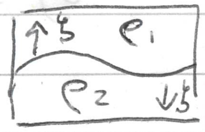
AH PS3 Q.3 Generalized wave dispersion relation to the Raleigh-Taylor Instability & Kelvin Helmholtz Instability

$$\boxed{\text{(1)} \quad \omega = k\,\frac{u_1\rho_1 + \rho_2 u_2}{\rho_1+\rho_2} \pm \sqrt{\frac{\rho_2-\rho_1}{\rho_1+\rho_2}\,gk - k^2\,\frac{\rho_1\rho_2(u_2-u_1)^2}{(\rho_1+\rho_2)^2}}}$$

Rayleigh-Taylor Instability

$u_1 = 0$ & $u_2 = 0$, $\rho_1 > \rho_2$



$$(1) \Rightarrow \omega = \pm\sqrt{\frac{\rho_2-\rho_1}{\rho_1+\rho_2}\,gk} = \pm i\sqrt{\frac{\rho_1-\rho_2}{\rho_1+\rho_2}\,gk}$$

since $\rho_1 > \rho_2$

$$\frac{\omega^2}{k} = \frac{\rho_2-\rho_1}{\rho_1+\rho_2}\,g = A_t\, g$$

Atwood number

Phase velocity, $v = \frac{\omega}{k} = \xi$

$k = \frac{2\pi}{\lambda}$, $\omega = \frac{2\pi}{T}$

Assuming frequency (period) or constant and wavenumber k changes

The displacement of the interface is given by

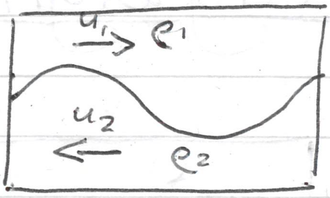
$$\xi = A e^{i(kx-\omega t)}$$

$$\Rightarrow \boxed{\xi = A e^{ikx}\, e^{+\sqrt{A_t g k}\, t}}$$

The interface grows exponentially rather than fading away with a growth rate $\sqrt{A_t g k}$

Kelvin - Helmholtz Instability

$\rho_1 = \rho_2 = \rho$



$$\omega = k \frac{u_1 + u_2 \cancel{\rho}}{\cancel{\rho}} \pm \sqrt{-k^2 \frac{\rho^2 (u_2 - u_1)^2}{(2\rho)^2}}$$

$$\cancel{\boxed{\omega = \pm i k^2 (u_1 + u_2) \pm \Delta u / \sqrt{\rho}}} \qquad \boxed{\omega = k(u_1 + u_2) \pm \tfrac{1}{2} i k \Delta u}$$

where $\Delta u = |u_2 - u_1|$

~~Phase velocity must be positive~~

$$\cancel{\Rightarrow \zeta = A e^{i(kx - \omega t)}}$$

$$\cancel{= A e^{ikx} e^{k^2 (u_1 + u_2) \Delta u / \sqrt{\rho}}}$$

The displacement of the interface is given by

$$\zeta = A e^{i(kx - \omega t)}$$

$$\cancel{= A e^{ik(x - (u_1 + u_2)t)} e^{\pm k \Delta u / \sqrt{\rho}\, t}}$$

Interface grows exponentially

$$\Rightarrow \boxed{\zeta = A e^{ik(x - (u_1 + u_2)t)} e^{+k \Delta u t / 2}}$$