ESERCIZI SCHEDA 10

ESERCIZIO 1

a) $$\bar{f} = \frac{\int_0^2 (3x^2+2x)\,dx}{2-0} = \frac{[x^3+x^2]_0^2}{2} = \frac{8+4}{2} = 6$$

b) $$\bar{f} = \frac{\int_0^\pi (\sin x + 3\cos x)\,dx}{\pi - 0} = \frac{[-\cos x + 3\sin x]_0^\pi}{\pi} = \frac{1+1}{\pi} = \frac{2}{\pi}$$

c) $$\bar{f} = \frac{\int_0^1 x^{\frac{7}{3}}}{1-0} = \frac{\left[\frac{3}{10}x^{\frac{10}{3}}\right]_0^1}{1} = \frac{3}{10}$$

ESERCIZIO 2

a) $$\int_{-1}^{1} (-2x^4 + 3x^5)\,dx = \left[-\frac{2}{5}x^5 + \frac{1}{2}x^6\right]_{-1}^{1} = -\frac{2}{5} + \frac{1}{2} - \frac{2}{5} - \frac{1}{2} = -\frac{4}{5}$$

b) $$\int_0^1 \left(\frac{5}{1+x^2} - 3\sinh x\right) dx = [5\arctan x - 3\cosh x]_0^1$$
$$= \frac{5}{4}\pi - 3\,\frac{e^2+1}{2e} - 0 - 1 = \frac{5}{4}\pi - 3\,\frac{e^2+1}{2e} - 1$$

ESERCIZIO 3

a) $$\int (e^x + \sqrt{x^3})\,dx = \int (e^x + x^{\frac{3}{2}})\,dx = e^x + \frac{2}{5}x^{\frac{5}{2}} + c \qquad e^x + \frac{2}{5}x^2\sqrt{x} + c$$

b) $$\int \left(\frac{1}{x} + \frac{1}{\sqrt{1-x^2}}\right) dx = \ln|x| + \arcsin x + c$$

c) $$\int \left(\frac{1}{\sqrt{x^2-1}} + \frac{1}{\sqrt{1+x^2}}\right) dx = \operatorname{settcosh} x + \operatorname{settsinh} x + c$$

ⓓ $$\int\left(\frac{1}{\cos^2 x}+\frac{1}{\cosh^2 x}\right)dx = \boxed{-\tan x+\tanh x + c}$$

ESERCIZIO 4

ⓐ $$\int \frac{\sin(\log x)}{x}\,dx = \boxed{-\cos(\log x)+c}$$

ⓑ $$\int \frac{\cos(\log(3x))}{x}\,dx = \int 3\,\frac{\cos(\log(3x))}{3x}\,dx = \boxed{\sin(\log(3x))+c}$$

ⓒ $$\int \frac{e^x\arctan(e^x)}{1+e^{2x}}\,dx = \frac{1}{2}\int \frac{2e^x\arctan(e^x)}{1+e^{2x}}\,dx = \boxed{\frac{1}{2}[\arctan(e^x)]^2+c}$$

ⓓ $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \boxed{-\log|\cos x|+c}$$

ⓔ $$\int xe^{x^2}dx = \frac{1}{2}\int 2xe^{x^2}dx = \boxed{\frac{e^{x^2}}{2}+c}$$

ⓕ $$\int \frac{3x}{1+x^2}\,dx = \frac{3}{2}\int\frac{2x}{1+x^2}\,dx = \frac{3}{2}\log|1+x^2|+c = \boxed{\frac{3}{2}\log(1+x^2)+c}$$

