ATOMIC FUNCTIONS

Definition

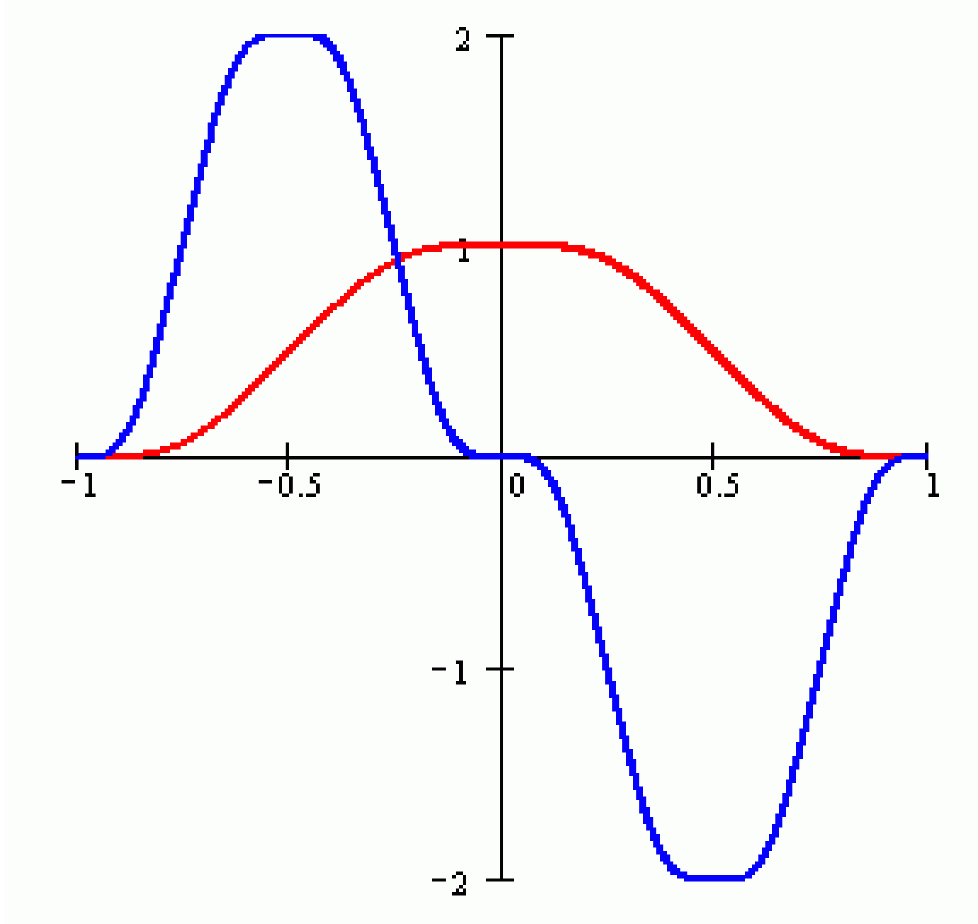
Atomic functions (AFs) are the compactly supported solutions to the following kind of functional-differential equations (FDEs)

$$\sum_{n=1}^{N} d_n y^{(n)}(x) = \sum_{m=1}^{M} c_m y(ax - b_m), \qquad (1)$$

where a, d_n, c_m, b_m are numerical parameters, and $|a| > 1$.

Function up(x)

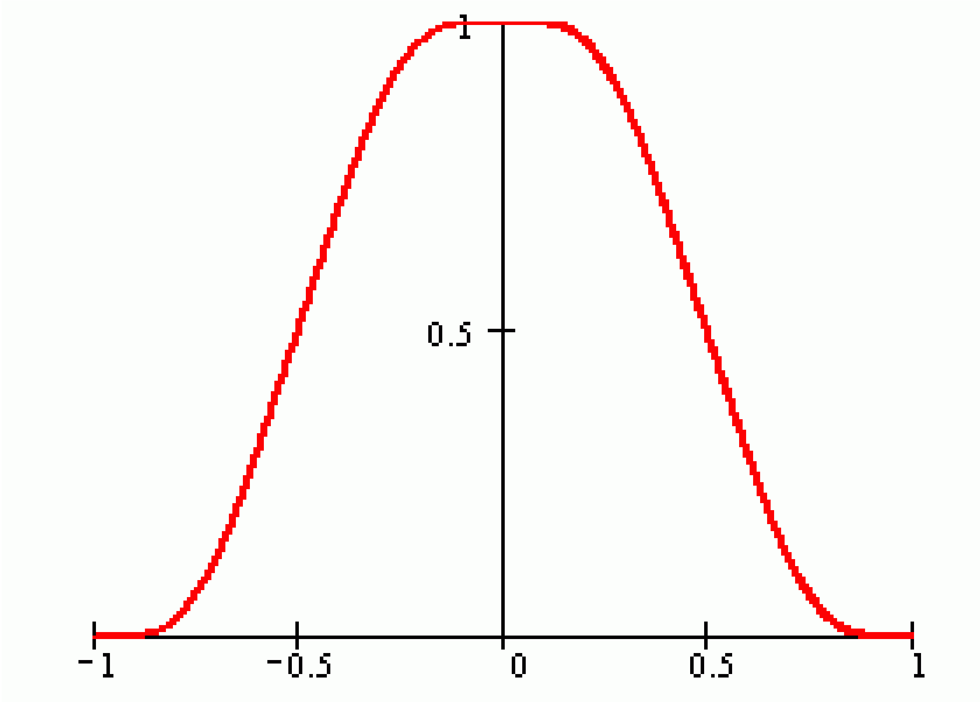
Function up(x) is the simplest and most important **AF** satisfying the following FDE:



$$\mathrm{up}'(x) = 2\,\mathrm{up}(2x+1) - 2\,\mathrm{up}(2x-1) \qquad (2)$$

Main properties of up(x)

