Murashko Artem SD20-01

Programming Paradigms Fall 2022 — Problem Sets

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1 Problem set №1

1. For each of the following λ-terms write down an α-equivalent term where all variables have different names:
   (a) $\lambda x.(\lambda y.x\ y)\ x$
   (b) $\lambda x.(\lambda x.x)\ x$
   (c) $\lambda x.\lambda y.x\ y$
   (d) $\lambda x.x\ (\lambda x.x)$
   (e) $\lambda x.(\lambda x.x)\ x$
   (f) $(\lambda x.\lambda y.y)\ z\ x$
2. Write down evaluation sequence for the following λ-terms:
   (a) $(\lambda x.\lambda y.x)\ y\ z$
   (b) $(\lambda x.\lambda y.x)\ (\lambda z.y)\ z\ w$
   (c) $(\lambda b.\lambda f.\lambda t.b\ t\ f)\ (\lambda f.\lambda t.t)$
   (d) $(\lambda s.\lambda z.s\ (s\ z))\ (\lambda b.\lambda f.\lambda t.b\ t\ f)\ (\lambda f.\lambda t.t)$
   (e) $(\lambda s.\lambda z.s\ (s\ z))\ (\lambda s.\lambda z.s\ (s\ z))\ (\lambda b.\lambda f.\lambda t.b\ t\ f)\ (\lambda f.\lambda t.t)$
3. Recall that with Church booleans we have the following encoding:
$$\begin{aligned}\mathsf{tru} &= \lambda t.\lambda f.t\\ \mathsf{fls} &= \lambda t.\lambda f.f\end{aligned}$$
   (a) Using only bare λ-calculus (variables, λ-abstraction and application), write down a λ-term for logical implication $\mathsf{implies}$ of two Church booleans.
   (b) Verify your implementation of $\mathsf{implies}$ by writing down evaluation sequence for the term $\mathsf{implies\ fls\ tru}$.
4. Recall that with Church numerals we have the following encoding:
$$\begin{aligned}\mathsf{c}_0 &= \lambda s.\lambda z.z\\ \mathsf{c}_1 &= \lambda s.\lambda z.sz\\ \mathsf{c}_2 &= \lambda s.\lambda z.s\ (s\ z)\\ \mathsf{c}_3 &= \lambda s.\lambda z.s\ (s\ (s\ z))\\ &\dots\end{aligned}$$
   (a) Using only bare λ-calculus (variables, λ-abstraction and application), write down a single λ-term for each of the following functions on natural numbers:
      i. $n \mapsto 2n+1$
      ii. $n \mapsto n^2+1$
      iii. $n \mapsto 2^n+1$
      iv. $n \mapsto 2^{n+1}$
   (b) Verify each your implementations of the functions above by writing down evaluation sequence for each of them, when applied to c_2

Solutions

1. For each of the following λ-terms write down an α-equivalent term where all variables have different names:

   (a) $\lambda x.(\lambda y.x\ y)\ x = \lambda x'.(\lambda y'.x'y')x'$
   (b) $\lambda x.(\lambda x.x)\ x = \lambda x'.(\lambda x''.x'').x'$
   (c) $\lambda x.\lambda y.x\ y = \lambda x'.\lambda y'.x'y'$
   (d) $\lambda x.x\ (\lambda x.x) = \lambda x'.x'\ (\lambda x''.x'')$
   (e) $\lambda x.(\lambda x.x)\ x = \lambda x'.(\lambda x''.x'')x'$
   (f) $(\lambda x.\lambda y.y)\ z\ x = (\lambda x'.\lambda y'.y')z'x''$

2. Write down evaluation sequence for the following λ-terms:

   (a) $(\lambda x.\lambda y.x)\ y\ z = (\lambda x.\lambda y'.x)y\,z \mapsto (\lambda y'.y)z \mapsto y$
   (b) $(\lambda x.\lambda y.x)\ (\lambda z.y)\ z\ w = (\lambda x.\lambda y'.x)(\lambda z'.y)z\,w \mapsto (\lambda y'.\lambda z'.y)z\,w \mapsto (\lambda z'.y)w \mapsto y$
   (c) $(\lambda b.\lambda f.\lambda t.b\ t\ f)\ (\lambda f.\lambda t.t) = (\lambda b.\lambda f.\lambda t.b\,t\,f)(\lambda f'.\lambda t'.t') \mapsto \lambda f.\lambda t.(\lambda f'.\lambda t'.t')t\,f \mapsto \lambda f.\lambda t.(\lambda t'.t')f \mapsto \lambda f.\lambda t.f$

