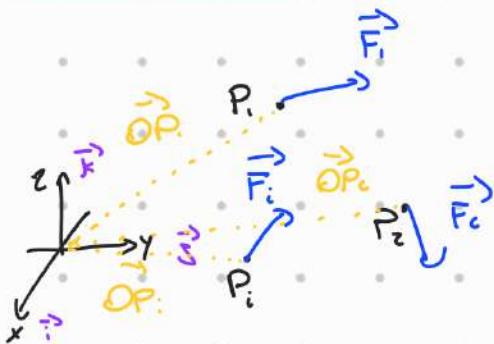


SISTEMI DI FORZE

FORZE PUNTUALI



La RISULTANTE delle forze è:

$$\vec{R} = \sum_i \vec{F}_i$$

Il MONTONE RISULTANTE rispetto a un punto è:

$$\vec{M}_o = \sum_i \vec{OP}_i \times \vec{F}_i$$

rispetto
all'origine

staticamente
equivalenti

Due sistemi di forze si dicono EQUIVALENTI se:

$$\begin{cases} \vec{R}' = \vec{R} \\ \vec{M}_o' = \vec{M}_o \end{cases}$$

dove essere lo
stesso PUNTO

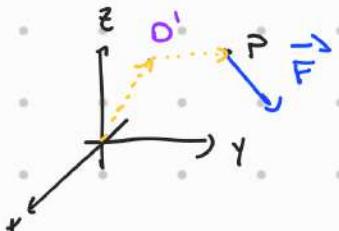
Un sistema è EQUILIBRATO

se:

$$\begin{cases} \vec{R} = \vec{0} \\ \vec{M}_o = \vec{0} \end{cases}$$

rispetto
al punto
genitivo

$$\begin{aligned} \vec{M}_o &= \sum_i \vec{OP}_i \times \vec{F}_i = \sum_i (\vec{OO'} + \vec{O'P}) \times \vec{F}_i \\ &= \vec{OO'} \times \sum_i \vec{F}_i + \sum_i \vec{O'P} \times \vec{F}_i \\ &= \vec{OO'} \times \vec{R} + \vec{M}_{O'} = \vec{M}_{O'} \end{aligned}$$



oss: calcolo del momento

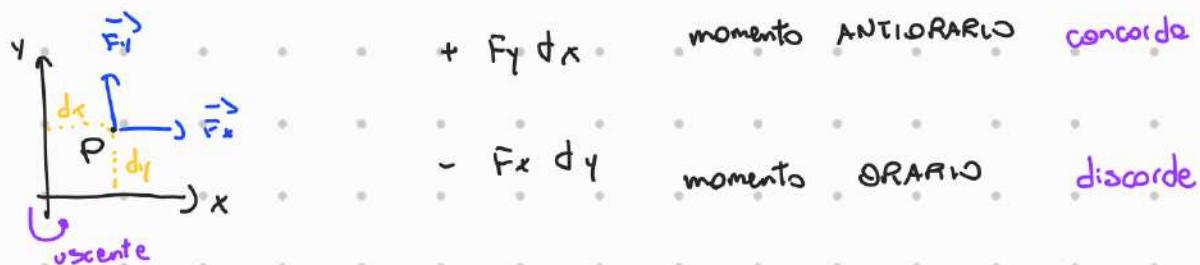
$$\vec{OP} = (dx, dy, dz)$$

$$\vec{F} = (F_x, F_y, F_z)$$

$$\vec{OP} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ dx & dy & dz \\ F_x & F_y & F_z \end{vmatrix} = \begin{matrix} \vec{i} (F_z dy - F_y dz) \\ -\vec{j} (F_z dx - F_x dz) \\ \vec{k} (F_x dy - F_y dx) \end{matrix}$$

nel sistema PIANO di forze vole:

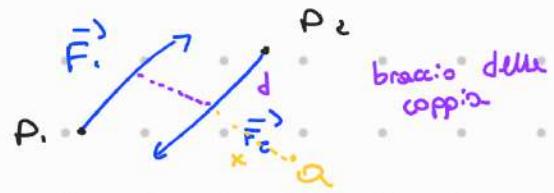
$$\overrightarrow{OP} \times \overrightarrow{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{dx}{dt} & \frac{dy}{dt} & 0 \\ F_x & F_y & 0 \end{vmatrix} = \vec{k} (F_y dx - F_x dy)$$



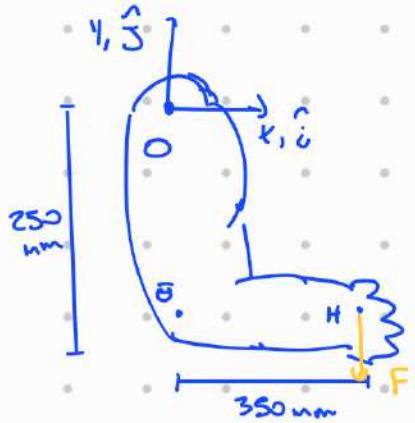
dx e dy sono rispettivamente : BRACCI delle forze \vec{F}_y e \vec{F}_x

def: si dicono COPPIA due forze se :

- hanno ugual modulo
- stessa direzione
- verso opposto



$$M_Q = F \cdot x - F (d+x) = -F \cdot d$$



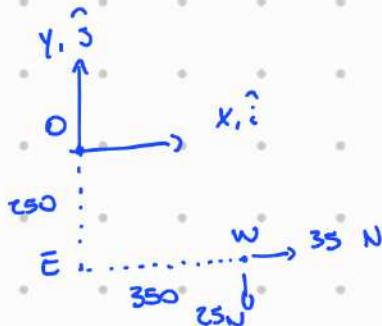
momento rispetto a O, E

$$\vec{OH} = (350, -250, 0) \quad \vec{OE} = (0, -250, 0)$$

$$\vec{EH} = \vec{OH} - \vec{OE} = (350, 0, 0)$$

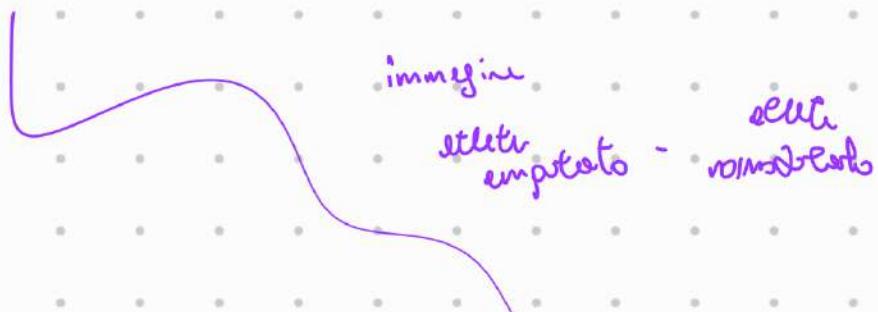
$$\vec{n}_O = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 350 & -250 & 0 \\ 0 & -250 & 0 \end{vmatrix} = (-350 \cdot 50) \hat{k}$$

$$\vec{n}_E = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 350 & 0 & 0 \\ 0 & -250 & 0 \end{vmatrix} = (-350 \cdot 50) \hat{k}$$



$$\vec{n}_O = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 350 & -250 & 0 \\ 35 & -25 & 0 \end{vmatrix} = \hat{k} (350(-45) + 35(250)) = 0$$

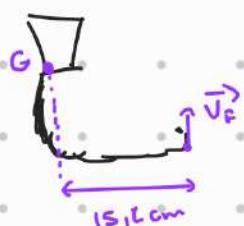
esempio: Sistema di forze staticamente equivalenti:



Dipende da:



- massa soggetto $B_N = 80 \text{ kg}$
- peso soggetto $B_W = B_N \cdot g = 784,8 \text{ N}$
- $V_F = 1,8 \text{ } B_W = 1411,6 \text{ N}$



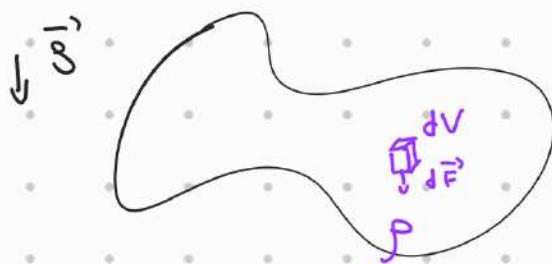
lo traslo G momento di trasporto
equivalente

$$I_C = 15,2 \cdot \frac{1411,6}{100} \text{ N.m}$$

Moduli di forza

forza puntuale

forza di massa (volume)



$$d\vec{F} = (\rho dV) \vec{g}$$

$$dF = \rho dV g$$

$$\hookrightarrow b = \frac{dF}{dV} = \rho g$$

forza di massa
[N/m³]

Oss: le unità di misura utilizzate sono:

Lunghezza mm

μm

Forza N

μN

Tempo s

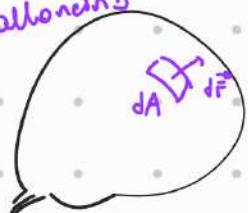
s

Massa t connellate

Kg

forze di superficie

palloncino

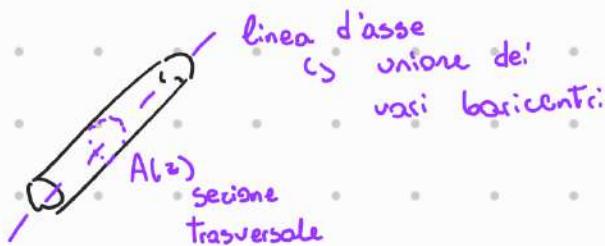


$$d\vec{F} = \rho d\vec{A}$$

$$\hookrightarrow \frac{dF}{dA} = \rho \quad \left[\frac{N}{mm^2} = MPa \right]$$

$$\left[\frac{N}{\mu m^2} = 17 Pa \right]$$

forze lineari

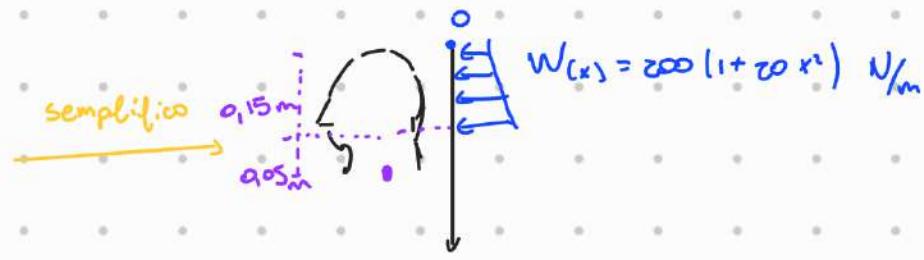


$$A dz \cdot \rho \cdot g = dF$$

$$\hookrightarrow q = \frac{dF}{dz} = A \rho g$$

forza per unità
di lunghezza

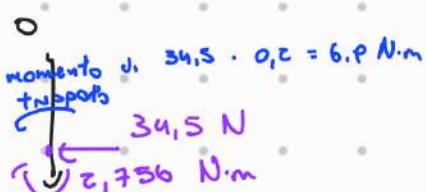
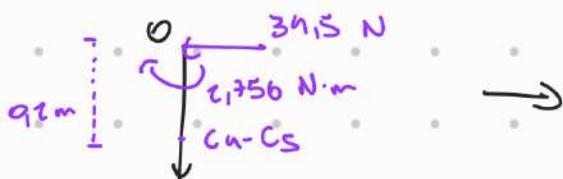
esempio:



$$R_0 = \int_0^{0.75} W(x) dx = \int_0^{0.75} 200(1+20x^2) dx = 200 \left[x + \frac{20}{3} x^3 \right]_0^{0.75} = 345 \text{ N}$$

$$M_0 = \int_0^{0.75} W(x) \cdot x dx = \int_0^{0.75} 200(x+20x^2) dx = 200 \left[\frac{x^2}{2} + 5x^3 \right]_0^{0.75} = 2,756 \text{ N.m}$$

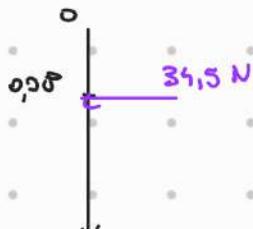
Il sistema equivalente è:



oss:

$$X_R = \frac{M_0}{R_0} = 0,08 \text{ m}$$

cioè



↳ se carico perpendicolare, risultante in questo punto ha momento nullo

Calcolo delle risultanti:

$$\bullet \begin{array}{c} \text{P} \\ \hline \text{l} \end{array}, x \quad R = \int_0^l p dx = pl \Rightarrow x_R = \frac{l}{2}$$

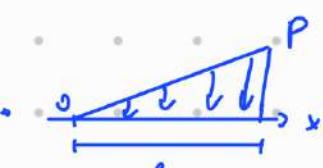
$$M_0 = \int_0^l p x dx = \frac{pl^2}{2}$$

$$R = \int_0^l \frac{p}{l} x dx = \frac{p}{2} l \Rightarrow x_R = \frac{2}{3} l$$

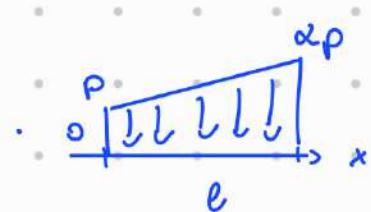
$$M_0 = \int_0^l \frac{p}{l} x^2 dx = \frac{pl^3}{3}$$

$$R = \int_0^l p + \frac{p(\alpha-1)}{l} x dx = pl + \frac{p(\alpha-1)}{2} l \Rightarrow x_R = \frac{1}{3} \frac{1+\alpha}{1+\alpha} l$$

$$M_0 = \int_0^l p x + \frac{p(\alpha-1)}{l} x^2 dx = \frac{pl^2}{2} + \frac{p(\alpha-1)}{3} l^3$$



$$R = \int_0^l p + \frac{p(\alpha-1)}{l} x dx = pl + \frac{p(\alpha-1)}{2} l \Rightarrow x_R = \frac{1}{3} \frac{1+\alpha}{1+\alpha} l$$

