TDT4171 - Assignment 2
Bayesian Networks

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1 The Monty Hall Problem

You are confronted with three doors A, B, and C. Behind exactly one of the doors there is $10 000. The money is yours if you choose the correct door. After you have made your first choice of door but still not opened it, an official comes in. He works according to two rules:

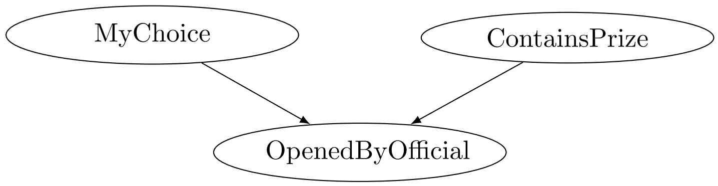
1. *He starts by opening a door. He knows where the prize is, and he is not allowed to open that door. Furthermore, he cannot open the door you have chosen. Hence, he opens a door with nothing behind.*

2. *Now there are two closed doors, one of which contains the prize. The official will ask you if you want to alter your choice (i.e., to trade your door for the other one that is not open).*

Should you do that?

I modelize the problem using a Bayesian Network with three different nodes representing the following door status: ContainsPrize, MyChoice, and OpenedByOfficial. These three events will be respectively written as random values Cp, Mc and Ob having values in $S = \{A, B, C\}$.

		$P(Mc)$
A	B	C
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$



		$P(Cp)$
A	B	C
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

		$P(Ob \mid Mc, Cp)$		
Mc	Cp	A	B	C
A	A	0	0.5	0.5
A	B	0	0	1
A	C	0	1	0
B	A	0	0	1
B	B	0.5	0	0.5
B	C	1	0	0
C	A	0	1	0
C	B	1	0	0
C	C	0.5	0.5	0

We get $P(Ob \mid Mc, Cp)$ table by enumerating all the cases:

1. We notice instantly that, because the official would not open the door designed by my choice or where the prize, we have:

$$\forall x, y \in S,\ P(Ob = x \mid Mc = x, Cp = y) = P(Ob = x \mid Mc = y, Cp = x) = 0$$

2. We also notice that if we choose the door where the prize is, then the official can choose uniformly random between the two other doors:

$$\forall x, y \in S,\ x \neq y,\ P(Ob = x \mid Mc = y, Cp = y) = 0.5$$

3. The last cases are when we did not choose the right door. Then the official has only one choice, because he knows where the prize is, and would not open the prize door:

$$\forall x, y, z \in S,\ x \neq y,\ x \neq z,\ y \neq z,\ P(Ob = x|Mc = y, Cp = z) = 1$$

Thus, we want to infer $P(Cp|Ob, Mc)$ ($\neq P(Cp|Ob)$).

Firstly, we can re-write $P(Ob|Mc, Cp)$ using the definition of conditional probabilities and noticing that MyChoice and ContainsPrize are independent,

$$P(Ob|Mc, Cp) = \frac{P(Mc, Ob, Cp)}{P(Mc, Cp)} = \frac{P(Mc, Ob, Cp)}{P(Mc)P(Cp)} \tag{1}$$

Secondly, we have two (useful) ways to write the joint distribution of our bayesian network:

$$P(Mc, Ob, Cp) = P(Mc)P(Ob|Mc)P(Cp|Ob, Mc) \tag{2}$$

$$= P(Cp)P(Mc|Cp)P(Ob|Mc, Cp) = P(Cp)P(Mc)P(Ob|Mc, Cp) \tag{3}$$

(2) and (3) allow us to obtain the form of our final result:

$$P(Cp|Ob, Mc) = \frac{P(Cp)P(Ob|Mc, Cp)}{P(Ob|Mc)} \tag{4}$$

We also have the law of total probability:

$$P(Ob|Mc) = \sum_{x \in S} P(Ob, Cp = x|Mc) = \sum_{x \in S} P(Cp = x)P(Ob|Mc, Cp = x)$$

Knowing that, $\forall x \in S,\ P(Cp = x) = \frac{1}{3}$,

$$P(Ob|Mc) = \frac{1}{3} \sum_{x \in S} P(Ob|Mc, Cp = x) \tag{5}$$

To compute, for a given $x \in S$, $P(Ob|Mc, Cp = x)$, we use the table of $P(Ob|Mc, Cp)$ obtained previously. We get the three following tables:

	$P(Ob\|Mc, Cp = A)$		
Mc	A	B	C
A	0	0.5	0.5
B	0	0	1
C	0	1	0

	$P(Ob\|Mc, Cp = B)$		
Mc	A	B	C
A	0	0	1
B	0.5	0	0.5
C	1	0	0

	$P(Ob\|Mc, Cp = C)$		
Mc	A	B	C
A	0	1	0
B	1	0	0
C	0.5	0.5	0

Using (5), we have the values of $P(Ob|Mc)$:

	$P(Ob\|Mc)$		
Mc	A	B	C
A	0	0.5	0.5
B	0.5	0	0.5
C	0.5	0.5	0

So we can now compute, using (4), the different values of $P(Cp|Ob, Mc)$ by the code given on 2:

		$P(Cp\|Mc, Ob)$		
Mc	Ob	A	B	C
A	A	0	0	0
A	B	$\frac{1}{3}$	0	$\frac{2}{3}$
A	C	$\frac{1}{3}$	$\frac{2}{3}$	0
B	A	0	$\frac{1}{3}$	$\frac{2}{3}$
B	B	0	0	0
B	C	$\frac{2}{3}$	$\frac{1}{3}$	0
C	A	0	$\frac{2}{3}$	$\frac{1}{3}$
C	B	$\frac{2}{3}$	0	$\frac{1}{3}$
C	C	0	0	0

