Appello 2 - soluzioni

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Analisi

1) a) $\dfrac{f(x,0)-f(0,0)}{x} = \dfrac{x^3/x^2 - 0}{x} = 1 \underset{x\to 0}{\longrightarrow} 1$

$\dfrac{f(0,y)-f(0,0)}{y} = \dfrac{-3y^3/y^2 - 0}{y} = -3 \longrightarrow -3$

$\Rightarrow \boxed{\nabla f(0,0) = (1,-3)}$

b) f diff in $(0,0) \Leftrightarrow$

$f(x,y) - f(0,0) = \nabla f(0,0)\cdot(x,y) + R(x,y)$

con $\displaystyle\lim_{(x,y)\to(0,0)} \frac{R(x,y)}{\sqrt{x^2+y^2}} = 0.$

È $f(x,y) - f(0,0) - \nabla f(0,0)(x,y) =$

$= \dfrac{x^3-3y^3}{x^2+y^2} - (x-3y) = \dfrac{x^3-3y^3-(x-3y)(x^2+y^2)}{x^2+y^2}$

$= \dfrac{\cancel{x^3}-\cancel{3y^3}-(\cancel{x^3}-3yx^2+xy^2-\cancel{3y^3})}{x^2+y^2}$

$= \dfrac{3yx^2-xy^2}{x^2+y^2} = \dfrac{xy}{x^2+y^2}(3x-y)$

ed è $\dfrac{\frac{xy}{x^2+y^2}(3x-y)}{\sqrt{x^2+y^2}} = \dfrac{xy(3x-y)}{(x^2+y^2)\sqrt{x^2+y^2}}$

Quindi f è differenziabile in $(0,0)$

$\Leftrightarrow \lim \dfrac{xy(3x-y)}{(x^2+y^2)\sqrt{x^2+y^2}} = 0.$

$$(x,y)\to(0,0) \qquad (x^2+y^2)\sqrt{x^2+y^2}$$

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Ora, su $y=x$ è $\dfrac{xy(3x-y)}{(x^2+y^2)\sqrt{x^2+y^2}} = \dfrac{2x^3}{2x^2\sqrt{2}|x|} \underset{x\to 0}{\not\to} 0$

$\Rightarrow$ $\boxed{f \text{ NON è differenziabile in } (0,0).}$

2. $yy' = 2x-1 \Rightarrow \dfrac{d}{dx}\left(\dfrac{1}{2}y^2\right) = 2x-1$
$= \dfrac{d}{dx}(x^2-x)$

$\Rightarrow \dfrac{1}{2}y^2(x) = x^2-x+c \qquad y^2(x) = 2x^2-2x+C$

$y(x) = \pm\sqrt{2x^2-2x+C}$.

È $y(0) = \pm\sqrt{C} = -1 \Leftrightarrow$ segno $-$ e

$C=1$:

$\boxed{y(x) = -\sqrt{2x^2-2x+1}}$, da cui

$\boxed{y(1) = -1.}$

3. $\displaystyle\int_{\{4x^2+y^2\le 1\}} |x|y^2\,dx\,dy = 2\int_{\substack{4x^2+y^2\le 1\\ x>0}} xy^2\,dx\,dy$

Poniamo $X=2x$, $Y=y$: $\begin{cases} x = \dfrac{X}{2} \\ y = Y \end{cases}$:

$$2\int_{\substack{X^2+Y^2\le 1\\ X>0}} \frac{X}{2}Y^2\,\frac{1}{2}\,dX\,dY = \frac{1}{2}\int_{\substack{X^2+Y^2\\ X>0}} XY^2\,dX\,dY$$

$$=$$

$$\left(\begin{array}{l} x=\rho\cos t \\ y=\rho\sin t, \end{array} \quad t\in[0,\pi], \quad \rho\in[0,1]\right)$$

$$\pi\,!$$

$$\frac{1}{2}\int_{\substack{t\in[0,\pi]\\ \rho\in[0,1]}} \rho^4\cos t\sin^2 t\, d\rho\, dt$$

$$= \frac{1}{2}\left(\int_0^1 \rho^4\, d\rho\right)\left[-\frac{1}{3}\sin^3 t\right]_0^{\pi} = \frac{1}{10}\cdot\frac{2}{3} = \boxed{\frac{1}{15}}$$