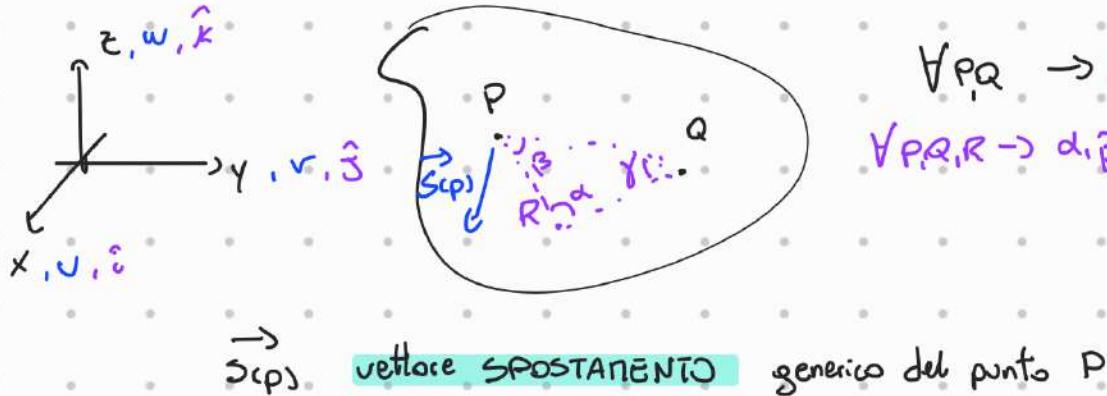


$$M_0 = \int_0^l p x + \frac{p(\alpha-1)}{l} x^2 dx = \frac{pl^2}{2} + \frac{p(\alpha-1)}{3} l^3$$

$$\begin{cases} x = p \cos \theta \\ y = p \sin \theta \end{cases} \quad R = \int \vec{F} dr = \int_0^{\pi} \vec{F} \frac{d\theta}{R} = R p \int_0^{\pi} (\cos \hat{i} + \sin \hat{j}) d\theta = R p [\sin \hat{i} - \cos \hat{j}]_0^{\pi} = R p (0 \hat{i} - (-1-1) \hat{j}) = 2Rp \hat{j}$$

$\vec{F} = F \cos \theta \hat{i} + F \sin \theta \hat{j}$
 $|\vec{F}| = p$
 $\vec{n}_o = 0$ tutti i buchi sono paralleli
 $p = (0, 0)$

Cinematica corpo rigido



$$\forall PQ \rightarrow |\vec{PQ}| = \text{cost}$$

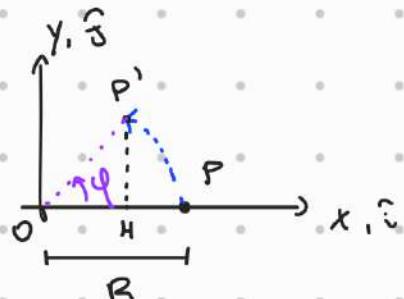
$$\forall PQR, R \rightarrow \alpha, \beta, \gamma = \text{cost}$$

$$\hookrightarrow \vec{s}(p) = v \hat{i} + r \hat{j} + w \hat{k} \quad \text{deve essere compatibile corpo rigido}$$

Traslazione

$$\vec{s}(p) = \vec{s} \quad \text{TUTTI i vettori spostamento sono PARALLELI e UGUALI}$$

Rotazione rispetto a un'asse \hat{k}



$\widehat{PP'}$ traiettoria del punto P

$\vec{PP'}$ spostamento del punto P

$$\vec{s}(p) = \vec{PP'}$$

$$\vec{PP'} = \vec{PH} + \vec{HP'} = (R_o - R \cos \varphi) \hat{i} + R \sin \varphi \hat{j}$$

se φ è abbastanza piccolo

$$\varphi \approx \tan \varphi \approx \alpha$$

$$\cos \varphi = 1$$

$$\vec{s}_{cp} \approx R\varphi \vec{j} = R\varphi \hat{k} \times \hat{i} = \varphi \hat{k} \times R\hat{i}$$

$\hat{k} \times \hat{i} = -\vec{j}$

Rotazione generale

$$\vec{s}_{cp} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \varphi_x & \varphi_y & \varphi_z \\ x_p & y_p & z_p \end{vmatrix}$$

rotazioni dei rispettivi assi
coordinate punto P

quindi:

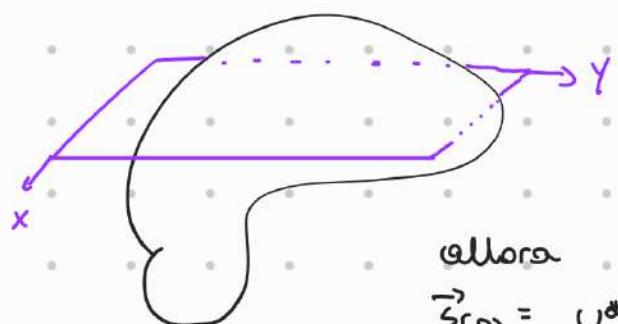
$$\vec{s}_{cp} = (\varphi_y z_p - \varphi_z y_p) \hat{i} - (\varphi_x z_p - \varphi_z x_p) \hat{j} + (\varphi_x y_p - \varphi_y x_p) \hat{k}$$

Rototraslazione

$$\vec{s}_{cp} = u^* \hat{i} + v^* \hat{j} + w^* \hat{k} + (\varphi_y z_p - \varphi_z y_p) \hat{i} - (\varphi_x z_p - \varphi_z x_p) \hat{j} + (\varphi_x y_p - \varphi_y x_p) \hat{k}$$

componente traslatoria
rotazione

SPOSTAMENTI PIANI



allora

$$\vec{s}_{cp} = u^* \hat{i} + v^* \hat{j} - \varphi_z y_p \hat{i} + \varphi_z x_p \hat{j}$$

$$\begin{cases} u_p = u^* - \varphi_z y_p \\ v_p = v^* + \varphi_z x_p \end{cases}$$

$\varphi_z = \varphi$ per semplicità

Per essere piano deve

$$\begin{cases} w^* = 0 \\ \varphi_x y_p - \varphi_y x_p = 0 \\ \Leftrightarrow \varphi_x = \varphi_y = 0 \end{cases}$$

consideriamo punto $Q \neq P$

$$\begin{cases} v_Q = v^* - \varphi y_Q \\ r_Q = r^* + \varphi x_Q \end{cases}$$

allora

$$\begin{cases} v_P - v_Q = -\varphi (y_P - y_Q) \\ r_P - r_Q = \varphi (x_P - x_Q) \end{cases}$$

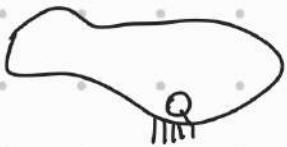
scritto questo

$$\begin{cases} v_P = v_Q - \varphi (y_P - y_Q) \\ r_P = r_Q + \varphi (x_P - x_Q) \end{cases}$$

VINCOLI puntuali nel piano xy

Vincoli ESTERNI

vincolo d'incastro



modello di trave



$$v_Q = r_Q = \varphi = 0$$

il vincolo ha MOLTEPLICITÀ 3

blocchia 3 gradi \Rightarrow libertà

$$\begin{cases} v_P = v_Q - \varphi (y_P - y_Q) \\ r_P = r_Q + \varphi (x_P - x_Q) \end{cases} \Rightarrow (v_P, r_P) = (0, 0)$$

vincolo di cerniera



$v_Q = r_Q = 0$ ha molteplicità 2 non esistono attriti

$$\begin{cases} v_p = v_Q - \varphi (y_p - y_Q) \\ r_p = r_Q + \varphi (x_p - x_Q) \end{cases} \Rightarrow \begin{cases} v_p = -\varphi (y_p - y_Q) \\ r_p = \varphi (x_p - x_Q) \end{cases}$$

vincolo di appoggio scorrevole (carrello)



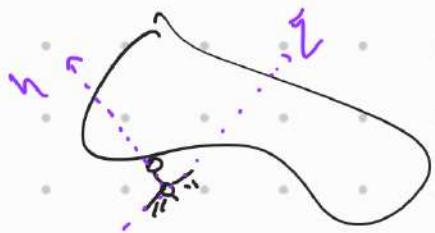
$v_\eta = 0$ molteplicità 1

$$\begin{cases} v_p = v_Q - \varphi (y_p - y_Q) \\ r_p = r_Q + \varphi (x_p - x_Q) \end{cases} \Rightarrow \begin{cases} v_p = v_Q - \varphi y_p \\ r_p = \varphi x_p \end{cases}$$

oss: il punto di rotazione è INDETERMINATO

- giace sulla retta perpendicolare allo scorrimento
- la retta passa per Q

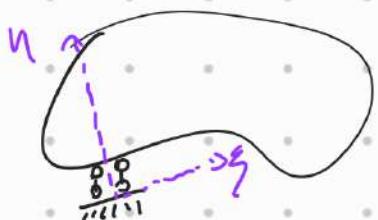
vincolo di biella



$$v_\eta = 0$$

è analogo al carrello

vincolo doppia biella (elio)

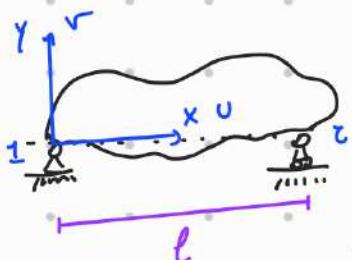


$$v_\eta = 0 \Rightarrow \text{ha molteplicità}$$

$$\varphi = 0$$

STUDIO CINEMATICO

STRUTTURE NON CORPO



Come può muoversi?

I gradi di libertà (g.d.l.) normalmente sono 3

Definiamo INCognite CINEMATICHE

\hookrightarrow scelgo il punto più vincolato

In questo caso scelgo il punto 1 (cerniera)

$$u_i, v_i, \varphi$$

equazioni di vincoli

$$\begin{cases} u_i = 0 \\ v_i = 0 \\ v_r = 0 \end{cases}$$

$v_r = 0 \rightarrow$ da riscrivere in funzione di φ

$$\begin{cases} u_i = 0 \\ v_i = 0 \\ v_r = v_i + \varphi \left(\frac{l}{x_c - x_i} \right) = 0 \end{cases}$$

$$\begin{cases} u_i = 0 \\ v_i = 0 \\ \varphi = 0 \end{cases}$$

il corpo è
FISSO

OSS:

: le equazioni si possono scrivere in forme matriciale

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ \varphi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

. le condizioni di vincolo sono

$$z+1=3$$

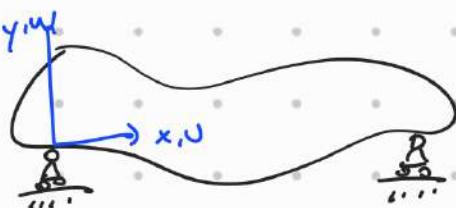
$$\text{cioè } gdl = cdv$$

inoltre è una STRUTTURA FISSA

} è una struttura
ISOCINEMATICA

ESERCIZI

1.



$$gdl - cdv = 3 - 2 = 1$$

$$gdl = 3 \Rightarrow cdv < gdl$$

$$cdv = 1+1=2$$

↳ sicuramente
è una ↳ non fissa

STRUTTURA LABILITÀ

↳ ha ALMENO un grado di libertà

scelgo il punto 1

$$u_1, v_1, \varphi$$

condizioni di vincolo

$$\begin{cases} v_1 = 0 \\ v_2 = 0 \end{cases}$$

$$\begin{cases} v_1 = 0 \\ 0 = v_1 + \varphi(x_2 - x_1) \end{cases}$$

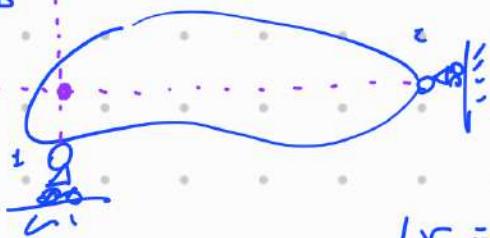
$$\begin{cases} v_1 = 0 \\ \varphi = 0 \end{cases}$$

quindi

$$\begin{cases} u_1 = u^* \\ v_1 = 0 \\ \varphi = 0 \end{cases}$$

arbitrario
grado di libertà 1

4.b



$$\text{osserv. } gdl - cdv = 1$$

incognite cinematiche v_i, r_i, φ

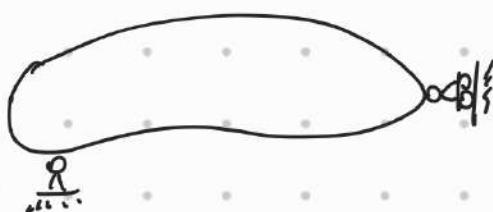
condizioni di vincolo

$$\begin{cases} r_i = 0 \\ v_i = 0 \end{cases}$$

$$v_r = v_i - \varphi (r_i - r_0)$$

$$\begin{cases} v_i = 0 \\ \dot{\theta} = v_i - \varphi (r_i - r_0) \end{cases}$$

$$\begin{cases} v_i = r_i \varphi \\ v_r = 0 \\ \varphi = \varphi^* \end{cases}$$



$$gdl = cdv$$

incognite cinematiche

$$v_i, v_r, \varphi$$

equazioni di vincolo

$$\begin{cases} v_i = 0 \\ v_r = 0 \\ v_r = 0 \end{cases}$$

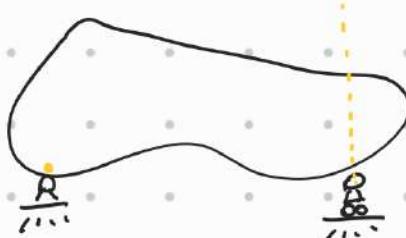
$$\begin{cases} v_i = 0 \\ v_r = 0 \end{cases}$$

$$\dot{\theta} = v_i - \varphi (r_i - r_0)$$

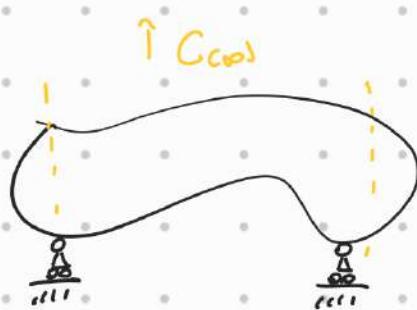
$$\begin{cases} v_i = 0 \\ v_r = 0 \\ \varphi = \varphi^* \end{cases}$$

oss:

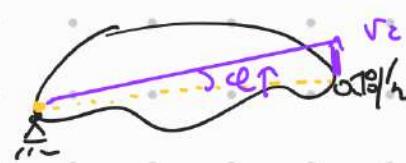
Se la struttura è LABILE, si può identificare il centro di rotazione assoluto