To answer the asked question, according to this table, we should change the door because there is an highest probability on the other one. Which can be written,

$$\forall x, y, z \in S,$$
$$x \neq y, x \neq z, y \neq z, \; P(Cp = x | Mc = y, Ob = z) = \frac{2}{3}$$
$$x \neq y, \; P(Cp = x | Mc = x, Ob = y) = \frac{1}{3}$$

2 Python Code

We first write the tabular corresponding to $P(Ob|Mc, Cp)$ and $P(Ob|Mc)$ which are written `A` and `B` in the following code on Figure 1 and then we compute $P(Cp|Mc, Ob)$.

```
import numpy as np

dic = { 0: 'A', 1: 'B', 2: 'C' }

A = np.array(
    [
        [0, .5, .5],
        [0, 0, 1],
        [0, 1, 0],
        [0, 0, 1],
        [.5, 0, .5],
        [1, 0, 0],
        [0, 1, 0],
        [1, 0, 0],
        [.5, .5, 0]
    ])

B = 1/2 * np.array(
    [
        [0, 1, 1],
        [1, 0, 1],
        [1, 1, 0]
    ]
)

def compute_proba(M, C, O):
    if M == O: return 0
    p = 1/3 * A[M * 3 + C, O] / B[M, O]
    return p

P = []
for M in range(3):
    for O in range(3):
        P.append([])
        for C in range(3):
            print(f"P(C={dic[C]} | O={dic[O]}, M={dic[M]}) = {compute_proba(M, C, O)}")
```

Figure 1: Python code to compute $P(Cb|Mc, Ob)$

The results of the code when its executed, are given by Figure 2.

```
P(C=A | O=A, M=A) = 0
P(C=B | O=A, M=A) = 0
P(C=C | O=A, M=A) = 0
P(C=A | O=B, M=A) = 0.3333333333333333
P(C=B | O=B, M=A) = 0.0
P(C=C | O=B, M=A) = 0.6666666666666666
P(C=A | O=C, M=A) = 0.3333333333333333
P(C=B | O=C, M=A) = 0.6666666666666666
P(C=C | O=C, M=A) = 0.0
P(C=A | O=A, M=B) = 0.0
P(C=B | O=A, M=B) = 0.3333333333333333
P(C=C | O=A, M=B) = 0.6666666666666666
P(C=A | O=B, M=B) = 0
P(C=B | O=B, M=B) = 0
P(C=C | O=B, M=B) = 0
P(C=A | O=C, M=B) = 0.6666666666666666
P(C=B | O=C, M=B) = 0.3333333333333333
P(C=C | O=C, M=B) = 0.0
P(C=A | O=A, M=C) = 0.0
P(C=B | O=A, M=C) = 0.6666666666666666
P(C=C | O=A, M=C) = 0.3333333333333333
P(C=A | O=B, M=C) = 0.6666666666666666
P(C=B | O=B, M=C) = 0.0
P(C=C | O=B, M=C) = 0.3333333333333333
P(C=A | O=C, M=C) = 0
P(C=B | O=C, M=C) = 0
P(C=C | O=C, M=C) = 0
```

Figure 2: Python result of $P(Cb|Mc, Ob)$ computation

3 GeNIe Result

Our results are confirmed if we design our bayesian network on GeNIe and put the values we compute as parameters for the nodes Mc and Ob. Figure 3 gives the result if we choose as evidence $Mc = A$ and $Ob = B$.

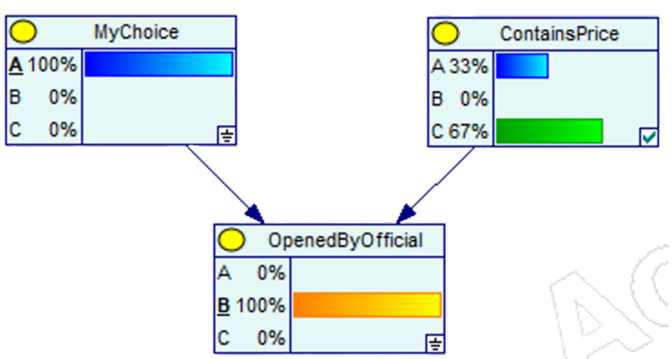


Figure 3: Result given by GeNIe if we choose $Mc = A$ and $Ob = B$

4 A more intuitive way

To go further, an intuitive way to understand that result is to understand that our probability on the door we choose is the same before and after that the official opened one of the two other doors. As the two others had a probability of $1 - 1/3$ to have the prize in it, and that the official knows where the prize is, thus the probability that the prize is on both doors is still the same, after the official opened one of both. As the prize is not behind the door that the official opened, thus the last door has a probability of $2/3$ to have the prize behind it. Hence the final result.

$$\begin{aligned}
x \neq y, x \neq z, y \neq z,\ P(Cp = x|Mc = y, Ob = z) &= 1 - P(\overline{Cp = x|Mc = y, Ob = z}) \\
&= 1 - P((Cp = y|Mc = y, Ob = z) \vee (Cp = z|Mc = y, Ob = z)) \\
&= 1 - (P(Cp = y|Mc = y, Ob = z) + P(Cp = z|Mc = y, Ob = z)) \\
&= 1 - P(Cp = y|Mc = y, Ob = z)
\end{aligned}$$

Yet, we can "feel" that $P(Cp = y|Mc = y, Ob = z) = P(Mc = y) = \frac{1}{3}$, because we first made a choice on a door y, thus the probability that we are not on the good door is $1 - P(Mc = y)$.

$$P(Cp = x|Mc = y, Ob = z) = 1 - \frac{1}{3} = \frac{2}{3}$$