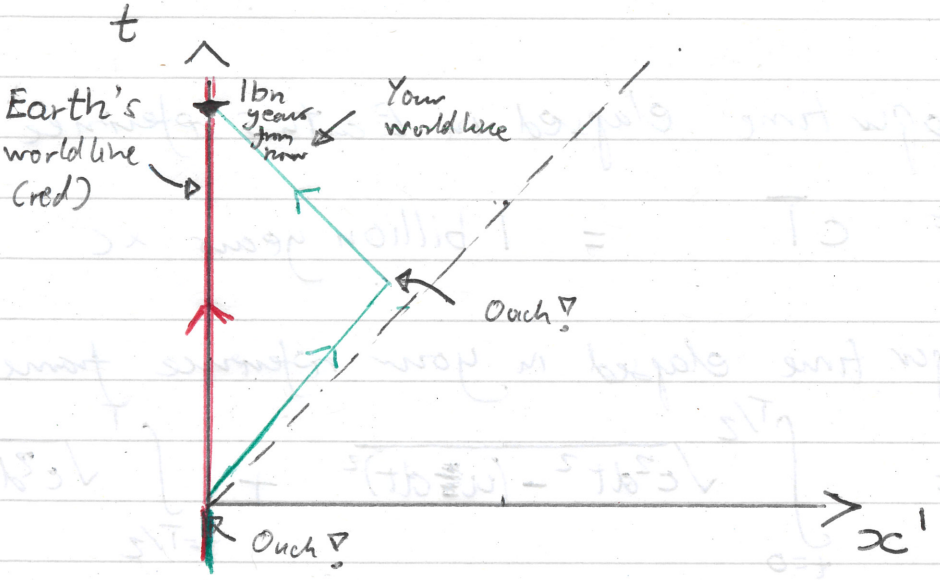
I want to see the future of Earth in a billion years, I have a rocketship with unlimited acceleration and unlimited fuel, what trajectory in space should I follow?

I suppose you don't haven't ~~eternity~~ also got immortality (unlimited lifetime) and unlimited patience, so you probably what your trip in your frame of reference to take no more than say twenty years.



~~Schwarzschild space time~~

$$ds^2 = \left(1 - \frac{}{}\right) dt^2 \cdots$$

Minkowski spacetime

Proper time

$$-d\tau^2 = ds^2 = -c^2dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

Time you experience

$$\Rightarrow \tau = \oint_{\text{Path, } \Gamma} \sqrt{c^2dt^2 - (dx^1)^2}$$

Earth worldline

$$x_E = (ct, 0, 0, 0)$$

Your worldline

$$x_{you} = \begin{cases} (ct, ut, 0, 0) & t \le T/2 \\ (ct, -ut, 0, 0) & t > T/2 \end{cases}$$

Proper time elapsed in Earth's reference frame

$$\tau_E = c\bar{T} = 1 \text{ billion years} \times c$$

Proper time elapsed in your reference frame

$$\tau_{you} = \int_{t=0}^{T/2} \sqrt{c^2dt^2 - (u\,dt)^2} + \int_{t=T/2}^{T} \sqrt{c^2dt^2 - (-u\,dt)^2}$$

$$= \sqrt{c^2 - u^2}\left(\int_{t=0}^{T/2} dt + \int_{t=T/2}^{T} dt\right)$$

$$= \sqrt{c^2 - u^2}\,T$$

If you want the journey to last no more than
$\tau_{you} = 20$ years

$$\frac{\tau_{you}}{\tau_E} = \frac{20}{1\times10^9} \qquad \sqrt{1 - u^2/c^2}$$

$$\Rightarrow \frac{u^2}{c^2} = 1 - (2\times10^{-8})^2$$

$$\frac{u}{c} = (1-(2\times10^{-9})^2)^{\frac{1}{2}}$$

$$\approx 1 - \frac{1}{2}\times 4\times10^{-16}$$

$$= 0.\underbrace{99\ldots9}_{*15\text{ nines}}8 \text{ times the speed of light}$$

$\Rightarrow$ To see the Earth in 1 billion years from now, if you are prepared to take 20 years ~~being accelerated~~ & undergo extreme forces of acceleration in a rocketship to get there, simply accelerate to

$$0.\underbrace{99\ldots9}_{15\times}8 \text{ times the speed of light } \sout{\text{for 10yrs}}$$

instantaneously, keep travelling in some direction for 10 years, then after 10 years, reverse your engines & accelerate to the same speed going ~~in the opposite~~ back to Earth instantaneously. Keep going for another 10 years & you will see Earth in 1 billion years! ∇