Connections and vector bundles

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1 Preliminaries

Let V and W be finite-dimensional real vector spaces. We fix the following algebraic notation:

1. $\Lambda^k(V^*)$ = alternating multilinear maps $V^k \to \mathbb{R}$.

2. $\mathrm{Alt}(V^k, W)$ = alternating multilinear maps $V^k \to W$.

3. $\mathrm{Hom}(V, W)$ = linear maps $V \to W$.

Fact 1. *Let V, W be real vector spaces. We have a natural linear isomorphism*

$$\begin{aligned} V^* \otimes W &\xrightarrow{\simeq} \mathrm{Hom}(V, W) \\ \lambda \otimes w &\mapsto \lambda(\bullet) w \end{aligned} \tag{1}$$

We have a perfect bilinear pairing

$$\begin{aligned} B : &\Lambda^k(V) \times \Lambda^k(V^*) \to \mathbb{R} \\ &(v_1, \ldots, v_k, \lambda^1, \ldots, \lambda^k) \mapsto \det(\lambda^i(v_j)) \end{aligned}$$

where the determinant here is given by the usual formula

$$\det(\lambda^i(v_j)) = \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \prod_i \lambda^i(v_{\sigma(i)}).$$

This perfect pairing yields another natural linear isomorphism:

Fact 2. *Let V be a real vector space. We have a natural linear isomorphism*

$$\begin{aligned} \Lambda^k(V^*) &\xrightarrow{\simeq} \big(\Lambda^k(V)\big)^* \\ x &\mapsto B(\bullet, x) \end{aligned} \tag{2}$$

Instead of taking linear functionals on V we could consider linear maps $V \to W$, and then Fact 2 extends immediately to the following:

Fact 3. *Let V, W be real vector spaces. We have a natural linear isomorphism*

$$\begin{aligned} \operatorname{Hom}(\Lambda^k(V), W) &\xrightarrow{\simeq} \operatorname{Alt}(V^k, W) \\ (A : \Lambda^k(V) \to W) &\mapsto (\widetilde{A} : V^k \to W) \end{aligned} \tag{3}$$

where $\widetilde{A}(v_1 \ldots, v_k) = A(v_1 \wedge \cdots \wedge v_k)$.

Fact 4. *Let M be a smooth manifold and let $K \subseteq U \subseteq M$, where K is closed and U is open. Then there exists a smooth function $\psi : M \to \mathbb{R}$ such that:*

(i) $0 \leq \psi \leq 1$ *on* M

(ii) $\operatorname{supp} \psi \subseteq U$

(iii) $\psi = 1$ *on* K

A smooth function ψ satisfying the conditions of Fact 4 is called a ***bump function***. The proof uses partitions of unity, a construction which itself hinges on the separability and second countability of manifolds. We will appeal to Fact 4 throughout this note whenever we need to use bump functions.

2 Vector bundles

Let M be a smooth manifold. A (real) ***vector bundle of rank-k*** over M is a smooth manifold E together with a surjective smooth map $\pi : E \to M$ satisfying the following conditions:

(i) For every $p \in M$, the fiber $E_p = \pi^{-1}(p)$ over p is a k-dimensional real vector space.

(ii) For every $p \in M$, there exists an open neighborhood U of p in M and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that

$$\pi_U \circ \Phi = \pi|_{\pi^{-1}(U)}$$

where $\pi_U : U \times \mathbb{R}^k \to U$ is the projection onto U, and for each $q \in U$ the restriction

$$\Phi|_{E_q} : E_q \to \{q\} \times \mathbb{R}^k \simeq \mathbb{R}^k$$

is a linear isomorphism.

In this case, E is called the ***total space*** of the vector bundle, M is the ***base space***, π is the ***projection map*** and each diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is called a ***local trivialization*** of E over U. In everything that follows we will use the notation $\pi^{-1}(U) = E|_U$ to denote the part of E which is fibered over $U \subseteq M$.

$$\begin{tikzcd} E|_U \arrow[r, "\Phi", "\simeq"'] \arrow[d, "\pi"'] & U \times \mathbb{R}^k \arrow[dl, "\pi_U"] \\ U & \end{tikzcd}$$

Remark. A few remarks about terminology and conventions:

- Technically speaking, here we are only considering *smooth* vector bundles. Of course, one could replace the condition "smooth manifold" with "topological space", then require only that π be a continuous map, and that the local trivializations merely be homeomorphisms, etc. But for our purposes everything will reside within the smooth category.
- In this note we will focus on real vector bundles, but most of this stuff works just as well for complex vector bundles, replacing $\mathbb{R}^k$ with $\mathbb{C}^k$.
- To be very explicit, we should say that a vector bundle is a triple (E, M, π) satisfying properties (i) and (ii) above, but instead we will often simply say that "$E \to M$ is a vector bundle" or "E is a vector bundle over M".

The first thing that one can observe from the definition is that, given a vector bundle $\pi : E \to M$, we have

$$\dim E = \dim M + \operatorname{rank} E$$

because any local trivialization gives a diffeomorphism between an open subset of E and a product $U \times \mathbb{R}^k$ where $\dim U = \dim M$ and $k = \operatorname{rank} E$.

Example 1 (Vector bundles)**.** Let M be a smooth manifold.

(a) The trivial bundle of rank-k over M is just the projection $\pi : M \times \mathbb{R}^k \to M$. In this case there is a global trivialization $\Phi : \pi^{-1}(M) = M \times \mathbb{R}^k \to M \times \mathbb{R}^k$ given by the identity map.

(b) The tangent bundle over M is $\pi : TM \to M$ given by $\pi(p, v) = p$ for every $v \in T_pM$, where TM is the disjoint union of tangent spaces across M,

$$TM = \bigsqcup_{p \in M} T_pM.$$

Given any smooth chart $\varphi : U \subseteq M \to \mathbb{R}^n$ with local coordinates (x^i) in U, define $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by

$$\Phi\left(\sum v^i \frac{\partial}{\partial x^i}\bigg|_p\right) = (p, (v^1, \ldots, v^n))$$

This is a local trivialization of TM over U. From this we also notice that the tangent bundle is of rank $n = \dim M$.

(c) The cotangent bundle over M is $\pi : T^*M \to M$ given by $\pi(p,\omega) = p$ for every $\omega \in T_p^*M$, where T^*M is the disjoint union of dual tangent spaces across M,

$$T^*M = \bigsqcup_{p\in M} T_p^*M.$$

Given any smooth chart $\varphi : U \subseteq M \to \mathbb{R}^n$ with local coordinates (x^i) in U, define $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by

$$\Phi\left(\sum a^i dx^i\big|_p\right) = (p, (a^1, \ldots, a^n))$$

This is a local trivialization of T^*M over U. From this we also notice that the cotangent bundle is of rank $n = \dim M$.

(d) Define an equivalence relation on $\mathbb{R}^2$ by setting $(x, y) \sim (x', y')$ if and only if $(x', y') = (x + n, (-1)^n y)$ for some $n \in \mathbb{Z}$. Let $E = \mathbb{R}^2/\!\sim$ be the quotient space with natural quotient map $q : \mathbb{R}^2 \to E$. For any $r > 0$, the image $q([0, 1]\times[-r, r])$ is a smooth compact manifold with boundary, called a Mobius band. Let $\varepsilon : \mathbb{R} \to \mathbb{S}^1$ be the usual covering of the circle and $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ the projection onto the first factor. Since $\varepsilon\circ\pi_1$ is constant on each equivalence class, it descends to a continuous map $\pi : E \to S^1$. This is a smooth real line bundle over S^1.

Example 2 (Differential of a smooth map)**.** The differential of a smooth map $\varphi : M \to N$ is a map between tangent bundles, $d\varphi : TM \to TN$ defined on fibers by the formula

$$d\varphi_p(v)(f) = v \cdot (f \circ \varphi)$$

for any $p \in M$, $v \in T_pM$ and $f \in C^\infty(N, \mathbb{R})$. The notation $v \cdot (f \circ \varphi)$ refers to the tangent vector v acting as a derivation on the smooth function $f \circ \varphi \in C^\infty(M; \mathbb{R})$. (A priori, this only defines a pointwise differential $d\varphi_p$ at each $p \in M$, but it will follow from the theory developed later that these glue together smoothly to produce a global map $d\varphi : TM \to TN$).

Fact 5. *Let $\pi : E \to M$ be a vector bundle. Then*

(a) π is a smooth submersion.

(b) π is an open quotient map.

(c) E is non-compact.

Proof. For (a), the quality of being a smooth submersion is a local one. Thus, around any point in M we take a local trivialization (U, Φ) so that $\pi|_{\pi^{-1}(U)} = \pi_U \circ \Phi$ and then for any point $q \in \pi^{-1}(U)$ we have

$$d\pi_q = d(\pi_U)_{\Phi(q)} \circ d\Phi_q$$

which is surjective as a composition of surjective maps (the derivative of π_U is a projection, and the derivative of Φ is an isomorphism). Hence π is a submersion on $\pi^{-1}(U)$, and we conclude that π is globally a submersion.

Assertion (b) follows immediately from (a) because a smooth submersion is an open map, and a continuous, surjective open map is also a quotient map.

For (c), take any open subset $U \subseteq M$ over which E is trivial so that $E|_U \simeq U \times \mathbb{R}^k$ – this submanifold of E is already non-compact, so there is no way E can be compact. ■

The following fact relates two local trivializations on their overlap.

Fact 6 (Transition functions). *Let $\pi : E \to M$ be a rank-k vector bundle over M, and let (ϕ, U) and (ψ, V) be two local trivializations of E with $U \cap V \neq \varnothing$. Then there exists a smooth map $\tau : U \cap V \to GL(k, \mathbb{R})$ such that*

$$\phi \circ \psi^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$$

has the form

$$\left(\phi \circ \psi^{-1}\right)(p, v) = (p, \tau(p)v).$$

Proof. Since ϕ and ψ are both local trivializations of the vector bundle, we have

$$\begin{aligned} &\pi_1 \circ \psi = \pi = \pi_1 \circ \phi \\ &\Rightarrow \pi_1 \circ \left(\phi \circ \psi^{-1}\right) = \pi_1 \end{aligned}$$

which implies that $\phi \circ \psi^{-1}$ acts as the identity on the first component, i.e.

$$\left(\phi \circ \psi^{-1}\right)(p, v) = (p, \sigma(p, v))$$

for some smooth map $\sigma : (U \cap V) \times \mathbb{R}^k \to \mathbb{R}^k$ in the second component. Moreover, for each fixed $p \in U \cap V$, the map

$$\begin{aligned} \mathbb{R}^k &\to \mathbb{R}^k \\ v &\mapsto \sigma(p, v) \end{aligned}$$

is an invertible linear map because both of ϕ and ψ are diffeomorphisms which restrict to linear isomorphisms on each fiber. Hence for each fixed $p \in U \cap V$ there exists an invertible matrix $\tau(p) \in GL(k, \mathbb{R})$ such that $\sigma(p, v) = \tau(p)v$.

It remains to check that the map $\tau : U \cap V \to GL(k, \mathbb{R})$ is smooth. Let (E_i) be a basis for $\mathbb{R}^k$ and let $\pi^i : \mathbb{R}^k \to \mathbb{R}$ denote the projection onto the ith coordinate, so that

$$\pi^i \left(\sum_{j=1}^{k} v^j E_j \right) = v^i.$$

Let $\tau^i_j : U \cap V \to \mathbb{R}$ be at each point $p \in U \cap V$ the (i, j)-entry of $\tau(p)$. Evidently then,

$$\tau^i_j(p) = \pi^j(\tau(p)E_i) \Rightarrow \tau^i_j = \pi^j \circ \sigma(\cdot, E_i)$$

Thus each function τ^i_j is smooth as a composition of smooth functions. Since matrix entries form global smooth coordinates for $GL(k, \mathbb{R})$, we conclude that τ is smooth. ∎

The smooth maps $\tau : U \cap V \to GL(k, \mathbb{R})$ defined in Fact 6 which relate two overlapping trivializations for $E \to M$ are called ***transition functions*** for the vector bundle.

Typically we want to define a vector bundle over a smooth manifold M by

$$E = \bigsqcup_{p \in M} E_p \quad \text{with} \quad \pi(p, v) = p$$

i.e. by taking a collection of vector spaces E_p, one for each point p of the base manifold, and then forming the disjoint union. In order to make such a set into a smooth vector bundle, we need to construct a manifold topology, a smooth structure, and local trivializations – in short we need to specify how all of the fibers glue together. The next thing we will prove is a shortcut: it suffices to construct local trivializations that are given by smooth transition functions on their overlap.

Lemma 1 (Vector bundle chart lemma)**.** *Let $E = \bigsqcup_{p\in M} E_p$, and define a map $\pi : E \to M$ by $\pi(p, v) = p$ for every $v \in E_p$. Suppose we are given the following data:*

(i) An open cover $\{U_\alpha\}_{\alpha\in A}$ for M.

(ii) For each $\alpha \in A$, a bijective map $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ which restricts to a linear isomorphism

$$\phi_\alpha|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$$

for every $p \in U_\alpha$.

(iii) For each $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \varnothing$, a smooth map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ such that

$$\begin{aligned}
&\phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k \\
&(\phi_\alpha \circ \phi_\beta^{-1})(p, v) = (p, \tau_{\alpha\beta}(p)v)
\end{aligned}$$

Then E admits a unique topology and smooth structure such that $\pi : E \to M$ is a rank-k vector bundle over M, with $\{(\phi_\alpha, U_\alpha)\}$ as the local trivializations.

Proof. Let $p \in M$ and choose some $U_\alpha \subseteq M$ containing p. Choose a smooth chart (V_p, φ_p) for M around p such that $V_p \subseteq U_\alpha$. Let $\widetilde{V}_p = \varphi_p(V_p) \subseteq \mathbb{R}^n$ and define $\widetilde{\varphi}_p = (\varphi_p \times \mathrm{id}) \circ \phi_\alpha : \pi^{-1}(V_p) \to \widetilde{V}_p \times \mathbb{R}^k$.

$$\pi^{-1}(V_p) \xrightarrow{\phi_\alpha} V_p \times \mathbb{R}^k \xrightarrow{\varphi_p\times\mathrm{id}} \widetilde{V}_p \times \mathbb{R}^k \subseteq \mathbb{R}^n \times \mathbb{R}^k$$

Then the collection of all such charts $\{(\pi^{-1}(V_p), \widetilde{\varphi}_p)\}_{p\in M}$ defines a smooth structure on E (the maximal smooth atlas generated by these charts). We will check the following necessary preconditions for $\pi : E \to M$ to be a smooth vector bundle:

- Each map ϕ_α is a diffeomorphism: with respect to the aforementioned smooth structure, each map $\widetilde{\varphi}_p$ is obviously a diffeomorphism, hence each map $\phi_\alpha = (\varphi_p \times \mathrm{id})^{-1} \circ \widetilde{\varphi}_p$ is a diffeomorphism as a composition of diffeomorphisms.

- π is smooth: take two charts (V_p, φ_p) for M and $(\pi^{-1}(V_p), \widetilde{\varphi}_p)$ for E, then the coordinate representation of π with respect to these charts is $\widehat{\pi} = \varphi_p \circ \pi \circ \widetilde{\varphi}_p^{-1}$. We can compute directly that,

$$\begin{aligned}
\widehat{\pi}(\varphi_p(x), y) &= \varphi_p\left(\pi\left(\widetilde{\varphi}_p^{-1}(\varphi_p(x), y)\right)\right) \\
&= \varphi_p\left(\pi\left(\phi_\alpha^{-1}\left(\left(\varphi_p^{-1} \times \mathrm{id}\right)(\varphi_p(x), y)\right)\right)\right) \\
&= \varphi_p\left(\pi\left(\phi_\alpha^{-1}(x, y)\right)\right) \\
&= \left(\varphi_p \circ \pi_{U_\alpha}\right)(x, y) \\
&= \varphi_p(x)
\end{aligned}$$

  which is to say that $\widehat{\pi}$ is the (smooth) projection $\widetilde{V}_p \times \mathbb{R}^k \to \widetilde{V}_p$. We conclude that π is smooth.

- Bundle structure: the fact that each ϕ_α is linear on fibers and satisfies $\pi|_{U_\alpha} = \pi$ follows immediately from hypothesis (ii). Hence $\{(\phi_\alpha, U_\alpha)_{\alpha\in A}$ form local trivializations for E and $\pi : E \to M$ is indeed a smooth vector bundle.

- Uniqueness of the smooth structure: any smooth structure satisfying the conditions of the lemma must include the charts $\{(\pi^{-1}(V_p), \widetilde{\varphi}_p)\}_{p\in M}$. Since our smooth structure is by definition the maximal smooth atlas generated by these charts, uniqueness follows.

■

$$\begin{tikzcd} \pi^{-1}(V_p) \arrow[r, "\pi"] \arrow[d, "\widetilde{\varphi}_p"'] & V_p \arrow[d, "\varphi_p"] \\ \widetilde{V}_p \times \mathbb{R}^k \arrow[r, "\widehat{\pi}"'] & \widetilde{V}_p \end{tikzcd} \qquad \begin{tikzcd} E|_{U_\alpha} \arrow[r, "\Phi_\alpha", "\simeq"'] \arrow[d, "\pi"'] & U_\alpha \times \mathbb{R}^k \arrow[dl, "\pi_{U_\alpha}"] \\ U_\alpha & \end{tikzcd}$$

Sticking with the above notation, note that for every $p \in M$ and vector $v \in E_p$:

$$\begin{aligned}(p, \tau_{\alpha\gamma}(p)v) &= \phi_\alpha \circ \phi_\gamma^{-1}(p, v) \\ &= \left(\phi_\alpha \circ \phi_\beta^{-1}\right) \circ \left(\phi_\beta \circ \phi_\gamma^{-1}\right)(p, v) \\ &= \left(\phi_\alpha \circ \phi_\beta^{-1}\right)(p, \tau_{\beta\gamma}(p)v) \\ &= (p, \tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p)v)\end{aligned}$$

which shows that $\tau_{\alpha\gamma} = \tau_{\alpha\beta} \circ \tau_{\beta\gamma}$. We summarize this observation with the following:

Fact 7. *Let $\pi : E \to M$ be a vector bundle of rank-k over M with local trivializations $\{(\phi_\alpha, U_\alpha)\}_{\alpha\in A}$. For each $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \varnothing$, let $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ denote the corresponding transition function. Then*

$$\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p) = \tau_{\alpha\gamma}(p)$$

for each $\alpha, \beta, \gamma \in A$ and $p \in U_\alpha \cap U_\beta \cap U_\gamma$.

Example 3 (Restriction of a vector bundle)**.** Let $\pi : E \to M$ be a rank-k vector bundle and $S \subseteq M$ an embedded submanifold. Define the restriction of E to S to be the set

$$E|_S = \bigsqcup_{p\in S} E_p \subseteq E$$

and define $\pi|_S : E|_S \to S$ by restricting π to S. Then any local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ of E over $U \subseteq M$ restricts to a diffeomorphism

$$\Phi|_U : (\pi|_S)^{-1}(U \cap S) \to (U \cap S) \times \mathbb{R}^k$$

which gives local trivializations for the restricted vector bundle $E|_S$ over S.