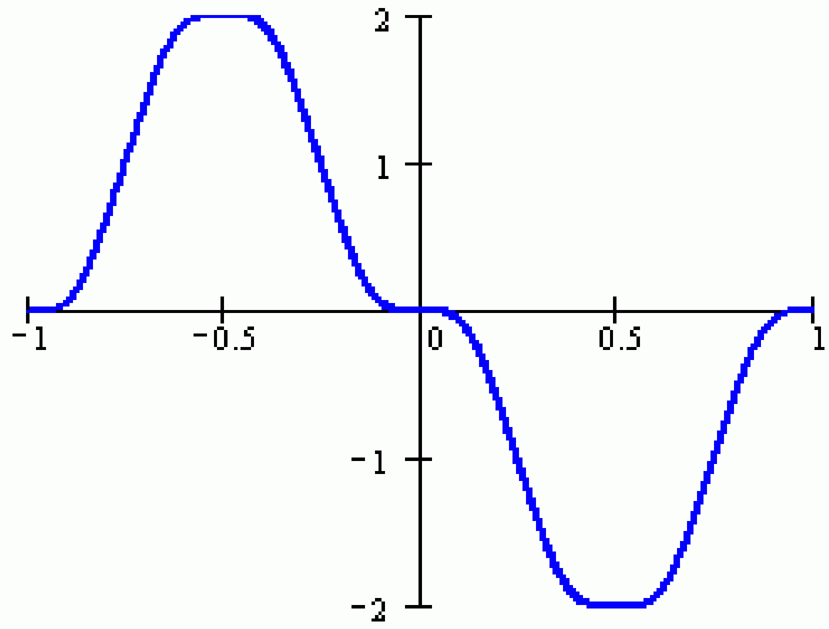
1. $\mathrm{up}(x) \in C^{\infty}(-\infty;\infty)$
2. $supp\,\mathrm{up}(x) = (-1,1)$
3. $\mathrm{up}(x)=\mathrm{up}(-x)$
4. $\mathrm{up}(0)=1$
5. $\int\limits_{-\infty}^{\infty} \mathrm{up}(x)dx = 1$
6. $\mathrm{up}'(x)>0$ on [-1,0], $\mathrm{up}'(x)<0$ on [0,1]
7. $\mathrm{up}(1-x)=1-\mathrm{up}(x)$



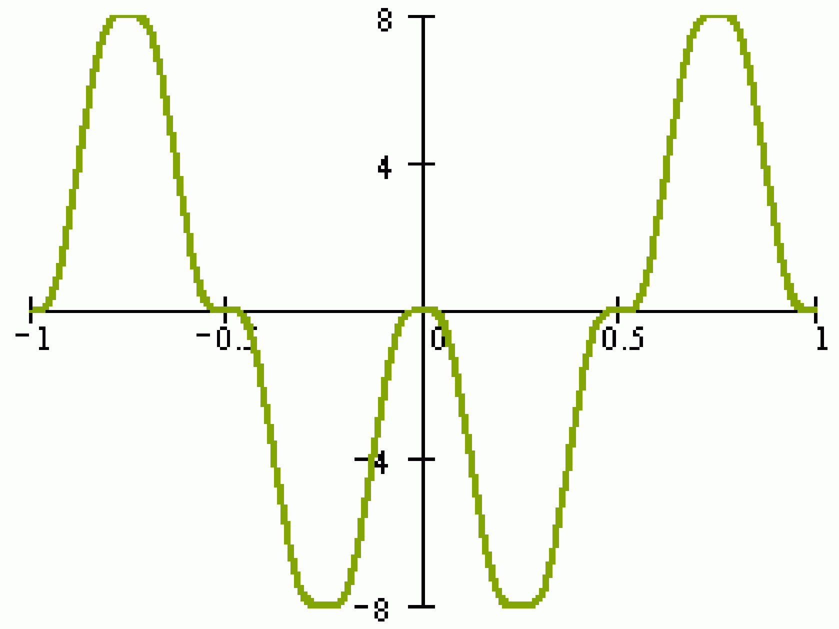
Derivatives of up(x)

$$\mathrm{up}^{(n)}(x) = 2^{n(n+1)/2} \sum_{k=1}^{2^n} \delta_k \, \mathrm{up}(2^n x + 2^n + 1 - 2k), \quad (4)$$

where $\delta_1 = 1, \ \delta_{2k} = -\delta_k, \ \delta_{2k-1} = \delta_k, \ k = 1, 2, \ldots$



$\mathrm{up}'(x)$



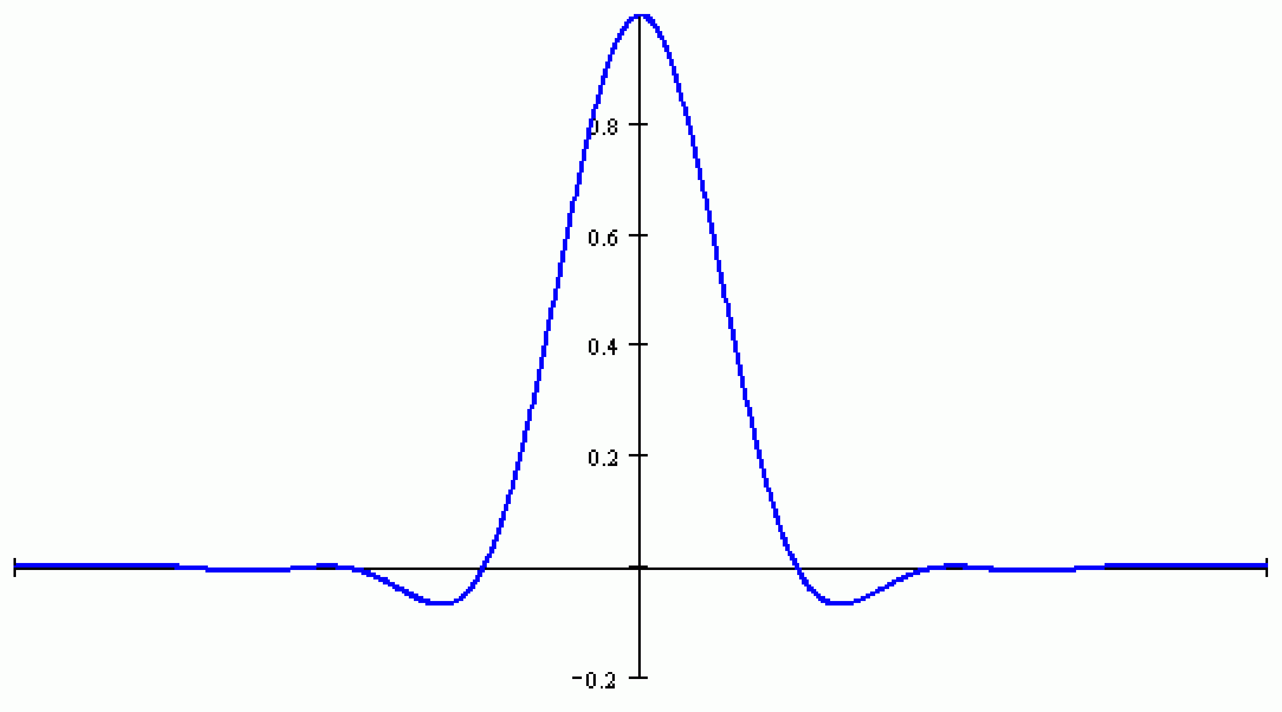
$\mathrm{up}''(x)$

Fourier transform of up(x)

$$F(p) = \prod_{k=1}^{\infty} \operatorname{sinc} \frac{p}{2^k}, \qquad (3)$$

where

$$\operatorname{sinc}(x) \equiv \frac{\sin x}{x}.$$



Computation of up(x)