ESERCIZIO 5

ⓐ $$\int \arctan x\,dx$$

Integro per parti $\begin{cases} f'(x) = 1 \\ g(x) = \arctan x \end{cases} \Rightarrow \begin{cases} f(x) = x \\ g'(x) = \frac{1}{1+x^2} \end{cases}$

$$\int \arctan x \, dx = x\arctan x - \int \frac{x}{1+x^2}\,dx$$

$$= x\arctan x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$$

$$= \boxed{x\arctan x - \frac{1}{2}\log(1+x^2) + c}$$

(b) $\int \arcsin x \, dx$

Integro per parti $\begin{cases} f'(x) = 1 \\ g(x) = \arcsin x \end{cases} \Rightarrow \begin{cases} f(x) = x \\ g'(x) = \frac{1}{\sqrt{1-x^2}} \end{cases}$

$$\int \arcsin x \, dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= x\arcsin x + \int \frac{-2x}{2\sqrt{1-x^2}}\,dx$$

$$= \boxed{x\arcsin x + \log(\sqrt{1-x^2}) + c}$$

Esercizio 6

$\int x e^x \cos x \, dx$

Integro per parti: $\begin{cases} f'(x) = e^x \\ g(x) = x\cos x \end{cases} \Rightarrow \begin{cases} f(x) = e^x \\ g'(x) = \cos x - x\sin x \end{cases}$

$$\int x e^x \cos x \, dx = x e^x \cos x - \int (\cos x - x\sin x) e^x \, dx$$

$$= xe^x\cos x - \int e^x\cos x\,dx + \int xe^x\sin x\,dx$$

Analizzo $\int xe^x\sin x\,dx$

Integro per parti: $\begin{cases} f'(x) = e^x \\ g(x) = x\sin x \end{cases} \Rightarrow \begin{cases} f(x) = e^x \\ g'(x) = \sin x + x\cos x \end{cases}$

$$\int xe^x\sin x\,dx = xe^x\sin x - \int e^x\sin x\,dx - \int xe^x\cos x\,dx$$

$$= xe^x\cos x - \int e^x\cos x\,dx + xe^x\sin x - \int e^x\sin x\,dx - \int xe^x\cos x\,dx$$

$$\Rightarrow \int xe^x\cos x\,dx = \frac{1}{2}\left(xe^x(\sin x + \cos x) - \int e^x\cos x\,dx - \int e^x\sin x\,dx\right)$$

Analizzo $\int e^x\cos x\,dx$

Integro per parti: $\begin{cases} f(x) = e^x \\ g'(x) = \cos x \end{cases} \Rightarrow \begin{cases} f'(x) = e^x \\ g(x) = \sin x \end{cases}$

$$\int e^x\cos x\,dx = e^x\sin x - \int e^x\sin x\,dx$$

$$= \frac{1}{2}\left(xe^x(\sin x + \cos x) - e^x\sin x + \cancel{\int e^x\sin x\,dx} - \cancel{\int e^x\sin x\,dx}\right)$$

$$= \boxed{\frac{e^x\left[x(\sin x + \cos x) - \sin x\right]}{2}}$$

ESERCIZIO 7

$$y = -x^2 + 4x + 5$$

Calcolo le intersezioni con l'asse x:

$$y=0 \iff -x^2+4x+5=0$$
$$\iff x^2-4x-5=0$$
$$\iff (x-5)(x+1)=0 \rightarrow x_1=-1,\ x_2=5$$

$$A=\int_{-1}^{5}(-x^2+4x+5)\,dx=\left[-\frac{x^3}{3}+2x^2+5x\right]_{-1}^{5}$$

$$=-\frac{125}{3}+50+25-\frac{1}{3}-2+5$$

$$=-\frac{126}{3}+78=-42+78=\boxed{36}$$

Esercizio 8

ⓐ $$\int\frac{6}{2x+9}\,dx=3\int\frac{2}{2x+9}\,dx=\boxed{3\ln|2x+9|}$$

ⓑ $$\int\frac{5}{x+6}\,dx=5\int\frac{1}{x+6}\,dx=\boxed{5\ln|x+6|}$$

ⓒ $$\int\frac{6}{3-2x}\,dx=-3\int\frac{-2}{3-2x}\,dx=\boxed{-3\ln|3-2x|}$$

Esercizio 9

ⓐ $$\int\frac{x+2}{x^2+x-6}\,dx=\int\frac{x+2}{(x+3)(x-2)}\,dx=\int\frac{A}{x+3}+\frac{B}{x-2}\,dx$$