Example 4 (Pullback bundles). Let $\pi : E \to N$ be a vector bundle and $\varphi : M \to N$ a smooth map. We want to construct a vector bundle over M which looks like E restricted to the image of φ. Following this heuristic, we can define $E'_p = E_{\varphi(p)}$ for every $p \in M$ and then

$$E' = \bigsqcup_{p \in M} E_{\varphi(p)}$$

with a projection map

$$\begin{aligned} \pi' &: E' \to M \\ u \in E_{\varphi(p)} &\mapsto p \in M \end{aligned}$$

Notice that $u \in E_{\varphi(p)}$ if and only if $\pi(u) = \varphi(p)$, so we can write E' in the following equivalent way:

$$E' = \{(p, u) \in M \times E : \pi(u) = \varphi(p)\} \subseteq M \times E$$

This turns out to be a well-defined vector bundle over M. Instead of writing E', the standard notation for this vector bundle is π^*E. Since the projection map projects onto the first coordinate of the product $M \times E$, we denote it by $\mathrm{pr}_1 : \pi^*E \to M$.

Example 5 (Dual bundles). Let $E \to M$ be a vector bundle. For any $p \in M$ we can consider the dual space

$$E_p^* = \{\text{linear functionals } E_p \to \mathbb{R}\}$$

These spaces glue together to assemble a smooth vector bundle $E^* \to M$ with the obvious projection map.

Example 6 (Tensor bundles). Let $E \to M$ and $F \to M$ be two vector bundles over M. For any $p \in M$ we can consider the vector space $E_p \otimes F_p$. These spaces glue together to assemble a smooth vector bundle $E \otimes F \to M$ with the obvious projection map.

Example 7 (Hom bundle). Let $E \to M$ and $F \to M$ be two vector bundles over M. For any $p \in M$ we can consider the vector space

$$\mathrm{Hom}(E_p, F_p) = \{\text{linear maps } \alpha : E_p \to F_p\}$$

These glue together to assemble a smooth vector bundle $\mathrm{Hom}(E, F) \to M$ with the obvious projection map. Moreover, we have a canonical isomorphism of vector bundles

$$\mathrm{Hom}(E, F) \simeq E^* \otimes F$$

which is obtained by gluing together the fiberwise linear isomorphisms $\mathrm{Hom}(E_p, F_p) \simeq E_p^* \otimes F_p$ (cf Fact)

3 Sections of vector bundles

Let $\pi : E \to M$ be a vector bundle. A ***section*** of E is a smooth map $\sigma : M \to E$ satisfying $\pi \circ \sigma = \mathrm{id}_M$. For every $p \in M$ we then have $\sigma(p) \in \pi^{-1}(p) = E_p$, so σ sends points in M to their respective fibers in E. One could also talk about sections with weaker regularity, e.g. continuous sections, but in this note we will always assume our sections are smooth unless stated otherwise.

It will often be useful to talk about sections that are defined only on an open subset of M – i.e. ***local sections*** of E. Explicitly, a local section of E over $U \subseteq M$ is a smooth map $\sigma : U \subseteq M \to E$ satisfying $\pi \circ \sigma = \mathrm{id}_U$, where $U \subseteq M$ is an open subset of M. We could express this more briefly by simply saying that σ is a section of $E|_U$. Whenever we want to emphasize the "local" distinction, we will refer to sections $\sigma : M \to E$ defined on all of M as ***global sections*** of E.

Example 8 (Sections of vector bundles)**.** Let M be a smooth manifold.

(a) Given any vector bundle $E \to M$, there is always at least one section: the zero section $\sigma : M \to E$ defined by $\sigma(x) = 0 \in E_x$ for every $x \in M$.

(b) The sections of the tangent bundle TM are precisely the vector fields on M. We use the notation $\mathfrak{X}(M) = \Gamma(TM)$.

(c) The sections of the cotangent bundle T^*M are precisely the differential 1-forms on M. We use the notation $\Omega^1(M) = \Gamma(T^*M)$.

(d) In general, sections of the tensor bundle $T^k(T^*M)$ are the covariant k-tensor fields on M.

(e) In general, sections of the alternating tensor bundle $\Lambda^k(T^*M)$ are precisely the differential k-forms on M, i.e. $\Omega^k(M) = \Gamma(\Lambda^k T^*M)$.

(f) Let $E = M \times \mathbb{R}^k$ be the rank-k trivial bundle over M. Then any smooth function $f : M \to \mathbb{R}^k$ defines a section $\sigma_f : M \to M \times \mathbb{R}^k$ given by $\sigma_f(x) = (x, f(x))$. Conversely, every section σ arises in this way from some smooth function $M \to \mathbb{R}^k$, namely $f = \pi_{\mathbb{R}^k} \circ \sigma$. Thus we have a one-to-one correspondence between sections of a trivial bundle and smooth functions on M.

Let $E \to M$ be a vector bundle and $U \subseteq M$ any open subset. Then the set of all local sections of $E|_U$ is an (infinite-dimensional) real vector space, and also a $C^\infty(U)$-module, which we denote by the symbol $\Gamma(E|_U)$. This algebraic structure comes from the operations happening at the level of fibers; explicitly, given local sections $s_1, s_2 \in \Gamma(E|_U)$ and $c, d \in \mathbb{R}$, the vector space structure is naturally given by

$$(cs_1 + ds_2)(p) = cs_1(p) + ds_2(p)$$

for every $p \in U$. The $C^\infty(U)$-module structure is naturally given by

$$(fs)(p) = f(p)s(p)$$

for any local section $s \in \Gamma(E|_U)$, any $f \in C^\infty(U)$, and $p \in U$. Evidently, all of the operations are happening at the level of fibers. We will use the symbol $\Gamma(E)$ to denote global sections of E.

Lemma 2 (Local section extension lemma)**.** *Let M be a smooth manifold and $\pi : E \to M$ a vector bundle over M. Let $A \subseteq M$ be a closed subset, and suppose that $\sigma : A \to E$ is a section which is smooth in the sense that it extends to a smooth local section in a neighborhood of each point on ∂A.*

Then σ extends to a global section in an arbitrarily small way. That is: for any open subset $U \subseteq M$ containing A, there exists a global section $\widetilde{\sigma} \in \Gamma(E)$ satisfying $\widetilde{\sigma}|_A = \sigma$ and $\operatorname{supp} \widetilde{\sigma} \subseteq U$.

Proof. Fix some open subset $U \subseteq M$ containing A. By the smoothness assumption on σ, we can find an open set $V \subseteq M$ such that $A \subseteq V \subseteq U$ and then extend σ to a local section $\sigma : V \subseteq M \to E$ on V. Specifically, the open set V is constructed by extending σ in a neighborhood of each point on ∂A, and then intersecting with U if necessary. Now since A is closed, we can find a smooth bump function $\psi : M \to \mathbb{R}$ for A supported on V, and then define

$$\widetilde{\sigma} = \begin{cases} \psi\sigma & \text{on } V \\ 0 & \text{on } M \setminus V \end{cases}$$

Evidently $\widetilde{\sigma}|_A = (\psi\sigma)|_A = \sigma$ because $\psi = 1$ identically on A, and $\operatorname{supp} \widetilde{\sigma} \subseteq V \subseteq U$. ∎

Let $E \to M$ be a rank-k vector bundle and $U \subseteq M$ an open subset. A ***local frame*** for E over U is an ordered k-tuple $(\sigma_1, \ldots, \sigma_k)$ of local sections over U such that, for every $p \in U$, $(\sigma_1(p), \ldots, \sigma_k(p))$ is a basis for the fiber E_p. If $U = M$ then we call it a ***global frame*** for E.

Local frames are closely related to local trivializations: we will prove now that they essentially carry the same information about the vector bundle. First of all, take a local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ for E – we shall construct a corresponding local frame for E over U. Let $(e_1, \ldots, e_k)$ denote the standard basis for $\mathbb{R}^k$ and for each $1 \leq i \leq k$ define maps

$$\sigma_i : U \subseteq M \to E$$
$$\sigma_i(x) = \Phi^{-1}(x, e_i)$$

Then each σ_i is smooth because Φ is a diffeomorphism, and

$$(\pi \circ \sigma_i)(x) = (\pi_U \circ \Phi) \circ \sigma_i(x) = \pi_U(x, e_i) = x$$

for every $x \in U$, so $\pi \circ \sigma_i = \mathrm{id}_U$ and σ_i is a local section for E over U. To see that $(\sigma_1(x), \ldots, \sigma_k(x))$ forms a basis for E_x, just note that the k-tuple $((x, e_i))$ forms a basis for $\{x\} \times \mathbb{R}^k$ and Φ restricts to a linear isomorphism

$$\Phi|_{E_x} : E_x \xrightarrow{\simeq} \{x\} \times \mathbb{R}^k$$

and $(\sigma_i(x))$ is the image of the basis $((x, e_i))$ under the inverse isomorphism $(\Phi|_{E_x})^{-1}$. Thus, (σ_i) is a local frame for E over U, called the ***local frame associated with the local trivialization*** Φ.

Notice that for $x \in U$ we can express any $v \in E_x$ in terms of this local frame as

$$v = \sum_i v^i \sigma_i(x)$$

for some $v^i \in \mathbb{R}$, and therefore the value of Φ on any fiber must be given by

$$\begin{aligned}\Phi|_{E_x}\left(\sum_i v^i \sigma_i(x)\right) &= \sum_i v^i \Phi|_{E_x}(\sigma_i(x)) \\ &= \sum_i v^i (x, e_i) \\ &= \sum_i (x, v^i e_i) \\ &= (x, (v^1, \dots, v^k))\end{aligned}$$

As a result, we see conversely that any local frame (σ_i) for E over U should uniquely determine a local trivialization of E according to this formula. We summarize the preceding discussion with the following key fact:

Fact 8 (Correspondence between local frames and trivializations)**.** *Let $\pi : E \to M$ be a rank-k vector bundle. Local frames are equivalent to local trivializations in the following sense:*

(i) Given a local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ for E we get a local frame $(\sigma_1, \dots, \sigma_k)$ for E on U, defined by $\sigma_i(x) = \Phi^{-1}(x, e_i)$ for every $x \in U$.

(ii) Given a local frame $(\sigma_1, \dots, \sigma_k)$ for E on an open subset $U \subseteq M$, we get a local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ satisfying

$$\Phi|_{E_x}\left(\sum_i v^i \sigma_i(x)\right) = (x, (v^1, \dots, v^k))$$

for every $x \in U$.

Proof. We have already proven (i) in the preceding discussion, so it only remains to prove (ii). Let's define a map $\Psi : U \times \mathbb{R}^k \to \pi^{-1}(U)$ by

$$\Psi(x, (v^1, \dots, v^k)) = \sum_i v^i \sigma_i(x).$$

We will show that Ψ is a diffeomorphism, so that the inverse Ψ^{-1} gives the desired local trivialization in (ii). In fact, since Ψ is clearly bijective it suffices to show that Ψ is a local diffeomorphism. To that end let $V \subseteq U$ be an arbitrary open subset of M such that we have a local trivialization $\Phi^V : \pi^{-1}(V) \to V \times \mathbb{R}^k$ for E. Since Φ^V is a diffeomorphism it will suffice to show that the map $\Phi^V \circ \Psi : V \times \mathbb{R}^k \to V \times \mathbb{R}^k$ is a diffeomorphism.

Restricting each σ_i to V we get smooth maps $\Phi^V \circ \sigma_i : V \to V \times \mathbb{R}^k$, hence we can write

$$(\Phi^V \circ \sigma_i)(x) = (x, (\sigma_i^1(x), \dots, \sigma_i^k(x)))$$

for some smooth functions $\sigma_i^j : V \to \mathbb{R}$. For each $x \in V$ we have

$$\begin{aligned}
(\Phi^V \circ \Psi)(x, (v^1, \dots, v^k)) &= \Phi^V \left(\sum_i v^i \sigma_i(x) \right) \\
&= \sum_i v^i \Phi^V(\sigma_i(x)) \\
&= \sum_i v^i (x, (\sigma_i^1(x), \dots, \sigma_i^k(x))) \\
&= \left(x, \left(\sum_i v^i \sigma_i^1(x), \dots, \sum_i v^i \sigma_i^k(x) \right) \right)
\end{aligned}$$

which is smooth since each function σ_i^j is smooth on V. Moreover, we can show that $(\Phi^V \circ \Psi)^{-1}$ is also smooth: for each $x \in V$ let $(\sigma_i^j(x))$ denote the invertible matrix determined by the smooth functions σ_i^j and let $(\tau_i^j(x)) = (\sigma_i^j(x))^{-1}$ denote the inverse matrix. Note that the functions τ_i^j are smooth on V because matrix inversion is a smooth operation. Thus $(\Phi^V \circ \Psi)^{-1}$ must satisfy

$$(\Phi^V \circ \Psi)^{-1}(x, (w^1, \dots, w^k)) = (x, (v^1, \dots, v^k))$$

whenever

$$(w^1, \dots, w^k) = (\sigma_i^j(x))(v^1, \dots, v^k)$$

which is to say that

$$\begin{aligned}
(\Phi^V \circ \Psi)^{-1}(x, (w^1, \dots, w^k)) &= (x, (\tau_i^j(x))(w^1, \dots, w^k)) \\
&= \left(x, \left(\sum_i w^i \tau_i^1(x), \dots, \sum_i w^i \tau_i^k(x) \right) \right)
\end{aligned}$$

and this is smooth since each function τ_i^j is smooth on V. In summary, we have shown that $\Phi^V \circ \Psi$ is a local diffeomorphism, hence Ψ is a local diffeomorphism. Since Ψ is also bijective we conclude that Ψ is a diffeomorphism. The inverse diffeomorphism $\Psi^{-1} : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is the desired local trivialization associated with the local frame. ■

Corollary 1. *A vector bundle is trivial if and only if it admits a global frame.*

Proof. By definition, a vector bundle $E \to M$ is trivial if and only if it admits a global trivialization $E \to M \times \mathbb{R}^k$, which, by the above correspondence, is equivalent to admitting a global frame. ■

Let M be an n-dimensional smooth manifold, $E \to M$ a rank-k vector bundle, and let (σ_i) be a local frame for E over an open subset $U \subseteq M$, with corresponding local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$. Let $s : U \subseteq M \to E|_U$ be any (not necessarily smooth) local section. With respect to the chosen local frame, we can write

$$s = \sum_i f^i \sigma_i$$

for some collection of functions $f^i : U \to \mathbb{R}$, called the component functions for s with respect to this local frame. We would like to show, as should be expected, that s is smooth if and only if its component functions are smooth. Take a smooth chart $\varphi : U \subseteq M \to \mathbb{R}^n$ for M, then we can represent s in local coordinates as

$$\widehat{s} = \pi_2 \circ \Phi \circ s \circ \varphi^{-1} : \varphi(V) \subseteq \mathbb{R}^n \to \mathbb{R}^k$$

and evidently s is smooth if and only if $\widehat{s}$ is smooth. For any $p \in U$ we calculate

$$\begin{aligned}
\widehat{s}(p) &= \pi_2(\Phi(s(\varphi^{-1}(p)))) \\
&= \pi_2\left(\Phi\left(\sum_i f^i(\varphi^{-1}(p))\sigma_i(\varphi^{-1}(p))\right)\right) \\
&= \pi_2(\varphi^{-1}(p), (f^1(\varphi^{-1}(p)), \ldots, f^k(\varphi^{-1}(p)))) \\
&= (f^1(\varphi^{-1}(p)), \ldots, f^k(\varphi^{-1}(p)))
\end{aligned}$$

hence $\widehat{s}$ is smooth if and only if the component functions $f^1, \ldots, f^k$ are smooth. We summarize this result in the following fact.

Fact 9. *Let $\pi : E \to M$ be a vector bundle and $s : M \to E$ a (not necessarily smooth) section. Let (σ_i) be a local frame for E on an open subset $U \subseteq M$. Then s is smooth on U if and only if its component functions with respect to (σ_i) are smooth.*

Fact 10. *Let $E_1, E_2 \to M$ be vector bundles. We have a $C^\infty(M)$-module isomorphism*

$$\Gamma(E_1 \otimes_{\mathbb{R}} E_2) \xrightarrow{\simeq} \Gamma(E_1) \otimes_{C^\infty(M)} \Gamma(E_2)$$

We introduce a slightly more general notion of section: sections along a smooth map. Given a vector bundle $\pi : E \to N$ and a smooth map $\varphi : M \to N$, a ***section of*** E ***along*** φ is a smooth map $s : M \to E$ such that $s(p) \in E_{\varphi(p)}$ for every $p \in M$. Of course, we can restrict to an open subset $U \subseteq M$ and consider *local* sections along φ, i.e. any smooth map $s : U \subseteq M \to E$ satisfying $s(p) \in E_{\varphi(p)}$ for every $p \in U$. We introduce the following notation:

$$\begin{aligned}
\Gamma_\varphi(E) &= \text{sections of } E \text{ along } \varphi \\
\Gamma_\varphi(U, E) &= \text{local sections of } E \text{ along } \varphi\text{, defined on } U \subseteq M
\end{aligned}$$

Heuristically, a section $s \in \Gamma_\varphi(E)$ is like a section of E restricted to the image of φ. Immediately from the definition we find a one-to-one correspondence

$$\Gamma_\varphi(E) \simeq \Gamma(\varphi^* E).$$

Indeed, if $\sigma \in \Gamma(\varphi^* E)$ is a section of the pullback bundle then σ is by definition a map

$$\sigma : M \to \varphi^*(E) \text{ with } s(p) \in (\varphi^* E)_p$$

for every $p \in M$, where the fibers are

$$(\varphi^* E)_p = \{p\} \times E_{\varphi(p)}.$$

Hence σ sends points $p \in M$ into $(\varphi^* E)_p$ if and only if $\mathrm{pr}_2 \circ \sigma$ sends points $p \in M$ into $E_{\varphi(p)}$, and the one-to-one correspondence follows by simply identifying any section $\sigma \in \Gamma(\varphi^* E)$ with the section $\mathrm{pr}_2 \circ \sigma \in \Gamma_\varphi(E)$. As a consequence of this correspondence, the space of sections along a map is not really anything new; all the previous results we've developed for the sections can be applied in the same way (for example, the construction of local frames).

Remark. Notice that by choosing $\varphi = \mathrm{id} : N \to N$ we find that

$$\Gamma_{\mathrm{id}}(E) = \Gamma(\mathrm{id}^* E) = \Gamma(E)$$

and so we recover the basic notion of a section of $E \to N$.

Example 9 (Sections along a smooth curve)**.** For example we could consider vector fields along a smooth curve $\gamma : \mathbb{R} \to N$. By definition this is a smooth map $X : (a, b) \subseteq \mathbb{R} \to TM$ satisfying $X(t) \in T_{\gamma(t)}M$ for every $t \in (a, b)$. The most obvious example of such a vector field is the curve's velocity $\gamma'(t)$ since $\gamma'(t) \in T_{\gamma(t)}M$ for every t.