Recurrent relations for *moments* of up(x)

$$a_0 = 1, \quad a_{2n} = \frac{(2n)!}{2^{2n}-1}\sum_{k=1}^{n}\frac{a_{2n}-2k}{(2n-2k)!(2k+1)!};$$

$$b_{2n+1} = \frac{1}{(n+1)2^{2n+3}}\sum_{k=0}^{n+1}\binom{2n}{2j}a_{2n+2},$$

where

$$a_{2n} = \int_{-1}^{1} x^{2n}\,\mathrm{up}(x)dx, \quad b_n = \int_{0}^{1} x^{n}\,\mathrm{up}(x)dx.$$

1. Values of up(x) *at binary rational points*

$$x = \frac{k}{2^n}$$

are rational numbers:

$$\operatorname{up}\left(\frac{1}{2^n}\right) = 1 - \frac{b_{n-1}}{2^{n(n-1)/2}(n-1)!};$$

$$\operatorname{up}\left(\frac{k}{2^n}\right) =$$

$$-\frac{2^{(-n^2+n+1)/2}}{n!}\sum_{j=1}^{k}\delta_j\sum_{l=0}^{\lfloor n/2\rfloor}\binom{n}{2l}\frac{a_{2l}}{4^l}\left(k-j+\frac{1}{2}\right)^{n-2l}$$

2. *Special rapidly convergent series*

$$\mathrm{up}(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{s_k} p_k}{2^{k(k-1)/2}} \sum_{j=0}^{k} \frac{b_{k-j-1}}{(k-j-1)!\, j!} \left(|x| \cdot 2^k - \left\lfloor |x| \cdot 2^k \right\rfloor \right)^j$$

where

$$b_{-1} = 1, \quad 0! = 1, \quad s_k = \sum_{j=1}^{k} p_j ,$$

$$p_k = \left\lfloor |x| \cdot 2^k \right\rfloor \bmod 2$$

3. *Normalized Legendre polynomials* $L_N(x)$:

$$\mathrm{up}(x) = \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} z_{2j,k} a_{2j-2k} \right) L_{2j}(x),$$

where

$$z_{j,k} = (-1)^k \binom{j}{k} \frac{(2j-2k)!\sqrt{2j+1}}{2^{j+1/2}\, j!(j-2k)!}, \quad (2k \le j),$$

$$L_n(x) = \sum_{k=0}^{[n/2]} z_{n,k} x^{n-2k}$$

4. *Fourier series*

$$\mathrm{up}(x) = \frac{1}{2} + \sum_{j=1}^{\infty}\left(\prod_{k=1}^{\infty}\operatorname{sinc}\frac{\pi(2j-1)}{2^k}\right)\cos\left[\pi(2j-1)x\right]$$

5. *Kotelnikov series*

$$\mathrm{up}(x) \approx \sum_{k=-2^n+1}^{2^n-1}\mathrm{up}\left(\frac{k}{2^n}\right)\operatorname{sinc}\left[\pi(2^n x-k)\right]$$

6. *Bernstein polynomial*

$$\mathrm{up}(x) \approx B_n(\mathrm{up};x) = \sum_{k=0}^{2^n}\binom{2^m}{k}\mathrm{up}\left(\frac{k}{2^n}\right)x^k(1-x)^{2^m-k}$$

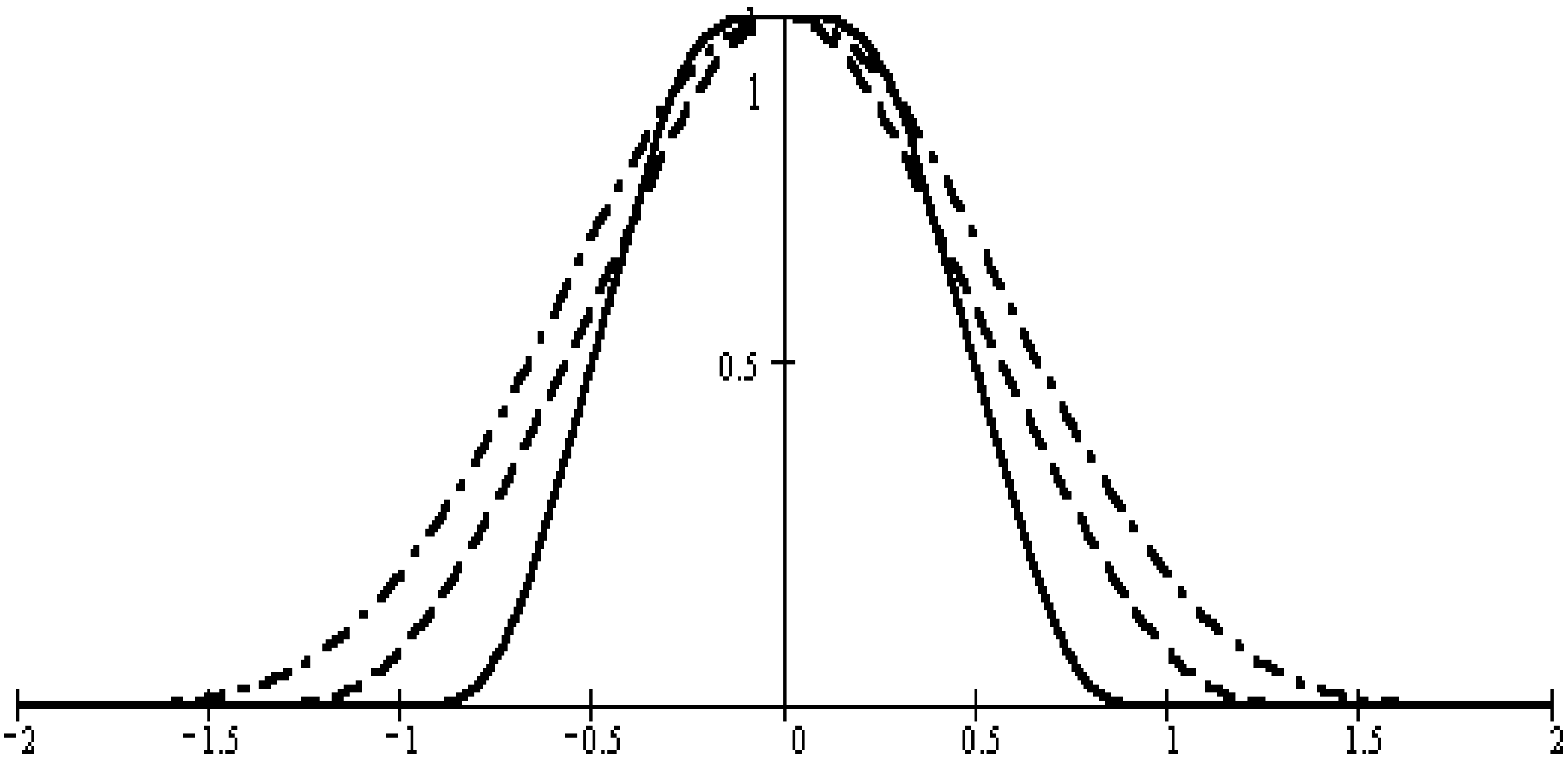
Functions $\mathrm{fup}_n(x)$

Recurrent functional-differential relations:

$$\mathrm{fup}_0(x) \equiv \mathrm{up}(x),$$

$$\mathrm{fup}_n'(x) = K\left[\mathrm{fup}_{n-1}\left(x - \frac{1}{2}\right) - \mathrm{fup}_{n-1}\left(x + \frac{1}{2}\right)\right]$$

$$supp\,\mathrm{fup}_n(x) = \left(-\frac{n+2}{2}, \frac{n+2}{2}\right)$$



up(x) solid line,

fup$_1$(x) dashed line,

fup$_2$(x) dashed-dotted line

Normalizing factor K is defined by one of the following conditions:

1. $\mathrm{fup}_n(0) = 1$;

2. $\sum_k \mathrm{fup}_n(x-k) \equiv 1$ (*partition of unity*);

3. $\int_{-\infty}^{\infty} \mathrm{fup}_n(x)dx = 1$.