$$x+2=(x-2)A+(x+3)B=Ax-2A+Bx+3B=(A+B)x+3B-2A$$

$$\Rightarrow\begin{cases}A+B=1\\3B-2A=2\end{cases}\quad\begin{cases}B=1-A\\3-3A-2A=2\end{cases}\quad\begin{cases}B=1-A=\frac{4}{5}\\A=\frac{1}{5}\end{cases}$$

$$\Rightarrow \int \frac{1}{5}\cdot\frac{1}{x+3}+\frac{4}{5}\cdot\frac{1}{x-2}\,dx = \frac{1}{5}\int\frac{1}{x+3}\,dx+\frac{4}{5}\int\frac{1}{x-2}\,dx$$

$$= \boxed{\frac{1}{5}\ln|x+3|+\frac{4}{5}\ln|x-2|}$$

(b)

$$\int\frac{4}{x^2-4}\,dx = \int\frac{4}{(x+2)(x-2)}\,dx$$

$$\frac{4}{(x+2)(x-2)} = \frac{A}{x+2}+\frac{B}{x-2} \iff (x-2)A+(x+2)B=4$$
$$\iff Ax-2A+Bx+2B=4$$
$$\iff (A+B)x+2B-2A=4$$

$$\begin{cases} A+B=0 \\ 2B-2A=4 \end{cases} \quad \begin{cases} A=-B \\ 2B+2B=4 \end{cases} \quad \begin{cases} A=-1 \\ B=1 \end{cases}$$

$$\int\frac{1}{x-2}-\frac{1}{x+2}\,dx = \ln|x-2|-\ln|x+2|+c$$

$$= \boxed{\ln\left|\frac{x-2}{x+2}\right|+c}$$

(c)

$$\int\frac{-6x}{9x^2+9x+3}\,dx$$

Il denominatore non ha radici, quindi provo a portarmi ad una forma "simil logaritmo".

$$-\frac{1}{3}\int\frac{6x}{3x^2+3x+1}\,dx = -\frac{1}{3}\int\frac{6x+3-3}{3x^2+3x+1}\,dx$$

$$= -\frac{1}{3}\left(\int\frac{6x+3}{3x^2+3x+1}\,dx-\int\frac{3}{3x^2+3x+1}\,dx\right)$$

$$= -\frac{1}{3}\ln\left(3x^2+3x+1\right)+\int\frac{1}{3x^2+3x+1}\,dx$$

Non ha zeri, quindi è sempre positivo

Porto il secondo integrale ad una forma "simil arcotangente":

$$3x^2+3x+1 = 3(x+\alpha)^2+\beta$$
$$\overset{!}{=} 3x^2+6\alpha x+3\alpha^2+\beta$$

$$\begin{cases} 6\alpha = 3 \\ 3\alpha^2+\beta = 1 \end{cases} \qquad \begin{cases} \alpha = \frac{1}{2} \\ \beta = 1-\frac{3}{4} = \frac{1}{4} \end{cases}$$

$$= -\frac{1}{3}\ln(3x^2+3x+1) + \int \frac{1}{3\left(x+\frac{1}{2}\right)^2+\frac{1}{4}}\,dx$$

$$= -\frac{1}{3}\ln(3x^2+3x+1) + 4\int \frac{1}{1+12\left(x+\frac{1}{2}\right)^2}\,dx$$

$$= -\frac{1}{3}\ln(3x^2+3x+1) + 4\int \frac{1}{1+\left(2\sqrt{3}x+\sqrt{3}\right)^2}\,dx$$