   (d) $(\lambda s.\lambda z.s\ (s\ z))\ (\lambda b.\lambda f.\lambda t.b\ t\ f)\ (\lambda f.\lambda t.t) = \lambda f.\lambda t.f$

   Let $A = (\lambda b.\lambda f.\lambda t.b\,t\,f)$,
   $B = (\lambda f.\lambda t.t)$

   Then,

   $(\lambda s.\lambda z.s\ (s\ z))\ (\lambda b.\lambda f.\lambda t.b\ t\ f)\ (\lambda f.\lambda t.t) = (\lambda s.\lambda z.s(s\ z))AB \mapsto$
   $\mapsto (\lambda z.A(A\,z))B \mapsto A(AB)$.

   $AB = (\lambda b.\lambda f.\lambda t.b\,t\,f)(\lambda f.\lambda t.t) \xmapsto{\text{see (2.c)}} \lambda f.\lambda t.f$
   $A(AB) = (\lambda b.\lambda f.\lambda t.b\,t\,f)(\lambda f.\lambda t.f) \mapsto$
   $\mapsto \lambda f.\lambda t.(\lambda f.\lambda t.f)t\,f \mapsto \lambda f.\lambda t.(\lambda t.t)f \mapsto \lambda f.\lambda t.f$

   Thus,

   $(\lambda s.\lambda z.s\ (s\ z))\ (\lambda b.\lambda f.\lambda t.b\ t\ f)\ (\lambda f.\lambda t.t) = \lambda f.\lambda t.f$

   (e) $(\lambda s.\lambda z.s\ (s\ z))\ (\lambda s.\lambda z.s\ (s\ z))\ (\lambda b.\lambda f.\lambda t.b\ t\ f)\ (\lambda f.\lambda t.t) = \lambda f.\lambda t.t$

   Let $A = (\lambda s.\lambda z.s\ (s\ z))$,
   $B = (\lambda b.\lambda f.\lambda t.b\ t\ f)$
   $C = (\lambda f.\lambda t.t)$

   Then,

   $(\lambda s.\lambda z.s\ (s\ z))\ (\lambda s.\lambda z.s\ (s\ z))\ (\lambda b.\lambda f.\lambda t.b\ t\ f)\ (\lambda f.\lambda t.t) = (\lambda s.\lambda z.s(s\ z))A\ B\ C \mapsto$
   $\mapsto (\lambda z.A(A\ z))BC \mapsto (A\,(AB))C = (\lambda s.\lambda z.s(s\ z))(AB))C \mapsto$
   $\mapsto (\lambda z.(A\,B)((AB)\,z))C \mapsto (AB)((AB)C)$

$AB = (\lambda s.\lambda z.s\ (s\ z))\, B \mapsto \lambda z.B\,(B\,z) = \lambda z.B((\lambda b.\lambda f.\lambda t.b\,t\,f)\,z) \mapsto$

$\mapsto \lambda z.B\,(\lambda f.\lambda t.z\,t\,f) = \lambda z.(\lambda b.\lambda f.\lambda t.b\,t\,f)(\lambda f.\lambda t.z\,t\,f) \mapsto \lambda z.\lambda f.\lambda t.(\lambda f.\lambda t.z\,t\,f)\,t\,f \mapsto$

$\mapsto \lambda z.\lambda f.\lambda t\,(\lambda t'.z\,t'\,t)\,f \mapsto \lambda z.\lambda f.\lambda t.z\,f\,t$

$(AB)C = (\lambda z.\lambda f.\lambda t.z\,f\,t)\,C \mapsto \lambda f.\lambda t.C\,f\,t = \lambda f.\lambda t.(\lambda f.\lambda t.t)\,f\,t \mapsto$

$\mapsto \lambda f.\lambda t.(\lambda t'.t)\,t \mapsto \lambda f.\lambda t.t$

$(AB)((AB)C) = (\lambda z.\lambda f.\lambda t.z\,f\,t)(\lambda f.\lambda t.t) \mapsto$