Recurrent convolution expressions:

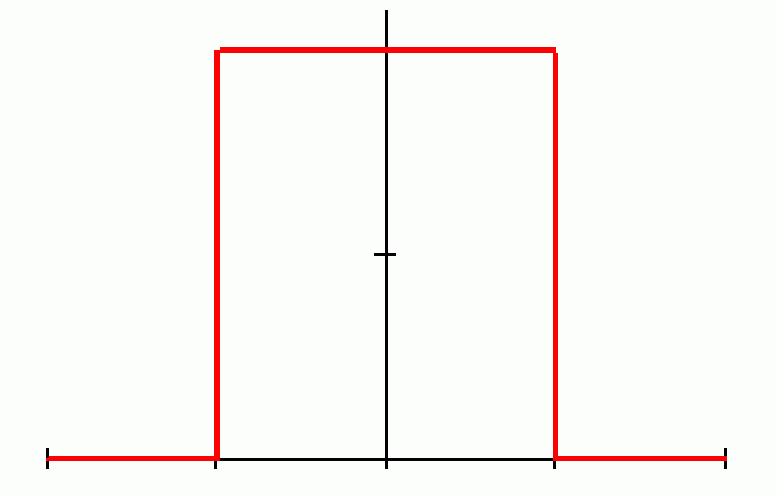
$$\mathrm{fup}_n(x) = K \cdot \mathrm{fup}_{n-1}(x) * B_0(x)$$

where “*” denotes the convolution operation:

$$f_1(x) * f_2(x) = \int_{-\infty}^{\infty} f_1(x) f_2(z-x)\,dz$$

and

$$B_0(x) = \begin{cases} 1, |x| \le 1/2 \\ 0, |x| > 1/2 \end{cases}$$



Fourier transform

$$F_n(p) = K\,\mathrm{sinc}^n\left(\frac{p}{2}\right)\prod_{k=1}^{\infty}\mathrm{sinc}\left(\frac{p}{2^k}\right)$$

Functional-differential equations

$$\mathrm{fup}_n'(x) =$$

$$= 2\sum_{k=0}^{n+2}\left[\binom{n+1}{k} - \binom{n+1}{k-1}\right]\mathrm{fup}_n[2x - k + (n+2)/2]$$

Evaluation of $\mathrm{fup}_n(x)$ via $\mathrm{up}(x)$

$$\mathrm{fup}_n(x) = \sum_{i=0}^{m+1} \alpha_i^{(m)} \,\mathrm{up}\left[\frac{1}{2^m}\left(x - 2^m + \frac{m}{2} + 1 - i\right)\right],$$

where

$$\alpha_0^{(m)} = 1, \quad \alpha_i^{(m)} = (-1)^i \binom{m+1}{i} - \sum_{j=0}^{i-1} \alpha_j^{(m)} \delta_{i-j+1},$$

$$\delta_1 = 1, \quad \delta_{2k} = -\delta_k, \quad \delta_{2k-1} = \delta_k.$$

Approximation by AFs $\mathrm{fup}_n(x)$

Consider approximation of a function

$$f(t) \in C^r[-\pi;\pi]$$

defined on the uniform mesh

$$\Delta_N: \quad t_i = ih, \quad h = \frac{\pi}{N}, \quad i = \overline{-N,N}$$

Suppose r is even and consider the basis of dilations and translations of the AF $\mathrm{fup}_r(x)$:

$$\varphi_{r,k}(t) \equiv \mathrm{fup}_r\left(\frac{t+\pi}{h} - k\right), \quad k \in \mathbb{Z}.$$

Supports of $\varphi_{r,k}(t)$ are defined by

$$\left|t-\tau_k\right| \le \frac{r+2}{2}h.$$

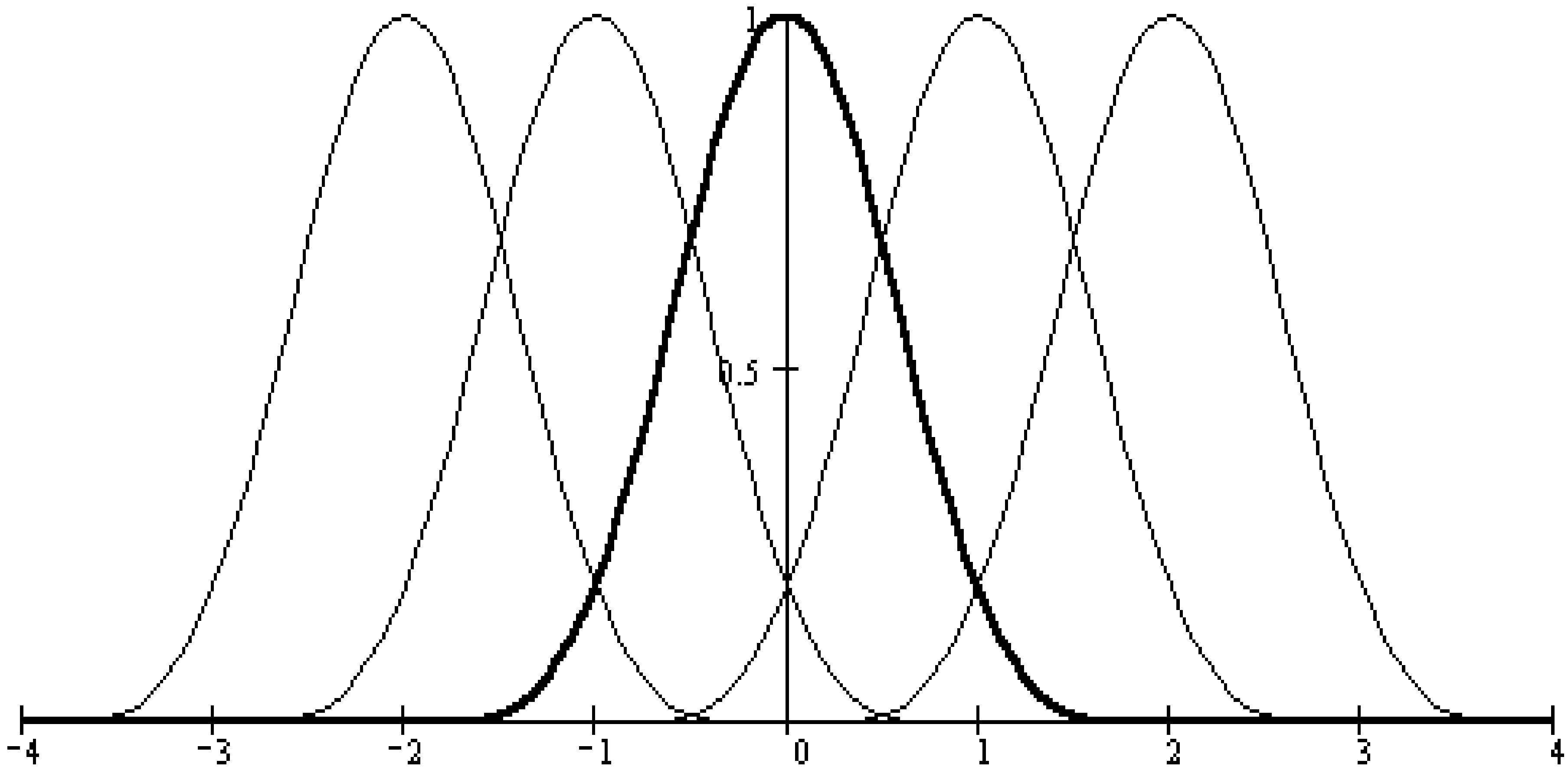
Atomic approximation of order r for $f(t)$