An easy way to construct a section of E along φ is to simply compose a section of E with φ: for any section $s \in \Gamma(E)$, the map $\widetilde{s} = s \circ \varphi$ defines a section of E along φ because $\widetilde{s(p)} = s(\varphi(p)) \in E_{\varphi(p)}$ for every $p \in M$. However, not every section of E along φ is of this form; in fact, sections of this form locally generate $\Gamma_\varphi(E)$ over $C^\infty(M)$. Take a local frame (σ_i) for E over some open subset $W \subseteq N$, then for every $q \in W$ and $u \in E_q$ we can write

$$u = \sum_i u^i \sigma_i(q)$$

for some $u^i \in \mathbb{R}$. In particular, taking $p \in \varphi^{-1}(W) \subseteq M$ and $q = \varphi(p) \in W$ we have

$$u = \sum_i u^i \sigma_i(\varphi(p))$$

for every $u \in E_{\varphi(p)}$, where $u^i \in \mathbb{R}$ depend smoothly on p. Thus the collection $(\sigma_i \circ \varphi)$ of local sections along φ (defined on $\varphi^{-1}(W) \subseteq M$) locally generate $\Gamma_\varphi(E)$ over $C^\infty(M)$, because for any section $\widetilde{s} \in \Gamma_\varphi(E)$ we can write (following the above)

$$\widetilde{s} = \sum_i f^i (\sigma_i \circ \varphi)$$

for some smooth functions f^i locally defined on $\varphi^{-1}(W) \subseteq M$. We summarize this as follows:

Fact 11. *Let $E \to N$ be a vector bundle and $\varphi : M \to N$ a smooth map. The space $\Gamma_\varphi(E)$ of sections along E is locally generated over $C^\infty(M)$ by the sections $\widetilde{s} = s \circ \varphi \in \Gamma_\varphi(E)$ where s is a local section of E. More precisely:*

> *For any $\widetilde{s} \in \Gamma_\varphi(E)$, we can find an open subset $U \subseteq M$, smooth functions $f^i \in C^\infty(U)$, and local sections $s_i \in \Gamma(E)$ such that*
>
> $$\widetilde{s}|_U = \sum_i f^i (s_i \circ \varphi) \tag{4}$$

4 Bundle homomorphisms

Let M be a smooth manifold and $\pi : E \to M$ and $\pi' : E' \to M$ two vector bundles over M. A ***bundle homomorphism*** between these vector bundles is a smooth map $F : E \to E'$ satisfying:

(i) $\pi = \pi' \circ F$.

(ii) For every $x \in M$, the restriction $F|_{E_x} : E_x \to E'_x$ is a linear map.

Note that condition (i) says that F must send the fiber E_x to the fiber E'_x, so condition (ii) makes sense.

Let $E, E' \to M$ be two vector bundles. A bijective bundle homomorphism $F : E \to E'$ whose inverse is also a bundle homomorphism is called a ***bundle isomorphism over f***, and we say that E and E' are isomorphic vector bundles.

Fact 12. *Suppose $E \to M$ and $E' \to M$ are vector bundles over a smooth manifold M, and $F : E \to E'$ is a bijective bundle homomorphism over M. Then F is a bundle isomorphism over M.*

In algebra we typically use the symbol $\mathrm{Hom}(A, B)$ to denote the set of morphisms between two objects of some category. In the case of vector bundles however, the symbol $\mathrm{Hom}(E, E')$ does *not* stand for the set of morphisms between two vector bundles E and E' over some manifold M. We use the symbol $\mathrm{Hom}(E, E')$ to denote the vector bundle over M whose fibers are spaces $\mathrm{Hom}(E_p, E'_p)$ of linear maps. That is,

$$\mathrm{Hom}(E, E') = \bigsqcup_{p \in M} \mathrm{Hom}(E_p, E'_p) \to M$$

In fact, a bundle homomorphism $E \to E'$ over M is exactly a section of the vector bundle $\mathrm{Hom}(E, E') \to M$. Indeed, a section $s \in \Gamma(\mathrm{Hom}(E, E'))$ is by definition a smooth map $s : M \to \mathrm{Hom}(E, E')$ such that $s(p) : E_p \to E'_p$ is a linear map for every $p \in M$, thus it determines a vector bundle homomorphism $E \to E'$ given by $s(p)$ on the fiber E_p. Conversely, any vector bundle homomorphism can be considered as a section of $\Gamma(\mathrm{Hom}(E, E'))$ in the same way. This establishes the following fact:

Fact 13. *For any vector bundles $E \to M$ and $E' \to M$ we have*

$$\Gamma(\mathrm{Hom}(E, E')) = \{\text{bundle homomorphisms } E \to E'\}$$

and in particular when $E = E'$ we have

$$\Gamma(\mathrm{End}(E)) = \{\text{bundle endomorphisms } E \to E\}.$$

When we defined the notion of a vector bundle, the first question we considered was: when does gluing together a collection of vector spaces E_p produce a genuine vector bundle? This question reflects the way vector bundles tend to arise in practice, and we formulated the vector bundle chart lemma (Lemma 1) to answer that question. In the same way, it is natural to wonder: when does gluing together a collection of linear maps $E_p \to E'_p$ between fibers produce a genuine bundle homomorphism $E \to E'$? First of all, note that the conditions (i) and (ii) from the definition of bundle homomorphism are already satisfied by assumption, so all that remains is smoothness.

Fact 14. *Let $\pi : E \to M$ and $\pi' : E' \to M$ be two vector bundles over M. Suppose that for every $p \in M$ we have a linear map $F_p : E_p \to E'_p$, and we define a map $F : E \to E'$ by $F|_{E_p} = F_p$ for every $p \in M$. Then F is smooth (hence a bundle homomorphism over M) if and only if*

$$s \in \Gamma(E|_U) \Rightarrow F \circ s \in \Gamma(E'|_U)$$

for every local section $s \in \Gamma(E|_U)$.

Proof. If F is smooth then obviously $F \circ s$ is smooth for every local section $s \in \Gamma(E|_U)$, so $F \circ s \in \Gamma(E'|_U)$. Conversely, let $p \in M$ and take an open neighborhood $U \subseteq M$ around p. Shrinking U if necessary, take local frames (e_i) for $E|_U$ and (e'_j) for $E'|_U$.

Since by assumption F maps smooth sections to smooth sections, we have $F \circ e_i \in \Gamma(E'|_U)$ for every i and thus we can find smooth component functions $f_i^j \in C^\infty(U)$ such that

$$F \circ e_i = \sum_j f_i^j e'_j \in \Gamma(E'|_U)$$

Let $\Phi : E|_U \to U \times \mathbb{R}^k$ be the local trivialization for E associated with the local frame (e_i). For any $v \in E|_U$ we can write

$$v = \sum_i v^i e_i(\pi(v))$$

for some components $v^i \in \mathbb{R}$, and by the construction from Fact 8, Φ satisfies

$$v = \sum_i v^i e_i(\pi(v)) = \Phi^{-1}(\pi(v), (v^1, \dots, v^k)).$$

For any $(x, b) \in U \times \mathbb{R}^k$ we calculate

$$\begin{aligned} \left(F \circ \Phi^{-1}\right)(x, b) &= F\left(\sum_i b^i e_i(x)\right) \\ &= \sum_i b^i F_x(e_i(x)) \\ &= \sum_i \sum_j b^i f_i^j(x) e'_j(x) \end{aligned}$$

which is smooth since all the involved functions are smooth. Hence $F \circ \Phi^{-1}$ is smooth and we conclude that F is smooth. ■

Now we will use the results of this section to extend the linear isomorphisms from Facts 7, 2, 3 to vector bundle isomorphisms.

Fact 15. *Let $E_1, E_2 \to M$ be vector bundles. Then we have bundle isomorphisms*

(i) $E_1^* \otimes E_2 \simeq \mathrm{Hom}(E_1, E_2)$

(ii) $\Lambda^k(E_1^*) \simeq (\Lambda^k E_1)^*$

(iii) $\mathrm{Hom}(\Lambda^k E_1, E_2) \simeq \mathrm{Alt}(E_1^k, E_2)$

Proof. We will prove (i) here. The other two isomorphisms follow a similar pattern, just with more arduous notation. By Fact 12 it suffices to construct a bijective bundle homomorphism

$$F : E_1^* \otimes E_2 \to \mathrm{Hom}(E_1, E_2)$$

Heuristically, in order to produce this map we should glue together the linear isomorphisms from Fact 7 along each fiber of $E_1^* \otimes E_2$. Thus, we take the natural linear isomorphisms

$$F_p : (E_1^*)_p \otimes (E_2)_p \xrightarrow{\simeq} \mathrm{Hom}((E_1)_p, (E_2)_p)$$

given by Fact 7 and define

$$F|_{(E_1^*)_p \otimes (E_2)_p} = F_p$$

for every $p \in M$. Then F is bijective and we just need to check that it's smooth. Take local frames (e_i) for E_1 and (f_j) for E_2 over some open subset $U \subseteq M$, and let $(e^i) = (e_i^*)$ be the dual frame for E_1^*. Thus, every smooth local section $s \in \Gamma(E_1^* \otimes E_2|_U)$ can be expressed as

$$s = \sum_{i,j} s_i^j e^i \otimes f_j$$

for some smooth functions $s_i^j \in C^\infty(U)$. By Fact 14 it suffices to show that $F \circ s$ is a smooth local section of $\mathrm{Hom}(E_1, E_2)$. But this is easy to see from the expression

$$\begin{aligned}
(F \circ s)(p) &= F_p(s(p)) \\
&= \sum_{i,j} s_i^j(p) F_p(e^i(p) \otimes f_j(p)) \\
&= \sum_{i,j} s_i^j(p) e^i(p)(\bullet) f_j(p) \in \mathrm{Hom}((E_1)_p, (E_2)_p)
\end{aligned}$$

because this is precisely an expression for $F \circ s$ in terms of a smooth local frame with smooth coefficient functions. ■

5 Subbundles

Let $\pi_E : E \to M$ be a vector bundle. A ***subbundle*** of E is a vector bundle $\pi_D : D \to M$ such that

(i) $D \subseteq E$ is an embedded submanifold.

(ii) $\pi_D = \pi_E|_D$.

(iii) For every $p \in M$, the fiber of D over p is $D_p = D \cap E_p$, a linear subspace of E_p with vector space structure inherited from E_p.

First of all, note that the statement that $D \to M$ is a vector bundle implies that all of the fibers of D are nonempty and that they have the same dimension. Moreover, notice that conditions (i)-(iii) essentially amount to demanding that the inclusion $i : D \hookrightarrow E$ be a bundle homomorphism.

Fact 16. *Let $\pi_E : E \to M$ be a vector bundle and $\pi_D : D \to M$ a subbundle. Then the inclusion map $i : D \hookrightarrow E$ is a bundle homomorphism over M.*

Proof. Condition (i) requiring that D be an embedded submanifold ensures that the inclusion is a smooth map between manifolds (a smooth embedding, in fact). The inclusion is linear on fibers because by condition (iii) of the definition the inclusion restricts to the linear inclusion $i|_{D_p} : D_p \hookrightarrow E_p$ on each fiber. Finally, condition (ii) is the same as saying that $\pi_E \circ i = \pi_D$. ∎

The most natural way to define a subbundle of $\pi_E : E \to M$ is to specify a linear subspace $D_p \subseteq E_p$ for each $p \in M$, and then define

$$D = \bigsqcup_{p \in M} D_p \subseteq E$$

with projection $\pi_D = \pi_E|_D$. The question arises: when does this construction genuinely define a subbundle of E? The following fact provides a convenient condition that one can check:

Fact 17 (Local frame criterion for subbundles)**.** *Let $\pi_E : E \to M$ be a vector bundle, and for every $p \in M$ let $D_p \subseteq E_p$ be a linear subspace with $\dim D_p = \ell$. Define*

$$D = \bigsqcup_{p \in M} D_p \quad \text{and} \quad \pi_D = \pi_E|_D$$

Then $\pi_D : D \to M$ is a smooth subbundle if and only if the following condition is satisfied:

For every $p \in M$ there exists an open neighborhood $U \subseteq M$ around p and smooth local sections $\sigma_1, \dots, \sigma_\ell \in \Gamma(E|_U)$ such that $(\sigma_1(q), \dots, \sigma_\ell(q))$ forms a basis for D_q for every $q \in U$. (5)

This condition says, in other words, that D is locally spanned by sections of E.

Proof. First suppose that $D \subseteq E$ is a smooth subbundle. Then around every $p \in M$ we can find a local trivialization (U, ϕ) for D, which gives us an associated local frame for D over U, say $(\tau_1, \dots, \tau_\ell)$ where each $\tau_j : U \subseteq M \to D$ is a local section of D. Thus the local sections for E that satisfy condition (5) can be defined as $\sigma_j = i \circ \tau_j : U \to E$.

Conversely, suppose that D satisfies condition (5), and say $\operatorname{rank} E = k$. Note that D satisifies conditions (ii) and (iii) in the definition of subbundle by hypothesis, so we just need to check that $D \subseteq E$ is an embedded submanifold and that $\pi_D : D \to M$ is a vector bundle.

To see that D is an embedded submanifold, take any $p \in M$ and let $(\sigma_1, \dots, \sigma_\ell)$ be an array of smooth local sections for E defined on an open neighborhood $U \subseteq M$ around p, as given by condition (5). Complete this array to get a smooth local frame $(\sigma_1, \dots, \sigma_\ell, \sigma_{\ell+1}, \dots, \sigma_k)$ for E on U, and let $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ be the associated local trivialization for E. Thus, by definition,

$$\phi(a^1\sigma_1(q) + \cdots + a^k\sigma_k(q)) = (q, (a^1, \dots, a^k))$$

for every $q \in U$. By definition of $(\sigma_1, \dots, \sigma_k)$ extending $(\sigma_1, \dots, \sigma_\ell)$, the diffeomorphism ϕ restricts to a diffeomorphism

$$\phi : D \cap \pi^{-1}(U) \xrightarrow{\cong} \{(q, (a^1, \dots, a^\ell, 0, \dots, 0))\} \subseteq U \times \mathbb{R}^k$$

so the fact that $\mathbb{R}^\ell$ is an embedded submanifold of $\mathbb{R}^k$ implies that $D \cap \pi^{-1}(U)$ is an embedded submanifold of $\pi^{-1}(U)$. Repeating this argument in a neighborhood of every point $p \in M$, we conclude that D is an embedded submanifold of E.

As for the assertion that $\pi_D : D \to M$ defines a vector bundle, construct local trivializations for D by

$$\begin{aligned} &\psi : D \cap \pi^{-1}(U) \to U \cap \mathbb{R}^\ell \\ &\psi(a^1\sigma_1(q) + \cdots + a^\ell\sigma_\ell(q)) = (q, (a^1, \ldots, a^k)) \end{aligned}$$

for every $q \in U$. We conclude that $\pi_D : D \to M$ is a vector bundle, hence a subbundle of E. ■

Example 10 (Subbundles)**.**

(a) Let M be a smooth manifold and X a nonvanishing smooth vector vfield on M. Then $D \subseteq TM$ given by $D_p = \operatorname{span}(X_p)$ for every $p \in M$ defines a smooth 1-dimensional subbundle of TM.

(b) Slightly more generally, a smooth k-dimensional subbundle of TM defined as the span of k pointwise linearly independent smooth vector fields on M,

$$D = \operatorname{span}\{X_1, \ldots, X_k\} \subseteq TM$$

is called a rank-k *distribution* on M.

(c) Tangent bundle of an immersed submanifold defines a subbundle of the restriction.

Linear algebra tells us that kernels and images of linear maps are linear subspaces. Thus it's natural to wonder whether taking a bundle homomorphism, restricting to each fiber, then gluing together the kernels (or images) will produce a subbundle. However, this can go wrong in a very trivial way: the kernels or images of a bundle homomorphism do not necessarily all have the same dimension (i.e. the homomorphism may not necessarily have the same rank on each fiber). Obviously, this would preclude us from combining the kernels or images to form a subbundle. This can be illustrated with a few simple examples:

Example 11.

(a) Consider the trivial line bundle $E = [0,1] \times \mathbb{R} \to [0,1]$ and define a bundle homomorphism

$$\phi : E \to E, \quad \phi(x,t) = (x, tx)$$

Then $(x,t) \in \operatorname{Ker} E$ means that (x,t) is a pair such that $tx = 0$. Thus we have two cases for the fibers of E,

$$\begin{cases} x = 0 \Rightarrow tx = 0 & \text{for any } t \in \mathbb{R} \\ x \neq 0 \Rightarrow tx = 0 & \text{for } t = 0 \text{ only} \end{cases}$$

so the kernel over $x = 0$ is 1-dimensional and the kernel over any $x \neq 0$ is 0-dimensional. It's clear from this example that $\operatorname{Ker}\phi$ is not a subbundle of ϕ (its fibers do not even have the same dimension) and, of course, ϕ does not have constant rank.

(b) Consider the trivial rank-2 bundle over the circle, $S^1 \times \mathbb{R}^2 \to S^1$. For each $\theta \in S^1$ define a linear map $f_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ on the θ-fiber by fixing e_1 and rotating e_2 through an angle of θ. Using these linear maps on the fibers, we get a bundle homomorphism

$$\begin{aligned} f &: S^1 \times \mathbb{R}^2 \to S^1 \times \mathbb{R}^2 \\ f(\theta, v) &= (\theta, f_\theta(v)) \end{aligned}$$

It's easy to check that

$$\begin{aligned} \theta = 0 &\Rightarrow \ker f_0 = \{y = -x\} \\ \theta = \pi &\Rightarrow \ker f_\pi = \{y = x\} \\ \theta \neq 0, \pi &\Rightarrow \ker f_\theta = 0 \end{aligned}$$

so f does not have constant rank, and $\ker f$ is not a subbundle.

To be precise: given two vector bundles $E \to M$ and $E' \to M$ over M and $F : E \to E'$ a bundle homomorphism over M, for each $p \in M$ the ***rank of*** F ***at*** p is the rank of the linear map $F|_{E_p} : E_p \to E'_p$. Say that F has ***constant rank*** if it has the same rank at every point $p \in M$.

Fortunately, it just so happens that the aforementioned failure of constant rank is the only way the kernels or images can fail to assemble a subbundle. This is the content of the following fact:

Fact 18. *Let $E \to M$ and $E' \to M$ be vector bundles over M and let $F : E \to E'$ be a vector bundle homomorphism. Then the following are equivalent:*

(i) F has constant rank.