$\mapsto \lambda f.\lambda t.(\lambda f.\lambda t.t)\,f\,t \mapsto \lambda f.\lambda t.(\lambda t'.t)\,t \mapsto$

$\mapsto \lambda f.\lambda t.t$

Thus,

$$(\lambda s.\lambda z.s\ (s\ z))\ (\lambda s.\lambda z.s\ (s\ z))\ (\lambda b.\lambda f.\lambda t.b\ t\ f)\ (\lambda f.\lambda t.t) = \lambda f.\lambda t.t$$

3. Recall that with Church booleans we have the following encoding:

$$\mathsf{tru} = \lambda t.\lambda f.t$$
$$\mathsf{fls} = \lambda t.\lambda f.f$$

(a) Using only bare λ-calculus (variables, λ-abstraction and application), write down a λ-term for logical implication implies of two Church booleans.

Let test $= \lambda a.\lambda b.\lambda c.a\,b\,c$, then logical or $=$ test x tru y

logical not $=$ test x fls tru

Since $x \to y = \neg x \vee y$, logical implication $=$

$= \lambda x.\lambda\ .$ test (test x fls tru) tru y

$= \lambda x.\lambda y.$test $((\lambda a.\lambda b.\lambda c.a\,b\,c)\,x$ fls tru$)$ tru y

$\mapsto \lambda x.\lambda y.$test $((\lambda b.\lambda c.x\,b\,c)$ fls tru$)$ tru y

$\mapsto \lambda x.\lambda y.$test $((\lambda c.x$ fls $c)$ tru$)$ tru y

$\mapsto \lambda x.\lambda y.$test $(x$ fls tru$)$ tru y

$\mapsto \lambda x.\lambda y.(\lambda a.\lambda b.\lambda c.a\,b\,c)\,(x$ fls tru$)$ tru $y \xrightarrow{\text{same steps as above}} \lambda x.\lambda y.(x$ fls tru$)$ tru y

Answer: implies $= \lambda x.\lambda y.(x$ fls tru$)$ tru y

(b) Verify your implementation of implies by writing down evaluation sequence for the term implies fls tru.

implies fls tru $= (\lambda x.\lambda y.(x \text{ fls tru}) \text{ tru } y) \text{ fls tru}$

$\mapsto (\lambda y.(\text{fls fls tru}) \text{ tru } y) \text{ tru}$

$\mapsto (\text{fls fls tru}) \text{ tru tru}$

$\mapsto \text{tru tru tru}$

$\mapsto \underline{\underline{\text{tru}}}$

4. Recall that with Church numerals we have the following encoding:

$$
\begin{aligned}
c_0 &= \lambda s.\lambda z.z \\
c_1 &= \lambda s.\lambda z.sz \\
c_2 &= \lambda s.\lambda z.s\ (s\ z) \\
c_3 &= \lambda s.\lambda z.s\ (s\ (s\ z)) \\
&\dots
\end{aligned}
$$

(a) Using only bare λ-calculus (variables, λ-abstraction and application), write down a single λ-term for each of the following functions on natural numbers:

i. $n \mapsto 2n + 1$
ii. $n \mapsto n^2 + 1$
iii. $n \mapsto 2^n + 1$
iv. $n \mapsto 2^{n+1}$

plus $= \lambda m.\lambda n.\lambda s.\lambda z.ms(nsz)$

scc $= \lambda n.\lambda s.\lambda z.s(nsz)$

times $= \lambda m.\lambda n.\lambda s.\lambda z.m(ns)z$

(i.) $n \mapsto 2n+1$

$2n+1 = \lambda n.\text{scc}(\text{plus } n\ n)$

$= \lambda n.\text{scc}((\lambda m.\lambda n.\lambda s.\lambda z.ms(nsz))n\ n)$

$\mapsto \lambda n.\text{scc}((\lambda n.\lambda s.\lambda z.ns(nsz))n)$

$\mapsto \lambda n.\text{scc}(\lambda s.\lambda z.ns(nsz))$

$= \lambda n.(\lambda n'.\lambda s'.\lambda z'.s'(n's'z'))(\lambda s.\lambda z.ns(nsz))$