$$\Phi_{N,r}(f;t)=\sum_{k=-N-r/2}^{N+r/2} c_k \varphi_{r,k}(t).$$

It is known that $\forall h>0\ \exists c_k$ *such that*

$$\left\|f(x)-\Phi_{N,r}(f;x)\right\|_{C[-\pi;\pi]} \le O(h^{r+1})$$

Basis $\mathrm{fup}_2\left(\dfrac{t+\pi}{h}-k\right)$:



SOLVING THE PROBLEM OF ELASTIC RING DYNAMICS WITH THE USE OF AFs $\mathrm{fup}_n(x)$

Consider the equation for free oscillations of an ideal inextensible rotating ring

$$\ddot{w}'' - \ddot{w} + \kappa^2 \left(w^{\mathrm{VI}} + 2w^{\mathrm{IV}} + w'' \right) = 0, \qquad (1)$$

where $w = w(\varphi)$ is the radial displacement.

Introduce the uniform mesh

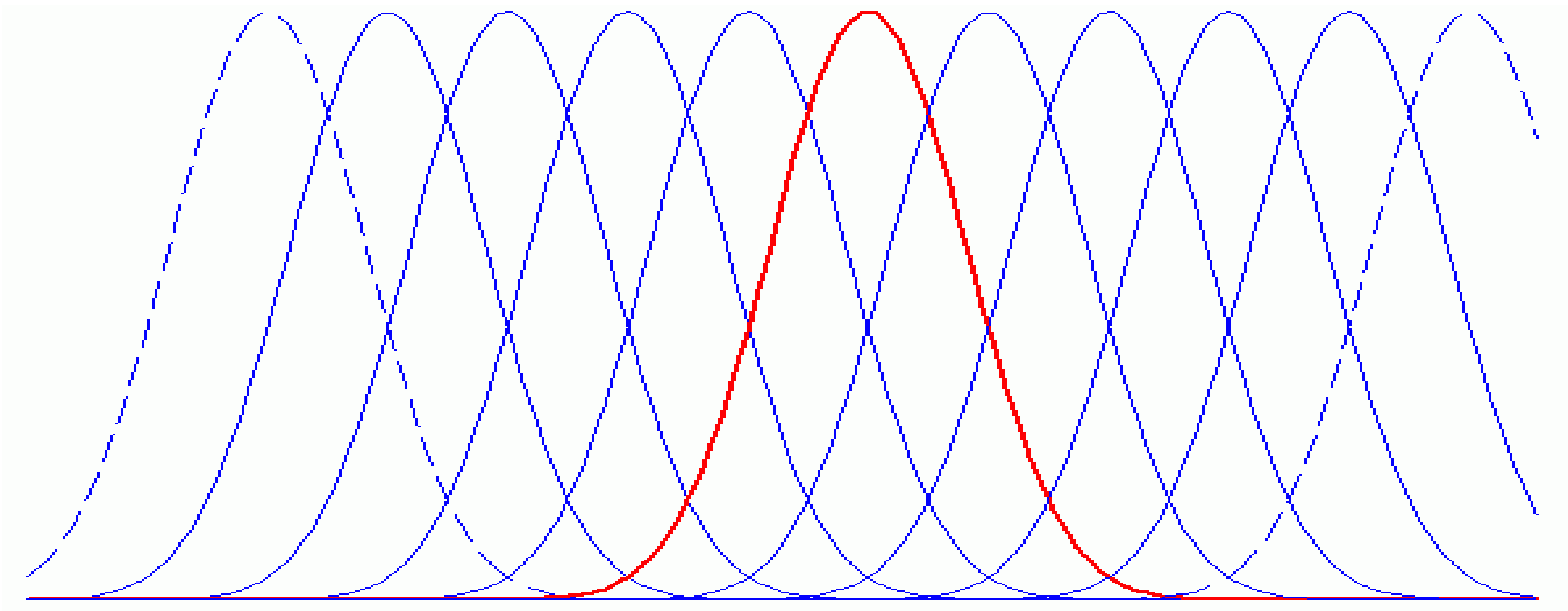
$$\varphi_i = ih;\ i = 0,1,\ldots,N;\ h = \frac{2\pi}{N}. \tag{2}$$

Since (1) is the sixth-order equation, then an approximate solution will be found in the form

$$w^*(z) = \sum_{j=-3}^{N+3} d_{j+3}\psi_j(z), \tag{3}$$

where

$$\psi_j(\varphi) = \mathrm{fup}_6\left(\frac{\varphi}{h} - j\right). \tag{4}$$



Undetermined coefficients d_i of expansion (3) and eigenvalues λ_k $(k = 0,1,...)$ are found from the generalized eigenvalue problem

$$\mathrm{Ad} = \lambda \mathrm{Bd},$$

where $A_{i,j}$ and B_i are defined by:

collocation conditions at nodes φ_i

$$A_{i,j} = \kappa^2 \left[\psi^{\mathrm{VI}}_{j-3}(\varphi_i) + 2\psi^{\mathrm{IV}}_{j-3}(\varphi_i) + \psi''_{j-3}(\varphi_i) \right],$$