(ii) $\operatorname{Ker} F$ *is a subbundle of E*

(iii) $\operatorname{Im} F$ *is a subbundle of E'*

Another method for constructing interesting subbundles:

Fact 19 (Orthogonal complement bundles)**.** *Let $M \subseteq \mathbb{R}^n$ be an immersed submanifold of Euclidean space, and let $D \subseteq T\mathbb{R}^n|_M$ be a rank-k subbundle. Define*

$$D^\perp = \bigsqcup_{p \in M} D_p^\perp \subseteq$$

Then $D^\perp \to M$ is a rank-$(n-k)$ subbundle of $T\mathbb{R}^n|_M$ called the **orthogonal complement** *of D.*

Proof. Gram-Schmidt process at the level of fibers plus the local frame criterion. ∎

Example 12 (The vertical bundle)**.** Let $\pi : E \to M$ be a vector bundle, and let $\pi_{TE} : TE \to E$ and $\pi_{TM} : TM \to M$ be projection maps for the tangent bundles. Then $d\pi : TE \to TM$ is a bundle homomorphism over π and it has constant rank because π is a submersion, so the kernel

$$\operatorname{Ker} d\pi = \bigsqcup_{u \in E} \ker d\pi_u \subseteq TE$$

is a submanifold of TE which we denote by $VE = \operatorname{Ker} d\pi$. Fact 18 tells us that this gives us a vector bundle over E

$$\pi_V = (\pi_{TE})|_{VE} : VE \to E$$

defined by $\pi_V(u, \xi) = u$ for every $u \in E$ and $\xi \in \ker d\pi_u$, and that this is a subbundle of $\pi_{TE} : TE \to E$. We call $\pi_V : VE \to E$ the ***vertical bundle*** of E. Note that the rank of the vertical bundle VE is equal to the rank of E: for any $u \in E$ the fiber $V_uE = \ker d\pi_u \subseteq T_uE$ has dimension

$$\dim \ker d\pi_u = \dim T_uE - \operatorname{rank} d\pi_u = \dim E - \dim T_{\pi(u)}M = \operatorname{rank} E$$

where we have used the fact that $d\pi_u$ has full rank because π is a submersion. Thus $\operatorname{rank} VE = \operatorname{rank} E$.

6 Hom-Gamma correspondence

Let $E \to M$ and $E' \to M$ be two vector bundles and let $\beta : \Gamma(E) \to \Gamma(E')$ be a linear operator between global section spaces. We define the following two notions:

1. β is a ***local operator*** if the action of β is locally determined, i.e. if for any two sections $s_1, s_2 \in \Gamma(E)$ we have

$$s_1|_U = s_2|_U \;\Rightarrow\; \beta(s_1)|_U = \beta(s_2)|_U$$

for every open subset $U \subseteq M$. Since β is linear, this is equivalent to saying that

$$s|_U = 0 \;\Rightarrow\; \beta(s)|_U = 0$$

for every open subset $U \subseteq M$.

2. β is a ***point operator*** if the action of β is determined pointwise, i.e. if for any two sections $s_1, s_2 \in \Gamma(E)$ we have

$$s_1(p) = s_2(p) \;\Rightarrow\; \beta(s_1)(p) = \beta(s_2)(p)$$

for every point $p \in M$. Since β is linear, this is equivalent to saying that

$$s(p) = 0 \;\Rightarrow \beta(s)(p) = 0$$

for every $p \in M$.

It's clear that every point operator is a local operator, but the converse is not true.

Example 13 (Local operators and point operators)**.**

(a) Differentiation on $\mathbb{R}$ is a local operator on smooth functions:

$$\frac{d}{dt} : \; C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$$
$$f(t) \mapsto f'(t)$$

because the slope of the tangent line at a point $(t_0, f(t_0))$ is determined by the values $f(t)$ for t in an arbitrarily small neighborhood around t_0. It's easy to construct simple examples showing that differentiation is *not* a point operator; for example, $f(t) = t$ and $g(t) = t^2$ both have $f(1) = 1 = g(1)$ but $f'(1) = 1 \neq 2 = g'(1)$.

(b) More generally, vector fields $X \in \Gamma(TM)$ acting as derivations on $C^\infty(M)$ are local operators but not point operators.

(c) The exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is a local operator but not a point operator.

In general, we should expect that operators involving differentiation will be local operators but *not* point operators. This is essentially because differentiation is not a $C^\infty(M)$-linear operation (instead, it follows the product rule).

Fact 20. *Let $E \to M$ and $E' \to M$ be two vector bundles, and let $\beta : \Gamma(E) \to \Gamma(E')$ be an $\mathbb{R}$-linear map. Then β acts pointwise if and only if it is $C^\infty(M)$-linear, meaning $\beta(fs) = f\beta(s)$ for every $f \in C^\infty(M)$ and $s \in \Gamma(E)$.*

Proof. Suppose that β is $C^\infty(M)$-linear.

- We will show first that β acts locally. Suppose $\sigma \equiv 0$ in some open subset $U \subseteq M$. Given $p \in U$, let $\psi \in C^\infty(M)$ be a smooth bump function supported in U with $\psi(p) = 1$. Then $\psi\sigma \equiv 0$ on M because σ vanishes inside U and ψ vanishes outside U. Since β is $C^\infty(M)$-linear we have

$$0 = \beta(\psi\sigma) = \psi\beta(\sigma)$$

  and evaluating at $p \in M$ yields $\beta(\sigma)(p) = 0$. Since this holds for every $p \in U$, we conclude that $\beta(\sigma) = 0$ on U and thus β acts locally.

- Now let's show that β actually acts pointwise. Suppose that $\sigma(p) = 0$. Let $(\sigma_1, \ldots, \sigma_k)$ be a local frame for E in some open neighborhood U of p in M. Write

$$\sigma = \sum_i f^i \sigma_i$$

  with respect to this frame; each $f^i : U \to \mathbb{R}$ is smooth. Since $\sigma(p) = 0$ we deduce that $f^i(p) = 0$ for each $1 \leq i \leq k$.

  By the local section extension lemma (Lemma 2), we can find smooth global sections $\widetilde{\sigma}_i \in \Gamma(E)$ extending σ_i, and smooth functions $F^i \in C^\infty(M)$ extending f^i, in a neighborhood of p. In particular $F^i(p) = 0$ for each $1 \leq i \leq k$. Thus $\sigma = \sum_i F^i \sigma_i$ in a neighborhood of p, and β is a local operator, so

$$\begin{aligned} \beta(\sigma)(p) &= \beta\left(\sum_i F^i \sigma_i\right)(p) \\ &= \sum_i F^i(p) \beta\left(\widetilde{\sigma}_i\right)(p) \\ &= 0 \end{aligned}$$

  (where in the second line we again used that β is $C^\infty(M)$-linear) and we conclude that β acts pointwise.

  Conversely, suppose that β acts pointwise and take any $f \in C^\infty(M)$. We want to show that $\beta(fs) = f\beta(s)$ for any $s \in \Gamma(E)$; that is to say,

$$\beta(fs)(p) = (f\beta(s))(p)$$

for every section s and $p \in M$. First of all, note that we have

$$(f\beta(s))(p) = f(p)\beta(s)(p) \tag{6}$$

simply by definition. Set $s_2 = fs - f(p)s \in \Gamma(E)$, then

$$s_2(p) = f(p)s(p) - f(p)s(p) = 0$$

implies that $\beta(s_2)(p) = 0$ since β acts pointwise. Thus

$$\beta(fs)(p) = \beta(f(p)s)(p) \tag{7}$$

for every $p \in M$, and we conclude that

$$\beta(fs)(p) \overset{(7)}{=} \beta(f(p)s)(p) = f(p)\beta(s)(p) \overset{(6)}{=} (f\beta(s))(p)$$

hence β is $C^\infty(M)$-linear as desired.

■

Remark. Here's how Fact 20 typically comes up. Say we have a point operator $\beta : \Gamma(E) \to \Gamma(E')$. Then it makes good sense to have β act directly on vectors in the fibers of E, because those vectors could be extended to sections in an arbitrary way. So, for example if we have a tensor field $A \in \Gamma(T^kT^*M)$, then (as one can readily verify) it defines a map $A : \mathfrak{X}^k(M) \to C^\infty(M)$ via the formula

$$A(X_1, \dots, X_k)(p) = A_p(X_1|_p, \dots, X_k|_p)$$

which is $C^\infty(M)$-linear in each component. By Fact 20 this map acts pointwise in each component, and consequently it makes good sense to write $A(v_1, \dots, v_k)$ for any vectors $v_1, \dots, v_k \in T_pM$.

Local operators are well-behaved with respect to restriction to open subsets.

Fact 21 (Restriction of a local operator)**.** *Local operators are well-behaved with respect to restriction to open subsets.*

We can glue together local operators defined on open subsets to produce a well-defined local operator which is defined globally.

Fact 22. *Let $E \to M$ and $E' \to M$ be vector bundles and let $\{U_i\}_{i \in I}$ be an open cover of M. Suppose we have a collection of local operators*

$$\{\beta_i : \Gamma(E|_{U_i}) \to \Gamma(E'|_{U_i}) : i \in I\}$$

such that

$$\beta_i|_{U_i \cap U_j} = \beta_j|_{U_i \cap U_j} \quad \textit{whenever} \quad U_i \cap U_j \neq \varnothing.$$

Then there exists a unique local operator $\beta : \Gamma(E) \to \Gamma(E')$ satisfying $\beta|_{U_i} = \beta_i$ for every $i \in I$.

Corollary 2. *Local operators can be reconstructed from all of their restrictions. They are completely determined by their local restrictions.*

Let $E \to M$ and $E' \to M$ be vector bundles with corresponding global section spaces $\Gamma(E)$ and $\Gamma(E')$. Any bundle homomorphism $F : E \to E'$ induces a map $F_* : \Gamma(E) \to \Gamma(E')$ given by $F_*(s) = F \circ s$; the latter is clearly a smooth section of E'. Moreover, F_* is clearly linear over $\mathbb{R}$ because F is linear on fibers; in fact, F_* is actually linear over $C^\infty(M)$ because for any $u_1, u_2 \in C^\infty(M)$ and $s_1, s_2 \in \Gamma(E)$ we calculate

$$\begin{aligned} F_*(u_1 s_1 + u_2 s_2) &= F \circ (u_1 s_1 + u_2 s_2) \\ &= u_1(F \circ s_1) + u_2(F \circ s_2) \\ &= u_1 F_*(s_1) + u_2 F_*(s_2) \end{aligned}$$

since F is linear on each fiber. As a result, we have defined a map

$$\left\{ \begin{matrix} \text{Bundle homomorphisms} \\ E \to E' \end{matrix} \right\} \xrightarrow{\Phi} \left\{ \begin{matrix} C^\infty - \text{linear maps} \\ \Gamma(E) \to \Gamma(E') \end{matrix} \right\} \tag{8}$$

given by $\Phi(F) = F_*$. In fact, we will show that this is a bijective correspondence; any $C^\infty(M)$-linear map between sections is induced by a unique bundle homomorphism. Therefore, we can write equation (8) symbolically as an isomorphism of $C^\infty(M)$-modules,

$$\Gamma(\mathrm{Hom}(E, E')) \simeq \mathrm{Hom}(\Gamma(E), \Gamma(E'))$$

where the left-hand follows from Fact 13, and on the right-side the "Hom" stands for morphisms in the category of $C^\infty(M)$-modules. In other words, interchanging the symbols Hom and Γ is mathematically justified. This is the reason for the name "Hom-Gamma correspondence".

Lemma 3. *Let $\pi : E \to M$ be a vector bundle, take any $p \in M$ and $v \in E_p$. Then there exists a global section $s \in \Gamma(E)$ with $s(p) = v$.*

Proof. It suffices to construct a local section $s : U \subseteq M \to E$ in a neighborhood U around p with $s(p) = v$, then we can use a smooth bump function supported on U to extend s and get the desired global section.

With this in mind, take a local trivialization $\phi : E|_U \to U \times \mathbb{R}^k$ in some neighborhood U of p, and take the associated local frame $(\sigma_1, \dots, \sigma_k)$ on U. Then we can express v uniquely in terms of this local frame as

$$v = v_1 \sigma_1(q) + \cdots v_k \sigma_k(q)$$

for some $q \in U$ and $v_i \in \mathbb{R}$. Define $s : U \subseteq M \to E|_U$ by

$$s(p) = v_1 \sigma_1(p) + \cdots + v_k \sigma_k(p)$$

for every $p \in U$. Clearly $s(p) = v$ and s is a smooth local section as a linear combination of smooth local sections. ■

In order to show that the above map Φ is a bijective correspondence, we need to show that each $C^\infty(M)$-linear map $\beta : \Gamma(E) \to \Gamma(E')$ comes from a unique bundle homomorphism. The first step is to show that this is true at the level of fibers, which is the content of the following lemma.

Lemma 4. *Let $\pi : E \to M$ and $\pi' : E' \to M$ be two vector bundles over M. Let $\beta : \Gamma(E) \to \Gamma(E')$ be a $C^\infty(M)$-linear map. Then for every $p \in M$ there exists a unique linear map $F_p : E_p \to E'_p$ such that*

$$F_p(s(p)) = \beta(s)(p)$$

for every $s \in \Gamma(E)$.

Proof. Fix $p \in M$ and $v \in E_p$. By Lemma 3 we can find a section $s \in \Gamma(E)$ with $s(p) = v$, so we can define $F_p(v) = \beta(s)(p)$. Note that F_p is well-defined (independently of the choice of s) because β is $C^\infty(M)$-linear (hence it acts pointwise). It is straightforward that:

- By definition, F_p satisfies the desired condition.
- F_p is uniquely defined by the desired condition.
- F_p is a linear map: for any $v_1, v_2 \in E_p$ and $c \in \mathbb{R}$ write $s_1(p) = v_1$ and $s_2(p) = v_2$ for some sections s_1 and s_2, then $(cs_1 + s_2)(p) = cv_1 + v_2$ and

$$\begin{aligned} F_p(cv_1 + v_2) &= \beta(cs_1 + s_2)(p) \\ &= c\beta(s_1)(p) + \beta(s_2)(p) \\ &= cF_p(v_1) + F_p(v_2) \end{aligned}$$

■

Now we can prove the aforementioned correspondence quite easily, just by piecing together the linear maps F_p given by Lemma 4.

Theorem 1 (Hom-Gamma correspondence theorem)**.** *The map Φ in diagram (8) given by $\Phi(F) = F_*$, sending any bundle homomorphism to its induced map on sections, defines a bijective correspondence*

$$\Gamma(\mathrm{Hom}(E, E')) \simeq \mathrm{Hom}(\Gamma(E), \Gamma(E')).$$

Proof. First we check that Φ is surjective. Take any $C^\infty(M)$-linear map $\beta : \Gamma(E) \to \Gamma(E')$, and for any $p \in M$ define $F_p : E_p \to E'_p$ as in Lemma 4. Then define $F : E \to E'$ by the formula

$$F(p, v) = F_p(v)$$

for every $p \in M$ and $v \in E_p$. By construction this map satisfies

$$F_*(s)(p) = (F \circ s)(p) = F(p, s(p)) = F_p(s(p)) = \beta(s)(p)$$

hence $\beta = F_* = \Phi(F)$ as claimed. To see that Φ is injective, suppose that $\Phi(F) = \Phi(F')$ i.e. $F_* = F'_*$, then for every $p \in M$ and $v \in E_p$ we find $s \in \Gamma(E)$ with $s(p) = v$, and determine that

$$F(v) = F(s(p)) = (F \circ s)(p) = F_*(s) = F'_*(s) = \cdots = F'(v)$$

so $F = F'$ and Φ is injective. ■

A 1-form $\omega \in \Omega^1(M)$ acts on tangent vectors to produce real numbers,

$$\omega : (p, v) \in T_pM \mapsto \omega_p(v) \in \mathbb{R}$$

so if we remove the specified point $p \in M$ from both sides, an implicit dependence upon $p \in M$ arises, and we get a map

$$\widetilde{\omega} : \mathfrak{X}(M) \to C^\infty(M)$$

where $\widetilde{\omega}(X)(p) = \omega_p(X_p)$. Thus the 1-form ω can be thought of as a map $\mathfrak{X}(M) \to C^\infty(M)$, and no information is lost from doing so. Note that, from this perspective, for any $f \in C^\infty(M)$ we have

$$\omega(fX)(p) = \omega_p(f(p)X_p) = f(p)\omega_p(X_p) = (f\omega)(X)(p)$$

so $\omega(fX) = f\omega(X)$ and ω is $C^\infty(M)$-linear. More generally, we say that a k-form $\omega \in \Omega^k(M)$ induces a map $\widetilde{\omega} : \mathfrak{X}(M)^k \to C^\infty(M)$ given by the formula

$$\widetilde{\omega}(X_1, \dots, X_k)(p) = \omega_p\left(X_1|_p, \dots, X_k|_p\right)$$

which is $C^\infty(M)$-multilinear. Conversely, this $C^\infty(M)$-multilinearity property is enough to establish that a map acting on vector fields comes from a tensor field; for this latter direction suppose we are given a $C^\infty(M)$-multilinear map $\widetilde{\omega} : \mathfrak{X}(M)^k \to C^\infty(M)$ and we want to construct a k-form $\omega \in \Omega^k(M)$ inducing it. The natural choice is to define $\omega_p \in \Lambda^k(T_p^*M)$ by the formula

$$\omega_p(v_1, \dots, v_k) = \widetilde{\omega}(V_1, \dots, V_k)(p)$$

where each V_i is a vector field on M extending $v_i \in T_pM$. The fact that this is well-defined independently of the choices of extensions is exactly where the $C^\infty(M)$-linearity assumption on $\widetilde{\omega}$ comes into play. We summarize all of this with the following fact:

Fact 23. *A map $\widetilde{A} : \mathfrak{X}(M)^k \to C^\infty(M)$ is induced by a smooth covariant k-tensor field $A \in \Gamma(T^k(T^*M))$ if and only if it is multilinear over $C^\infty(M)$.*

Proof. One direction is clear: a smooth covariant k-tensor field induces such a $C^\infty(M)$-multilinear map. Conversely, suppose that we are given a $C^\infty(M)$-multilinear map $\widetilde{A} : \mathfrak{X}(M)^k \to C^\infty(M)$. Since $\widetilde{A}$ is $C^\infty(M)$-multilinear, each of its components $\widetilde{A}_i : \mathfrak{X}(M) \to C^\infty(M)$ acts pointwise by Fact 20. Thus we can define a tensor field $A : M \to T^kT^*M$ by

$$A_p(v_1, \dots, v_k) = \widetilde{A}(V_1, \dots, V_k)(p)$$

for any $p \in M$ and $v_1, \dots, v_k \in T_pM$, where V_i is a smooth global vector field on M extending v_i. This definition is independent of the choices of extensions because $\widetilde{A}$ acts pointwise in each component, and we conclude that $\widetilde{A}$ is induced by the tensor field A. ■

In everything that follows we will simply identify a tensor field A with its associated $C^\infty(M)$-multilinear map $\widetilde{A}$ and denote them both by A. We will pass freely between the two ways of thinking about the map.

Fact 24. *A map $A : \mathfrak{X}(M)^k \to C^\infty(M)$ is induced by a differential k-form $A \in \Omega^k(M)$ if and only if it is alternating and multilinear over $C^\infty(M)$.*

Fact 25. *A map*

$$A : \Omega^1(M)^h \times \mathfrak{X}(M)^k \to C^\infty(M)$$

is induced by a mixed (h, k)-tensor field if and only if it is multilinear over $C^\infty(M)$.