$\mapsto \lambda n.\lambda s'.\lambda z'.s'((\lambda s.\lambda z.ns(nsz))s'z')$

$\mapsto \lambda n.\lambda s'.\lambda z'.s'((\lambda z.ns'(ns'z))z')$

$\mapsto \lambda n.\lambda s'.\lambda z'.s'(ns'(ns'z'))$

$= \lambda n.\lambda s.\lambda z.s(ns(nsz))$

(i.i) $n^2+1 = \lambda n.\, scc\,(times\; n\; n)$
$$= \lambda n.\, scc((\lambda m.\lambda n.m\,(plus\; n)\, c_0)\; n\; n)$$
$$\mapsto \lambda n.\, scc\,(\lambda n.n\,(plus\; n)\, c_0)n)$$
$$\mapsto \lambda n.\, scc\,(n\,(plus\; n)\, c_0)'$$
$$\mapsto \lambda n.\,(\lambda n.\lambda s.\lambda z.s\,(n\; s\; z)\,(n\,(plus\, n)\; c_0))$$
$$\mapsto \lambda n.\,\lambda s.\lambda z.\, s((\underbrace{n\,(plus\, n)\, c_0}_{n \text{ times}})\, s\;\, z)$$

(i.i.i)
$$2^n = \underbrace{2\cdot 2\cdot \ldots \cdot 2}_{n} = \underbrace{c_2(times\; c_2(\ldots(times\; c_2\; c_1))}_{n}) =$$
$$= \lambda n.n\,(times\; c_2\;\; c_1)$$
$$2^n+1 = \lambda n.\, scc\,(n\,(times\;\, c_2\; c_1))$$

(i.v)
$$2^{n+1} = \lambda n.\, scc(n)(times\, c_2\; c_1)$$

(b) Verify each your implementations of the functions above by writing down evaluation sequence for each of them, when applied to c_2

$$c_2 = \lambda s.\lambda z\; s(s\; z)$$

(i) $2n+1$
$$(\lambda n.\lambda s.\lambda z.s\,(n s(n s z)))\, c_2 \mapsto \lambda s.\lambda z.s\,(c_2 s\;\,(c_2\, s\; z)) =$$
$$= \lambda s.\lambda z.\, s\,(c_2 s\,((\lambda s.\lambda z\; s(s\; z))s\; z)) \mapsto$$
$$\mapsto \lambda s.\lambda z.s\,(c_2\, s\,((\lambda z.s\,(s\, z))\, z)) \mapsto$$
$$\mapsto \lambda s.\lambda z\; .s\,(c_2 s\,(s\,(s\; z)) =$$
$$= \lambda s.\lambda z.s\,((\lambda s.\lambda z\; s(s\; z))s\,(s\,(s\, z))$$
$$\mapsto \lambda s.\, \lambda z.s\,(\,(\lambda z\, .s\,(s\; z))\;\;\; s\,(s\, z))$$
$$\mapsto \lambda s.\lambda z.\, s\,(s\,(s\; s)(s\; z))$$
$$\mapsto \lambda s.\, \lambda z.\, s(s\,(s\;(s\;(s\; z))) = c_5\;\; ,q.e.d$$
$$(2\cdot 2+1=5)$$

(c.i) n^2+1

$(\lambda n.\ scc\ (times\ n\ n))\ c_2 \mapsto$

$\mapsto scc\ (times\ c_2\ c_2)$

$\cdot\ times\ c_2 c_2 = (\lambda m.\lambda n.\ \lambda s.\lambda z.\ m\,(n\ s)\ z)\ c_2\, c_2 \mapsto$

$\mapsto (\lambda n.\lambda s.\lambda z.\,c_2\,(n\ s)\ z)\,c_2 \mapsto \lambda s.\lambda z.\ c_2\,(c_2\, s)\ z \overset{*}{=}$

$\cdot\ c_2 s = (\lambda s.\lambda z\, s\,(s\ z))\ s = \lambda z.\, s\,(s\ z)$

$\overset{*}{=} \lambda s.\lambda z.\,(\lambda s.\lambda z.\,s\,(s\ z))\,(\lambda z.\,s\,(s\ z))\,z \mapsto$

$\mapsto \lambda s.\lambda z.\,(\lambda z.\,(\lambda z.\,s\,(s\ z))((\lambda z.\,s\,(s\ z)\ z))\ z \mapsto$

$\mapsto \lambda s.\lambda z.\,(\lambda z.\,(\lambda z.\,s\,(s\ z))\,(s\,(s\ z)))\ z \mapsto$

