Art

Advanced Particle Physics 2017 Q.4
The Dirac Equation

i In classical mechanics, the Hamiltonian, the conserved energy is

$$H = \underbrace{\frac{p^2}{2m}}_{K.E.} + \underbrace{V}_{P.E.} \qquad \boxed{0}$$

In quantum mechanics particles must obey

$$\hat{H}\psi = i\frac{\partial \psi}{\partial t} \qquad \boxed{a}$$

ψ ← Wavefunction of particle.

where H becomes an operator $\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}$

$\hat{p}^2 = -\nabla^2$, $\hat{p} = -i\nabla$

Relativity (special) tells us that particles must obey the Klein-Gordon equation if they have mass.

$$\hat{H}^2\psi = \nabla^2\psi - m^2\psi = \frac{\partial^2\psi}{\partial t^2} \qquad \boxed{b}$$

Dirac wanted a Hamiltonian which is linear in energy & momentum and satisfied quantum mechanics (a) and special relativity (b)

Note, (b) comes from the relativistic energy relation

$$E^2 = p^2 + m^2$$

Assuming $\hat{H} = \underline{\alpha} \cdot \underline{p} + \beta m$

$$\hat{H} \psi = i \frac{\partial \psi}{\partial t}$$

$$(i\alpha \cdot \nabla + \beta m) \psi = i \frac{\partial \psi}{\partial t} \qquad \boxed{1}$$

$$\Rightarrow \quad \frac{\partial^2 \psi}{\partial t^2} = i \frac{\partial}{\partial t} (\alpha \cdot \nabla + \beta m) \psi$$

$$- i (\alpha \cdot \nabla + \beta m) \frac{\partial \psi}{\partial t}$$

m, β, α constant

$$\Rightarrow (b) \quad \nabla^2 \psi - m^2 \psi = -i \frac{\partial}{\partial t} (\alpha \cdot \nabla + \beta m) \psi$$

$$- i (\alpha \cdot \nabla + \beta m) \frac{\partial \psi}{\partial t}$$

In tensor notation

$$\nabla = \partial_\mu \qquad \mu = (1, 2, 3)$$

$$\frac{\partial}{\partial t} = \partial_0$$

$$\Rightarrow \quad \partial_\mu \partial^\mu \psi - m^2 \psi = -i (\alpha \partial_\mu + \beta m) \partial_0 \psi$$

The Klein Gordon equation is

$$H^2 \psi = (\alpha_i p_i + \beta m)(\alpha^j p^j + \beta m)$$

$$= \alpha_i \alpha_j p_i p_j + (\alpha_i p_i \beta + \beta \alpha_j p_j) m$$

pto

$$+ \quad \beta^2 m^2$$

$$\hat{H}^2\psi = \alpha_i\alpha_j p_i p_j + (\alpha_i\beta + \beta\alpha_i) p_i m + \beta^2 m^2.$$

$$= (\alpha_i\alpha_j + \alpha_j\alpha_i) p_i p_j + (\alpha_i^2 p_i^2) + (\alpha_i\beta + \beta\alpha_i) p_i m + \beta^2 m \quad \text{for } i > j$$

relativistic

Compare with energy relation $E^2 = p^2 + m^2$ (1)

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 0$$

$$\alpha_i\alpha_i = \delta_{ij}$$

$$\alpha_i\beta + \alpha_i\beta = 0$$

$$\beta^2 = 1$$

The gamma matrices are defined as a set of matrices in a 4-vector.

$$\gamma^\mu = (\beta, \beta\underline{\alpha})$$

Multiply (1) by β

$$i\beta\frac{\partial\psi}{\partial t} = (i\beta\alpha\cdot\nabla + \beta^2 m)\psi$$

$$\Rightarrow i\left(+\beta\frac{\partial}{\partial t} + \beta\alpha\cdot\nabla\right)\psi = \beta^2 m\psi$$

$$= i\gamma^\mu\partial_\mu\psi = \beta^2 m\psi$$

$\beta^2 = 1$

$$\Rightarrow \boxed{i\gamma^\mu\partial_\mu\psi = m\psi}$$

$$i\gamma^\mu \partial_\mu \psi = m\psi$$

$$= i\gamma^0 \partial_t \psi + i\gamma^j \partial_j \psi = m\psi$$

$$= i\beta \psi + i\beta \nabla \psi = m\psi$$

iii <u>Claim</u>

$\hat{S} = \frac{1}{2}\gamma^5\gamma^0\underline{\gamma}$ satisfies $[\hat{H},\hat{S}] = \gamma^0\underline{\gamma}\times\underline{\nabla}$

$\hat{H} = i\gamma^\mu \partial_\mu$

<u>x - component</u>

$\hat{H}\psi = \frac{\partial\psi}{\partial t}$

$[\hat{H},\hat{S}] = \hat{H}\hat{S} - \hat{S}\hat{H}$

$= (i\gamma^\mu\partial_\mu)\frac{1}{2}\gamma^5\gamma^0\underline{\gamma} - \frac{1}{2}\gamma^5\gamma^0\underline{\gamma}(i\gamma^\mu\partial_\mu)$

γ are constant & $\underline{\gamma}$ commutes

$= \frac{1}{2}i\gamma^\mu\gamma^5\gamma^0\underline{\gamma}\,\partial_\mu - \frac{1}{2}i\gamma^5\gamma^0\gamma^\mu\underline{\gamma}\partial_\mu$

