alpha, e_\beta), e_j)$$

1. Index theory for Dirac operators
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Heat equation proof of the local index theorem

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Consider the operator trace of the (chiral) heat operator

$$\mathrm{Tr}(e^{-tD^- D^+}) \quad = \quad \sum_j \langle e^{-tD^- D^+} \psi_j^+, \psi_j^+ \rangle$$

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Define the supertrace of the heat operator

$$\mathrm{Str}(e^{-tD^2}) := \mathrm{Tr}(e^{-tD^-D^+}) - \mathrm{Tr}(e^{-tD^+D^-}).$$

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Renormalized trace defect

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Cylindrical (Melrose)

$$^{b}\mathrm{Tr}[P, Q] = \frac{-1}{2\pi i} \oint_{S^1} \mathrm{Tr}\left(\frac{\partial P_\xi}{\partial \xi} Q_\xi\right) d\xi.$$

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Periodic + family + $\mathbb{Z}_2$-graded (T. 2025)

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EP & cylindrical family transgression formulas

EP & cylindrical family transgression formulas

End-periodic

Cylindrical

EP & cylindrical family transgression formulas

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$$\frac{d}{dt}\,{}^{R}\mathrm{ch}(A_t) = -\,{}^{R}\mathrm{Str}\left[A_t, \dot{A}_t e^{-A_t^2}\right] + \alpha_1(t) + \alpha_2(t)$$

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$$(*) \quad -\frac{1}{2}\widehat{\eta}_{\mathrm{ep}}(t) = \alpha_1(t) + \beta_1(t) = \alpha_1(t) - \frac{1}{2\pi i}\oint_{|\xi|=1} t^{1/2}\,\mathrm{Str}\left(c\ell(d_{Y/B}f)\dot{A}_t(Y)(\xi)e^{-A_t^2(Y)(\xi)}\right)\frac{d\xi}{\xi}$$

EP & cylindrical family transgression formulas

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Melrose-Piazza (1997) first step: $\frac{d}{dt}\,{}^b\mathrm{ch}(A_t) = -\,{}^b\mathrm{Str}[A_t, \dot{A}_t e^{-A_t^2}]$.

EP & cylindrical family transgression formulas

Melrose-Piazza (1997) first step: $\dfrac{d}{dt}\,{}^{b}\mathrm{ch}(A_t) = -\,{}^{b}\mathrm{Str}[A_t, \dot{A}_t e^{-A_t^2}]$.

Proof. By Duhamel's formula, the derivative of the heat kernel is

$$\frac{d}{dt} e^{-\mathbb{A}_t^2} = -\int_0^1 e^{-s\mathbb{A}_t^2} \cdot \frac{d\mathbb{A}_t^2}{dt} \cdot e^{-(1-s)\mathbb{A}_t^2}\, ds. \tag{13.10}$$

^{b}Str defect formula

* The indicial family of $\mathbb{A}_t^2$ is even in λ, so applying (12.5) to commute the leading term, from left to right, no boundary terms arise. Thus

$$\frac{d}{dt}\; b\text{-}\mathrm{Ch}(\mathbb{A}_t) = -\; b\text{-}\mathrm{STr}\left(\frac{d\mathbb{A}_t^2}{dt} e^{-\mathbb{A}_t^2}\right). \tag{13.11}$$

Since $\mathbb{A}_t$ is odd and commutes with $\exp(-\mathbb{A}_t^2)$, this can be written as a supercommutator:

$$\frac{d}{dt}\; b\text{-}\mathrm{Ch}(\mathbb{A}_t) = -\; b\text{-}\mathrm{STr}[\mathbb{A}_t, \frac{d\mathbb{A}_t}{dt} e^{-\mathbb{A}_t^2}]. \tag{13.12}$$

Now, $d\mathbb{A}_t/dt$ is a fibre operator so (12.5) can be applied to give

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$$\int_0^1 {}^R\mathrm{Str}[e^{-sA_t^2}, \partial_t(A_t^2)e^{-(1-s)A_t^2}]\, ds = \alpha_1(t) + \alpha_2(t).$$

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Recall that we have, by the supertrace defect formula,

$$\alpha_1(t) \quad = \quad \frac{-1}{2\pi i} \int_0^1 \oint_{S^1} \mathrm{Str}\left(\partial_\xi(e^{-sA_t^2(\xi)}) \cdot \partial_t(A_t^2(\xi)) \cdot e^{-(1-s)sA_t^2(\xi)}\right) d\xi\, ds$$

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$$\begin{aligned}
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&\stackrel{?}{=} \frac{-1}{2\pi i}\int_0^1 \oint_{S^1} \mathrm{Str}(\mathrm{odd}\cdot\mathrm{even}\cdot\mathrm{even})\frac{d\xi}{\xi}\, ds \\
&= 0
\end{aligned}$$

Complication in the end-periodic case

End-periodic

$$A_t(\xi) = A_t(Y) - \ln(\xi)\delta_t m_0(df)$$

Cylindrical

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Proposition (T., 2025)

$$\begin{aligned}[A(Y), m_0(df)] &= m_0\left(\nabla^{T^*Y} df\right) - 2\nabla^{\mathbb{E}_Y,0}_{(d_{Y/B}f)^\#} \\ &= 0 \quad \text{iff the periodic end is cylindrical}\end{aligned}$$

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$$e^{-A_t^2(\xi)} = ???$$

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$$A_t^2(\xi) = A_t^2(Y) - \ln(\xi)\delta_t\underbrace{\cancel{[A(Y), m_0(dr)]}}_{0} - t\ln(\xi)^2|d_{Y/B}r|^2$$

$$= A_t^2(Y) - t\ln(\xi)^2$$

$$e^{-A_t^2(\xi)} = e^{-A_t^2(Y)}e^{-t\ln(\xi)^2}$$

Proposition (T., 2025)

$$[A(Y), m_0(df)] = m_0\left(\nabla^{T^*Y}df\right) - 2\nabla^{\mathbb{E}_Y,0}_{(d_{Y/B}f)^\#}$$

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