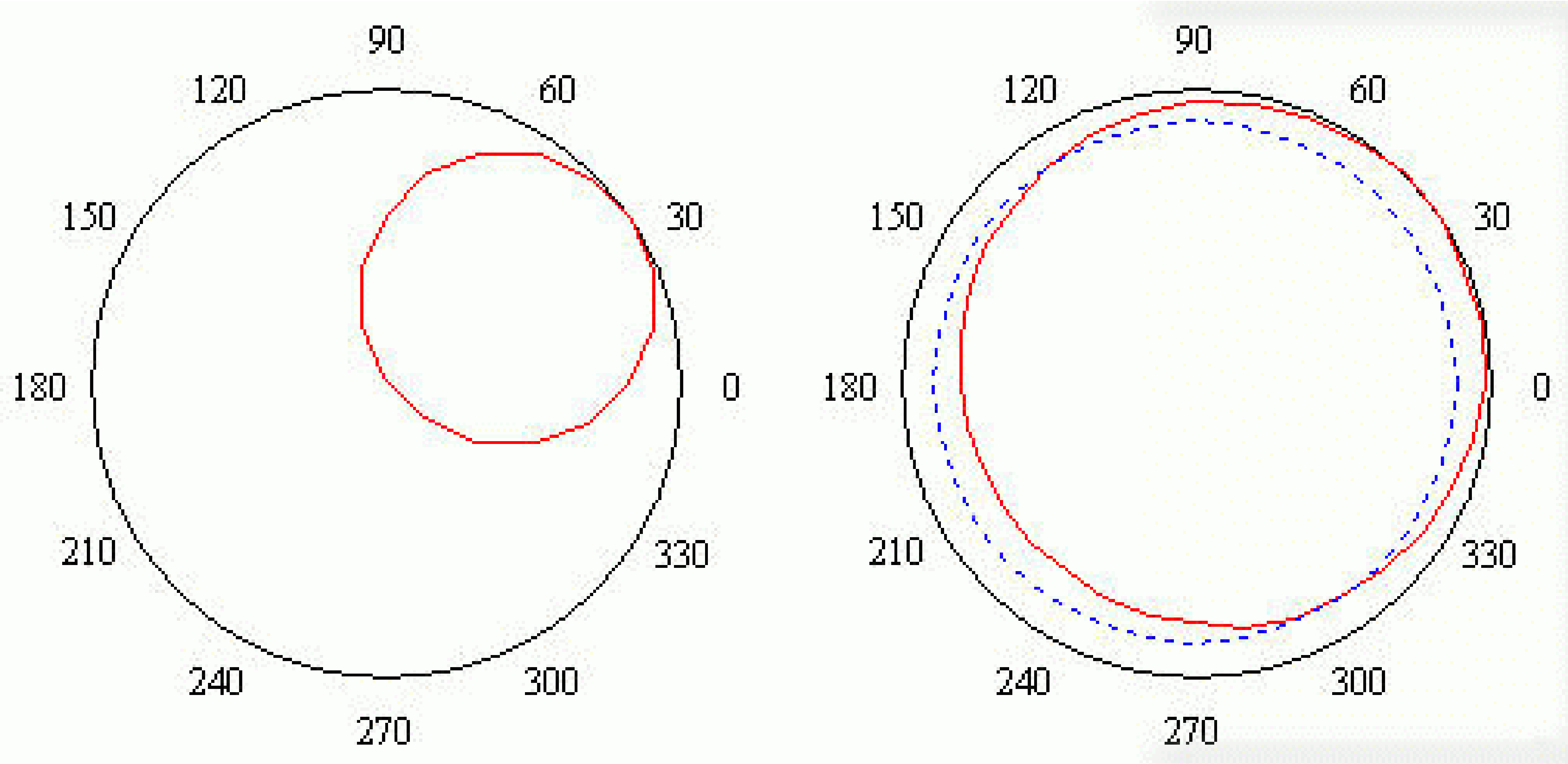
$$B_i = \psi_{j-3}(\varphi_i) - \psi''_{j-3}(\varphi_i),$$

$$(i = 0,1,\ldots,N;\ j = 0,1,\ldots,N+6);$$

and periodicity conditions

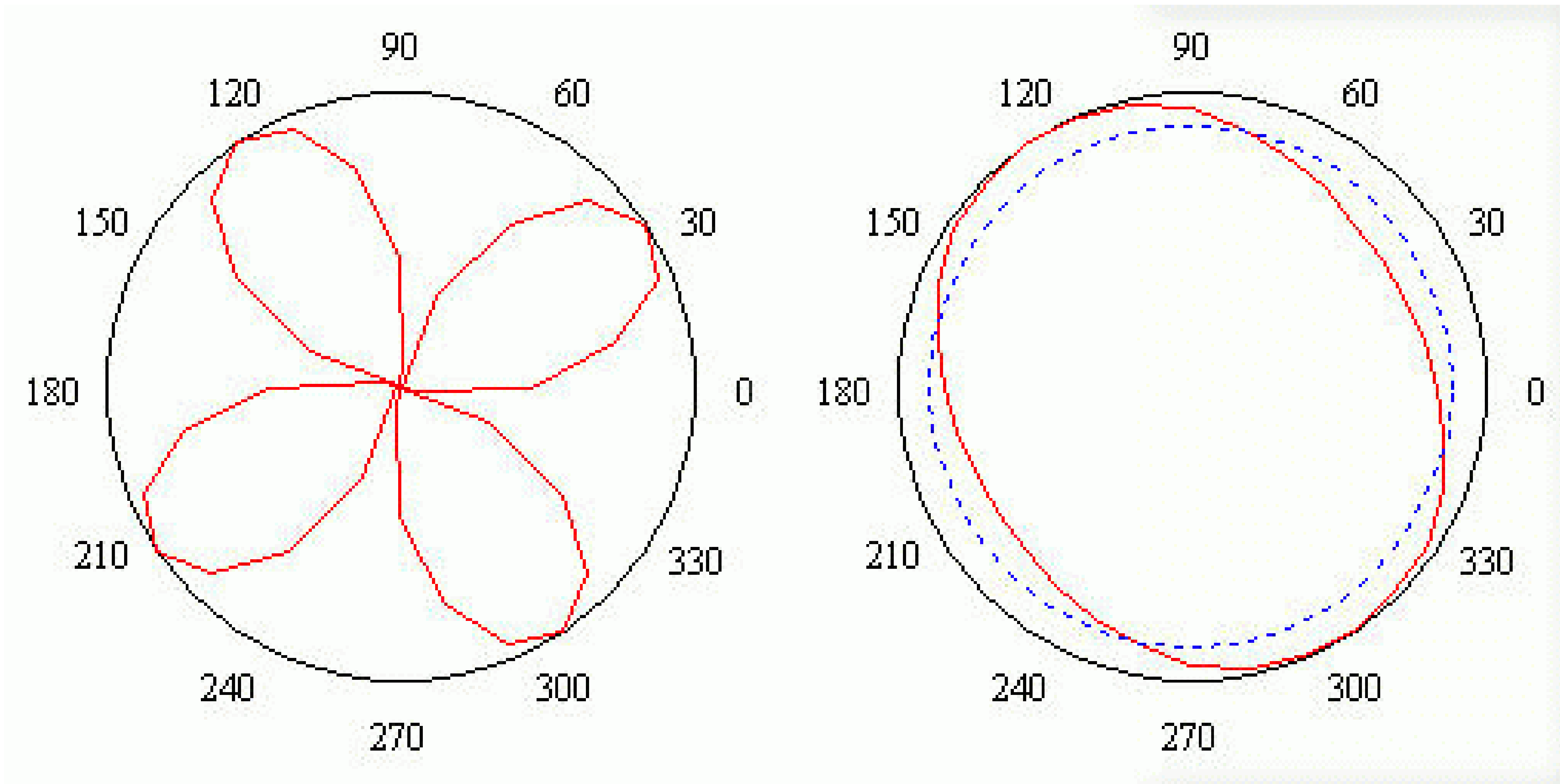
$$A_{N+q,j} = \psi^{(q)}_{j-3}(\varphi_N) - \psi^{(q)}_{j-3}(\varphi_0);\ b_{N+q,j} = 0,$$