$= 0$

$\gamma^5\gamma^0 = -\gamma$

Commutation relations

(1st term) $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$

$\Rightarrow \gamma^\mu\gamma^5\gamma^0 = -\gamma^5\gamma^\mu\gamma^0$

$= +\gamma^5\gamma^0\gamma^\mu$

$\Rightarrow [\hat{H},\hat{S}] = \frac{1}{2}i\gamma^5\gamma^0\gamma^\mu\underline{\gamma}\partial_\mu - \frac{1}{2}i\gamma^5\gamma^0\gamma^\mu\underline{\gamma}\partial_\mu$

$= 0$

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$$\hat{S}^2 = \left(\tfrac{1}{2}\gamma^5\gamma^0\not{\gamma}\right)^2$$

$$= \left(\tfrac{1}{2}\right)^2 (\gamma^5\gamma^0\not{\gamma})\cdot(\gamma^5\gamma^0\not{\gamma})$$

$$= \tfrac{1}{4}\gamma^5\gamma^0\gamma^5\gamma^0\not{\gamma}\cdot\not{\gamma}$$

$$\{\gamma^5, \gamma^\mu\} = \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0$$

$\Rightarrow \gamma^5$ & γ^μ anti-commute.

$$\Rightarrow \hat{S}^2 = -\tfrac{1}{4}\gamma^0\gamma^5\gamma^5\gamma^0 \underbrace{\not{\gamma}\cdot\not{\gamma}}_{\text{Scalar so commute}}$$

$(\gamma^5)^2 = 1$

$$\Rightarrow \quad = -\tfrac{1}{4}\gamma^0\gamma^0\not{\gamma}\cdot\not{\gamma}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \Rightarrow \gamma^0\gamma^0 = -1$$

$$\Rightarrow \quad = +\tfrac{1}{4}\not{\gamma}\cdot\not{\gamma}$$

$$= +\tfrac{1}{4}\left(\gamma^1\gamma^1 + \gamma^2\gamma^2 + \gamma^3\gamma^3\right)$$

$S(S+1)$

$= \frac{1}{2}\frac{3}{2}$

$$= +\tfrac{1}{4}\times 3$$

$$\Rightarrow \boxed{\hat{S}^2 = \tfrac{3}{4}}$$

Compare with $\hat{S}^2\psi = S(S+1)\psi$

$\Rightarrow \boxed{S = \tfrac{1}{2}}$ Dirac eq^n describes a spin $\frac{1}{2}$ particle

✓ Measuring momenta of charged particles:

- Apply a magnetic field perpendicular to the plane of motion of the particles.

$$R = \frac{p_T}{B}$$

p_T: Transverse momentum

B: Magnetic field.

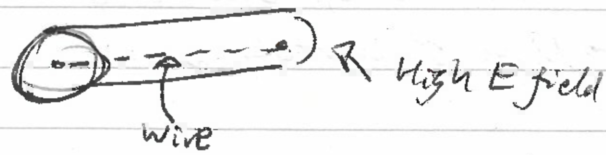
R: Radius of curvature

- To measure positions of particles along trajectory use one of ~~the~~ (or several of) the following

↳ ~~Scil~~ Scintillation counter – high sensitivity, limited resolution

↳ Bubble chamber ~~Resolution ~ 50μm~~

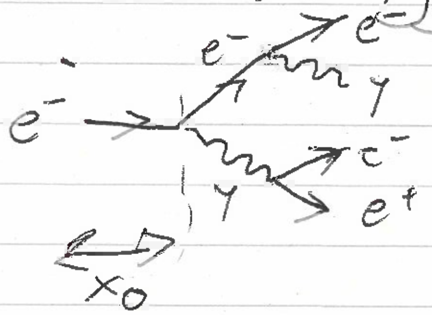
↳ Silicon strip detector – Resolution ~ 50μm

↳ 

– Resolution ~100μm

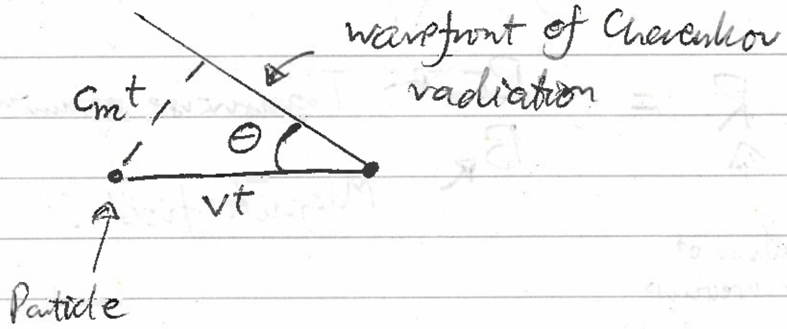
- Charged leptons (e^-s especially) produce electron showers, hadrons produce hadron showers.

Measure ~~energy~~ from the radiation length, X_0



vi • Can measure energy from radiation length (see V.)

• Can measure velocity from Cherenkov radiation



Angle of radiation emitted relative to momentum

$$\sin\theta = \frac{c_m}{v}$$

c_m ← speed of light in medium

v ← velocity of particle

→ Can determine mass of particle.

? • If beam ~~is mono-energetic~~ is composed of particles with same speed can determine mass from radius of curvature