7 Bundle-valued differential forms

In this section we extend the notion of differential form to that of vector-valued and bundle-valued differential forms. Let M be a smooth manifold. Differential k-forms on M are sections $\omega \in \Omega^k(M) = \Gamma(\Lambda^k T^*M)$, i.e. smooth maps

$$\begin{aligned} \omega :& M \to \Lambda^k(T^*M) \\ & p \mapsto \omega_p \in \Lambda^k(T^*_pM) \end{aligned}$$

i.e. the values are (alternating covariant) k-tensors $\omega_p : (T_pM)^k \to \mathbb{R}$ on tangent spaces. But we could define an analogous map whose values lie in any vector space (instead of just $\mathbb{R}$), or on the fibers of any vector bundle. This idea leads immediately to the concept of bundle-valued differential forms: take any vector bundle $E \to M$, then we consider in analogous fashion smooth maps

$$\begin{aligned} \alpha :& M \to \Lambda^k(T^*M) \otimes E \\ & p \mapsto \alpha_p \in \Lambda^k(T^*_pM) \otimes E_p \end{aligned}$$

i.e. the values are (alternating covariant) k-tensors $\alpha_p : (T_pM)^k \to E_p$. In other words, α is a section

$$\alpha \in \Gamma(\Lambda^k(T^*M) \otimes_{\mathbb{R}} E) \simeq \Omega^k(M) \otimes_{C^\infty(M)} \Gamma(E).$$

These sections are called ***E-valued differential k-forms on M***, or E-valued forms in brief. We denote the space of such sections by the notation

$$\Omega^k(M; E) = \Gamma(\Lambda^k T^*M \otimes E) \simeq \Omega^k(M) \otimes \Gamma(E).$$

Remark. Notice the subscripts in the tensor products above. We emphasize that the first tensor product is taken between $\mathbb{R}$-modules, whereas the second tensor product is taken between $C^\infty(M)$-modules, and the equivalence is between $C^\infty(M)$-modules (see Fact 10). In general we will omit these subscripts, hopefully the meaning will be clear from the context.

In the same way that (real-valued) differential forms act on vector fields to produce functions, bundle-valued differential forms act on vector fields to produce sections. Namely, α induces a smooth alternating map

$$\begin{aligned} &\alpha : \mathfrak{X}(M)^k \to \Gamma(E) \\ &\alpha(X_1, \dots, X_k)(p) = \alpha_p(X_1|_p, \dots, X_k|_p) \in E_p \end{aligned}$$

which is clearly $C^\infty(M)$-linear in each of its components, for any $f \in C^\infty(M)$ we have

$$\begin{aligned} \alpha(fX_1, \dots, X_k)(p) &= \alpha_p(f(p)X_1|_p, \dots, X_k|_p) \\ &= f(p)\alpha_p(X_1|_p, \dots, X_k|_p) \end{aligned}$$

i.e. $\alpha(fX_1, \dots, X_k) = f\alpha(X_1, \dots, X_k)$, and the same argument holds for every other component. The following Fact shows that these two ways of looking at α are completely equivalent, so there's no harm in denoting them by the same symbol.

Fact 26. *Let $E \to M$ be any vector bundle and suppose we have an alternating $\mathbb{R}$-linear map*

$$\widetilde{A} : \mathfrak{X}(M)^k \to \Gamma(E)$$

then $\widetilde{A}$ is induced by a section $A \in \Omega^k(M; E)$ if and only if $\widetilde{A}$ is $C^\infty(M)$-multilinear.

Proof. Suppose we have an alternating $C^\infty(M)$-multilinear map $\widetilde{A} : \mathfrak{X}(M)^k \to \Gamma(E)$. Then by Fact 20 each of its components $\widetilde{A}_i : \mathfrak{X}(M) \to \Gamma(E)$ is a point operator, so for every $p \in M$ we can define an alternating multilinear map

$$\begin{aligned} &A_p : (T_pM)^k \to E_p \\ &A_p(v_1, \dots, v_k) = \widetilde{A}(V_1, \dots, V_k)(p) \end{aligned}$$

where each V_i is a vector field extending $v_i \in T_pM$. This definition is independent of the choices of extension because $\widetilde{A}$ acts pointwise in each component. Thus

$$A_p \in \mathrm{Alt}((T_pM)^k, E_p) \simeq \Lambda^k(T_p^*M) \otimes E_p$$

so $\widetilde{A}$ is induced by the section

$$\begin{aligned} A : &M \to \Lambda^k(T^*M) \otimes E \\ &p \mapsto A_p \in \Lambda^k(T_p^*M) \otimes E_p \end{aligned}$$

Note that we have used the linear isomorphisms from Facts 7, 2, 3 to get the isomorphism

$$\mathrm{Alt}((T_pM)^k, E_p) \simeq \mathrm{Hom}(\Lambda^k T_pM, E_p) \simeq (\Lambda^k T_pM)^* \otimes E_p \simeq \Lambda^k(T_p^*M) \otimes E_p.$$

Conversely, suppose that we have a section $A \in \Gamma(\Lambda^k(T^*M) \otimes E)$. Then A induces a smooth, alternating map

$$\begin{aligned} &\widetilde{A} : \mathfrak{X}(M)^k \to \Gamma(E) \\ &\widetilde{A}(X_1, \dots, X_k)(p) = A_p(X_1|_p, \dots, X_k|_p) \in E_p \end{aligned}$$

and it's easy to see that A is $C^\infty(M)$-multilinear. ∎

Let's see how this works in local coordinates. Given $\alpha \in \Omega^k(M; E)$, take local frames (dx^I) for $\Lambda^k(T^*M)$ and (e_j) for E over $U \subseteq M$. Then over U we can write

$$\alpha = \sum_{I,j} \alpha_I^j dx^I \otimes e_j$$

for some smooth coefficient functions $\alpha_I^j \in C^\infty(U)$. Equivalently, we can write

$$\alpha = \sum_j \omega^j \otimes e_j$$

for some k-forms $\omega^j \in \Omega^k(U)$. This is convenient, for example, when we do not wish to specify a local frame for $\Lambda^k(T^*M)$. The E-valued k-forms looking like $\omega \otimes s$ where $\omega \in \Omega^k(M)$ and $s \in \Gamma(E)$ are called ***decomposable***. Thus, the local coordinate expressions above tell us that $\Omega^k(M;E)$ is locally generated by decomposable k-forms. In local coordinates, the action on vector fields looks like

$$\alpha(X_1,\dots,X_k) = \sum_{I,j} \alpha_I^j dx^I(X_1,\dots,X_k)e_j \in \Gamma(E).$$

In the proof of Fact 26 we stuck to the level of fibers because it felt most natural to me personally, but the space $\Omega^k(M;E)$ can be identified with the space of alternating $C^\infty(M)$-multilinear maps $\mathfrak{X}(M)^k \to \Gamma(E)$ through a sequence of global isomorphisms. Namely, we have

$$\begin{aligned}\Omega^k(M;E) &= \Gamma(\Lambda^k(T^*M)\otimes E) \;\text{ by definition}\\ &\simeq \Gamma(\mathrm{Hom}(\Lambda^k TM, E)) \;\text{ by Fact 15}\\ &\simeq \mathrm{Hom}(\Gamma(\Lambda^k TM),\Gamma(E)) \;\text{ by Theorem 1}\end{aligned}$$

and this latter space is exactly the space of alternating $C^\infty(M)$-multilinear maps $\mathfrak{X}(M)^k \to \Gamma(E)$.

Example 14 (Bundle-valued forms)**.**

(a) If $E = M\times\mathbb{R}$ is the rank-1 trivial bundle, then $\Gamma(E) = C^\infty(M)$ and therefore $\Omega^\bullet(M;E) \simeq \Omega^\bullet(M)\otimes C^\infty(M) = \Omega^\bullet(M)$. So, unsurprisingly, we recover the old notion of a differential form.

(b) Slightly more generally, if V is any finite-dimensional real vector space and $E = M\times V$ is the trivial bundle with fibers given by $E_p \simeq V$, then $\Gamma(E) = C^\infty(M;V)$ i.e. smooth maps on M with values in V. Thus, $\alpha\in\Omega^k(M;V)$ is a vector-valued differential form

$$\begin{aligned}\alpha : M &\to \Lambda^k(T^*M)\otimes(M\times V)\\ p &\mapsto \alpha_p \in \Lambda^k(T_p^*M)\otimes V\end{aligned}$$

i.e. $\alpha_p : (T_pM)^k \to V$ (multilinear, alternating) for every $p\in M$

(c) Let $\mathfrak{g}$ be any finite-dimensional Lie algebra. In particular $\mathfrak{g}$ is a finite-dimensional vector space, so we can apply (b) and consider the vector bundle $E = M\times\mathfrak{g}$ with fibers given by $E_p \simeq \mathfrak{g}$. Then $\alpha\in\Omega^k(M;\mathfrak{g})$ is a Lie algebra-valued differential form

$$\begin{aligned}\alpha : M &\to \Lambda^k(T^*M)\otimes(M\times\mathfrak{g})\\ p &\mapsto \alpha_p \in \Lambda^k(T_p^*M)\otimes\mathfrak{g}\end{aligned}$$

i.e. $\alpha_p : (T_pM)^k \to \mathfrak{g}$ (multilinear, alternating) for every $p\in M$. Here is one special thing that happens when working with Lie algebra-valued differential forms: since $\mathfrak{g}$ is equipped with a bilinear form (the Lie bracket $[\cdot,\cdot]$), we can define a wedge product between Lie algebra-valued forms, given by

$$\begin{aligned}\wedge : \Omega^k(M;\mathfrak{g})\times\Omega^\ell(M;\mathfrak{g}) &\to \Omega^{k+\ell}(M;\mathfrak{g})\\ (\alpha\otimes X)\wedge(\beta\otimes Y) &= (\alpha\wedge\beta)\otimes[X,Y]\end{aligned}$$

for every $\alpha,\beta\in\Omega^\bullet(M)$ and $X,Y\in C^\infty(M;\mathfrak{g})$. This is special because in general there is no natural wedge product on bundle-valued forms (see the discussion below).

(d) Let $E \to M$ be a vector bundle and $\operatorname{End} E \to M$ the associated bundle of endomorphisms. Then an endomorphism-valued form is a section $A \in \Omega^k(M; \operatorname{End} E)$,

$$\begin{aligned} A : M &\to \Lambda^k(T^*M) \otimes \operatorname{End} E \\ p &\mapsto A_p \in \Lambda^k(T_p^*M) \otimes \operatorname{End} E_p \end{aligned}$$

i.e. $A_p : (T_pM)^k \to \operatorname{End} E_p$ (multilinear, alternating) for every $p \in M$.

(e) The most important objects we study in these notes are covariant derivatives (aka connections) which are linear operators acting on bundle-valued forms, $\nabla : \Omega^k(M; E) \to \Omega^{k+1}(M; E)$. We will see in the next few sections that the *curvature* of a connection can be considered as an endomorphism-valued 2-form $\nabla^2 \in \Omega^2(M; \operatorname{End} E)$ in the sense that its action on any E-valued form is given by wedge product with an endomorphism-valued 2-form.

Now that we have the notion of a bundle-valued differential form, we would like to extend the usual constructions to this setting. Most importantly the wedge product and the the exterior derivative.

In general there is no natural way to perform a wedge product of two bundle-valued differential forms, because there's no natural product on an arbitrary vector space (or on the fibers of a vector bundle). We can perform the usual exterior product on the "differential form part" of a bundle-valued form, which gives us one-sided wedge products (defined first on decomposable elements)

$$\begin{aligned} \wedge : \Omega^k(M; E) \times \Omega^\ell(M) &\to \Omega^{k+\ell}(M; E) \\ (\omega \otimes s) \wedge \theta &= (\omega \wedge \theta) \otimes s \\ \wedge : \Omega^k(M) \times \Omega^\ell(M; E) &\to \Omega^{k+\ell}(M; E) \\ \theta \wedge (\omega \otimes s) &= (\theta \wedge \omega) \otimes s \end{aligned}$$

for any $\omega, \theta \in \Omega^\cdot(M)$ and $s \in \Gamma(E)$, and then extended linearly. In fact, we do have a natural wedge product operation for $\operatorname{End}(E)$-valued differential forms, because here composition provides a natural product on endomorphisms:

$$\begin{aligned} \wedge : \Omega^k(M; \operatorname{End} E) \times \Omega^\ell(M; \operatorname{End} E) &\to \Omega^{k+\ell}(M; \operatorname{End} E) \\ (\omega \otimes \phi) \wedge (\theta \otimes \psi) &= (\omega \wedge \beta) \otimes (\phi \circ \psi) \end{aligned}$$

for every $\omega, \theta \in \Omega^\cdot(M)$ and $\phi, \psi \in \Gamma(\operatorname{End} E)$. In words, this wedge product acts by the usual exterior product in the k-form component, and by composition in the sectional component. We are remembering here that elements of $\Gamma(\operatorname{End} E)$ – sections of the endomorphism bundle – are in fact bundle endomorphisms $E \to E$ (by Fact 13).

Remark. The commutator gives a natural bracket on each fiber of $\operatorname{End} E \to M$, so we could define a wedge product on endomorphism-valued forms by

$$(\omega \otimes \phi) \wedge (\theta \otimes \psi) = (\omega \wedge \theta) \otimes [\phi, \psi]$$

which is of course different from the wedge product given by $(\omega \wedge \theta) \wedge \phi \circ \psi$. In these notes we will not use the wedge product given by the commutator bracket on $\operatorname{End} E$ and we will always mean composition in the $\operatorname{End} E$ component. In general, when reading any literature that uses endomorphism-valued forms, one should just be careful about how the wedge product is defined.

Considering endomorphism-valued differential forms also allows us to define another one-sided wedge product for $\Omega^\bullet(M;E)$, this time on the left-side. We define

$$\begin{aligned}\wedge : \Omega^k(M;\operatorname{End} E) \times \Omega^\ell(M;E) &\to \Omega^{k+\ell}(M;E)\\ (\omega \otimes \phi) \wedge (\theta \otimes s) &= (\omega \wedge \theta) \otimes \phi(s)\end{aligned}$$

for every $\omega, \theta \in \Omega^\bullet(M), \phi \in \Gamma(\operatorname{End} E)$ and $s \in \Gamma(E)$.

Fact 27. *The right-sided and left-sided wedge products*

$$\begin{aligned}\wedge : \Omega^k(M) \times \Omega^\ell(M;E) &\to \Omega^{k+\ell}(M;E)\\ \wedge : \Omega^k(M;\operatorname{End} E) \times \Omega^\ell(M;E) &\to \Omega^{k+\ell}(M;E)\end{aligned}$$

are compatible in the sense that

$$(A \wedge \omega) \wedge \alpha = A \wedge (\omega \wedge \alpha)$$

for every $A \in \Omega^\bullet(M;\operatorname{End} E), \omega \in \Omega^\bullet(M)$, and $\alpha \in \Omega^\bullet(M;E)$.

Proof. This follows directly by applying the definitions to decomposable forms. To wit,

$$\begin{aligned}((\omega_1 \otimes \varphi) \wedge (\omega_2 \otimes s)) \wedge \omega_3 &= ((\omega_1 \wedge \omega_2) \otimes \varphi(s)) \wedge \omega_3\\ &= (\omega_1 \wedge \omega_2 \wedge \omega_3) \otimes \varphi(s)\end{aligned}$$

and similarly on the other side,

$$\begin{aligned}(\omega_1 \otimes \varphi) \wedge ((\omega_2 \otimes s) \wedge \omega_3) &= (\omega_1 \otimes \varphi) \wedge ((\omega_2 \wedge \omega_3) \otimes s)\\ &= (\omega_1 \wedge \omega_2 \wedge \omega_3) \otimes \varphi(s).\end{aligned}$$

■

We also note that the left and right wedge products of $\Omega^\bullet(M)$ with $\Omega^\bullet(M;E)$ are anticommutative in the sense that

$$\omega \wedge \alpha = (-1)^{k\ell} \alpha \wedge \omega \tag{9}$$

for every $\omega \in \Omega^k(M)$ and $\alpha \in \Omega^\ell(M;E)$. To summarize, we have natural wedge products for bundle-valued forms such that:

(i) $\Omega^\bullet(M;\operatorname{End} E)$ is an algebra over $C^\infty(M)$.

(ii) $\Omega^\bullet(M;E)$ is a left $\Omega^\bullet(M;\operatorname{End} E)$-module.

(iii) $\Omega^\bullet(M;E)$ is a left and right $\Omega^\bullet(M)$-module.

An ***antiderivation*** of E-valued forms is a local operator $D : \Omega^\bullet(M;E) \to \Omega^\bullet(M;E)$ satisfying the following two conditions:

(i) There exists a number $k \in \mathbb{N}$ called the ***degree*** of D such that D restricts to a local operator $D : \Omega^\ell(M;E) \to \Omega^{\ell+k}(M;E)$ for every $\ell \in \mathbb{N}$.

(ii) D satisfies the product rule with respect to the wedge product, i.e.

$$D(\omega \wedge \alpha) = d\omega \wedge \alpha + (-1)^{k\ell}\omega \wedge D\alpha$$
$$D(\alpha \wedge \omega) = D\alpha \wedge \omega + (-1)^{k\ell}\alpha \wedge d\omega$$

for every $\omega \in \Omega^\ell(M)$ and $\alpha \in \Omega^\bullet(M; E)$.

Note that an antiderivation is by assumption a linear map between vector spaces, because this is built into the definition of a "local operator". The "anti" part refers to the $(-1)^k$ factor, arising from the antisymmetry of the wedge product.

Example 15 (Antiderivations)**.**

(a) The Lie derivative of differential forms is an antiderivation of degree 0 because for any vector field $X \in \mathfrak{X}(M)$ it satisfies

$$\mathcal{L}_X : \Omega^\ell(M) \to \Omega^\ell(M)$$
$$\mathcal{L}_X(\omega \wedge \theta) = \mathcal{L}_X(\omega) \wedge \theta + \omega \wedge \mathcal{L}_X\theta$$

for every $\ell \in \mathbb{N}$, and $\omega, \theta \in \Omega^\bullet(M)$.

(b) The exterior derivative of differential forms is an antiderivation of degree $+1$ because it satisfies

$$d : \Omega^\ell(M) \to \Omega^{\ell+1}(M)$$
$$d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^{\deg \omega}\omega \wedge d\theta$$

for every $\ell \in \mathbb{N}$ and $\omega, \theta \in \Omega^\bullet(M)$.

(c) The interior multiplication is an antiderivation of degree -1. For any vector field $X \in \mathfrak{X}(M)$, interior multiplication with X is given by

$$i_X : \Omega^\ell(M) \to \Omega^{\ell-1}(M)$$
$$i_X(\omega)(V_1, \dots, V_{\ell-1}) = \omega(X, V_1, \dots, V_{\ell-1})$$

and it satisfies the product rule

$$i_X(\omega \wedge \theta) = (i_X\omega) \wedge \theta + (-1)^{\deg \omega}\omega \wedge (i_X\theta)$$

for every $\omega, \theta \in \Omega^\bullet(M)$.

An antiderivation of $\Omega^\bullet(M; E)$ is completely determined by its action on the degree zero bundle-valued forms (i.e. sections of E).