$$= -\frac{1}{3}\ln(3x^2+3x+1) + \cancel{4}^{2}\cdot\frac{1}{\cancel{2}\sqrt{3}}\int \frac{2\sqrt{3}}{1+\left(2\sqrt{3}x+\sqrt{3}\right)^2}\,dx$$

$$= \boxed{-\frac{1}{3}\ln(3x^2+3x+1) + \frac{2\sqrt{3}}{3}\arctan\left(2\sqrt{3}x+\sqrt{3}\right)+c}$$

d) $$\int \frac{3x+16}{6x^2+x-2}\,dx = \int \frac{3x+16}{6\left(x+\frac{2}{3}\right)\left(x-\frac{1}{2}\right)}\,dx = \int \frac{3x+16}{(3x+2)(2x-1)}\,dx$$

$$\Delta = 1+4\cdot 2\cdot 6 = 49$$
$$x_{1,2} = \frac{-1\pm 7}{12} \Rightarrow x_1 = -\frac{2}{3}, \quad x_2 = \frac{1}{2}$$

$$\frac{3x+16}{(3x+2)(2x-1)} = \frac{A}{3x+2} + \frac{B}{2x-1}$$

$$\Rightarrow 3x+16 = (2x-1)A + (3x+2)B$$

$$3x+16 = 2Ax - A + 3Bx + 2B$$

$$3x+16 = (2A+3B)x + (2B-A)$$

$$\Rightarrow \begin{cases} 2A+3B=3 \\ 2B-A=16 \end{cases} \qquad \begin{cases} 4B-32+3B=3 \\ A=2B-16 \end{cases}$$

$$\begin{cases} 7B=35 \\ A=2B-16 \end{cases} \qquad \begin{cases} B=5 \\ A=-6 \end{cases}$$

$$\int \frac{-6}{3x+2} + \frac{5}{2x-1}\,dx = -2\int \frac{3}{3x+2}\,dx + \frac{5}{2}\int \frac{2}{2x-1}\,dx$$

$$= \boxed{-2\log|3x+2| + \frac{5}{2}\log|2x-1| + c}$$

Esercizio 10

ⓐ

$$\int \frac{8}{x^2+25+10x}\,dx = \int \frac{8}{(x+5)^2}\,dx = -8\int -\frac{1}{(x+5)^2}\,dx$$

$$= \boxed{-\frac{8}{x+5} + c}$$

ⓑ

$$\int \frac{3}{16x^2-40x+25}\,dx = \int \frac{3}{(4x-5)^2}\,dx = -\frac{3}{4}\int -\frac{4}{(4x-5)^2}\,dx$$

$$= \boxed{-\frac{3}{4(4x-5)} + c}$$

c)

$$\int \frac{-5x-32}{x^2-14x+49}\,dx = \int \frac{-5x-32}{(x-7)^2}\,dx$$

$$\frac{-5x-32}{(x-7)^2} = \frac{A}{(x-7)^2} + \frac{B}{x-7}$$

$$\Leftrightarrow -5x-32 = A + (x-7)B$$

$$-5x-32 = Bx + (A-7B)$$

$$\begin{cases} B=-5 \\ A-7B=-32 \end{cases} \qquad \begin{cases} B=-5 \\ A=-67 \end{cases}$$

$$\int \frac{-67}{(x-7)^2} + \frac{-5}{x-7}\,dx = 67\int -\frac{1}{(x-7)^2}\,dx - 5\int \frac{1}{x-7}\,dx$$

$$= \boxed{\frac{67}{x-7} - 5\log|x-7| + c}$$

d)

$$\int \frac{x+1}{9x^2+12x+4}\,dx = \int \frac{x+1}{(3x+2)^2}\,dx$$

$$\frac{x+1}{(3x+2)^2} = \frac{A}{3x+2} + \frac{B}{(3x+2)^2}$$

$$\Leftrightarrow x+1 = (3x+2)A + B$$

$$x+1 = 3Ax + (2A+B)$$

$$\begin{cases} 3A=1 \\ 2A+B=1 \end{cases} \qquad \begin{cases} A=\frac{1}{3} \\ B=1-2A=\frac{1}{3} \end{cases}$$