NON ha
centro di rotazione



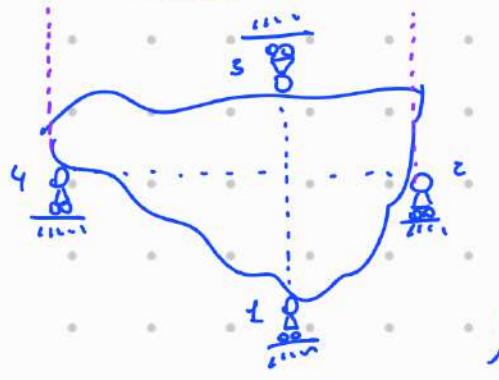
punto di intersezione improprio,
ha centro di rotazione



$$v_r = \varphi l$$

per case

$T_{(Co)}$



osservo $gdl < cdv$

incognite cinematiche U_1, V_1, φ

condizioni di vincolo

$$V_1 = 0$$

$$V_2 = 0$$

$$V_3 = 0$$

$$V_4 = 0$$

$$V_2 = V_1 + \varphi (x_2 - x_1)$$

$$V_3 = V_1 + \varphi (x_3 - x_1)$$

$$V_4 = V_1 + \varphi (x_4 - x_1)$$

$$\begin{cases} U_1 = U^* \\ V_1 = 0 \\ \varphi = 0 \end{cases}$$

ha un grado
di libertà



osservo $gdl = cdv$

incognite cinematiche U_1, V_1, φ

condizioni di vincolo

$$\begin{cases} U_1 = 0 \\ V_1 = 0 \\ V_\eta = 0 \end{cases}$$

dove

$$U_c = -V_\eta \sin \alpha$$

$$V_c = V_\eta \cos \alpha$$

$$U_c = U_1 - \varphi (y_c - y_1)$$

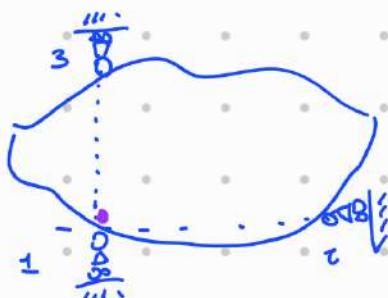
$$V_c = V_1 + \varphi (x_c - x_1)$$

$\alpha \neq \frac{\pi}{2}$

$$\Rightarrow \begin{cases} U_1 = 0 \\ V_1 = 0 \\ \varphi = 0 \end{cases} \text{ è fisso} \Rightarrow gdl = cdv \text{ isocinematico}$$

$$\alpha = \frac{\pi}{2} \quad \begin{cases} U_1 = 0 \\ V_1 = 0 \\ \varphi = \varphi^* \end{cases} \text{ ha un grado} \\ \text{di libertà}$$

3.



osservo $gdl = cdv$

incognite cinematiche U_1, V_1, φ

condizioni di vincolo

$$\begin{cases} V_1 = 0 \\ U_2 = 0 \\ V_3 = 0 \end{cases}$$

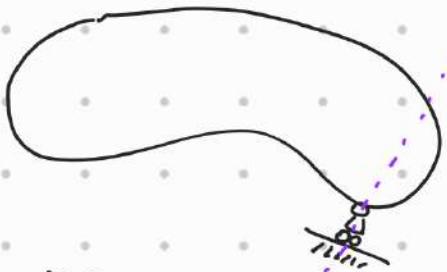
$$\begin{cases} V_1 = 0 \\ 0 = U_1 - \varphi (y_c - y_1) \\ 0 = V_1 + \varphi (x_c - x_1) \end{cases}$$

$$\begin{cases} U_1 = 0 \\ V_1 = 0 \\ \varphi = \varphi^* \end{cases}$$

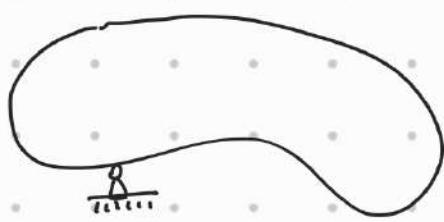
Analisi statica struttura monocoque piano xy

REAZIONI VINCOLARI

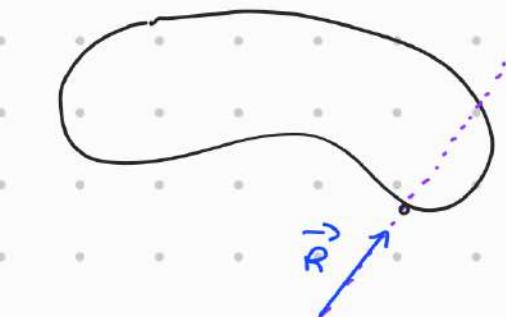
- Appoggio scorrevole / biella



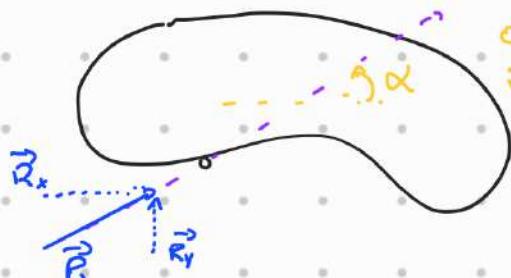
Cerniera



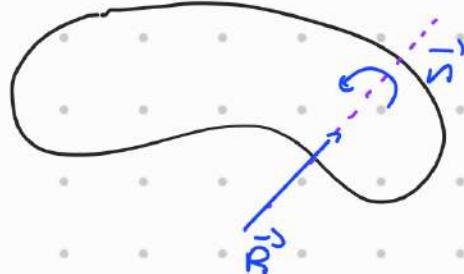
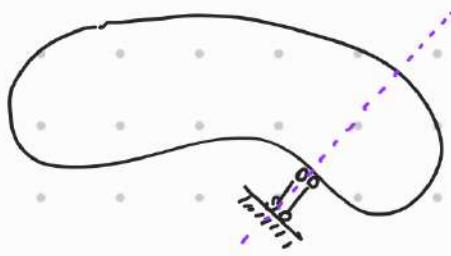
Incastro



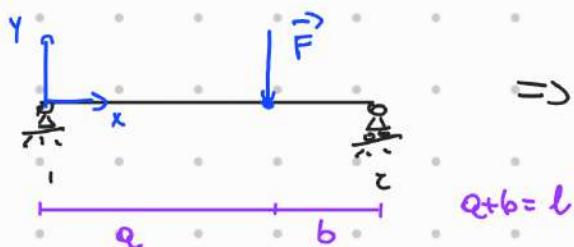
angolo indeterminato



- Doppia biella



ANALISI STATICA



- sistema principale / diagramme corps libres
- force esterne attive
- force esterne e vincolari

abbiamo INCOGNITE STATICHE

H_1, V_1, V_L

nel piano xy

si chiamano

EQUAZIONI CARDINALI

equazioni di equilibrio

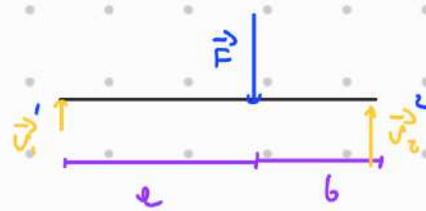
$$\begin{cases} H_1 = 0 \\ V_1 + V_L - F = 0 \\ -F \cdot a + V_L \cdot l = 0 \end{cases}$$

$$\begin{cases} H_1 = 0 \\ V_1 + V_L = F \\ V_L \cdot l = F \cdot a \end{cases}$$

$$\begin{cases} H_1 = 0 \\ V_1 = F(1 - \frac{a}{l}) \\ V_L = \frac{a}{l} F \end{cases}$$

quindi

$$\begin{cases} H_1 = 0 \\ V_1 = \frac{b}{l} F \text{ la soluzione è} \\ V_L = \frac{a}{l} F \text{ UNICA} \end{cases}$$



ossi in forme matriciali è

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & l \end{bmatrix} \begin{bmatrix} H_1 \\ V_1 \\ V_L \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ F \cdot a \end{bmatrix} \rightarrow \begin{array}{l} \text{vettore termini} \\ \text{forze attive} \end{array}$$

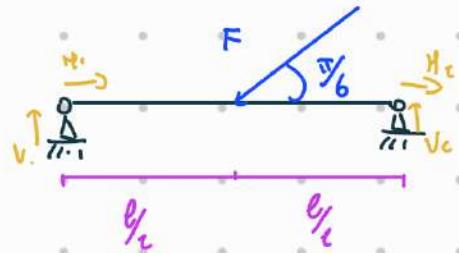
\hookrightarrow vettore
componenti vincolari

la struttura è staticamente DETERMINATA

la struttura è in equilibrio rispetto a \Rightarrow

struttura

QUALSIASI SISTEMA \Rightarrow ISOSTATICA



equazioni di equilibrio statico

$$\begin{cases} H_1 + H_2 - F \cos \frac{\pi}{6} = 0 \\ V_1 + V_2 - F \sin \frac{\pi}{6} = 0 \\ -F \sin \frac{\pi}{6} \cdot \frac{l}{2} + V_2 \cdot l = 0 \end{cases}$$

rispetto al punto 1

$$\begin{cases} H_1 + H_2 = \frac{\sqrt{3}}{2} F \\ V_1 = \frac{F}{4} \\ V_2 = \frac{F}{4} \end{cases}$$

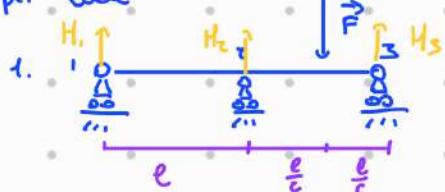
infinite soluzioni \rightarrow cdv > gde
↳ staticamente indeterminata

oss: c'è in EQUILIBRIO per qualsiasi sistema di forze

↳ struttura lassa

) IPERSTATICA

per caso



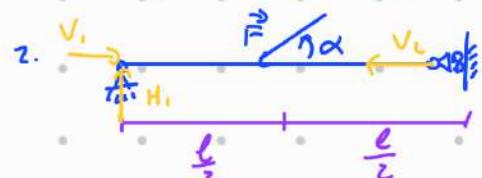
incognite vincolari: H_1, H_2, H_3

equazioni di equilibrio

$$\begin{cases} H_1 + H_2 + H_3 = F \\ H_2 \cdot \frac{l}{3} - \frac{3}{2} F \cdot \frac{l}{3} + 2H_3 \cdot \frac{l}{3} = 0 \end{cases}$$

$$\begin{cases} H_1 + H_2 + H_3 = F \\ H_2 + 2H_3 = \frac{3}{2} F \end{cases}$$

staticamente indeterminata qualsiasi sistema



incognite cinematiche H_1, V_1, V_2

eq. di equilibrio

$$\begin{cases} V_1 - F \cos \alpha - V_2 = 0 \\ H_1 - F \sin \alpha = 0 \\ -F \sin \alpha \cdot \frac{l}{2} - H_1 \cdot l = 0 \end{cases}$$

$\alpha = \frac{\pi}{4}$

$$\begin{cases} V_1 - \frac{\sqrt{2}}{2} F - V_2 = 0 \\ H_1 = \frac{\sqrt{2}}{2} F \\ H_1 = \frac{\sqrt{2}}{4} F \end{cases}$$

Non ha soluzione

statischamente indeterminata

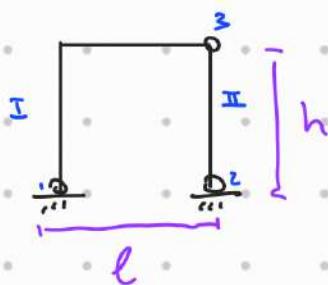
qualsiasi sistema



$$\begin{cases} N_1 - F_N \sin \alpha = 0 \\ H_1 + F_N \cos \alpha = 0 \\ \frac{l}{2} F - l H_1 = 0 \end{cases} \quad \begin{cases} V_1 = -\frac{F}{2} \tan \alpha \\ F_N = \frac{-F}{2 \cos \alpha} \\ H_1 = \frac{F}{2} \end{cases}$$

staticamente determinata
 problema:
 sistema

Studio cinematico di struttura multycorpo piano XY



$$gde = 2 \cdot 3 = 6$$

$$cdv = 2 + 2 + c = 6$$

se fixe e sostituta

incognite cinematiche

$$u_1, v_1, \varphi_I, u_2, v_2, \varphi_{II}$$

equazioni di vincolo

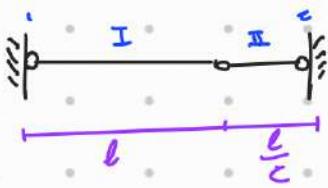
$$\begin{cases} u_1 = 0 \\ v_1 = 0 \\ u_c = 0 \\ v_c = 0 \\ u_3^I = u_3^{II} \\ v_3^I = v_3^{II} \end{cases}$$

$$\begin{cases} u_1 = 0 \\ v_1 = 0 \\ u_c = 0 \\ v_c = 0 \\ u_1 - \varphi_I h = u_c - \varphi_c h \\ v_c + \varphi_{II} l = v_2 \end{cases}$$

$$\begin{aligned} u_3^I &= u_1 - \varphi_I (y_3 - y_1) \\ &\stackrel{!}{=} u_1 - \varphi_I h \\ u_3^{II} &= u_2 - \varphi_{II} (y_3 - y_2) \\ &\stackrel{!}{=} u_c - \varphi_c h \\ v_3^I &= v_1 + \varphi_I (x_3 - x_1) \\ &\stackrel{!}{=} v_1 + \varphi_I l \\ v_3^{II} &= v_2 + \varphi_{II} (x_3 - x_2) \\ &\stackrel{!}{=} v_2 \end{aligned}$$

$$\begin{cases} u_1 = 0 \\ v_1 = 0 \\ u_c = 0 \\ v_c = 0 \\ \varphi_I = 0 \\ \varphi_{II} = 0 \end{cases}$$

la struttura è
 FISSA
 \Rightarrow sostituita
 $cdv = gde$



$$gde = 6$$

incognite cinematiche $u_i, v_i, \dot{\varphi}_I, u_c, v_c, \dot{\varphi}_c$

equazioni d: vincolo

$$\begin{cases} u_i = 0 \\ v_i = 0 \\ u_c = 0 \\ v_c = 0 \\ u_3^I = u_3^{II} \\ v_3^I = v_3^{II} \end{cases}$$

$$\begin{cases} u_i = 0 \\ v_i = 0 \\ u_c = 0 \\ v_c = 0 \\ u_3^I - \dot{\varphi}_I (y_3 - y_i) = u_3^{II} - \dot{\varphi}_{II} (y_3 - y_c) \\ v_i + \dot{\varphi}_I (x_3 - x_i) = v_c + \dot{\varphi}_{II} (x_3 - x_c) \end{cases}$$

$$\begin{cases} u_i = 0 \\ v_i = 0 \\ u_c = 0 \\ v_c = 0 \\ \dot{\varphi}_I \cdot 0 = \dot{\varphi}_{II} \cdot 0 \\ \dot{\varphi}_I \cdot l = -\frac{l}{2} \dot{\varphi}_{II} \end{cases}$$

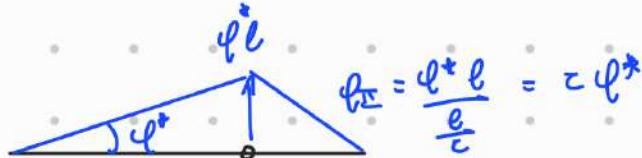
$$\begin{cases} u_i = 0 \\ v_i = 0 \\ u_c = 0 \\ v_c = 0 \end{cases}$$

ha un grado

d: labilità

$$\dot{\varphi}_I = \dot{\varphi}^*$$

$$\dot{\varphi}_{II} = -c \dot{\varphi}^*$$



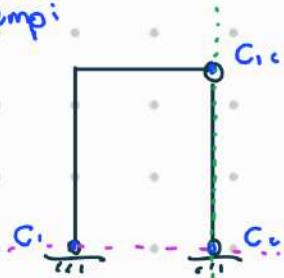
è molto pericoloso, consigliato

TÖRENI DELLE CATENE CINEMATICHE:

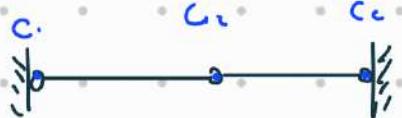
I teorema: per un sistema con $z > 0$ corpi, la condizione necessaria e SUFFICIENTE per la labilità è che siano ALLINEATI i CENTRI DI ROTAZIONE assoluti e relativi riferiti a qualsiasi coppia di punti.