Fact 28. *Suppose $D_1, D_2 : \Omega^\bullet(M; E) \to \Omega^\bullet(M; E)$ are two antiderivations of the same degree. If D_1 and D_2 agree on $\Gamma(E)$, then $D_1 = D_2$.*

Proof. Say the common degree of D_1 and D_2 is $k \in \mathbb{N}$. Let (e_i) be a local frame for E over some subset $U \subseteq M$, and let $\alpha \in \Omega^\ell(U; E)$ be any bundle-valued form defined locally on U. Thus we can express α in terms of the local frame as

$$\alpha = \sum_i \omega^i \otimes e_i$$

for some ℓ-forms $\omega^i \in \Omega^\ell(U)$. Let D_1^U denote the operator D_1 restricted to $\Gamma(E|_U)$ as in Fact 21, and similarly with D_2^U. Using the product rule together with the fact that D_1 and D_2 coincide on sections of E, we have

$$\begin{aligned} D_1^U(\alpha) &= \sum_i D_1(\omega^i \otimes e_i) \\ &= \sum_i d\omega^i \otimes e_i + (-1)^{k\ell}\omega^i \otimes D_1(e_i) \\ &= \sum_i d\omega^i \otimes e_i + (-1)^{k\ell}\omega^i \otimes D_2(e_i) \\ &= D_2^U(\alpha) \end{aligned}$$

thus $D_1^U = D_2^U$. Note that in the first line we have applied D_1 to the local sections $\omega^i \otimes e_i$ since we can extend them to global sections arbitrarily and D_1 doesn't care how we do it because it's a local operator. Moreover, since D_1 is a local operator it is completely determined by the restrictions D_1^U to open subsets of M, i.e. D_1 is the unique operator obtained by gluing together the various operators D_1^U (by Fact 22). We conclude that $D_1 = D_2$. ■

Corollary 3. *Any local operator $\Gamma(E) \to \Omega^1(M; E)$ extends uniquely to a degree $+1$ antiderivation of $\Omega^\bullet(M; E)$.*

The only antiderivation we really care about in this note is the one which is uniquely determined by a connection/covariant derivative on a vector bundle. So we don't yet have any purpose for these facts about antiderivations, but they will be necessary background material for the next section. Antiderivations fit more thematically with bundle-valued forms anyway, so I decided to include them here.

We also don't have any use for the wedge products just yet, but they will play an important role in calculations involving the curvature.

8 Connections

Given a vector bundle $E \to M$ and a local frame (e_i) for E over $U \subseteq M$, we can define an exterior derivative acting on sections of E which directly generalizes the usual exterior derivative acting on forms. Namely, given any section $s \in \Gamma(E)$, write $s = \sum_i s^i e_i$ with respect to this frame and define

$$\begin{aligned} &d : \Gamma(E|_U) \to \Gamma(E|_U) \otimes \Gamma\left(T^*M|_U\right) \\ &ds = \sum_{i,j} \frac{\partial s^i}{\partial x^j} e_i \otimes dx^j \end{aligned} \tag{10}$$

We emphasize that this operator is only defined relative to the specified local frame. The expression (10) should take a vector field $X \in \Gamma(TM|_U)$ and yield a local section of E via the formula

$$ds(X) = d_X s = \sum_{i,j} X^j \frac{\partial s^i}{\partial x^j} e_i = \sum_i X(s^i) e_i \in \Gamma(E|_U). \tag{11}$$

Notice that d is $\mathbb{R}$-linear in both X and s, and for any $f \in C^\infty(U)$ we have

$$\begin{aligned}
d_{fX}(s) &= \sum_{i,j} f X^i \frac{\partial s^j}{\partial x^i} e_j = f d_X s \\
d_X(fs) &= \sum_{i,j} X^i \frac{\partial}{\partial x^i}(f s^j) e_j \\
&= \sum_{i,j} X^i \frac{\partial f}{\partial x^i} s^j e_j + f X^i \frac{\partial s^j}{\partial x^i} e_j \\
&= X(f)s + f d_X s
\end{aligned}$$

so d is C^∞-linear in X and satisfies a product rule in s. Therefore, we have

$$\begin{aligned}
d_X s &= \sum_{i,j} X^i d_{\partial_i}(s^j e_j) \\
&= \sum_{i,j} X^i \frac{\partial s^j}{\partial x^i} e_j + s^j d_{\partial_i} e_j
\end{aligned}$$

and comparing this with equation (11) we find that

$$d_{\partial_i} e_j = 0$$

for every i, j, which is to say that $de_j = 0$ for every j. Thus the sections (e_i) are "constant" with respect to this derivative operator. In fact, the requirement that d be C^∞-linear in X, that it satisfy the product rule in s, and that $de_j = 0$ for every j, are all enough to completely characterize d.

The operator d is an example of a *connection* on the vector bundle E. The general notion of a connection is obtained by taking the linearity properties of d and allowing the sections $d_{\partial_i}(e_j)$ to be nonzero, i.e. to depend upon any arbitrary collection of smooth coefficient functions. Before properly defining the notion of connection, let's take a more global look at the structure of a linear operator of the form $\beta : \Gamma(E) \to \Gamma(E) \otimes \Gamma(T^*M)$. Writing

$$\Gamma(E) \otimes \Gamma(T^*M) = \Gamma(E) \otimes \Omega^1(M) \simeq \Omega^1(M; E)$$

we see that β sends any section $s \in \Gamma(E)$ to an E-valued form $\beta(s) \in \Gamma(E) \otimes \Omega^1(M)$. Thus Fact 26 tells us that $\beta(s)$ is a $C^\infty(M)$-linear map acting on vector fields,

$$\beta(s) : \mathfrak{X}(M) \to \Gamma(E).$$

As a result, we can equivalently consider β as a map

$$\begin{aligned}
\beta &: \Gamma(E) \times \mathfrak{X}(M) \to \Gamma(E) \\
\beta(s, X) &= \beta(s)X = \beta_X s
\end{aligned}$$

which is $\mathbb{R}$-linear in the s argument and $C^\infty(M)$-linear in the X argument. This discussion motivates the following definition: a ***connection*** on a vector bundle $E \to M$ is a map

$$\nabla : \Gamma(E) \to \Gamma(E) \otimes \Gamma(T^*M)$$

written as $\nabla(s, X) = \nabla_X s$, such that

(i) ∇ is $C^\infty(M)$-linear in the X argument.

(ii) ∇ is $\mathbb{R}$-linear in the s argument.

(iii) ∇ satisfies the product rule in the s argument:

$$\nabla_X(fs) = (Xf)s + f\nabla_X s$$

for every $f \in C^\infty(M)$.

$\nabla_X s$ is also called the ***covariant derivative*** of s along X. By the preceding discussion we can also equivalently regard ∇ as a map

$$\nabla : \Gamma(E) \times \mathfrak{X}(M) \to \Gamma(E).$$

satisfying properties (i)-(iii).

Fact 29. *Let $E \to M$ be a vector bundle. A connection ∇ on E is (equivalent to) an $\mathbb{R}$-linear local operator*

$$\nabla : \Omega^0(M; E) \to \Omega^1(M; E)$$

satisfying the product rule

$$\nabla(fs) = df \otimes s + f\nabla s$$

for every $f \in C^\infty(M)$ and $s \in \Gamma(E)$.

Proof. The only thing that remains to be shown is that a connection is a local operator. Suppose $s \in \Gamma(E)$ such that $s|_U = 0$ on some open subset $U \subseteq M$. We want to show that $(\nabla s)|_U = 0$. Take any $p \in U$, and choose an open subset $p \in V \subseteq U$ and a smooth bump function $\chi : M \to \mathbb{R}$ with

$$\chi|_V = 1 \text{ and } \operatorname{supp} \chi \subseteq U$$

then $d\chi_p = 0$, and $\chi s = 0$ identically on M, hence

$$\begin{aligned} 0 &= \nabla(\chi s)(p) \\ &= d\chi_p \otimes s(p) + \chi(p)(\nabla s)(p) \\ &= (\nabla s)(p) \end{aligned}$$

and since this holds for every $p \in U$ we conclude that $(\nabla s)|_U = 0$, thus ∇ is a local operator. ∎

Let's see what a connection ∇ looks like in local coordinates. Take a local frame (∂_i) for TM and (e_j) for E over $U \subseteq M$ and write

$$X = \sum_{i=1}^{n} X^i \partial_i \quad \text{and} \quad s = \sum_{j=1}^{r} s^j e_j$$

where $n = \dim M$ and $r = \operatorname{rank} E$. Then we write

$$\nabla_{\partial_i} e_j = \sum_{k=1}^{r} \Gamma_{ij}^k e_k$$

for some collection of smooth functions $\Gamma_{ij}^k \in C^\infty(U)$. We suggested in our motivating discussion above that allowing these functions to be nonzero is what distinguishes the connection ∇ from the naive derivative operator d, and being able to select these functions arbitrarily gives us the most general way to differentiate sections of a vector bundle. Locally, these functions uniquely define the connection ∇ on E. With respect to these local coordinates, the connection ∇ looks like

$$\begin{aligned}
\nabla_X s &= \sum_{i=1}^{n} X^i \nabla_{\partial_i} \left(\sum_{j=1}^{r} s^j e_j \right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{r} X^i \nabla_{\partial_i} (s^j e_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{r} X^i \left(s^j \nabla_{\partial_i} e_j + \frac{\partial s^j}{\partial x^i} e_j \right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{r} \sum_{k=1}^{r} \left(X^i s^j \Gamma_{ij}^k e_k \right) + \sum_{i=1}^{n} \sum_{j=1}^{r} X^i \frac{\partial s^j}{\partial x^i} e_j \\
&= \sum_{i=1}^{n} \sum_{j=1}^{r} \sum_{k=1}^{r} \left(X^i s^j \Gamma_{ij}^k e_k \right) + \sum_{j=1}^{r} X(s^j) e_j
\end{aligned}$$

Renaming the index of the last summation from j to k and combining the sums into one, we obtain

$$\nabla_X s = \sum_{i,j,k} \left(X(s^k) + X^i s^j \Gamma_{ij}^k \right) e_k. \tag{12}$$

Example 16 (Flat connection on a trivial vector bundle)**.** The simplest connection is the exterior derivative acting on smooth functions. Let M be a smooth manifold and let $E = M \times \mathbb{R} \to M$ the rank-1 trivial bundle over M. The exterior derivative is an operator from $\Gamma(E) = C^\infty(M; \mathbb{R})$ to $\Omega^1(M; E) = \Omega^1(M)$, given by

$$\begin{aligned}
&d : C^\infty(M; \mathbb{R}) \to \Omega^1(M) \\
&df(X) = d_X(f) = Xf
\end{aligned}$$

This is called the flat connection because it has zero curvature (which in this case just means that $d^2 = 0$). Similarly, when $E = M \times \mathbb{R}^k$ is the rank-k trivial bundle over

M, we have a trivial connection defined by applying the exterior derivative to each component:

$$\begin{aligned}\nabla_X &: C^\infty(M;\mathbb{R}^k) \to C^\infty(M;\mathbb{R}^k)\\ \nabla_X(f_1,\dots,f_k) &= (Xf_1,\dots,Xf_k)\end{aligned}$$

Example 17 (Flat connection on a nontrivial vector bundle)**.** Let $E \to M$ be any vector bundle and suppose we fix local frames (e_i) for E and (dx^j) for T^*M over $U \subseteq M$. At the beginning of this section in equation (10) we defined an operator

$$\begin{aligned}d &: \Gamma(E|_U) \to \Gamma(E|_U) \otimes \Gamma\left(T^*M|_U\right)\\ ds &= \sum_{i,j} \frac{\partial s^i}{\partial x^j} e_i \otimes dx^j\end{aligned}$$

This is the flat connection on $E|_U \to U$, generalizing the previous example because it's essentially just the exterior derivative acting on the bundle $E|_U = U \times \mathbb{R}^k$ which is trivial over U. However, this operator is *only well-defined locally* and does not define a global connection on $E \to M$.

A vector bundle is called *flat* if it admits a globally defined flat connection. Every vector bundle is *locally flat* because d is always locally a connection with $d^2 = 0$.

We will show that every connection ∇ on $E \to M$ *locally* looks like the flat connection d plus some matrix of 1-forms A.

Example 18 (Euclidean connection)**.** Define the ***Euclidean connection*** on $T\mathbb{R}^n$ as the map

$$\begin{aligned}\overline{\nabla} &: \Gamma(T\mathbb{R}^n) \times \Gamma(T\mathbb{R}^n) \to \Gamma(T\mathbb{R}^n)\\ \overline{\nabla}_X Y &= X(Y^1)\partial_1 + \cdots + X(Y^n)\partial_n\end{aligned}$$

whose value at each point $p \in \mathbb{R}^n$ is the vector

$$\begin{aligned}\left(\overline{\nabla}_X Y\right)(p) &= \overline{\nabla}_{X_p} Y\\ &= X_p(Y^1)\partial_1\big|_p + \cdots + X_p(Y^n)\partial_n\big|_p\end{aligned}$$

Notice that this is just the usual directional derivative operator from vector calculus.

Example 19 (Tangential connection on a Euclidean submanifold)**.** Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold and let $\pi^T : T\mathbb{R}^n \to TM$ denote the orthogonal projection onto TM. The ***tangential connection*** on TM is the operator ∇^T given by

$$\nabla^T_X(Y) = \pi^T\left(\overline{\nabla}_{\widetilde{X}} \widetilde{Y}\right)$$

for every pair of smooth vector fields $X, Y \in \mathfrak{X}(M)$, extended to smooth vector fields $\widetilde{X}$ and $\widetilde{Y}$ in an open subset of $\mathbb{R}^n$. We note that:

- For any $p \in M$, we have by definition

  $$\left(\nabla^T_X Y\right)(p) = \pi^T\big|_p \left(\overline{\nabla}_{X_p} \widetilde{Y}\right)$$

  which is precisely the tangential component of the directional derivative of $\widetilde{Y}$ in the direction of X_p.

- $\nabla^T_X(Y)$ is a smooth vector field on M because π^T is a smooth bundle homomorphism and $\overline{\nabla}_{\widetilde{X}}\widetilde{Y}$ is a smooth vector field on $\mathbb{R}^n$.

Fact 30 (Existence of connections)**.** *Let M be a smooth manifold of dimension n and let $E \to M$ be any vector bundle of rank r. Then E admits infinitely many connections, each one locally determined by a choice of smooth connection coefficients $\{\Gamma^k_{ij} : 1 \le i \le n, 1 \le j, k \le r\}$.*

Proof. Cover M with coordinate charts $\{U_\alpha\}$ which locally trivialize E. Then we can define a connection ∇^α on each U_α just by selecting smooth connection coefficients $\Gamma^k_{ij} : U_\alpha \to \mathbb{R}$ – then ∇^α is completely determined by the formula (12) above. Choosing a partition of unity $\{\psi_\alpha\}$ subordinate to $\{U_\alpha\}$ and patching together all the various local guys gives us a connection

$$\nabla_X s = \sum_\alpha \psi_\alpha \nabla^\alpha_X s.$$

Let's check that this does indeed define a connection on E.

- ∇ is $\mathbb{R}$-linear in s and $C^\infty(M)$-linear in X immediately from the corresponding properties for the ∇^α.
- As for the product rule, take any $f \in C^\infty(M)$ and calculate

$$\begin{aligned}
\nabla_X(fs) &= \sum_\alpha \psi_\alpha \nabla^\alpha_X(fs) \\
&= \sum_\alpha \psi_\alpha \left(f\nabla^\alpha_X s + X(f)s\right) \\
&= \sum_\alpha \psi_\alpha f\nabla^\alpha_X s + \sum_\alpha \psi_\alpha X(f)s \\
&= f\sum_\alpha \psi_\alpha \nabla^\alpha_X s + X(f)Y\sum_\alpha \psi_\alpha \\
&= f\nabla_X s + X(f)s
\end{aligned}$$

  where the last line follows from the fact that $\sum_\alpha \psi_\alpha = 1$.

■

Remark. Since any choice of smooth connection coefficients locally determine a connection on E, the preceding fact shows that connections are quite plentiful. On the other hand, there's not really any canonical choice of connection on E. However, we will show that for a vector bundle equipped with a Riemannian metric there is a canonical choice of connection (the Levi-Civita connection) modelled on the Euclidean connection $\overline{\nabla}$. More generally, a "natural" connection on a vector bundle can also arise by trying to construct one with "minimal curvature" (for some suitable quantitative measurement of curvature). This leads to the notion of *Yang-Mills connection*.

Let's return to equation (12), the local coordinate expression for a connection ∇ on a vector bundle $E \to M$. Notice that the first term in this expression is simply the exterior derivative associated with the local frame (e_k) over $U \subseteq M$. Similarly, we want to interpret the second term as a map $\Gamma(E|_U) \to \Gamma(E|_U) \otimes \Gamma(T^*M|_U)$.

$$\nabla_X s = \underbrace{\sum_k X(s^k)e_k}_{d_X s} + \underbrace{\sum_{i,j,k} X^i s^j \Gamma^k_{ij} e_k}_{?}$$

This leads precisely to the *connection 1-forms* associated with ∇. Namely, for every each $j.k \in \{1, \ldots, r\}$ we define a 1-form on U by

$$A^k_j = \sum_{i=1}^n \Gamma^k_{ij} dx^i \in \Omega^1(U)$$

and the collection $\{A^k_j\}^r_{j,k=1}$ are called the ***connection 1-forms*** associated with ∇. They are defined locally over U, with respect to the chosen local frame (dx^i) for T^*M. It is computationally convenient to consider the 1-forms A^k_j as the entries of an $r \times r$ matrix,

$$A = \left(A^k_j\right)_{jk}$$

and with this notation we can consider A as an operator acting on sections via matrix multiplication:

$$\begin{aligned} &A : \Gamma(E|_U) \to \Gamma(E|_U) \otimes \Gamma(T^*M|_U) \\ &As = \sum_{j,k=1}^r A^k_j s^j \otimes e_k = \sum_{j,k=1}^r \left(\sum_{i=1}^n \Gamma^k_{ij} s^j\right) dx^i \otimes e_k \end{aligned}$$

for any $s = \sum_\ell s^\ell e_\ell \in \Gamma(E|_U)$. Evidently, As acts on a vector field $X \in \mathfrak{X}(U)$ by the formula

$$As(X) = A_X s = \sum_{j,k=1}^r \sum_{i=1}^n \Gamma^k_{ij} s^j X^i e_k$$

which is precisely the second term in equation (12). As a result, the local coordinate expression tells us that the connection can be locally decomposed as

$$\nabla = d + A$$

where d is the flat connection i.e. the exterior derivative with respect to a local frame (e_k) for E over $U \subseteq M$, and A represents the matrix of connection 1-forms with respect to a local frame (dx^i) for T^*M over U.