$$\int \frac{1}{3}\cdot\frac{1}{3x+2} + \frac{1}{3}\cdot\frac{1}{(3x+2)^2}\,dx = \frac{1}{9}\int \frac{3}{3x+2}\,dx - \frac{1}{9}\int -\frac{3}{(3x+2)^2}\,dx$$

$$= \frac{1}{9}\log|3x+2| - \frac{1}{9(3x+2)} + c$$

ESERCIZIO 11

ⓐ

$$\int \frac{1}{x^2+3}\,dx = \frac{1}{3}\int \frac{1}{\frac{x^2}{3}+1}\,dx = \frac{1}{3}\int \frac{1}{1+\left(\frac{x}{\sqrt{3}}\right)^2}\,dx$$

Non ha zeri.

$$= \frac{\sqrt{3}}{3}\int \frac{1}{\sqrt{3}}\cdot\frac{1}{1+\left(\frac{x}{\sqrt{3}}\right)^2}\,dx = \frac{\sqrt{3}}{3}\arctan\left(\frac{x}{\sqrt{3}}\right)+c$$

ⓑ

$$\int \frac{5}{4x^2+12x+10}\,dx = \int \frac{5}{4x^2+12x+9+1}\,dx = 5\int \frac{1}{(2x+3)^2+1}\,dx$$

$\Delta = 144-160<0$

$\Rightarrow$ Non ha zeri.

$$= \frac{5}{2}\int \frac{2}{(2x+3)^2+1}\,dx = \frac{5}{2}\arctan(2x+3)+c$$

ⓒ

$$\int \frac{x}{x^2+2x+4}\,dx = \frac{1}{2}\int \frac{2x}{x^2+2x+4}\,dx = \frac{1}{2}\int \frac{2x+2-2}{x^2+2x+4}\,dx =$$

$\Delta = 4-16<0$

$\Rightarrow$ Non ha zeri.

$$= \frac{1}{2}\int \frac{2x+2}{x^2+2x+4}\,dx + \frac{1}{2}(-2)\int \frac{1}{x^2+2x+1+3}\,dx =$$

$$= \frac{1}{2}\log|x^2+2x+4| - \int \frac{1}{(x+1)^2+3}\,dx =$$

$$= \frac{1}{2}\log(x^2+2x+4) - \frac{1}{3}\int \frac{1}{\left(\frac{x+1}{\sqrt{3}}\right)^2+1}\,dx =$$

$$= \frac{1}{2}\log(x^2+2x+4) - \frac{\sqrt{3}}{3}\int \frac{\frac{1}{\sqrt{3}}}{1+\left(\frac{x+1}{\sqrt{3}}\right)^2}\,dx$$

$$= \frac{1}{2}\log(x^2+2x+4) - \frac{\sqrt{3}}{3}\arctan\left(\frac{x+1}{\sqrt{3}}\right) + c$$

d) $$\int \frac{18x+3}{9x^2+6x+2}\,dx = \int \frac{18x+6-3}{9x^2+6x+2}\,dx$$

$\Delta = 36-72<0$
Non ha zeri

$$= \int \frac{18x+6}{9x^2+6x+2}\,dx - \int \frac{3}{9x^2+6x+1+1}\,dx$$

$$= \log|9x^2+6x+2| - \int \frac{3}{(3x+1)^2+1}\,dx$$

$$= \log(9x^2+6x+2) - \arctan(3x+1) + c$$

Esercizio 12

$$F(x) = \int_1^x \left(\frac{1}{t} + t^3 + \frac{1}{1+t^2}\right) dx \rightsquigarrow \mathrm{dom}f = [1, +\infty)$$

$$F'(x) = \frac{1}{t} + t^3 + \frac{1}{1+t^2} = \frac{1+t^2+t^3+t^5+t}{t(1+t^2)}$$

*) $t^5+t^3+t^2+t+1$ strettamente crescente in quanto somma di funzioni crescenti.

