Particle Physics 2015 Q.2

Why is Dirac's Equation
$i\gamma^\mu \partial_\mu \psi = m\psi$?

~~Dirc~~

Classically total energy (Hamiltonian)

$$H = \frac{p^2}{2m} + V$$ BUT • No quantum mechanics [1]

+Q.M. Quantum mechanically, Schrödinger's eq^n says

$$\hat{H}\psi = i\frac{\partial\psi}{\partial t}$$ BUT • Non-relativistic • No spin • No antiparticles [2]

+ RELATIVITY Dirac proposed the Hamiltonian is linear in both momentum and energy since in relativity time and space are equivalent.

$$\Rightarrow H = \underset{\text{constant}}{\alpha} \cdot p + \underset{\text{constant}}{\beta} m$$ [3]

Relativity theory says

$$E^2 = p^2c^2 + m^2c^4$$ Relativistic energy relation [4]

Let's use units where $c = 1$ & switch to quantum mechanical description.

$$p \Rightarrow \hat{p} = -i\nabla \qquad E \to \hat{E} = i\frac{\partial}{\partial t}$$ [*]

$$\Rightarrow -\frac{\partial^2}{\partial t^2}\psi = (\nabla^2 + m^2)\psi$$ [5]

[6]

$$\Rightarrow \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\psi + m^2\psi = 0$$ BUT • No spin • Negative sol^{n}s for probability allowed.

vector form [7]

Klein-Gordon eq^n → tensor form

$$\boxed{\partial^\mu \partial_\mu \psi + m^2\psi = 0}$$

p.t.o.

Substitute

$$\hat{H} = \underline{\alpha} \cdot \underline{p} + \beta m \quad = \alpha_i p_i + \beta m \qquad \boxed{8}$$

→ Schrödinger's eqⁿ

$$\hat{H}\psi = i\frac{\partial \psi}{\partial t}$$

$$(\underline{\alpha} \cdot \underline{p} + \beta m)\psi = i\frac{\partial \psi}{\partial t} \qquad \boxed{9a}$$

(*) → $$= (-i\underline{\alpha} \cdot \nabla + \beta m)\psi = i\frac{\partial \psi}{\partial t} \qquad \boxed{9b}$$

→ K-G eqⁿ

$$H^2 = (\alpha_i p_i + \beta m)(\alpha_j p_j + \beta m)$$

scalar element of a vector

scalar

$$= \alpha_i \alpha_i p_i p_i + \alpha_i p_i \beta m + \beta m \alpha_i p_i$$

$$+ \beta^2 m^2 + \cancel{\alpha_i p_i \alpha_i p_j}$$

$$+ (\alpha_i p_i \alpha_j p_j + \alpha_j p_j \alpha_i p_i)$$

(where $j > i$ in the final term)

$$= \alpha_i^2 p_i^2 + (\alpha_i \beta + \beta \alpha_i) p_i m$$

$$+ (\alpha_i \alpha_j + \alpha_j \alpha_i) p_i p_j + \beta^2 m^2 \qquad \boxed{10}$$

Compare with relativistic energy relation.

$$E^2 = p^2 + m^2$$

pto.

$\Rightarrow$ In order to satisfy ~~quantum mechanics~~ relativity

$$\alpha_i \beta + \beta \alpha_i = 0 \qquad \boxed{11}$$

AND $$\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \qquad \boxed{12}$$

AND $$\beta^2 = 1 \qquad \boxed{13}$$

AND $$\alpha_i^2 = 1 \qquad \boxed{14}$$

So α_i and β are matrices.

Let $\gamma^\mu = (\beta, \beta \underline{\alpha})$

$$\beta = \gamma^0$$

$$\alpha_i = \frac{\gamma^i}{\beta}$$

Momentum 3-vector

$$E = ~~\text{[illegible]}~~ (m, \underline{p})$$

$$\Rightarrow \hat{H}^2 = \gamma^\mu E^\mu$$

$\beta \times (9b)$

$$\beta(-i\underline{\alpha} \cdot \nabla + \beta m)\psi = i\beta \frac{\partial \psi}{\partial t}.$$

$$\Rightarrow (-i\underline{\gamma} \cdot \nabla + (\gamma^0)^2 m)\psi = i\gamma^0 \frac{\partial \psi}{\partial t}$$

$$\Rightarrow i\left(\gamma^0 \frac{\partial}{\partial t} + \underline{\gamma} \cdot \nabla\right)\psi = m\psi$$

$$\partial_\mu = \frac{\partial}{\partial t} + \nabla$$

pto.

$$= \boxed{i\gamma^{\mu}\partial_{\mu}\psi = m\psi} \quad \text{Dirac's Eq}^{n}.$$