$\mapsto \lambda s.\lambda z.\,(\lambda z.\ s\,(s\,(s\,(s\ z))))\ z \mapsto$

$\mapsto \lambda s.\lambda z.\,s\,(s\,(s\,(s\ z))) = c_4$

Thus,

$scc\,(times\ c_2 c_2) = scc\,(c_4) = c_5$, q.e.d

$(n^2+1 = 2^2+1 = 5)$

(c.i.i) 2^n+1

$(\lambda n.\ scc\,(n\,(times\ c_2\ c_1)))\ c_2 \mapsto scc\,(c_2\,(times\ c_2\ c_1))$

$\cdot\ c_2\,(times\ c_2\ c_1) \mapsto\ c_2 \cdot c_2 = (\lambda s.\lambda z.\,s\,(s\ z))\,c_2 \mapsto$

$\mapsto \lambda z.\,c_2\,(c_2\ z) = \lambda z.\,(\lambda s.\lambda z'.\,s\,(s\ z'))\,((\lambda s.\lambda z'.\,s\,(s\ z'))\,z) \mapsto$

$\mapsto \lambda z.\,(\lambda s.\lambda z'.\,s\,(s\ z'))\,(\lambda z''.\,z\,(z\ z'')) \mapsto$

$\mapsto \lambda z.\,\lambda z'\,\lambda z''.\ z\,(z\ z'')\,(\lambda z''.\,z\,(z\ z'')\,z')$

$\mapsto \lambda z.\,\lambda z'.\ z\,(z\,(\lambda z''.\,z\,(z\ z'')\,z') \mapsto$

$\mapsto \lambda z.\,\lambda z'.\ z\,(z\,(z\,(z\ z'))) = \lambda s.\lambda z.\,s\,(s\,(s\,(s\ z))) = c_4$

$\cdot\ scc(c_4) = c_5$, q.e.d

$(2^2+1 = 4+1 = 5)$

i.v $\quad 2^{n+1}$

$(\lambda n.\, scc(n)(\text{times}\, c_2\, c_1))\, c_2 \longmapsto$

$\longmapsto scc(c_2)(\text{times}\, c_2\, c_1) \longmapsto$

$\longmapsto c_3\, c_2 = (\lambda s.\lambda z.\, s(s(s\, z)))\, c_2$

$\longmapsto \lambda z.\, c_2(c_2(c_2\, z)) \longmapsto$

$\longmapsto \lambda z.\, \lambda s.\, \lambda z'.\, s(s\, z')(\lambda s.\, \lambda z'.\, s(s\, z')(\lambda s.\lambda z'.\, s(s\, z')z))$

$\longmapsto \lambda z.\, \lambda s.\, \lambda z'.\, s(s\, z')(\lambda s.\, \lambda z'.\, s(s\, z')\, \lambda z'.\, z(z\, z')) \longmapsto$

$\longmapsto \lambda z.\, \lambda z'.\, \lambda s.\, s(s\, z'')\, \lambda z''.\, z(z\, z')(\lambda s.\, z''.\, s(s\, z'')\, \lambda z'.\, z(z\, z'')z$

$\longmapsto \lambda z.\lambda z'.\lambda z''.\, \lambda z''.\, z(z\, z''')(\lambda z'''.\, z(z\, z''')z'')(\lambda s.\lambda z'.\, s(s\, z'')\, \lambda z''.\, z(z\, z')z') \longmapsto$

$\longmapsto \lambda z.\, \lambda z'.\, z(z(\lambda z''.\, z(z\, z')(\lambda s.\lambda z''.\, s(s\, z'')\lambda z''.\, z(z\, z'')z z')$

$\longmapsto \lambda z.\, \lambda z_0.\, z(z(z(\lambda z''.\, z(z\, z')(\lambda z''.\, z(z\, z'')z')))) \longmapsto$

$\longmapsto \lambda z.\, \lambda z'.\, z(z(z(z(z(\lambda z''.\, z(z\, z'')z'))))) \longmapsto$

$\longmapsto \lambda z.\, \lambda z'.\, z(z(z(z(z(z(z(z\, z')))))))=$

$= \lambda s.\, \lambda z.\, s(s(s(s(s(s(s(s\, z))))))) = c_8$, q.e.d

$(2^{2+1} = 2^3 = 8)$