Sostituendo 1, si ottiene 5 $\Rightarrow t^5+t^3+t^2+t+1>0 \quad \forall x \in \mathrm{dom}f$

*) $t>0 \ \forall x \in \mathrm{dom}f$

*) $1+t^2>0 \ \forall x \in \mathrm{dom}f$

$\Rightarrow F'(x)>0 \ \forall x \in \mathrm{dom}f \Rightarrow F(x)$ è strettamente crescente

$\Rightarrow x = 1$ è punto di minimo

$$F(1) = \int_1^1 \left(\frac{1}{t} + t^3 + \frac{1}{1+t^2} \right) dx = 0 \Rightarrow$$ minimo assoluto: $(1, 0)$

ESERCIZIO 13

$$F(x) = \int_1^x e^t \log t \, dt \qquad \mathrm{dom} f = [1, +\infty)$$

$$F'(x) = e^x \log x$$

$$F''(x) = e^x \log x + e^x \cdot \frac{1}{x} = e^x \, \frac{1 + x \log x}{x}$$

Studio il segno della derivata seconda:

$$\cancel{e^x} \cdot \frac{1 + x \log x}{x} \geq 0 \iff \frac{1 + x \log x}{x} \geq 0$$

$1 + x\log x$ è strettamente crescente e in 1 vale 1, quindi: $1 + x \log x > 0 \ \forall x \in \mathrm{dom} f$

$x > 0 \ \forall x \in \mathrm{dom} f$

$\Rightarrow F''(x) > 0 \ \forall x \in \mathrm{dom} f$

$\Rightarrow F(x)$ è convessa $\forall x \in \mathrm{dom} f$

ESERCIZIO 14

$$F(x) = \int_0^x e^t \log(1+t^2)\,dt$$

$$T_{3,0}(x) = F(0) + F'(0)\cdot x + F''(0)\cdot \frac{x^2}{2!} + F'''(0)\cdot \frac{x^3}{3!} + o(x^3)$$

$$F(0) = \int_0^0 e^t \log(1+t^2)\,dt = 0$$

$$F'(x) = e^x \log(1+x^2) \Rightarrow F'(0) = e^0 \log 1 = 0$$

$$F''(x) = e^x \log(1+x^2) + e^x \cdot \frac{2x}{1+x^2} \Rightarrow F''(0) = 0$$

$$F'''(x) = e^x \log(1+x^2) + e^x \cdot \frac{2x}{1+x^2} + e^x \frac{2(1+x^2) - 2x\cdot 2x}{(1+x^2)^2}$$

$$= e^x \log(1+x^2) + e^x \frac{2x}{1+x^2} + e^x \frac{2-2x^2}{(1+x^2)^2} \Rightarrow F'''(0) = 2$$

$$= 2\frac{x^3}{3!} + o(x^3) = \boxed{\frac{x^3}{3} + o(x^3)}$$

ESERCIZIO 15

Questa si può risolvere con un' equazione differenziale.

problema di Cauchy:

$$\begin{cases} f'(x) = 3x^2 + 9x + 18 \\ f(0) = 0 \end{cases}$$

Scrivo la derivata come rapporto di differenziali:

$$\frac{dy}{dx} = 3x^2 + 9x + 18$$

$\Leftrightarrow \quad dy = (3x^2 + 9x + 18)\,dx$

$\Leftrightarrow \int dy = \int (3x^2 + 9x + 18)\,dx$

$y + C_y = x^3 + \frac{9}{2}x^2 + 18x + C_x$

$y = x^3 + \frac{9}{2}x^2 + 18x + C$

$f(0) = 0: \quad f(0) = C = 0 \quad \Rightarrow \quad \boxed{y = x^3 + \frac{9}{2}x^2 + 18x}$