4. $\nabla f(x,y)=(2x+3x^2y,\ 2y+x^3)=(0,0)$

$$\Leftrightarrow \begin{cases} x(2+3xy)=0 \\ 2y+x^3=0 \Rightarrow y=-x^3/2 \end{cases} \qquad x(2-3x^4/2)=0$$

$$\Rightarrow (x=0 \ \text{ o } \ x=\pm\sqrt[4]{\frac{4}{3}} \ \text{ e } \ y=-x^3/2$$

Quindi i punti critici sono

- $x=0$ e $y=0$
- $x=\sqrt[4]{\frac{4}{3}}$, $y=-\frac{1}{2}\frac{4^{3/4}}{3^{3/4}}$
- $x=-\sqrt[4]{4/3}$ $y=\frac{1}{2}\frac{4^{3/4}}{3^{3/4}}$

$$\mathrm{Hess}\, f(x,y)=\begin{pmatrix} 2+6xy & 3x^2 \\ 3x^2 & 2 \end{pmatrix}$$

$$\det \mathrm{Hess}\, f(x,y)=4+12xy-6x^4$$

Nei pti critici:

$$\det \mathrm{Hess}\, f(x,y)=4-12x^4$$

- $(0,0)$: $2>0$, $\det \text{Hess} = 4>0$: $(0,0)$ <u>min loc. stretto</u>
- $\pm\left(\sqrt[4]{4/3}, -\frac{1}{2}\sqrt[4]{\frac{4^3}{3^3}}\right)$: $2+6xy = 2-3\cdot\sqrt[4]{\frac{4^4}{3^4}} = 2-3\cdot\frac{4}{3} < 0$

$\det \text{Hess} = 4+12xy-6x^4 = 4-12x^4 = 4-12\frac{4}{3} = 4-16 = -12 < 0$

$\Rightarrow$ entrambi punti di <u>sella</u>.

Probabilità

1. (a) $X \sim B(800, 0.001)$

(b) $Y \sim Po(800 \times 0.001) = Po(0.8)$

$P(X=0) \approx P(Y=0) = e^{-0.8}$.

2. X_i = peso della cassa i

La probabilità cercata è

$P(X_1 + \dots + X_{49} < 9800)$

$X_1 + \dots + X_{49} \approx N(\mu, \sigma^2)$ (Th. c. Limite)

con $\mu = 49 \times 205 = 10.045$
$\sigma^2 = 49 \times 15^2$

$P(X_1 + \dots + X_{49} < 9800) \approx P(10.045 + 7\cdot 15 Z < 9800)$

$= P(7\cdot 15 Z < -245) = P\left(Z < -\frac{245}{7\cdot 15}\right) = \Phi\left(-\frac{35}{15}\right) = \Phi\left(-\frac{7}{3}\right)$

$= 1 - \Phi\left(\frac{7}{3}\right) = 1 - 0,9893 = 0.0107$.

3. (a) $f_X(x) = \begin{cases} 0 & x \le 0 \\ 6e^{-2x} \underbrace{\int_0^{\infty} e^{-3y}\,dy} & x > 0 \end{cases} = \begin{cases} 0 & x \le 0 \\ 2e^{-2x} & x > 0 \end{cases}$

$$f_Y(y) = \begin{cases} 0 & \text{se } y \le 0 \\ 6e^{-3y} \underbrace{\int_0^{\infty} e^{-2x}\,dx}_{\frac{1}{2}} & y > 0 \end{cases} = \begin{cases} 0 & y \le 0 \\ 3e^{-3y} & y > 0 \end{cases}$$

"1/3"

X, Y sono indipendenti perché

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \quad \forall x, y.$$

(b) Si riconosce che $X \sim Exp(2)$,
$Y \sim Exp(3)$.

(c)
$$P(X+Y<1) = \int_{\substack{0<x \\ 0<y \\ x+y<1}} 6e^{2x} e^{-3y}\,dx\,dy$$

$$= 6 \int_0^1 e^{-2x} \int_0^{1-x} e^{-3y}\,dy\,dx$$

$$= 6 \int_0^1 e^{-2x} \left[-\frac{1}{3} e^{-3y} \right]_{y=0}^{y=1-x} dx$$

$$= 2 \int_0^1 e^{-2x} \left(1 - e^{-3(1-x)} \right) dx$$

$$= 2 \int_0^1 e^{-2x} - e^{-3} e^{x}\,dx$$

$$= 2\left[-\frac{1}{2}e^{-2x} - e^{-3}e^{x}\right]_0^1$$

$$= 2\left(-\frac{1}{2}e^{-2} - e^{-2} + \frac{1}{2} + e^{-3}\right) = -3e^{-2} + 1 + 2e^{-3} \approx 0.69$$