Next we will derive a few useful formulas related to the matrix A. First of all, note that we can equivalently consider A itself as a 1-form with matrix coefficients in $M(r, C^\infty(U))$, i.e. $A \in M(r, C^\infty(U)) \otimes \Omega^1(U)$. Thus A is called the ***the connection 1-form*** associated with the connection ∇. To see this very explicitly, suppose for simplicity that $n = r = 2$ and write

$$\begin{aligned} A &= \begin{bmatrix} A^1_1 & A^1_2 \\ A^2_1 & A^2_2 \end{bmatrix} \\ &= \begin{bmatrix} \Gamma^1_{11} dx^1 + \Gamma^1_{21} dx^2 & \Gamma^1_{12} dx^1 + \Gamma^1_{22} dx^2 \\ \Gamma^2_{11} dx^1 + \Gamma^2_{21} dx^2 & \Gamma^2_{12} dx^1 + \Gamma^2_{22} dx^2 \end{bmatrix} \\ &= \begin{bmatrix} \Gamma^1_{11} & \Gamma^1_{12} \\ \Gamma^2_{11} & \Gamma^2_{12} \end{bmatrix} dx^1 + \begin{bmatrix} \Gamma^1_{21} & \Gamma^1_{22} \\ \Gamma^2_{21} & \Gamma^2_{22} \end{bmatrix} dx^2. \end{aligned}$$

Thus we will write

$$A = \sum_{i=1}^{n} a_i dx^i$$

where each $a_i \in M(r, C^\infty(U))$ is the matrix given by

$$(a_i)_{uv} = \Gamma^u_{iv}$$

One useful thing about this expression is that it becomes clear how to perform operations like dA (take entry-wise derivatives of each a_i) and $A \wedge A$ (perform the wedge product of 1-forms as usual). Explicitly, we have

$$\begin{aligned} dA &= \sum_{i,j=1}^{n} \frac{\partial a_j}{\partial x^i} dx^i \wedge dx^j \\ &= \frac{1}{2} \sum_{i<j} \left(\frac{\partial a_j}{\partial x^i} - \frac{\partial a_i}{\partial x^j} \right) dx^i \wedge dx^j \end{aligned} \tag{13}$$

and

$$\begin{aligned} A \wedge A &= \sum_{i,j=1}^{n} a_i a_j dx^i \wedge dx^j \\ &= \frac{1}{2} \sum_{i<j} (a_i a_j - a_j a_i) dx^i \wedge dx^j \\ &= \frac{1}{2} \sum_{i<j} [a_i, a_j] dx^i \wedge dx^j. \end{aligned} \tag{14}$$

Before moving on from this topic and using the decomposition to study the connection, we should make note of one more equivalent way of thinking about the connection matrix A ... In this way, we can view the connection matrix as an endomorphism-valued 1-form, i.e. $A \in \Omega^1(M; \operatorname{End} E)$.

The local decomposition $\nabla = d + A$ is useful for understanding the structure of various induced connections on associated bundles.

Example 20 (Induced connection on direct sum)**.** Let $E \to M$ be a vector bundle with connection ∇^E and $F \to M$ another vector bundle with connection ∇^F. Define a connection ∇ on $E \oplus F \to M$ by the formula

$$\nabla_X(r, s) = (\nabla^E_X r, \nabla^F_X s)$$

for every $r \in \Gamma(E)$ and $s \in \Gamma(F)$.

Example 21 (Induced connection on dual bundle)**.** Let $E \to M$ be a vector bundle with connection ∇. We want to define a connection on the dual bundle $E^* \to M$ which is dual to or induced by ∇ in some sense. Note that the bilinear pairing between E and E^* extends to a pairing between sections,

$$\begin{gathered} (\bullet, \bullet) : \Gamma(E) \times \Gamma(E^*) \to C^\infty(M) \\ (s, \eta)(p) = \eta_p(s_p) \in \mathbb{R} \end{gathered}$$

and this can in turn be extended to pairings between bundle-valued forms,

$$\begin{aligned}
(\bullet,\bullet) &: \Omega^k(M;E)\times\Gamma(E^*)\to\Omega^k(M)\\
&(s\otimes\omega,\eta)=\eta(s)\omega\\
(\bullet,\bullet) &: \Gamma(E)\times\Omega^k(M;E^*)\to\Omega^k(M)\\
&(s,\eta\otimes\omega)=\eta(s)\omega
\end{aligned}$$

i.e. just by pairing the section component of the bundle-valued form with the dual. The ***dual connection*** on E^* is the operator $\nabla^*:\Gamma(E^*)\to\Omega^1(M;E^*)$ defined by the relation

$$d(s,\eta)=(\nabla s,\eta)+(s,\nabla^*\eta)$$

for every $s\in\Gamma(E)$ and $\eta\in\Gamma(E^*)$. Applying both sides of this equation to any vector field $X\in\mathfrak{X}(M)$, we see that this is equivalent to saying that $\nabla^*:\Gamma(E^*)\times\mathfrak{X}(M)\to\Gamma(E^*)$ is given by

$$(\nabla^*_X\eta)s=X(\eta(s))-\eta(\nabla_X s).$$

Let (e_j) be a local frame for E, and let (e_j^*) denote the dual frame for E^* so that $(e_i,e_j^*)=\delta_{ij})$. With respect to these coordinates we decompose the connections as $\nabla=d+A$ and $\nabla^*=d^*+A^*$.We want to determine the local coordinate expression for A^* and figure out its relation with A. We calculate

$$\begin{aligned}
0&=d(e_i,e_j^*)\\
&=(\nabla e_i,e_j^*)+(e_i,\nabla^*e_j^*)\\
&=(de_i,e_j^*)+(Ae_i,e_j^*)+(e_i,d^*e_j^*)+(e_i,A^*e_j^*)\\
&=(Ae_i,e_j^*)+(e_i,A^*e_j^*)
\end{aligned}$$

because $de_i=0$ and $d^*e_j^*=0$ for every i,j. Moreover,

$$(Ae_i,e_j^*)=\left(\sum_k A_i^k e_k,e_j^*\right)=\sum_k e_j^*(e_k)A_i^k=A_i^j$$

and similarly

$$(e_i,A^*e_j^*)=\left(e_i,\sum_k (A^*)_j^k e_k^*\right)=\sum_k e_k^*(e_i)(A^*)_j^k=(A^*)_j^i.$$

Thus $0=A_i^j+(A^*)_j^i$, which is to say that

$$A^*=-A^T$$

i.e. the matrix A^* is the negative transpose A. Relating this to the connection coefficients, we have

$$\nabla^*_{\partial_i}e_j^*=\sum_k\left(-\Gamma_{ik}^j\right)e_k^*$$

where Γ_{ij}^k are the connection coefficients of ∇.

Example 22 (Induced connection on tensor product)**.** Let $E \to M$ be a vector bundle with connection ∇^E and $F \to M$ another vector bundle with connection ∇^F. Define a connection $\nabla^{\otimes}$ on $E \otimes F \to M$ by the formula

$$\nabla^{\otimes}(r \otimes s) = (\nabla^E r) \otimes s + r \otimes (\nabla^F s)$$

for every decomposable section $r \otimes s \in \Gamma(E \otimes F)$, and then extending linearly.

Example 23 (Induced connection on Hom bundle)**.** Let $E \to M$ be a vector bundle with connection ∇^E and $F \to M$ another vector bundle with connection ∇^F. Define a connection $\widetilde{\nabla}$ on the homomorphism bundle $\mathrm{Hom}(E,F) \to M$ by the formula

$$\begin{aligned}
&\widetilde{\nabla} : \Gamma(\mathrm{Hom}(E,F)) \times \mathfrak{X}(M) \to \Gamma(\mathrm{Hom}(E,F)) \\
&(\widetilde{\nabla}_X \phi)(r) = \nabla^F_X(\phi(r)) - \phi(\nabla^E_X r)
\end{aligned}$$

where $\phi : \Gamma(E) \to \Gamma(F)$ and $r \in \Gamma(E)$. We want to show that, under the bundle isomorphism $\mathrm{Hom}(E,F) \simeq E^* \otimes F$, the connection $\widetilde{\nabla}$ is exactly the induced connection on $E^* \otimes F$. First of all, note that under the aforementioned bundle isomorphism any section $\phi \in \Gamma(\mathrm{Hom}(E,F))$ is uniquely identified with a section $\eta \otimes s \in \Gamma(E^*) \otimes \Gamma(F)$ such that

$$\phi(r) = \eta(r)s \in \Gamma(F)$$

for every $r \in \Gamma(E)$. Thus we have

$$\begin{aligned}
(\widetilde{\nabla}_X \phi)(r) &= \nabla^F_X(\phi(r)) - \phi(\nabla^E_X r) \\
&= \nabla^F_X(\eta(r)s) - \eta(\nabla^E_X r)s \\
&= \big(X(\eta(r))s + \eta(r)\nabla^F_X s\big) - \eta(\nabla^E_X r)s
\end{aligned}$$

On the other hand let $\nabla^{\otimes}$ denote the induced connection on $E^* \otimes F$, by definition

$$\nabla^{\otimes}_X(\eta \otimes s) = (\nabla^*_X \eta) \otimes s + \eta \otimes \big(\nabla^F_X s\big)$$

which acts on $r \in \Gamma(E)$ by

$$\begin{aligned}
\nabla^{\otimes}_X(\eta \otimes s)(r) &= [(\nabla^*_X \eta) r]s + \eta(r)\nabla^F_X s \\
&= [X(\eta(r)) - \eta\left(\nabla^E_X r\right)]s + \eta(r)\nabla^F_X s.
\end{aligned}$$

Comparing these equations for $\widetilde{\nabla}$ and $\nabla^{\otimes}$, we see that they are exactly the same after the identification $\phi = \eta \otimes s$.

Example 24 (Pullback connections)**.** Let $E \to N$ be a vector bundle with connection ∇ and $\varphi : M \to N$ a diffeomorphism. Let $\varphi^* E \to M$ denote the pullback of E along φ. We can define a natural connection ∇^{φ} on $\varphi^* E \to M$ by the formula

$$\begin{aligned}
&\nabla^{\varphi} : \Gamma(\varphi^* E) \times \mathfrak{X}(M) \to \Gamma(\varphi^* E) \\
&\nabla^{\varphi}_X(s \circ \varphi) = (\nabla_{\varphi_* X} s) \circ \varphi
\end{aligned}$$

for any $s \in \Gamma(E)$. Here we are using the fact that the space of sections of $\varphi^*(E)$ (equivalently, sections of E along φ) is locally generated by sections of the form $s \circ \varphi$ for

$s \in \Gamma(E)$ (see Fact 11). Thus it makes good sense to define ∇^φ by this formula locally, and then extend it linearly. Let's check that this does indeed define a connection:

∇^φ is $C^\infty(M)$-linear in the vector field argument. Given $f \in C^\infty(M)$, write $f = \widetilde{f} \circ \varphi$ so that

$$\begin{aligned}\nabla^\varphi_{fX}(s \circ \varphi) &= (\nabla_{\varphi_*(fX)} s) \circ \varphi \\ &= (\nabla_{\widetilde{f}\varphi_* X} s) \circ \varphi \\ &= (\widetilde{f} \nabla_{\varphi_* X} s) \circ \varphi \\ &= f(\nabla_{\varphi_* X} s) \circ \varphi\end{aligned}$$

as expected. Moreover, ∇^φ satisfies the product rule in the section argument. Given $f = \widetilde{f} \circ \varphi \in C^\infty(M)$ we have

$$\begin{aligned}\nabla^\varphi_X(f(s \circ \varphi)) &= \nabla^\varphi_X((\widetilde{f}s) \circ \varphi) \\ &= (\nabla_{\varphi_* X}(\widetilde{f}s)) \circ \varphi \\ &= ((\varphi_* X)(\widetilde{f})s + \widetilde{f}\nabla_{\varphi_* X} s) \circ \varphi \\ &= (Xf)(s \circ \varphi) + f\nabla^\varphi_X(s \circ \varphi)\end{aligned}$$

which completes the proof.

As in Example 23, given a vector bundle $E \to M$ with connection ∇, we get an induced connection on the endomorphism bundle $\operatorname{End} E \to M$,

$$\widetilde{\nabla} : \Omega^0(M; \operatorname{End} E) \to \Omega^1(M; \operatorname{End} E)$$

which is defined by the formula

$$\begin{aligned}&\widetilde{\nabla}_X \phi : \Gamma(E) \to \Gamma(E) \\ &(\widetilde{\nabla}_X \phi)(s) = \nabla_X(\phi(s)) - \phi(\nabla_X s)\end{aligned}$$

for every $\phi \in \Gamma(\operatorname{End} E)$ and $s \in \Gamma(E)$. We have a convenient local expression for $\widetilde{\nabla}$ which exhibits the relation between ∇ and $\widetilde{\nabla}$ more clearly:

Fact 31. *Let $E \to M$ be a vector bundle with connection ∇, with local decomposition $\nabla = d + A$. Let $\widetilde{\nabla}$ denote the induced connection on the endomorphism bundle* $\operatorname{End} E \to M$*. For any $\phi \in \Gamma(\operatorname{End} E)$, we have*

$$\widetilde{\nabla}\phi = d\phi + [A, \phi] = d\phi + A\phi - \phi A$$

where the Lie bracket is given at the level of fibers by commutator of matrix multiplication.

Proof. Write $\nabla = d + A$ with respect to a local frame (e_i) for E, and let (e^i) denote the dual frame for E^*. Under the isomorphism $\operatorname{End} E \simeq E \otimes E^*$, write $\phi = \sum_{i,j} \phi^i_j e_i \otimes e^j$. By definition we have

$$\nabla e_j = \sum_k A^k_j \otimes e_k \quad \text{and} \quad \nabla^* e^j = -\sum_k A^j_k \otimes e^k$$

because $de_j = 0$ and $de^j = 0$ for every j. We calculate

$$\begin{aligned}
\widetilde{\nabla}\phi &= \sum_{i,j} \widetilde{\nabla}\left(\phi^i_j e_i \otimes e^j\right) \\
&= \sum_{i,j} \nabla(\phi^i_j e_i) \otimes e^j + \phi^i_j e_i \otimes \nabla^*(e^j) \\
&= \sum_{i,j} \left(d(\phi^i_j) \otimes e_i \otimes e^j + \phi^i_j \nabla(e_i) \otimes e^j + \phi^i_j e_i \otimes \nabla^*(e^j)\right) \\
&= \sum_{i,j} \left(d(\phi^i_j) \otimes e_i \otimes e^j + \sum_k \phi^i_j A^k_i \otimes e_k \otimes e^j - \sum_k \phi^i_j A^j_k \otimes e_i \otimes e^k\right) \\
&= d\phi + A\phi - \phi A
\end{aligned}$$

as desired. Notice that the distinction between $A\phi$ and ϕA is simply whether we sum over the columns of A or the rows of A. The notation $d\phi$ here refers to the exterior derivative (determined by a local frame) acting on a section of $\operatorname{End} E$. ■

Now by Fact 28 (and the corollary following from it), there is a unique degree $+1$ graded antiderivation of $\Omega^\bullet(M;E)$ extending the connection ∇ from sections to E-valued forms of any degree. We denote this antiderivation by the same symbol ∇,

$$\nabla : \Omega^\ell(M;E) \to \Omega^{\ell+1}(M;E)$$

for any $\ell \in \mathbb{N}$. By definition, it satisfies the product rules

$$\begin{aligned}
\nabla(\omega \wedge \alpha) &= d\omega \wedge \alpha + (-1)^{\deg \omega}\omega \wedge \nabla\alpha \\
\nabla(\alpha \wedge \omega) &= (\nabla\alpha) \wedge \omega + (-1)^{\deg \omega}\alpha \wedge d\omega
\end{aligned}$$

for every $\omega \in \Omega^\bullet(M)$ and $\alpha \in \Omega^\bullet(M;E)$. The only antiderivation that we care about in this note is this exterior derivative of bundle-valued forms uniquely determined by a connection. In summary we have the following fact:

Fact 32. *Let $E \to M$ be a vector bundle with connection ∇. There exists a unique graded antiderivation $\nabla : \Omega^\bullet(M;E) \to \Omega^\bullet(M;E)$ which coincides with ∇ on sections of E. In other words, ∇ extends the action of the connection to bundle-valued forms of any degree.*

To be precise, ∇ acts on decomposable E-valued forms according to the formula

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge (\nabla s)$$

where $\omega \in \Omega^k(M)$ and $s \in \Gamma(E)$.

By Fact 32 the induced connection $\widetilde{\nabla}$ on $\operatorname{End} E \to M$ extends to a covariant derivative operator

$$\widetilde{\nabla} : \Omega^k(M; \operatorname{End} E) \to \Omega^{k+1}(M; \operatorname{End} E)$$

Using the local expression for $\widetilde{\nabla}$ given in Fact 31, we can derive a similar local expression for the operator $\widetilde{\nabla}$.

Fact 33. *Let $E \to M$ be a vector bundle with connection ∇, with local decomposition $\nabla = d + A$. Let $\widetilde{\nabla}$ denote the induced connection on $\operatorname{End} E \to M$. The operator $\widetilde{\nabla} : \Omega^\ell(M; \operatorname{End} E) \to \Omega^{\ell+1}(M; \operatorname{End} E)$ satisfies*

$$\widetilde{\nabla}\alpha = d\alpha + A \wedge \alpha - (-1)^\ell \alpha \wedge A$$

for every $\alpha \in \Omega^\ell(M; \operatorname{End} E)$.