II Teorema: per un sistema con 3 o più corpi, la condizione necessaria e SUFFICIENTE per la labilità è che siano ALLINEATI i CENTRI DI ROTAZIONE per qualsiasi terza

esempi:

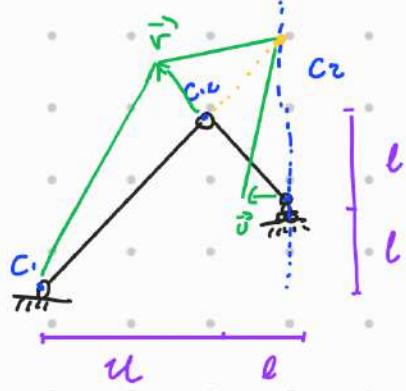


NON sono allineati
↳ è una struttura
FISSA



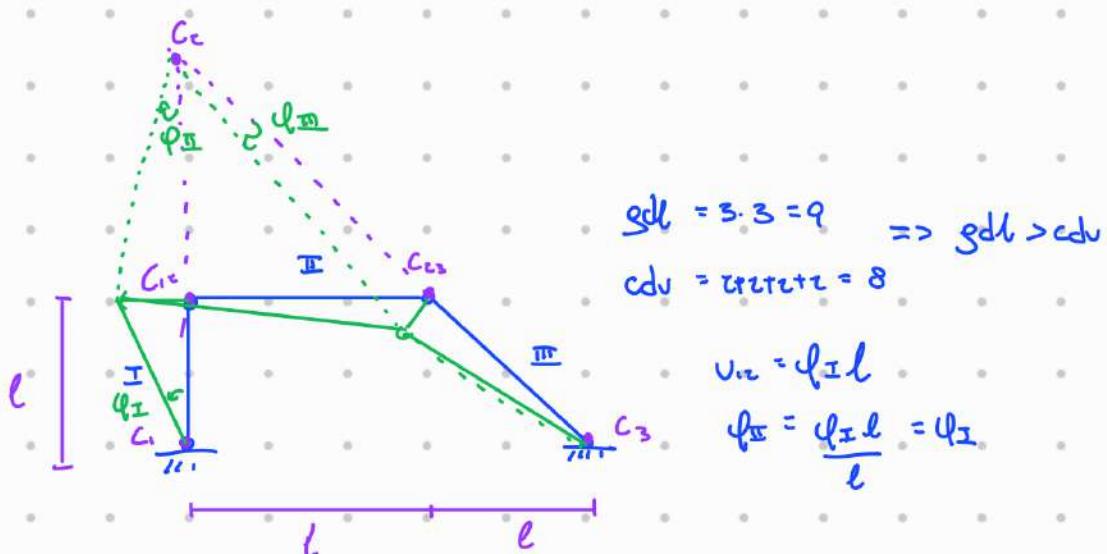
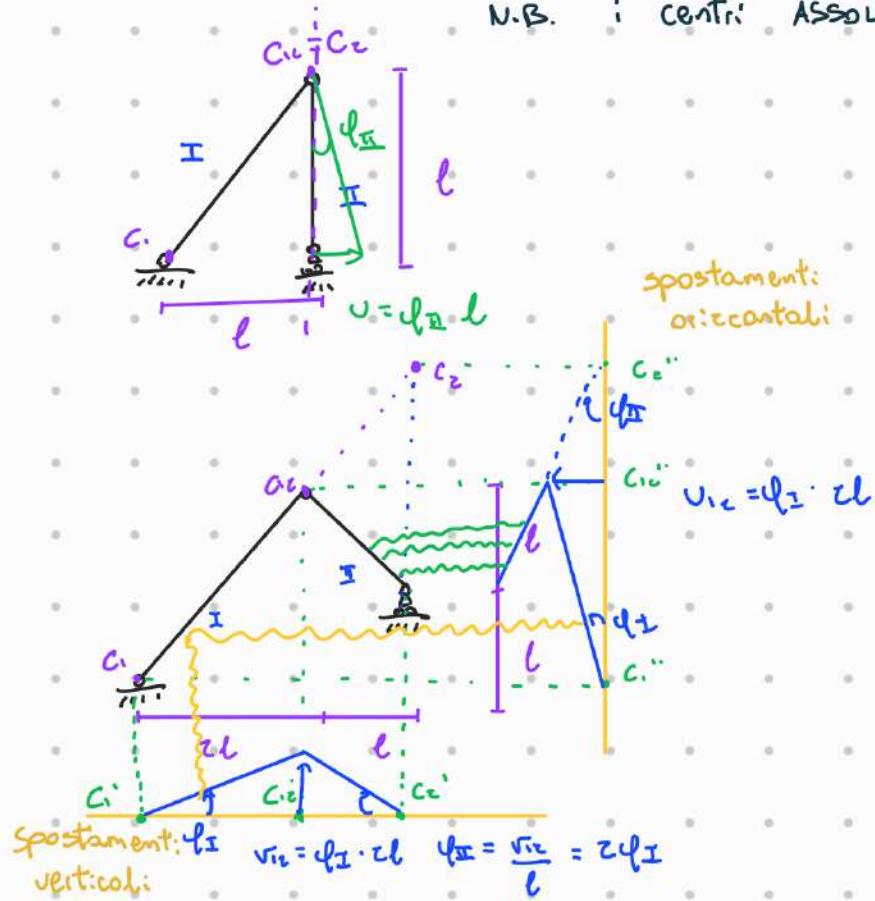
C1, C2, C3 sono ALLINEATI
↳ la struttura è labile

↳ i centri ASSOLUTI → non si spostano
i centri RELATIVI → si spostano

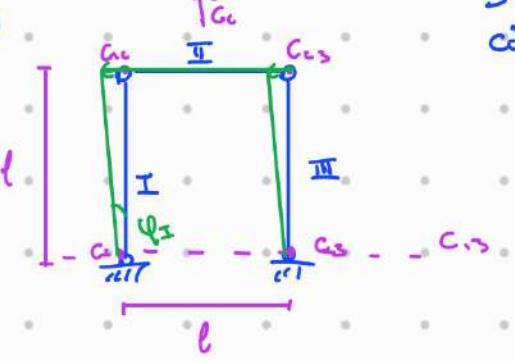


approccio grafico

N.B. i centri ASSOLUTI NON TRASLANO



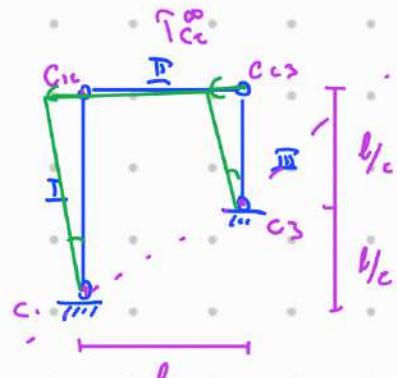
per caso



$$gdL = 9 \Rightarrow gdL > cdL$$

$$U_{12} = f_1 l$$

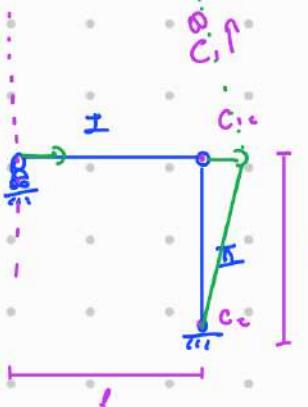
$$U_{23} = U_{12} = f_1 l = \phi_1 l = \phi_1 \cdot l \Rightarrow \phi_1 = f_1$$



$$gdL = 9 \Rightarrow gdL > cdL$$

$$U_{12} = \phi_1 l_c$$

$$U_{23} = U_{12} = \phi_1 l_c = \phi_1 \cdot l_c = \phi_{II} \frac{l}{c}$$
$$\Rightarrow \phi_{III} = c \phi_{II}$$

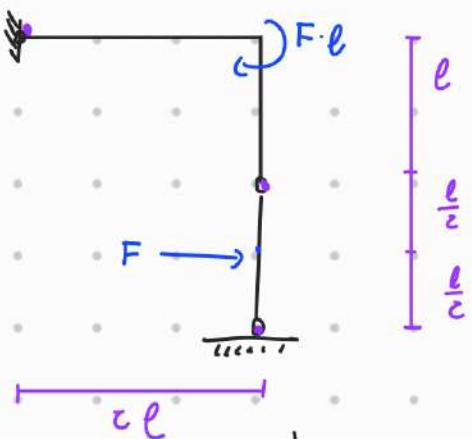


$$gdL = 6 \Rightarrow gdL > cdL$$

$$cdL = 3$$

Analsi statica strutture multicorpo

piano xy



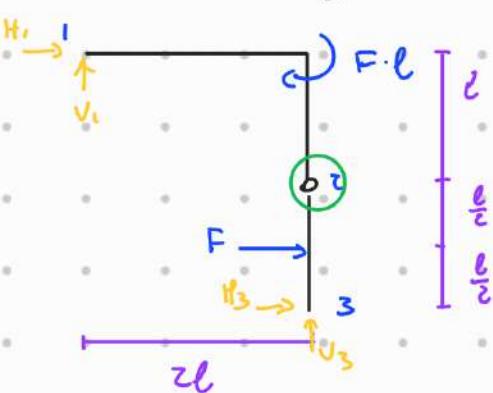
$$gdl = 6 \Rightarrow cdv = gdl$$

$$cdv = 6$$

è fisso perché i centri non sono allineati:

è una struttura ISOSTATICA

diagramme corpo libero



incognite statiche

$$H_1 + H_3 + F = 0$$

$$V_1 + V_3 = 0$$

$$1: -F \cdot l + \frac{3}{c} F \cdot l + 2 H_3 \cdot l + 2 V_3 \cdot l = 0$$

$$2: F \frac{l}{c} + H_3 \cdot l = 0$$

equazione ausiliaria
d'equilibrio

(del corpo 2-3)

$$H_1 = -\frac{F}{2}$$

$$V_1 = -V_3$$

$$2V_3 = F - \frac{3}{c} F + \frac{F}{c} \cdot c$$

$$U_3 = -\frac{F}{c}$$

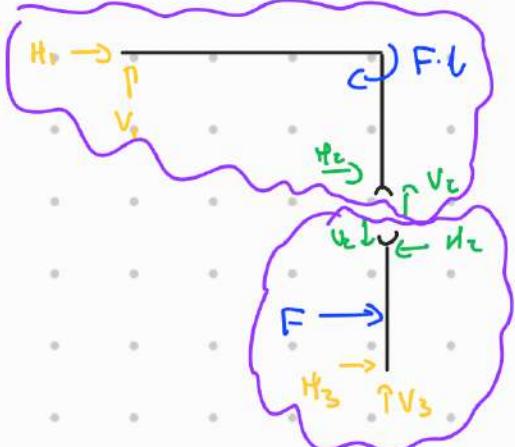
$$H_1 = -\frac{F}{2}$$

$$V_1 = -\frac{F}{4}$$

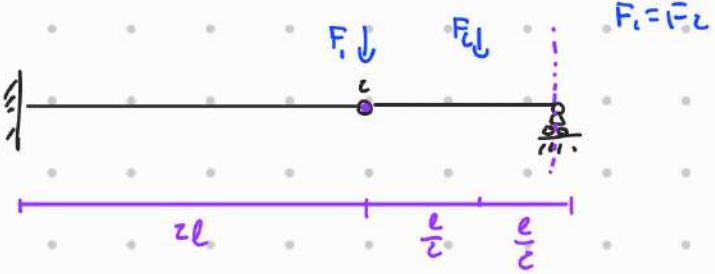
$$H_3 = -\frac{F}{2}$$

$$V_3 = \frac{F}{4}$$

modo alternativo



6 incognite
6 equazioni



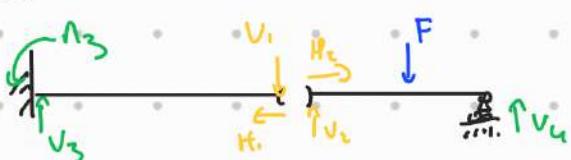
$\text{g}l = 6 = \text{cdv}$
è fiso \rightarrow isostatico

OSS: vale la SOVRAPPPOSIZIONE DEGLI EFFETTI

considero F_L



diventa



osserva subito

$$H_C = 0 \Rightarrow H_L = 0$$

calcolo i momenti:

cassella: $-V_L \cdot l + F \frac{l}{2} = 0 \Rightarrow V_L = \frac{F}{2} = V_4 = V_1$

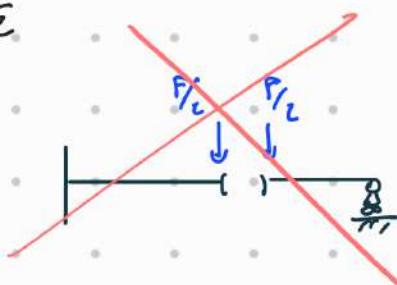
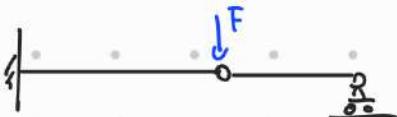
equilibrio dei momenti: di corpo 1 - cerniere

$$-H_3 + V_1 \cdot zL = 0 \Rightarrow H_3 = F \cdot l$$

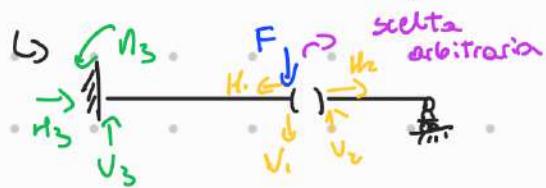
equilibrio forze verticali: corpo 1 - cerniere

$$V_3 - V_1 = 0 \Rightarrow V_3 = V_1 = \frac{F}{2}$$

considero F_L



No!!!!