$$(q = 0,1,\ldots,5;\ j = 0,1,\ldots,N+6).$$



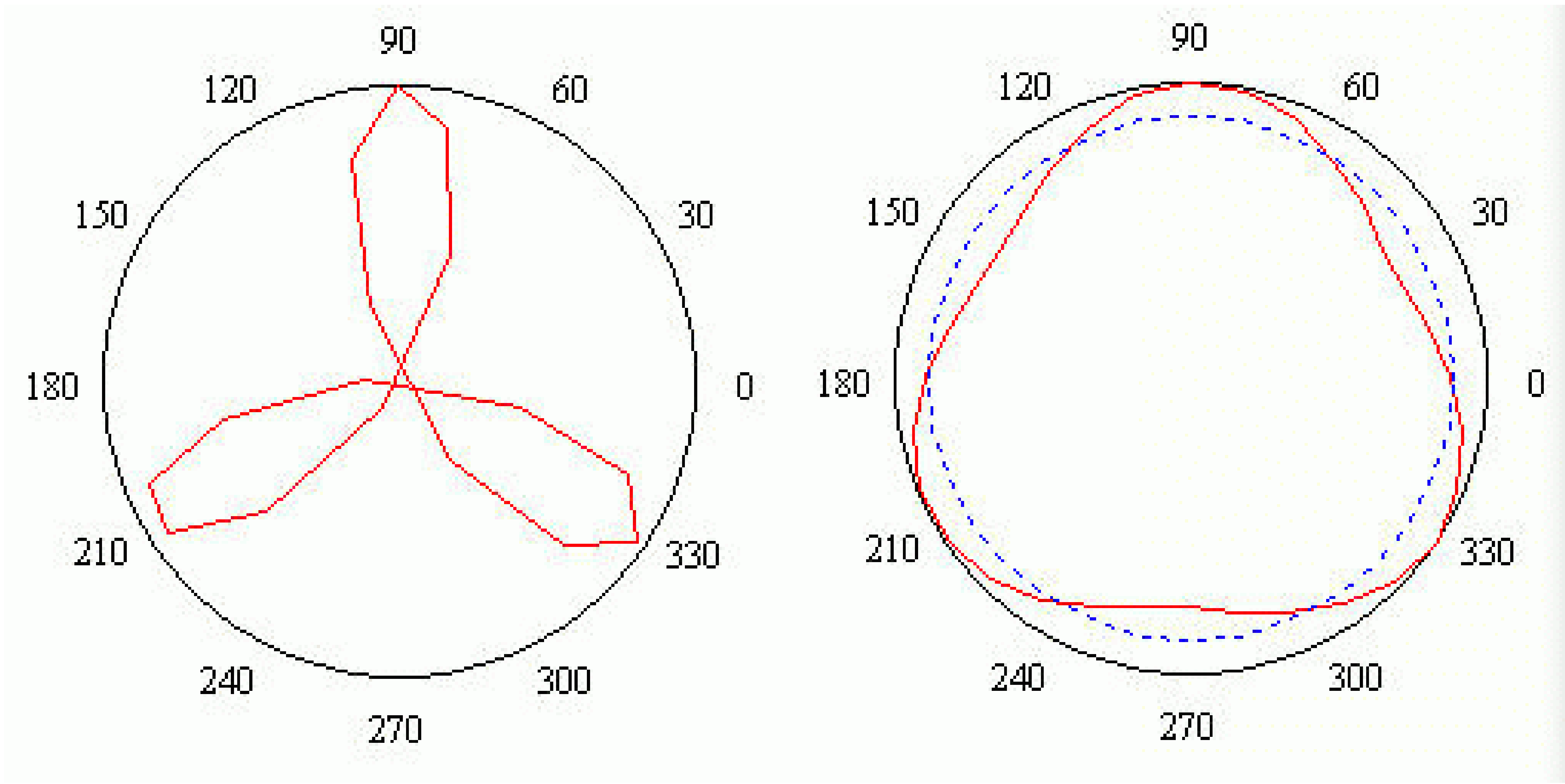
$\delta w(\varphi)$ $w(\varphi)$

I



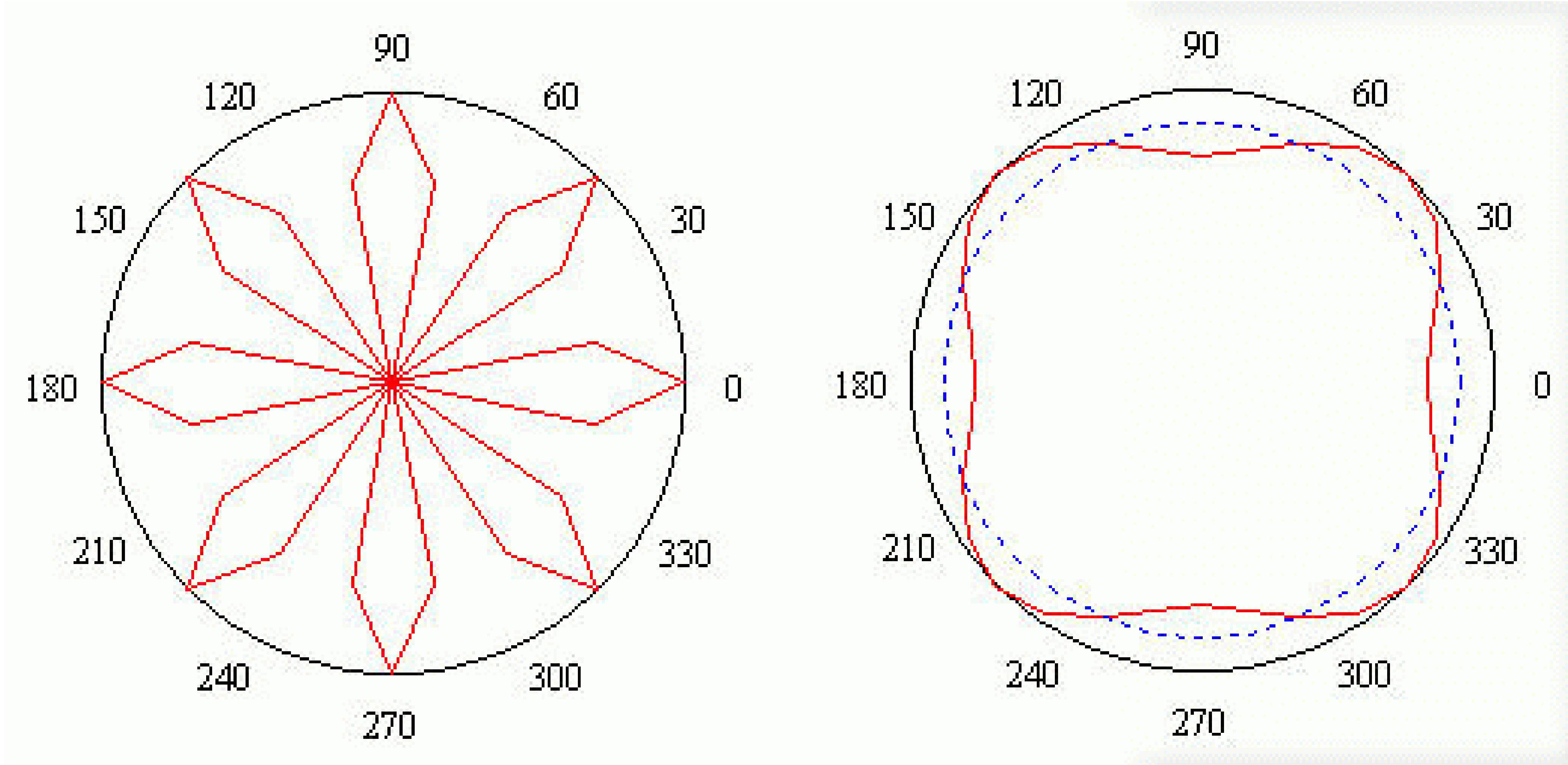
$\delta w(\varphi)$ $w(\varphi)$

II



$\delta w(\varphi)$ $w(\varphi)$

III



$\delta w(\varphi)$ $w(\varphi)$

IV

The Atomic Functions

AF, support	FDE, Fourier transform
up(x), $[-1, 1]$	$y'(x) = 2y(2x+1) - 2y(2x-1)$, $y(p) = \prod_{k=1}^{\infty} \operatorname{sinc}\left(\frac{p}{2^k}\right)$, $(\operatorname{sinc}(x) \equiv \sin x / x)$
$h_a(x)$ $(a>1)$, $\left[-\frac{1}{a-1}, \frac{1}{a-1}\right]$	$y'(x) = \frac{a^2}{2}\left[y(ax+1) - y(ax-1)\right]$, $y(p) = \prod_{k=1}^{\infty} \operatorname{sinc}\left(\frac{p}{a^k}\right)$
cup(x), $[-2, 2]$	$y''(x) = 4\left[y(2x+1) - 2y(2x) + y(2x-1)\right]$, $y(p) = \prod_{k=1}^{\infty} \operatorname{sinc}^2\left(\frac{p}{2^k}\right)$

$\mathrm{fup}_n(x)$, $\left[-\frac{n+2}{2}, \frac{n+2}{2}\right]$	$y'(x) = 2^{n+1}\sum_{k=0}^{n+2}\left(C_{n+1}^{k} - C_{n+1}^{k-1}\right)y\left(2^{n-1}x - \frac{2(k-1)-n}{2^{n+2}}\right)$, $y(p) = \mathrm{sinc}^n\left(\frac{p}{2}\right)\prod_{k=1}^{\infty}\mathrm{sinc}\left(\frac{p}{2^k}\right)$
$\mathrm{Fup}_n(x)$, $\left[-\frac{n+2}{2^{n+1}}, \frac{n+2}{2^{n+1}}\right]$	$y'(x) = 2\sum_{k=0}^{n+2}\left(C_{n+1}^{k} - C_{n+1}^{k-1}\right)y\left(2x - \frac{2(k-1)-n}{2^{n+2}}\right)$, $y(p) = \mathrm{sinc}^{n+1}\left(\frac{p}{2^{n+1}}\right)\prod_{k=n+2}^{\infty}\mathrm{sinc}\left(\frac{p}{2^k}\right)$
