TDT4171 - Assignment 3
Probabilistic Reasoning Over Time

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1 Exercise 1

Describe the umbrella world as an HMM:

- *What is the set of unobserved variable(s) for a given time-slice t (denoted X_t in the book)?*

  The set of unobserved variables for a given time slice t contains a single state variable $Rain_t \in \{0, 1\}$. We have $Rain_t = 0$ if it does not rain and $Rain_t = 1$ if it does.

- *What is the set of observable variable(s) for a given time-slice t (denoted E_t in the book)?*

  The set of observable variables for a given time slice t contains a single evidence variable $Umbrella_t \in \{0, 1\}$. We have $Umbrella_t = 0$ we don't see the umbrella and $Umbrella_t = 1$ if we see it.

- *Present the dynamic model $P(X_t|X_{t-1})$ and the observation model $P(E_t|X_t)$ as matrices.*

  The dynamic model $P(X_t|X_{t-1})$, or transition model, is given by (1). We then have to choose the values inside the matrix. The values given here have been taken from the book [RN10].

$$P(X_t|X_{t-1}) = \begin{bmatrix} P(X_t = 0|X_{t-1} = 0) & P(X_t = 1|X_{t-1} = 0) \\ P(X_t = 0|X_{t-1} = 1) & P(X_t = 1|X_{t-1} = 1) \end{bmatrix} = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix} \tag{1}$$

  The observation model $P(E_t|X_t)$, or sensor model, is given by (2). Same here, the values were taken from the same book.

$$P(E_t|X_t) = \begin{bmatrix} P(E_t = 0|X_t = 0) & P(E_t = 1|X_t = 0) \\ P(E_t = 0|X_t = 1) & P(E_t = 1|X_t = 1) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix} \tag{2}$$

- *Which assumptions are encoded in this model? Are the assumptions reasonable for this particular domain? (See 14.1 Time and Uncertainty on Page 479).*

  The first assumption is that the system is stationary. In other words, for all values of t, $P(X_t|X_{t-1})$ is identical, so it values depends of the distribution at any step t and the initial distribution. It defines a first-order Markov process. Mathematically, it means that:

$$\forall t,\ P(X_t|X_{0,t-1}) = P(X_t|X_{t-1})$$

  The second assumption is the sensor Markov assumption, which considers that the observed variables E_t only depends on current state of unobserved variable X_t. Mathematically, it means that we have,

$$\forall t,\ P(E_t|X_{0,t}) = P(E_t|X_t)$$

  Both assumptions are given modelled by Figure 1.

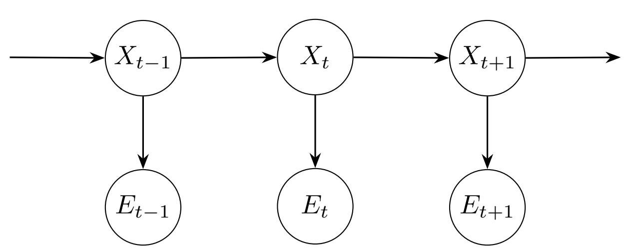

Figure 1: 1-st order model Hidden Markov Model of umbrella world

2 Exercise 2

Implement filtering using the Forward operation (see Equation 14.5 on Page 485 and Equation 14.12 on Page 492) by programming. The forward operation can be done with matrix operations in the HMM.

The equation (14.5) from the book give us the basic equation used in the Forward operation:

$$P(X_{t+1}|e_{1:t+1}) = \alpha P(e_{t+1}|X_{t+1}) \sum_{x_t} P(X_{t+1}|x_t)P(x_t|e_{1:t})$$

We can re-write this equation, by (14.12):

$$f_{1:t+1} = \alpha O_{t+1} T^T f_{1:t}$$

Where,

$$\begin{aligned} f_{1:t} &= P(X_t|e_{1:t}) \\ O_t &= \begin{bmatrix} P(e_t|X_t = 0) & 0 \\ 0 & P(e_t|X_t = 1) \end{bmatrix} \\ T &= (T_{i,j})_{i,j} = (P(X_t = j|X_{t-1} = i)_{i,j} = P(X_t|X_{t-1}) \\ \alpha &= \text{ Normalization constant} \end{aligned}$$

I implemented the algorithm as the Figure 2. If normalized is on true, it gives us the list of

$$(P(X_k = 0|e_{1:k}, P(X_k = 1|e_{1:k})_{k=1,\dots,t}$$

```
import numpy as np

O = np.array(
    [ np.array([0.8, 0.2]),
     np.array([0.1 ,0.9]),
    ])
T = np.array(
    [ np.array([0.7, 0.3]),
     np.array([0.3 ,0.7]),
    ])

def forward_operation(p_x0, e1_t, normalized = True):
    p_x0 = np.array([1-p_x0, p_x0])
    p_x1 = T.dot(p_x0)
    tf = len(e1_t)
    p_unnormalized = [ p_x1 ]
    p_normalized = [ p_x1 ]
    alpha = []
    for t in range(tf):
        p_unnormalized.append(O[:,e1_t[t]] * T.dot(p_unnormalized[t]))

        alpha = 1 / np.sum(p_unnormalized[t+1])
        p_normalized.append(p_unnormalized[t+1]*alpha)
    return p_normalized if normalized else p_unnormalized
```

Figure 2: Forward computation of $P(X_t|e_{1:t})$ prediction

- *Verify your implementation by calculating $P(X_2|e_{1:2})$, where $e_{1:2}$ is the evidence that the umbrella was used both on day 1 and day 2. The desired result is that the probability of rain at day 2 (after the observations) is 0.883.*

  My implementation gives me $P(X_2|e_{1:2}) = 0.8833570412517782 \approx 0.883$.

- *Use your program to calculate the probability of rain at day 5 given the following sequence of observations:*

  $$e1:5 = \{Umbrella_1 = true, Umbrella_2 = true, Umbrella_3 = false, Umbrella_4 = true, Umbrella_5 = true\}$$

  Document your answer by showing all normalized forward messages (in the book, the un-normalized forward messages are denoted $f_{1:k}$ for $k = 1, 2, ..., 5$) in the PDF report.

Our implementation gives us the following normalized forward messages:
`[0.81818182, 0.88335704, 0.19066794, 0.730794, 0.86733889]`

References

[RN10] Stuart J Russell and Peter Norvig. *Artificial intelligence a modern approach*. London, 2010.