osservo

$$H_L = 0 \Rightarrow H_3 = 0$$

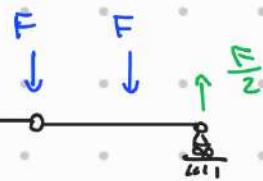
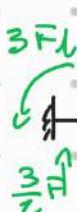
ep. momento: $V_L = 0 \Rightarrow V_1 = 0$

quindi: equilibri:

$$\begin{cases} H_3 = 0 \\ V_3 = F \\ -F \cdot d + H_3 = 0 \end{cases} \Rightarrow H_3 = F \cdot d$$

momenti:

sommiamo



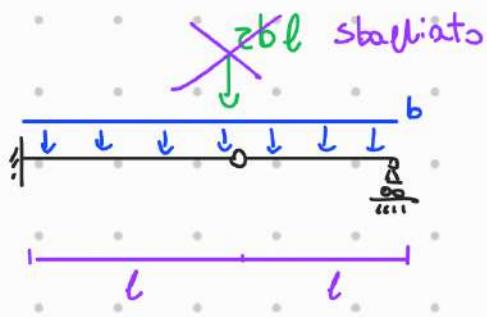
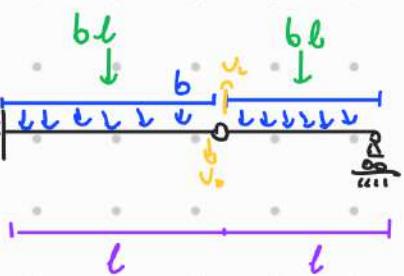
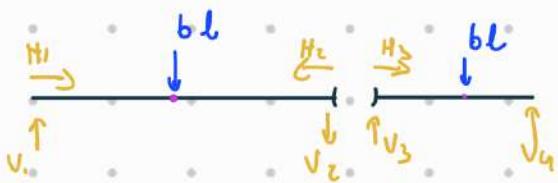


diagramma corpo libero



$$\begin{cases} \text{gdl} = \text{cd} \\ \text{piso} \end{cases}$$

↳ isostatico

Osseruo:

$$H_3 = 0 \Rightarrow H_1 = H_2 = 0 \Rightarrow H_4 = 0$$

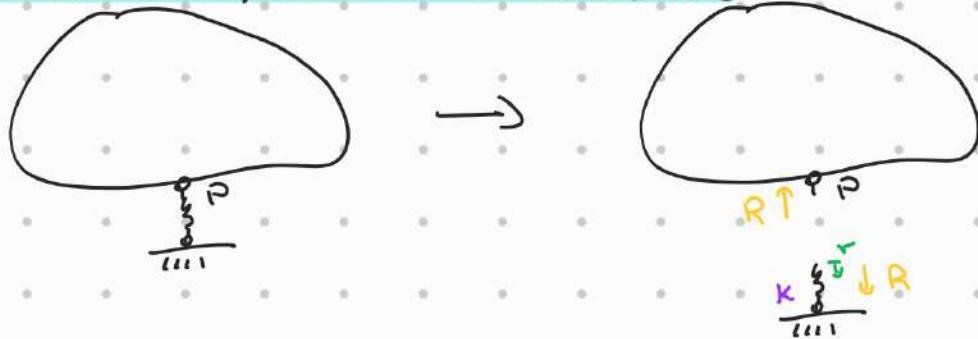
$$\begin{aligned} \text{momento } V_4: \quad & b l \cdot \frac{l}{2} - V_3 l = 0 \\ & \Rightarrow V_3 = \frac{b l}{2} \Rightarrow v_c = -\frac{b l}{2} \end{aligned}$$

eq. x

$$V_1 - b l - V_c = 0 \Rightarrow V_1 = \frac{b l}{2}$$

VINCOLI CEDEVOLI ELASTICAMENTE

- Molla estensionile / biella cedevole elasticamente



$$\text{assumo molla lineare} \rightarrow \text{RIGIDEZZA } K \Rightarrow V = \frac{1}{K} R$$

- Incastro cedevole elasticamente

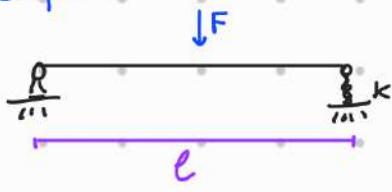


l'incastro si oppone alla rotazione

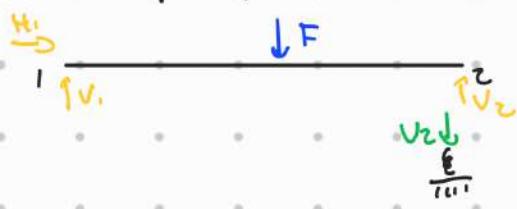
$$\text{RIGIDEZZA rotazionale } K_\phi \Rightarrow \phi = \frac{1}{K_\phi} M$$

↳ radianti (adimensionali)

esempi:



sistema principale



$$\left\{ \begin{array}{l} H_1 = 0 \\ V_1 + V_2 = F \\ -F \frac{l_1}{2} - V_2 l = 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} H_1 = 0 \\ V_1 = F/l \\ V_2 = F/l \end{array} \right.$$

oss:

i vincoli non dipendono dalla rigidità

cedimento punto z

$$v = \frac{F}{zk} \quad \text{verso il basso}$$

diventa



NON c'è una labilità

$$\varphi = \frac{v}{l} = \frac{F}{zk} l \ll 1$$

oss:
se l'angolo è troppo GRANDE,
la nostra teoria non vale

$$\hookrightarrow v \ll l$$

oss: se gli elementi CEDIVOLI sono LINEARI, vale la sovrapposizione degli effetti



rimane

$$\begin{cases} H_1 = 0 \\ V_1 = F \\ \wedge = F \cdot l_1 \end{cases}$$

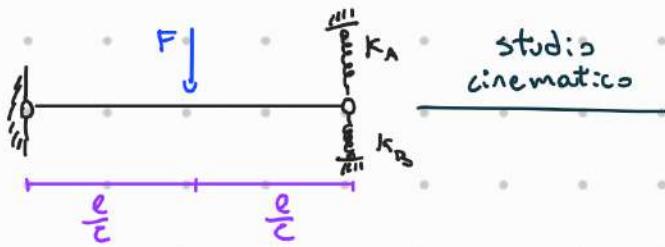
cedimento

$$y = \frac{F \cdot l \cdot h}{K}$$



3. METODI DELLE FORZE

ASSUMO struttura IPERSTATICA



fisso e gdl < cdv
↳ struttura iperstatica

Sistema principale



X è INCOGNITA IPERSTATICA

→ garantisce continuità del sistema.

$\rightarrow X=0$
NON è compatibile

↳ equazione di congruenza

$$\text{assumo } \sigma_c^{(1)} = \sigma_c^{(2)}$$

Sistema 2)

$$\sigma_c = \frac{X}{K_B} \quad \text{verso } U \text{ basso}$$

Sistema 2)



$$\text{momenti: } -\frac{F \cdot l}{c} + (X + R_A) \cdot l = 0 \Rightarrow R_A = \frac{F \cdot l - X}{l}$$

$$\text{quindi: } \sigma_c = \frac{R_A}{K_A} = \frac{\frac{F \cdot l - X}{l}}{K_A} \quad \text{verso } U \text{ basso}$$

equazione di congruenza:

$$V_c^D = V_c^S \Rightarrow \frac{F/c - x}{K_A} = \frac{x}{K_B}$$

$$x \left(\frac{1}{K_A} + \frac{1}{K_B} \right) = \frac{F/c}{K_A}$$

$$x \left(\frac{K_A + K_B}{K_A K_B} \right) = \frac{F/c}{K_A}$$

$$R_B = x = \frac{K_B}{K_A + K_B} \cdot \frac{F/c}{K_A}$$

quindi:

$$R_A = \frac{F/c}{K_A + K_B} \quad R_B = \frac{K_B}{K_A + K_B} \cdot \frac{F/c}{K_A}$$

C'è

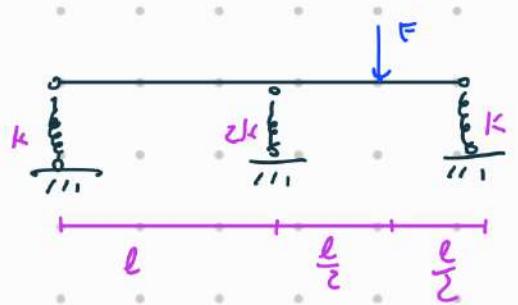
$$R_A + R_B = F/c$$

Oss:
i vincoli dipendono
dalla RIGIDEZZA

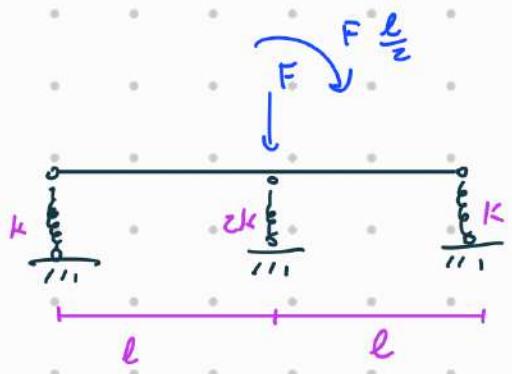
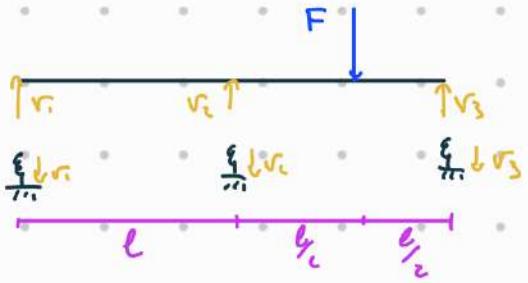
Oss:

Le molle sono in PARALLELO

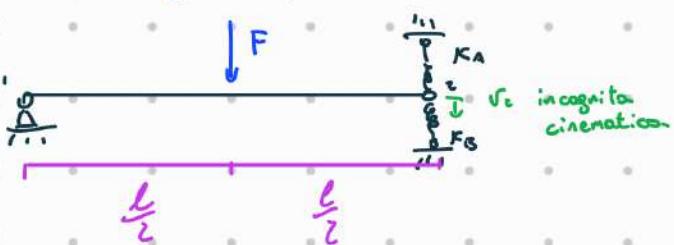
$$\frac{1}{K_A + K_B}$$



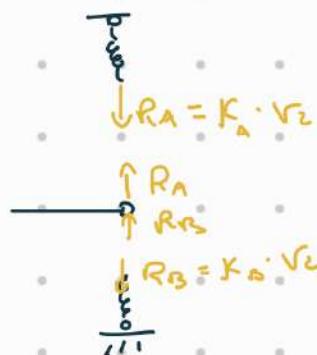
sistema principale



Metodo degli spostamenti:



quindici incognite
cinematiche



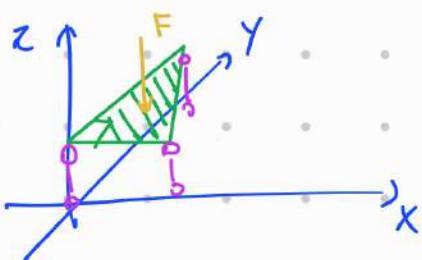
$$-F \frac{l}{2} + (R_A + R_B) l = 0$$

$$\Rightarrow R_A + R_B = \frac{F}{2} \Rightarrow K_A v_2 + K_B s_2 = \frac{F}{2}$$

$$\Rightarrow s_2 = \frac{F/2}{K_A + K_B}$$

Oss: più vincoli: ci sono 3, ci sono più calcoli e equazioni da fare

$C_A > A$



Determinare le sollecitazioni
e le tensioni
nelle sezioni

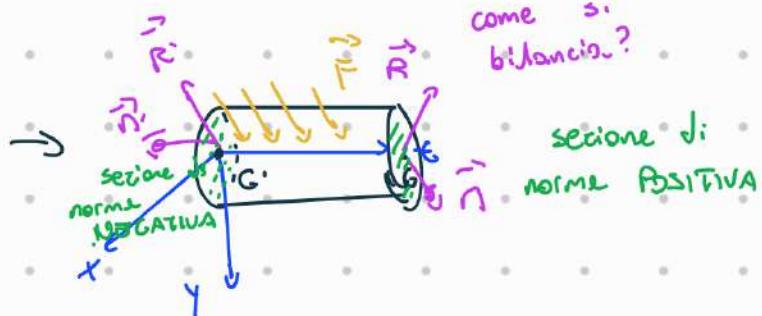
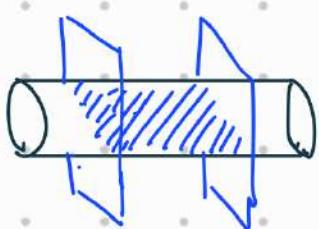
RECAP

CINEMATICA	STRUTTURA	STATICIA
<p>fissa</p> <p>↳</p> <p>ISO CINEMATICA</p>	 <p>GDL = 3 CDV = 2 + 1 = 3</p>	<p>ISOSTATICA</p> <p>↳</p> <p>staticamente DETERMINATA</p> <p>valido per tutti i sistemi di forze</p>
<p>IPER CINEMATICA</p> <p>↳</p> <p>labile</p>	 <p>condizione sufficiente</p> <p>GDL = 3 CDV = 2</p>	<p>IPOSTATICA</p> <p>staticamente DETERMINATA (non in generale)</p> <p>in equilibrio (non in generale)</p>
<p>fissa</p> <p>↳</p> <p>IPO CINEMATICA</p>	 <p>GDL = 3 CDV = 4</p>	<p>IPERSTATICA</p> <p>staticamente INDETERMINATA</p> <p>in equilibrio per qualsiasi sistema di forze</p>
<p>labile</p> <p>↳</p> <p>STRUCTURE DEGENERATE</p>	 <p>GDL = 3 CDV = 3</p>	<p>staticamente INDETERMINATA (in generale)</p> <p>non in equilibrio (in generale)</p>