Proof. Write $\nabla = d + A$ with respect to a local frame (e_i) for E and let (e^j) denote the dual frame for E^*. Take any decomposable form $\alpha = \omega \otimes \phi \in \Omega^\ell(M; \operatorname{End} E)$ and calculate

$$\begin{aligned}
\widetilde{\nabla}\alpha &= \widetilde{\nabla}(\omega \otimes \phi) \\
&= d\omega \otimes \phi + (-1)^\ell \omega \wedge \widetilde{\nabla}\phi \\
&= d\omega \otimes \phi + (-1)^\ell \omega \wedge (d\phi + A\phi - \phi A) \\
&= d\omega \otimes \phi + (-1)^\ell \omega \wedge (d\phi) + (-1)^\ell \omega \wedge (A\phi) - (-1)^\ell \omega \wedge (\phi A).
\end{aligned}$$

Now it's straightforward to identify the first term in this equation as

$$d\alpha = d(\omega \otimes \phi) = d\omega \otimes \phi + (-1)^\ell \omega \wedge (d\phi)$$

and writing $A = \sum_i a_i \otimes dx^i$ the second term in the equation is

$$\begin{aligned}
\omega \wedge (A\phi) &= \sum_i (\omega \wedge dx^i) \otimes a_i \phi \\
&= \sum_i (-1)^\ell (dx^i \wedge \omega) \otimes a_i \phi \\
&= \sum_i (-1)^\ell (dx^i \otimes a_i) \wedge (\omega \otimes \phi) \\
&= (-1)^\ell A \wedge \alpha.
\end{aligned}$$

So all that remains is to show that $\alpha \wedge A = \omega \wedge (\phi A)$. This one seems a little bit trickier than the previous two terms, so we inspect it in local coordinates. We have

$$\begin{aligned}
\omega \wedge (\phi A) &= \omega \wedge \left(\sum_{ijk} \phi^i_j A^j_k \otimes e_i \otimes e^k \right) \\
&= \omega \wedge \left(\sum_{ijkm} \phi^i_j \Gamma^j_{mk} dx^m \otimes e_i \otimes e^k \right) \\
&= \sum_{ijkm} \phi^i_j \Gamma^j_{mk} (\omega \wedge dx^m) \otimes e_i \otimes e^k
\end{aligned}$$

although it's not immediately clear that this is the same as $\alpha \wedge A$, direct calculation shows that it is indeed:

$$\begin{aligned}
\alpha \wedge A &= \sum_i (\omega \wedge dx^i) \otimes (\phi a_i) \\
&= \sum_i (\omega \wedge dx^i) \otimes \left(\sum_{jkm} \phi^k_j (a_i)^j_m e_k \otimes e^m \right) \\
&= \sum_{ijkm} \phi^k_j \Gamma^j_{im} (\omega \wedge dx^i) \otimes e_k \otimes e^m
\end{aligned}$$

permuting indices and comparing the two equations shows that $\omega \wedge (\phi A) = \alpha \wedge A$ exactly as claimed. ∎

The following product rule will also be useful in the next section:

Fact 34. *For any $\theta \in \Omega^k(M; \operatorname{End} E)$ and $\alpha \in \Omega^\ell(M; E)$ we have*

$$\nabla(\omega \wedge \theta) = (\widetilde{\nabla}\theta) \wedge \alpha + (-1)^k \theta \wedge \nabla\alpha$$

Proof. The formula follows directly by writing down all of the definitions, calculating both sides of the equation on decomposable forms, and then comparing. It's a good exercise for getting a hang of all the definitions and how they interact. Here are the details: for any decomposable forms $\theta \in \Omega^k(M; \operatorname{End} E)$ and $\alpha \in \Omega^\ell(M; E)$, write

$$\theta = \omega \otimes \phi \quad \text{and} \quad \alpha = \eta \otimes s$$

where $\omega \in \Omega^k(M)$, $\eta \in \Omega^\ell(M)$, $\phi \in \Gamma(\operatorname{End} E)$ and $s \in \Gamma(E)$. Then by definition,

$$\begin{aligned} \nabla(\theta \wedge \alpha) &= \nabla((\omega \wedge \eta) \otimes \phi(s)) \\ &= d(\omega \wedge \eta) \otimes \phi(s) + (-1)^{k+\ell}(\omega \wedge \eta) \wedge \nabla_{(\bullet)}(\phi(s)). \end{aligned} \tag{15}$$

On the other side of the equation we have

$$\begin{aligned} \widetilde{\nabla}\theta \wedge \alpha &= \widetilde{\nabla}(\omega \otimes \phi) \wedge (\eta \otimes s) \\ &= (d\omega \otimes \phi) \wedge (\eta \otimes s) + (-1)^k (\omega \wedge \widetilde{\nabla}\phi) \wedge (\eta \otimes s) \\ &= (d\omega \wedge \eta) \otimes \phi(s) + (-1)^k (\omega \wedge \eta) \otimes (\widetilde{\nabla}\phi)(s) \\ &= (d\omega \wedge \eta) \otimes \phi(s) + (-1)^k (\omega \wedge \eta) \otimes (\nabla_{(\bullet)}\phi(s) - \phi(\nabla_{(\bullet)} s)) \end{aligned}$$

as well as

$$\begin{aligned} \theta \wedge (\nabla\alpha) &= (\omega \otimes \phi) \wedge \left(d\eta \otimes s + (-1)^\ell \eta \wedge \nabla s\right) \\ &= (\omega \wedge d\eta) \otimes \phi(s) + (-1)^\ell (\omega \wedge \eta) \otimes \phi(\nabla_{(\bullet)} s). \end{aligned} \tag{16}$$

In all of these equations we have used the notation $(\bullet)$ to denote the vector field argument. Expanding the product rule $d(\omega \wedge \eta)$ in equation (15) and multiplying equation (16) by $(-1)^k$, we end up with the desired formula. ∎

Given any $\alpha \in \Omega^\ell(M; E)$, it's also useful to understand the action of the derivative $\nabla\alpha \in \Omega^{\ell+1}(M; E)$ on vector fields.

Fact 35. *Let $E \to M$ be a vector bundle with connection ∇. For any $\alpha \in \Omega^\ell(M; E)$, the map*

$$\nabla\alpha : \mathfrak{X}(M)^{\ell+1} \to \Gamma(E)$$

is given by

$$\begin{aligned} (\nabla\alpha)(X_0, \ldots, X_\ell) = &\sum_{i=0}^{\ell} (-1)^i \nabla_{X_i} \left(\alpha\left(X_0, \ldots, \widehat{X}_i, \ldots, X_\ell\right)\right) \\ &+ \sum_{0 \le i < j \le \ell} (-1)^{i+j} \alpha\left([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, X_\ell\right) \end{aligned}$$

Just like the exterior derivative of real-valued differential forms, the covariant derivative induced by a connection commutes with pullbacks.

Fact 36. *Let $E \to N$ be a vector bundle with connection ∇ and $\varphi : M \to N$ a smooth map. For any $\alpha \in \Omega^{\boldsymbol{\cdot}}(N; E)$ we have*

$$\varphi^*(\nabla\alpha) = \nabla(\varphi^*\alpha)$$

i.e. the exterior derivative commutes with pullbacks.

9 Curvature of a connection

Let $E \to M$ be a vector bundle with connection ∇. Given vector fields $X, Y \in \mathfrak{X}(M)$ and a section $s \in \Gamma(E)$ we define a map

$$\begin{aligned}
&R^\nabla : \mathfrak{X}(M)^2 \times \Gamma(E) \to \Gamma(E) \\
&R^\nabla(X,Y)(s) = \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s
\end{aligned}$$

called the ***curvature of the connection*** ∇. We say that ∇ is a ***flat connection*** if $R^\nabla = 0$.

Example 25 (Flat connection)**.** The exterior derivative on differential forms is a flat connection because its curvature is $d^2 = 0$.

Fact 37. *The curvature R^∇ is $C^\infty(M)$-linear in each of its arguments, and antisymmetric with respect to its vector field arguments. Therefore, the curvature is an antisymmetric, endomorphism-valued operator*

$$R^\nabla : \mathfrak{X}(M)^2 \to \Gamma(\operatorname{End} E).$$

i.e. $R^\nabla \in \Omega^2(M; \operatorname{End} E)$.

Proof. The fact that the curvature is $\mathbb{R}$-linear is clear by the $\mathbb{R}$-linearity of the connection ∇. So it suffices to check homogeneity with respect to $C^\infty(M)$. Take any $f \in C^\infty(M)$, $X, Y \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$. Then

$$\begin{aligned}
R^\nabla(fX,Y)s &= \nabla_{fX}\nabla_Y s - \nabla_Y\nabla_{fX}s - \nabla_{[fX,Y]}s \\
&= f\nabla_X\nabla_Y s - \nabla_Y\left(f\nabla_X s\right) - \left(\nabla_{f[X,Y]}s - \nabla_{Y(f)X}s\right) \\
&= f\nabla_X\nabla_Y s - (Y(f)\nabla_X s + f\nabla_Y\nabla_X s) - f\nabla_{[X,Y]}s + Y(f)\nabla_X s \\
&= f\nabla_X\nabla_Y s - f\nabla_Y\nabla_X s - f\nabla_{[X,Y]}s \\
&= fR^\nabla(X,Y)s.
\end{aligned}$$

A similar procedure shows that the curvature is also $C^\infty(M)$-linear in the Y-variable and the s-variable. It's also easy to see that the curvature is antisyummetric with respect to X and Y because interchanging these arguments yields

$$\begin{aligned}
R^\nabla(Y,X) &= \nabla_Y\nabla_X - \nabla_X\nabla_Y - \nabla_{[Y,X]} \\
&= \nabla_Y\nabla_X - \nabla_X\nabla_Y - \nabla_{-[X,Y]} \\
&= \nabla_Y\nabla_X - \nabla_X\nabla_Y + \nabla_{[X,Y]} \\
&= -R^\nabla(X,Y).
\end{aligned}$$

Since the curvature is $C^\infty(M)$-multilinear, for every pair of vector fields $X, Y \in \mathfrak{X}(M)$ we have a module endomorphism $R^\nabla(X,Y) : \Gamma(E) \to \Gamma(E)$, which is to say that

$$R^\nabla(X,Y) \in \mathrm{End}(\Gamma(E)) \simeq \Gamma(\mathrm{End}\, E)$$

so R^∇ is an $\mathrm{End}(E)$-valued 2-form on M. ■

Another consequence of Fact 37 is that R^∇ is a point operator, so it makes sense to apply the curvature operator pointwise to the fibers of E. Thus for any $p \in M$, $v, w \in T_pM$ and $\xi \in E_p$ we can make sense of the expression

$$R^\nabla(v,w)(\xi) \in E_p.$$

There is a simple relation between the curvature of a connection and the curvature of any pullback:

Fact 38. *Let $E \to N$ be a vector bundle with connection ∇ and let $\varphi : M \to N$ be a smooth map. Let ∇^φ denote the pullback connection on the pullback bundle $\varphi^*E \to M$. For every $p \in M$, $v, w \in T_pM$ and $\xi \in E_{\varphi(p)}$ we have*

$$R^{\nabla^\varphi}(v,w)(\xi) = R^\nabla(d\varphi_p(v), d\varphi_p(w))(\xi)$$

which is to say that $R^{\nabla^\varphi} = \varphi^* R^\nabla$.

Since it's easy to take the pullback whenever we want to consider the slightly more general case of sections along a map φ, we will just focus on the notationally simpler situation where $\varphi = \mathrm{id}$.

Let's return to the exterior derivative operator ∇ induced by a connection ∇ on a vector bundle $E \to M$. A key fact about the exterior derivative acting on real-valued differential forms is that it satisfies $d \circ d = 0$, which is to say that differential forms assemble a chain complex

$$\cdots \to \Omega^{j-1}(M) \xrightarrow{d^j} \Omega^j(M) \xrightarrow{d^{j+1}} \Omega^{j+1}(M) \to \cdots$$

and hence we can form the de Rham cohomology groups (really $\mathbb{R}$-vector spaces)

$$H^j_{dR}(M) = \ker d^{j+1} / \operatorname{im} d^j$$

which provide information about the topology of M. A natural question to investigate is whether the exterior derivative ∇ exhibits the same behavior, and whether we can get a chain complex of bundle-valued forms, etc. In the following theorem we will show that the curvature R^∇ is precisely the obstruction against $(\Omega^\bullet(M;E), \nabla)$ forming a chain complex.

Theorem 2. *Let $E \to M$ be a vector bundle with connection ∇. For any fixed $k \geq 0$, consider the sequence*

$$\Omega^k(M;E) \xrightarrow{\nabla} \Omega^{k+1}(M;E) \xrightarrow{\nabla} \Omega^{k+2}(M;E)$$

For every $\alpha \in \Omega^k(M;E)$ we have

$$\nabla \circ \nabla(\alpha) = R^\nabla \wedge \alpha$$

thus $\nabla \circ \nabla = 0$ if and only if $R^\nabla = 0$, if and only if ∇ is a flat connection.

Proof. First consider the case $k = 0$, so that $\alpha = s \in \Gamma(E)$. Then for every $X, Y \in \mathfrak{X}(M)$ we have by Fact 35,

$$\begin{aligned}(\nabla \circ \nabla)(s)(X, Y) &= \nabla_X(\nabla s(Y)) - \nabla_Y(\nabla s(X)) - \nabla s([X, Y]) \\ &= \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \\ &= R^\nabla(X, Y)s\end{aligned}$$

which is to say that $(\nabla \circ \nabla)s = R^\nabla \wedge s$. Now for the general case $k \geq 0$, it suffices to take a decomposable element $\alpha = \omega \otimes s$ and compute

$$\begin{aligned}(\nabla \circ \nabla)\alpha &= (\nabla \circ \nabla)(\omega \otimes s) \\ &= \nabla(d\omega \otimes s) + (-1)^k \nabla(\omega \wedge \nabla s) \\ &= d^2\omega \otimes s + (-1)^{k+1} d\omega \wedge \nabla s \\ &\qquad + (-1)^k \left(d\omega \wedge \nabla s + (-1)^k \omega \wedge (\nabla \circ \nabla)s\right) \\ &= \omega \wedge (\nabla \circ \nabla)s \\ &= \omega \wedge (R^\nabla \wedge s)\end{aligned}$$

and now the result follows from the observation that

$$\omega \wedge (R^\nabla \wedge s) = (R^\nabla \wedge s) \otimes \omega = R^\nabla \wedge (s \otimes \omega) = R^\nabla \wedge \alpha$$

where we have used graded anticommutativity of the wedge product from equation (9), together with the compatibility result of Fact 27. ■

Choose local frames (∂_i) for TM and (e_k) for E. Write

$$R^\nabla(\partial_i, \partial_j)e_k = \sum_\ell R^\ell_{ijk} e_\ell$$

for some smooth coefficient functions R^ℓ_{ijk}. Directly from the definition of the curvature, we calculate

$$\begin{aligned}R^\nabla(\partial_i, \partial_j)e_k &= \nabla_{\partial_i}(\nabla_{\partial_j} e_k) - \nabla_{\partial_j}(\nabla_{\partial_i} e_k) - \nabla_{[\partial_i, \partial_j]} e_k \\ &= \nabla_{\partial_i}\left(\sum_\ell \Gamma^\ell_{jk} e_\ell\right) - \nabla_{\partial_j}\left(\sum_\ell \Gamma^\ell_{ik} e_\ell\right) - 0 \\ &= \sum_\ell \nabla_{\partial_i}\left(\Gamma^\ell_{jk} e_\ell\right) - \nabla_{\partial_j}\left(\Gamma^\ell_{ik} e_\ell\right) \\ &= \sum_\ell \left(\partial_i \Gamma^\ell_{jk} e_\ell + \Gamma^\ell_{jk} \nabla_{\partial_i} e_\ell\right) - \left(\partial_j \Gamma^\ell_{ik} e_\ell + \Gamma^\ell_{ik} \nabla_{\partial_j} e_\ell\right) \\ &= \sum_\ell \left(\partial_i \Gamma^\ell_{jk} e_\ell + \Gamma^\ell_{jk} \sum_m \Gamma^m_{i\ell} e_m\right) - \left(\partial_j \Gamma^\ell_{ik} e_\ell + \Gamma^\ell_{ik} \sum_m \Gamma^m_{j\ell} e_m\right) \\ &= \sum_{\ell,m} \left(\partial_i \Gamma^\ell_{jk} - \partial_j \Gamma^\ell_{ik} + \Gamma^m_{jk}\Gamma^\ell_{im} - \Gamma^m_{ik}\Gamma^\ell_{jm}\right) e_\ell\end{aligned}$$

and therefore

$$R^\ell_{ijk} = \sum_m \partial_i \Gamma^\ell_{jk} - \partial_j \Gamma^\ell_{ik} + \Gamma^m_{jk}\Gamma^\ell_{im} - \Gamma^m_{ik}\Gamma^\ell_{jm}.$$

Looking at the structure of this equation, we notice that the first two terms involve derivatives of the connection coefficients and the last two terms together look like a commutator formed by the connection coefficients. We have already encountered objects of this type, namely the derivative dA and the wedge product $A \wedge A$ of the connection 1-form A associated with the connection ∇. In fact,

$$\begin{aligned}
\frac{\partial a_j}{\partial x^i}(e_k) &= \sum_\ell \partial_i \Gamma^\ell_{jk} e_\ell \\
\frac{\partial a_i}{\partial x^j}(e_k) &= \sum_\ell \partial_j \Gamma^\ell_{ik} e_\ell \\
[a_i, a_j]e_k &= \sum_\ell \left(\sum_m \Gamma^\ell_{im}\Gamma^m_{jk} - \Gamma^\ell_{jm}\Gamma^m_{ik} \right) e_\ell
\end{aligned}$$

where the objects on the left act on e_k via matrix multiplication. Therefore

$$\begin{aligned}
R^\nabla(\partial_i, \partial_j)e_k &= \sum_\ell R^\ell_{ijk} e_\ell \\
&= \sum_{\ell,m} \left(\partial_i \Gamma^\ell_{jk} - \partial_j \Gamma^\ell_{ik} + \Gamma^m_{jk}\Gamma^\ell_{im} - \Gamma^m_{ik}\Gamma^\ell_{jm} \right) e_\ell \\
&= \left(\frac{\partial a_j}{\partial x^i} - \frac{\partial a_i}{\partial x^j} + [a_i, a_j] \right) e_k
\end{aligned}$$

and by equations (13) and (14) this is to say that

$$(R^\nabla \wedge s)(\bullet, \bullet) = R^\nabla(\bullet, \bullet)s = (dA + A \wedge A)s$$

for any section $s \in \Gamma(E)$. This gives us the local decomposition

$$\nabla \circ \nabla = R^\nabla \wedge (\bullet) = dA + A \wedge A$$

which is an equality of operators $\Omega^0(M; E) \to \Omega^2(M; E)$. Note that the first two expressions make sense acting on any E-valued k-form, but the last expression only makes sense acting on sections of E. We summarize the preceding discussion with the following fact:

Fact 39 (Curvature in local coordinates)**.** *Let $E \to M$ be a vector bundle with connection ∇. In local coordinates, the curvature R^∇ has components*

$$R^\ell_{ijk} = \sum_m \partial_i \Gamma^\ell_{jk} - \partial_j \Gamma^\ell_{ik} + \Gamma^m_{jk}\Gamma^\ell_{im} - \Gamma^m_{ik}\Gamma^\ell_{jm}$$

and we have a local decomposition

$$\nabla \circ \nabla = R^\nabla \wedge (\bullet) = dA + A \wedge A$$

in terms of the connection 1-form A associated with the connection.

For a vector bundle $E \to M$ with connection ∇ and induced connection $\widetilde{\nabla}$ on $\operatorname{End} E \to M$, the curvature 2-form is an element $R^\nabla \in \Omega^2(M; \operatorname{End} E)$ and so we can consider the derivative $\widetilde{\nabla}(R^\nabla) \in \Omega^3(M; \operatorname{End} E)$. In fact, this derivative is zero.

Fact 40 (Bianchi identity)**.** *Let $E \to M$ be a vector bundle with connection ∇ and let $\widetilde{\nabla}$ denote the induced connection on* $\operatorname{End} E \to M$*. Then $\widetilde{\nabla}(R^\nabla) = 0$.*

Proof. Take any $\alpha \in \Omega^\bullet(M; E)$. We're going to use Theorem 2 to calculate $(\nabla)^3(\alpha)$ in two different ways. First we have

$$(\nabla)^3(\alpha) = (\nabla)^2(\nabla\alpha) = R^\nabla \wedge \nabla\alpha$$

and the other hand we also have (using the product rule from Fact 34)

$$\begin{aligned}
(\nabla)^3(\alpha) &= \nabla\left((\nabla)^2(\alpha)\right) \\
&= \nabla(R^\nabla \wedge \alpha) \\
&= \widetilde{\nabla}(R^\nabla) \wedge \alpha + (-1)^2 R^\nabla \wedge \nabla\alpha \\
&= \widetilde{\nabla}(R^\nabla) \wedge \alpha + (\nabla)^3(\alpha)
\end{aligned}$$

and therefore $\widetilde{\nabla}(R^\nabla) \wedge \alpha = 0$ for every α. We conclude that $\widetilde{\nabla}(R^\nabla) = 0$ as claimed. ∎