$\Xi_n(x)$, $[-1,1]$	$y'(x) = 2^{-n}(n+1)^{(n+1)}\sum_{k=0}^{n+2}(-1)^k C_n^k y[(n+1)x - 2k + n]$, $y(p) = \prod_{k=1}^{\infty}\mathrm{sinc}^n\left(\frac{p}{(n+1)^k}\right)$

$y_k(x)$, $[-1,1]$	$$y'(x)-ky(x)=\frac{2e^{-k/2}}{\mathrm{shc}(k/2)}y(2x+1)-\frac{2e^{k/2}}{\mathrm{shc}(k/2)}y(2x-1),$$ $$(\mathrm{shc}(x)\equiv \mathrm{sh}\,x/x),$$ $$y(p)=\prod_{n=1}^{\infty}\frac{\mathrm{shc}\left(k2^{-1}+ip2^{-n}\right)}{\mathrm{shc}(k/2)}$$
$\pi_m(x)$, $[-1,1]$	$$y'(x)=a\left[y\left(x_1(m)\right)+\sum_{k=2}^{2m-1}(-1)^k y\left(x_k(m)\right)-y\left(x_{2m}(m)\right)\right],$$ $$\left(x_k(m)=2mx+2m-2k+1,\ x\in R^1,\ k=\overline{1,2m}\right),$$ $$y(p)=\prod_{k=1}^{m}\frac{\sin\left(\frac{(2m-1)t}{(2m)^k}\right)+\sum_{\nu=2}^{m}(-1)^{\nu}\sin\left(\frac{(2m-2\nu+1)t}{(2m)^k}\right)}{(3m-2)t\big/(2m)^k}$$

$g_{k,h}(x)$, $[-h, h]$	$$y''(x)+k^2y(x)=ay(3x+2h)-by(3x)+ay(3x-2h),$$ $$a=\frac{3}{2}\frac{k^2}{1-\cos\left(\frac{2kh}{3}\right)},\ b=2a\cos\left(\frac{2kh}{3}\right),$$ $$y(p)=\prod_{j=1}^{\infty}\frac{\frac{k^2}{1-\cos(2kh/3)}\left(\cos\left(\frac{p2h}{3^j}\right)-\cos\left(\frac{2kh}{3}\right)\right)}{k^2-p^2/9^{j-1}}$$
$up_n(x)$, $[-1, 1]$	$$y'(x)=2\sum_{k=1}^{n}\left[y(2nx+2n-2k+1)-y(2nx-2k+1)\right],$$ $$y(p)=\prod_{k=1}^{\infty}\frac{\sin^2\left(np(2n)^{-k}\right)}{np(2n)^{-k}\sin(p(2n)^{-k})}$$