STATICA DELLA TRAVE



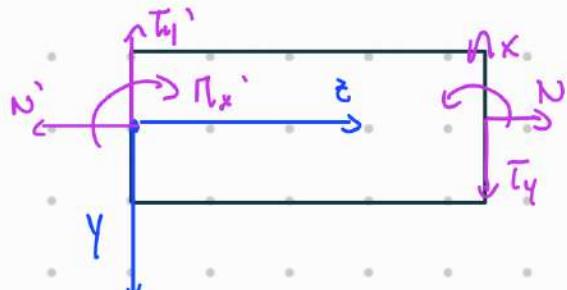
immagino di isolare delle sezioni



scompongo

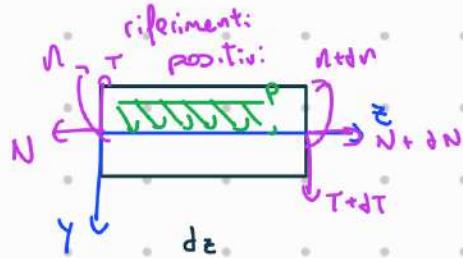
force	nome	simbolo
R_z	sollecitazione ASSIALE	N
R_x, R_y	sollecitazione TAGLIANTE	T_x, T_y
M_t	momento TORCENTE	M_t
M_x, M_y	momento FLETTENTE	M_f

Noi consideriamo principalmente il PIANO yz



convenzione dei segni (per lato)

unico momento nell'asse x



equilibrio delle forze direzione z:

$$N + dN - N + p \cos \alpha \cdot dz = 0$$

$$\hookrightarrow \frac{dN}{dz} = -t = -p \cos \alpha \quad \text{ep. indennità dell' equilibrio}$$

equilibrio delle forze direzione y

$$T + dT - T + p \sin \alpha \cdot dz = 0$$

$$\frac{dT}{dz} = -q$$

$$\text{dove } q = p \sin \alpha$$

equilibrio dei momenti: per $z + dz$:

$$N + dN - N - T dz +$$

$$q dz \cdot \frac{dz}{z} = 0$$

$$\frac{dM}{dz} = T$$

$\frac{1}{z} q dz^2 \Rightarrow$ ordine superiore

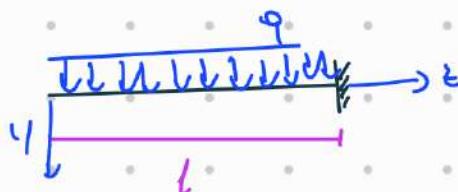
trascuro

quindi:

$$\left\{ \begin{array}{l} \frac{dN}{dz} = -t \\ \frac{dT}{dz} = -q \\ \frac{dM}{dz} = T \end{array} \right. \Rightarrow$$

$$\left\{ \begin{array}{l} \frac{dN}{dz} = -t \\ \frac{d^2 M}{dz^2} = \frac{dT}{dz} = -q \end{array} \right.$$

esempio

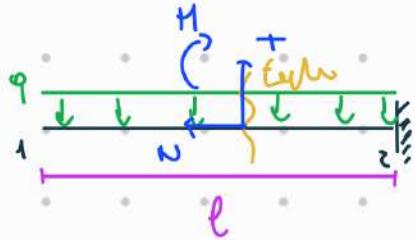


Determinare N, M, T

$$\frac{dN}{dz} = 0 \rightarrow N(z) = C = 0$$

Definito condizioni di carico

DIAGRAMMI DI SOLLECITAZIONE



$$\frac{dN}{dz} = 0 \rightsquigarrow N(z) = C$$

↳ condizioni al contorno

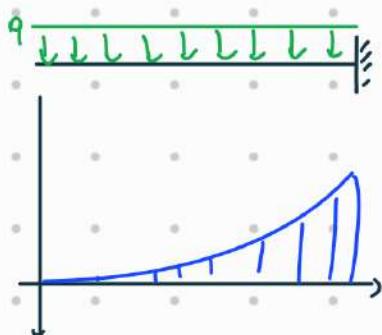
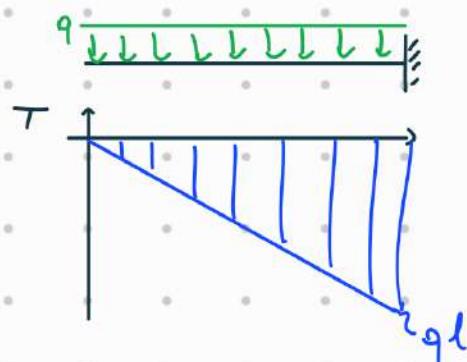
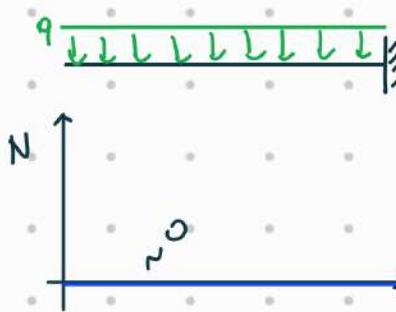
$$N(0) = 0 \rightsquigarrow N(z) = 0$$

$$\begin{cases} \frac{d^2M}{dz^2} = -q \\ T = \frac{dM}{dz} = -qz + C_1 \end{cases} \Rightarrow M = -\frac{1}{2}qz^2 + C_1 z + C_2$$

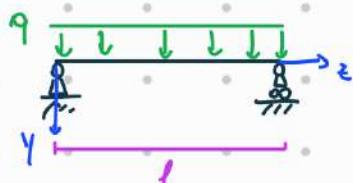
condizioni al contorno

$$\begin{cases} T(0) = C_1 = 0 \\ N(0) = C_2 = 0 \end{cases} \Rightarrow \begin{cases} T(z) = -qz \\ M(z) = -\frac{1}{2}qz^2 \end{cases}$$

Diagrammi:



esempio



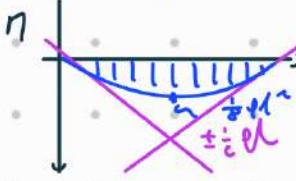
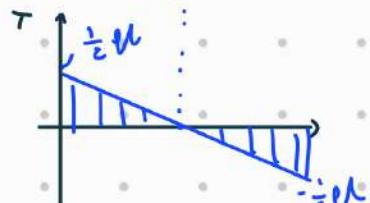
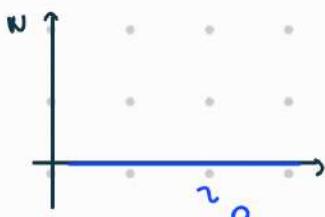
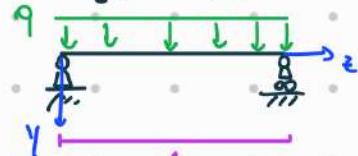
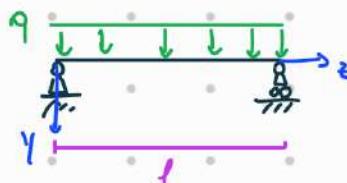
$$N(z) = C$$

$$(\Rightarrow N(l) = 0 \Rightarrow N(z) = 0)$$

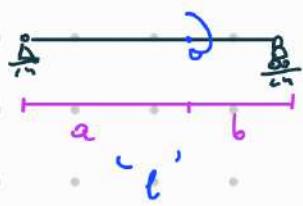
condizioni al contorno

$$\begin{cases} T = -qz + C_1 \\ N = -\frac{1}{2}qz^2 + C_1 z + C_2 \end{cases}$$

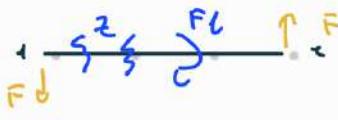
$$\begin{cases} N(0) = 0 \\ N(l) = 0 \end{cases} \Rightarrow \begin{cases} N(z) = 0 \\ T(z) = -qz + \frac{1}{2}qz^2 \\ M(z) = -\frac{1}{2}qz^2 + \frac{1}{2}qz^2 \end{cases}$$



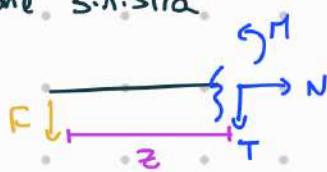
esempio Metodo dell'equilibrio



\rightarrow



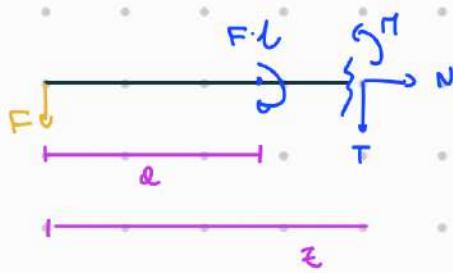
Regione sinistra



$z < a$

$$\begin{cases} N=0 \\ N+F_z=0 \\ T+F=0 \end{cases}$$

$$\begin{cases} N=0 \\ N=-F_z \\ T=-F \end{cases}$$

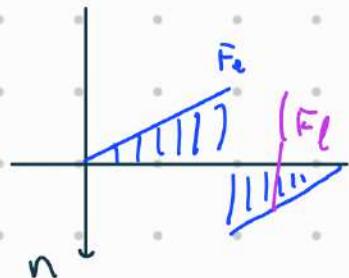
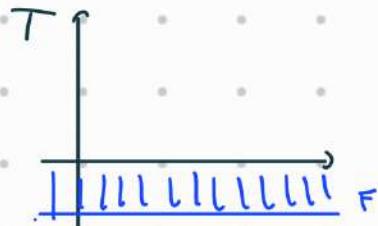


$$\begin{cases} N=0 \\ T+F=0 \\ N-Fl+F_z=0 \end{cases}$$

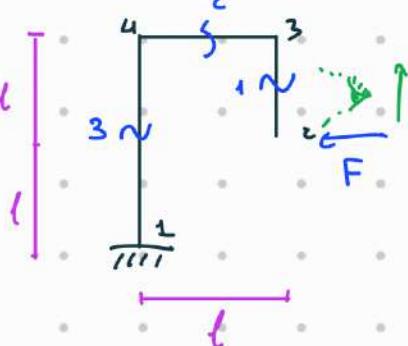
$$\begin{cases} N=0 \\ T=-F \\ N=F(l-z) \end{cases}$$

$$\Rightarrow M(l)=0$$

$$N(a)=F \cdot b$$



esempio



①



$$\begin{cases} N=0 \\ T-F=0 \\ N-F_z=0 \end{cases}$$

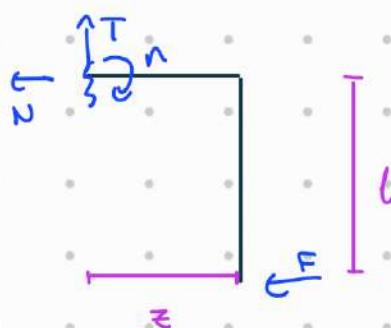
$$\begin{cases} N=0 \\ T=F \\ N=F \cdot z \end{cases}$$

OSS:



NON ci sono forze concentrate
il momento si CONSERVA

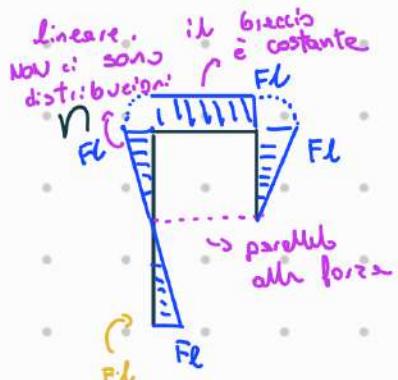
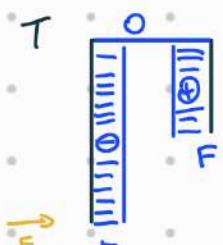
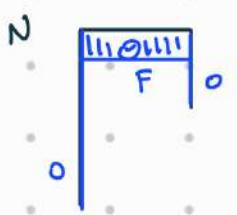
②



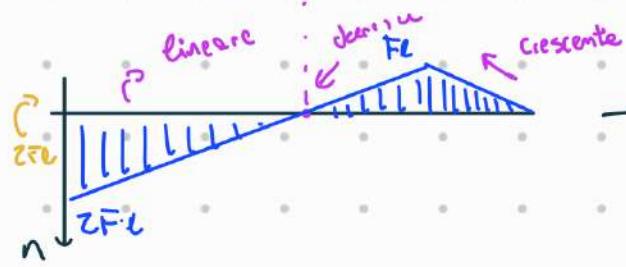
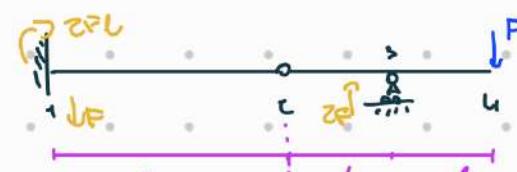
$$\begin{cases} N+F=0 \\ T=0 \\ N+Fl=0 \end{cases}$$

$$\begin{cases} N=-F \\ T=0 \\ N=-Fl \end{cases}$$

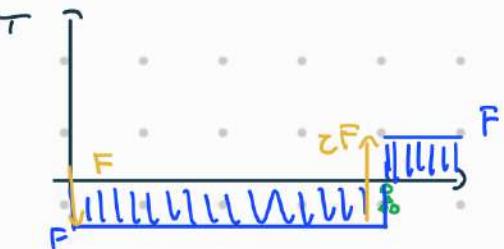
Rappresentazione dei diagrammi



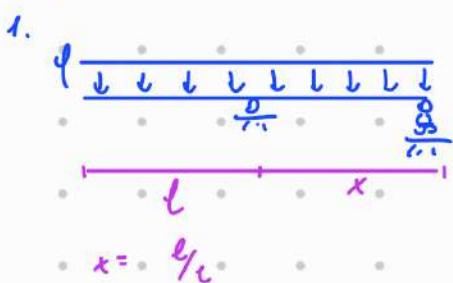
esercizio



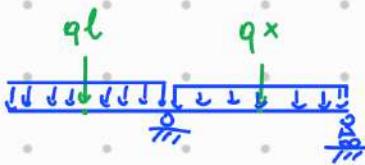
$$\rightarrow \frac{dN}{dx}$$



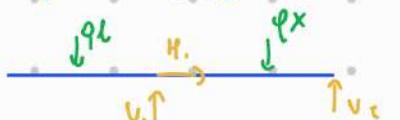
per caso



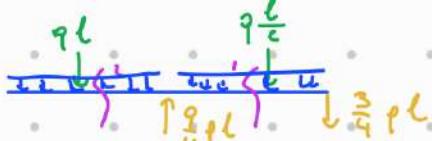
somma
degli effetti:



sistema libero



$$\begin{cases} H_1 = 0 \\ V_1 + V_2 = q(l+x) \\ q\frac{x^2}{2} + q(l+x)\frac{l}{2} = V_1 \cdot x \end{cases}$$



$$\begin{cases} H_1 = 0 \\ V_1 = \frac{q}{4}ql^2 \quad \text{cioè} \\ V_2 = -\frac{3}{4}ql^2 \end{cases}$$

$$\textcircled{1} \quad N = 0$$

$$\begin{cases} N = 0 \\ T - qz - \frac{3}{4}ql = 0 \\ T + q\frac{z^2}{2} + \frac{3}{4}qlz = 0 \end{cases}$$

$$\begin{cases} N = 0 \\ T = q(z + \frac{3}{4}l) \\ T = -q\frac{z^2}{2} - \frac{3}{4}qlz \end{cases}$$

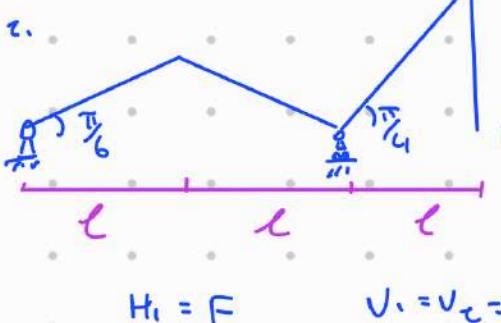
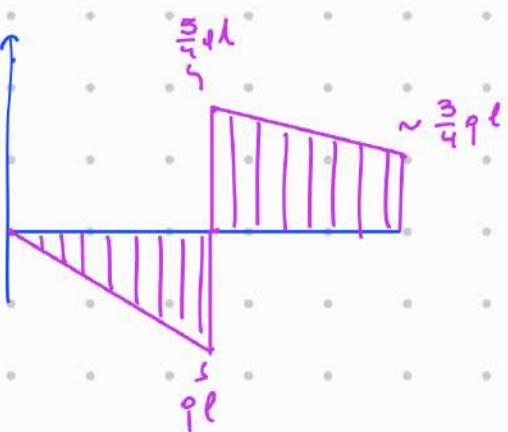
②

$$\begin{cases} N=0 \\ T - q \frac{l}{2} - \frac{3}{4}ql + \frac{q}{4}ql^2 - qz = 0 \\ M + q \frac{l}{2} \cdot (z + \frac{l}{4}) + \frac{3}{4}ql^2(z + \frac{l}{2}) - \frac{q}{4}ql^2 z + q \frac{z^2}{2} = 0 \end{cases}$$

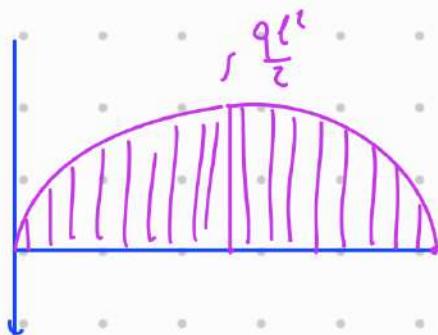
$$\begin{cases} N=0 \\ T = q(z - l) \\ M = -q \frac{z^2}{2} + qlz - \frac{1}{2}l^2 z^2 \end{cases}$$

Diagramma sollecitazione

①

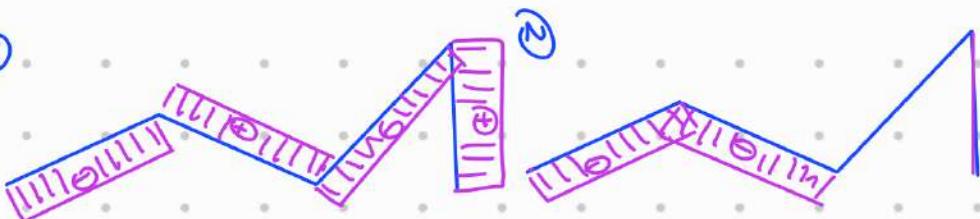
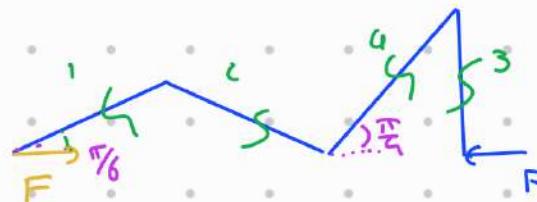


②

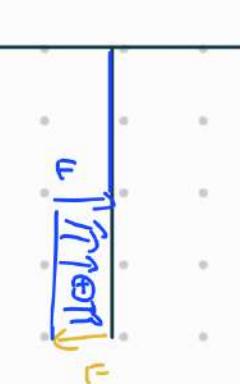
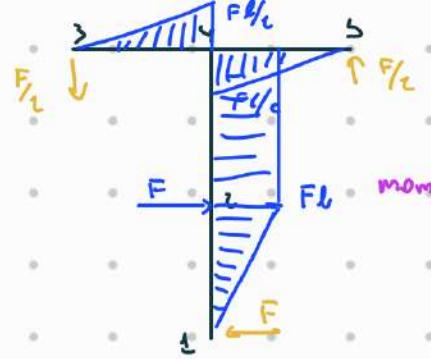
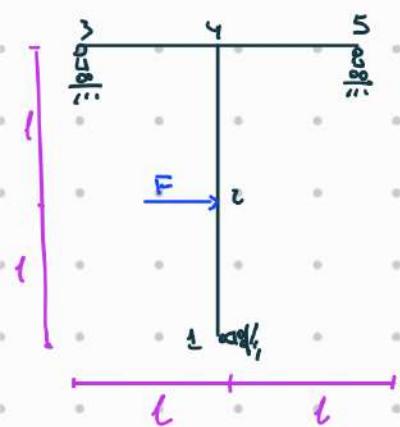


sistema
libero

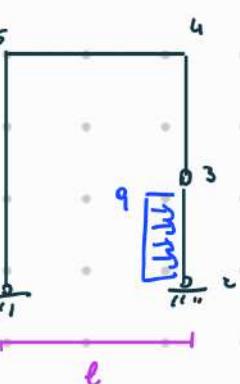
- | | | | |
|----|----------------------------|---------------------------|-----------------------------------|
| 1. | $N = -\frac{\sqrt{2}}{2}F$ | $T = -Fz$ | $M = -\frac{F}{2}z^2$ |
| 2. | $N = -\frac{\sqrt{2}}{2}F$ | $T = Fz$ | $M = -F(z - \frac{l}{2})$ |
| 3. | $N = 0$ | $T = -F$ | $M = -Fz$ |
| 4. | $N = -\frac{\sqrt{2}}{2}F$ | $T = \frac{\sqrt{2}}{2}F$ | $M = -F(l - \frac{\sqrt{2}}{2}z)$ |



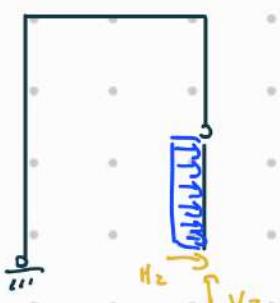
③



è una struttura statica



struttura isostatica



eq. momento: cerniere 3

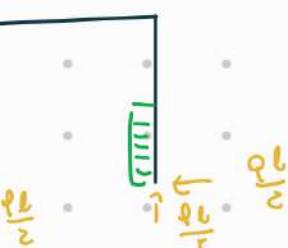
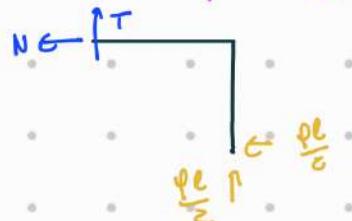
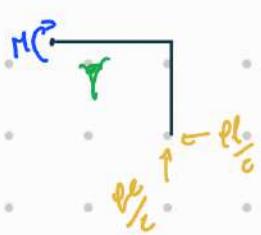
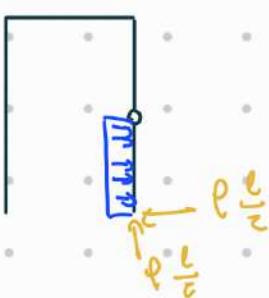
$$q \frac{l^2}{2} + M_0 \cdot l = 0 \\ \Rightarrow M_0 = -q \frac{l^2}{2}$$

eq momento: cerniere 1

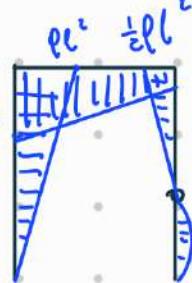
$$-q l \cdot \frac{l}{2} + V_U \cdot l \Rightarrow V_U = q \frac{l}{2}$$

Punto 1)

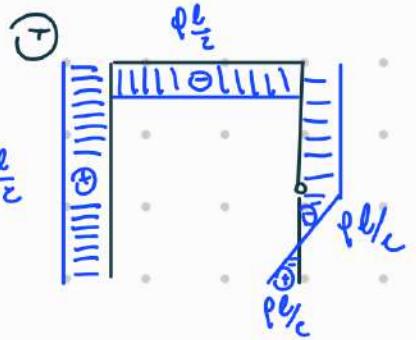
riprendo lo studio della struttura / controlli



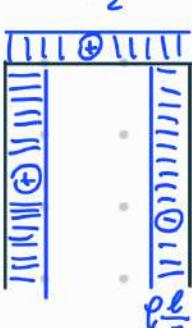
(R)



$$\frac{q l^2}{2}$$



(T)

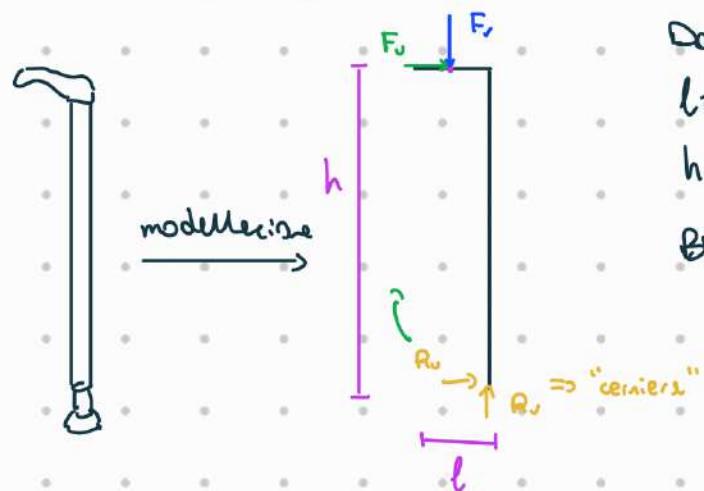


$\frac{q l}{2}$

$$\frac{q l}{2}$$

esempio

bastone biomedico (stampella)



Dati:

$$l = 120 \text{ mm}$$

$$h = 85 \text{ cm}$$

$$BW = 800 \text{ N} \quad F_v = 800 \text{ N}$$

equilibrio verticale

$$R_v - F_v = 0 \Rightarrow R_v = F_v = 800 \text{ N}$$

equilibrio momenti: suppose:

$$F_v \frac{l}{2} - F_v \cdot h = 0 \Rightarrow F_v = \frac{F_v}{2} \frac{l}{h} = 60 \text{ N}$$

$$\text{fondi} \quad F = \sqrt{F_v^2 + F_v^2} = 802,2 \text{ N} \approx F_v$$

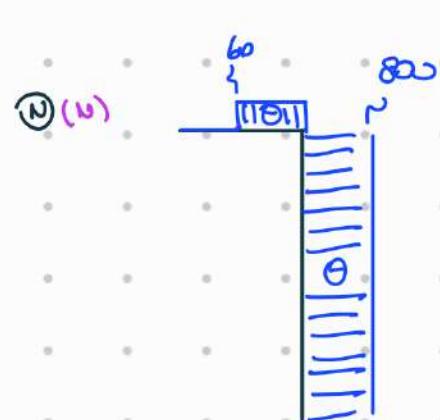
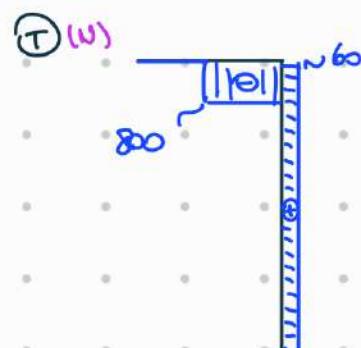
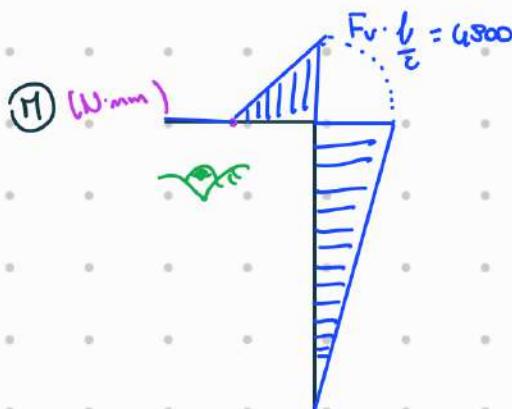
approssimazione
aceptabile

equilibrio orizzontale

$$F_v + R_v = 0 \Rightarrow R_v = -F_v = -60 \text{ N}$$

$$\text{coll. effetti} \rightarrow \mu = \frac{|F_v|}{F_v} = 0,075$$

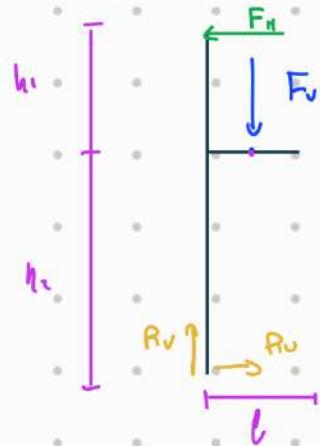
Diagrammi di sollecitazione (all'esame niente formule, solo numeri!)



Stampelle canadese



modellizzazione



Dati:

$$B_N = 800 \text{ N} \quad N \quad F_V = 800 \text{ N}$$

$$l = 12 \text{ cm}$$

$$h_2 = 80 \text{ cm}$$

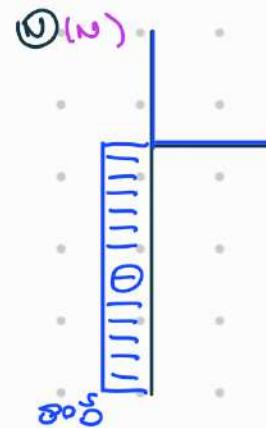
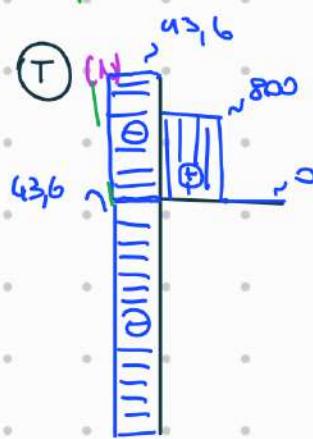
$$h_1 = 30 \text{ cm}$$

$$\text{el. verticale} \quad R_V = F_V$$

$$\text{el. momenti} \quad \text{spese} \rightarrow$$

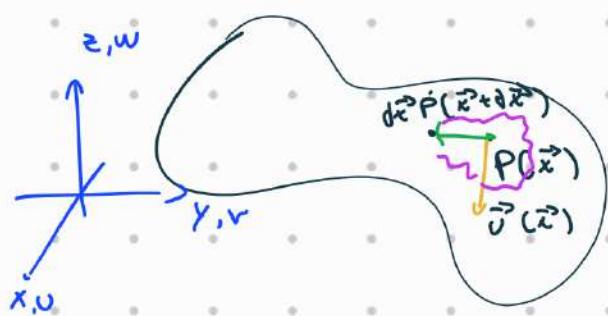
$$-F_V \frac{l}{2} + (h_1 \cdot h_2) F_H = 0 \Rightarrow F_H = \frac{l}{z(h_1+h_2)} F_V = 43,6 \text{ N}$$

diagrammi:



Analisi delle deformazioni

CONTINUIO TRIDIMENSIONALE



Come si muovono i punti attorno a P?

$$\vec{U}(\vec{x} + d\vec{x}) = \vec{U}(\vec{x}) + \frac{\partial \vec{U}}{\partial \vec{x}} d\vec{x}$$

gradiente

espansione di Taylor
del primo ordine → movimenti
PICCOLI

$$= \vec{U}(\vec{x}) + \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} d\vec{x}$$

$$d\vec{x} = \vec{U}(\vec{x}) + \nabla \vec{U} \cdot d\vec{x}$$

osservo che

$$\vec{U}(\vec{x} + d\vec{x}) - U(\vec{x}) = \nabla \vec{U} \cdot d\vec{x}$$

$$\nabla \vec{U} = \frac{1}{2} (\nabla \vec{U} - \nabla^T \vec{U}) + \frac{1}{2} (\nabla \vec{U} + \nabla^T \vec{U})$$

parte antisimmetrica
||

parte simmetrica
||

$\bar{\omega}$
tensori delle
ROTAZIONI infinitesime

$\bar{\epsilon}$
tensori delle
DEFORMAZIONI infinitesime

quindi

$$U(\vec{x} + d\vec{x}) = \vec{U}(\vec{x}) + \bar{\omega} d\vec{x} + \bar{\epsilon} d\vec{x}$$

$$\text{II3} = \frac{1}{2} \begin{bmatrix} 0 & \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} & \frac{\partial v}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} & 0 & \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial x} - \frac{\partial v}{\partial z} & \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} & 0 \end{bmatrix}$$

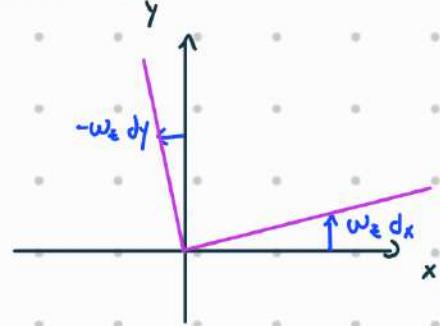
$$= \frac{1}{2} \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix}$$

matrice J₁
rotazione infinitesima

esempio:

$$\text{II3} = \frac{1}{2} \begin{bmatrix} 0 & -w_z & 0 \\ w_z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

rotazione
att. ass. z



$$\text{II3} \cdot \vec{dx} = \begin{bmatrix} 0 & -w_z & 0 \\ w_z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} -w_z dy \\ w_z dx \\ 0 \end{bmatrix}$$

$$\text{II4} = \frac{1}{2} \begin{bmatrix} z \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} & z \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} & \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} & z \frac{\partial w}{\partial x} \end{bmatrix}$$

simmetrico

NON sono
DILATAZIONI

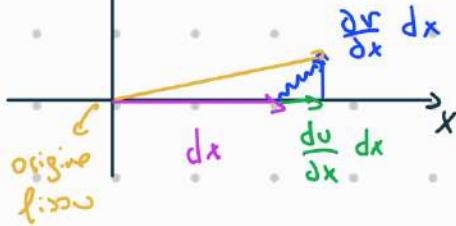
$$= \begin{bmatrix} \epsilon_x & \frac{1}{2} \gamma_{yx} & \frac{1}{2} \gamma_{zx} \\ \frac{1}{2} \gamma_{xy} & \epsilon_y & \frac{1}{2} \gamma_{zy} \\ \frac{1}{2} \gamma_{xz} & \frac{1}{2} \gamma_{yz} & \epsilon_z \end{bmatrix}$$

ϵ_i : componenti DIRETTE /
 ϵ_{03} : componenti di dilatazione

γ_{ij} : componenti di scorrimenti
"ds":

esempio: componente diretta

$$d\vec{t} = (dx, 0, 0)$$



$$\begin{aligned} |\vec{v}| &= \sqrt{\left(\frac{\partial v}{\partial x} dx\right)^2 + \left(\frac{\partial u}{\partial x} dx\right)^2} \\ &= \sqrt{\left(\frac{\partial v}{\partial x}\right)^2 dx^2 + \left(\frac{\partial u}{\partial x}\right)^2 dx^2} \\ &= \sqrt{\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2} dx \end{aligned}$$

$$f\left(\frac{\partial u}{\partial x}\right) = \int_0^1 1 + \frac{\partial u}{\partial x} dt$$

$$f'\left(\frac{\partial u}{\partial x}\right) = \frac{1}{\int_0^1 1 + \frac{\partial u}{\partial x} dt} \Rightarrow \begin{aligned} f(0) &= 1 \\ f'(0) &= 1 \end{aligned}$$

approssimazione

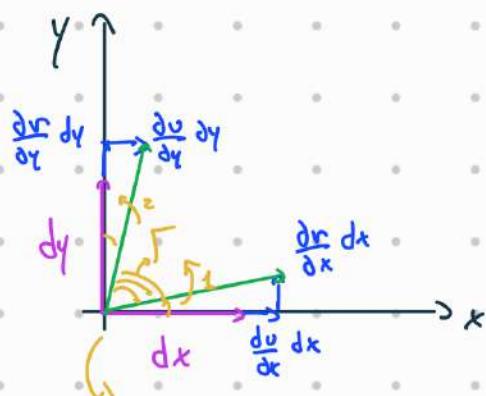
$$f\left(\frac{\partial u}{\partial x}\right) = f(0) + f'(0) \frac{\partial u}{\partial x} = 1 + \frac{\partial u}{\partial x}$$

può essere

$$|\vec{v}| = |dx| \left(1 + \frac{\partial u}{\partial x}\right) \Rightarrow \frac{\partial u}{\partial x} = \frac{|\vec{v}| - |dx|}{|dx|}$$

variazione relativa
di lunghezze infinitesimali

esempio: scorrimento angolare



$$J_1 = \frac{\frac{\partial r}{\partial x} dx}{dx + \frac{\partial u}{\partial x} dx} = \frac{\frac{\partial r}{\partial x}}{1 + \frac{\partial u}{\partial x}}$$

$$J_2 = \frac{\frac{\partial u}{\partial y} dy}{dy + \frac{\partial v}{\partial y} dy} = \frac{\frac{\partial u}{\partial y}}{1 + \frac{\partial v}{\partial y}}$$

$$\frac{\pi}{2} - J_2 - J_1 = \frac{\pi}{2} - \underbrace{\left(\frac{\partial r}{\partial x} + \frac{\partial u}{\partial y}\right)}_{J_{xy}}$$

tangente $\rightarrow \tan J_i \approx J_i$

J_{xy}

Quel è l'utilità del tensore $\bar{\epsilon}$?

- Considero versore generico \vec{n}

$$\vec{n} \cdot \bar{\epsilon} \cdot \vec{n} = [n_1 \ n_2 \ n_3] \begin{bmatrix} \epsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \epsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \epsilon_z \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

componente j: dilatazione
nella direzione \vec{n}

- Considero versori ortogonali: fra loro $\vec{m} \cdot \vec{n} = 0$

allora $\frac{1}{2}\gamma_{mn} = \vec{m} \cdot \bar{\epsilon} \cdot \vec{n}$

- Identificazione delle DIREZIONI PRINCIPALI, cioè:

\vec{n} versore generico t.e.

$$\bar{\epsilon} \cdot \vec{n} = \epsilon_n \cdot \vec{n}$$

↪ scalare

$$(\bar{\epsilon} - \epsilon_n I) \vec{n} = \vec{0} \rightarrow \text{possibili tipici stati}$$

degli autovettori e autovettori

gli autovettori $\epsilon_1, \epsilon_2, \epsilon_3$ sono detti deformazioni PRINCIPALI

↪ convenzione $\epsilon_1 > \epsilon_2 > \epsilon_3 \rightarrow$ valori massimi/minimi possibili
di deformazione

oss: Dato che $\bar{\epsilon}$ è simmetrico

↪ è sicuramente diagonalizzabile

e gli autovettori sono ORTOGONALI

OSS: $\vec{e} \neq \vec{0} \Leftrightarrow \det(\bar{\epsilon} - \epsilon_n I) = 0$

$$\text{cioè } \epsilon_n^3 - I_1 \epsilon_n^2 + I_2 \epsilon_n = 0$$

Def: I_1, I_2, I_3 sono det: **INVARIANTI** delle deformazione

$$I_1 = \text{tr}(\bar{\epsilon}) = \epsilon_x + \epsilon_y + \epsilon_z$$

$$I_2 = \frac{1}{2} (I^2 - \text{tr}(\bar{\epsilon}^2)) = \epsilon_x \epsilon_y + \epsilon_y \epsilon_z + \epsilon_z \epsilon_x - \frac{1}{4} (\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2)$$

$$I_3 = \det(\bar{\epsilon})$$

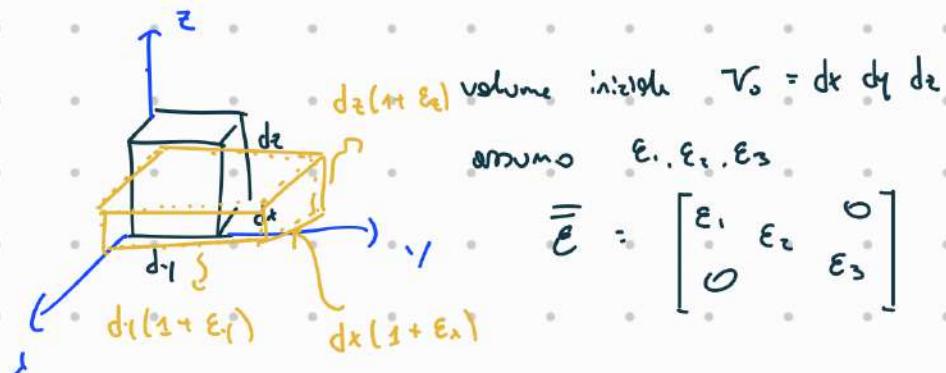
nelle direzioni principali valgono:

$$I_1 = \epsilon_1 + \epsilon_2 + \epsilon_3$$

$$I_2 = \epsilon_1 \epsilon_2 + \epsilon_2 \epsilon_3 + \epsilon_3 \epsilon_1$$

$$I_3 = \epsilon_1 \epsilon_2 \epsilon_3$$

Interpretazione geometrica del primo invarianto



$$\left. \begin{aligned} \lambda_x &= 1 + \epsilon_x \\ \lambda_y &= 1 + \epsilon_y \\ \lambda_z &= 1 + \epsilon_z \end{aligned} \right\} \text{DILATAZIONI}$$

$$\text{volume finale } V = (1 + \epsilon_x) dx \cdot (1 + \epsilon_y) dy \cdot (1 + \epsilon_z) dz$$

$$\rightarrow V = dx dy dz \cdot (1 + \epsilon_x) (1 + \epsilon_y) (1 + \epsilon_z)$$

$$= V_0 (1 + \epsilon_x + \epsilon_y + \epsilon_z + \cancel{\epsilon_x \epsilon_y + \epsilon_y \epsilon_z + \epsilon_z \epsilon_x + \epsilon_x \epsilon_y \epsilon_z})$$

$$\approx V_0 (1 + \epsilon_x + \epsilon_y + \epsilon_z)$$

$$= V_0 (1 + I)$$

$$\Rightarrow I_1 = \frac{V - V_0}{V_0}$$

variazione del volume

\hookrightarrow deformazioni di ordine superiore \rightarrow trascurabili

$$\epsilon_m = \frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{3} = \frac{I_1}{3}$$

deformazione MEDIA

Componente deviatorica del tensore di deformazione

$$\bar{\bar{\epsilon}}_{dev} = \bar{\bar{\epsilon}} - \epsilon_m \bar{\bar{I}} \quad \text{tensore identità}$$

$$\downarrow \quad \text{tr}(\bar{\bar{\epsilon}}_{dev}) = \text{tr}(\bar{\bar{\epsilon}}) - 3\epsilon_m = 0$$

esercizi:

Calcolo delle parte deviatoria

$$\bar{\bar{\epsilon}} = \begin{bmatrix} 0,01 & 0,02 & 0 \\ 0,02 & 0,01 & 0 \\ 0 & 0 & 0,015 \end{bmatrix}$$

$$I_1 = \epsilon_x + \epsilon_y + \epsilon_z = 0,045$$

$$\bar{\bar{\epsilon}}_{dev} = \bar{\bar{\epsilon}} - \frac{I_1}{3} \bar{\bar{I}} = \begin{bmatrix} 0,01 & 0,02 & 0 \\ 0,02 & 0,01 & 0 \\ 0 & 0 & 0,015 \end{bmatrix} - \begin{bmatrix} 0,015 & 0 & 0 \\ 0 & 0,015 & 0 \\ 0 & 0 & 0,015 \end{bmatrix}$$

$$= \begin{bmatrix} 0,005 & 0,02 & 0 \\ 0,01 & -0,005 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{tr}(\bar{\bar{\epsilon}}_{dev}) = 0$$

oss: Questa teoria è applicata LOCALMENTE (in un intorno),

se la deformazione è ORTOGENEA si può applicare GLOBALMENTE.

Calcolo delle deformazioni principali e delle direzioni principali:

$$\bar{\bar{\epsilon}} = \begin{bmatrix} 0 & 0,02 & 0 \\ 0,02 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\hookrightarrow \epsilon_3$ è sicuramente
direzione principale (non ha scostamenti)

$$\det(\bar{\bar{\epsilon}} - \epsilon \bar{\bar{I}}) = \begin{bmatrix} -s & 0,02 & 0 \\ 0,02 & -s & 0 \\ 0 & 0 & s \end{bmatrix} = -s(s^2 - 0,04s^2) = 0$$

$$\epsilon_1 = 0,02 \quad \epsilon_2 = 0 \quad \epsilon_3 = -0,02 \quad \text{deformazioni principali}$$

Calcolo delle direzioni:

convenzionalmente
 $\epsilon_1 \geq \epsilon_2 \geq \epsilon_3$

$$(\bar{\bar{\epsilon}} - \epsilon_1 \bar{\bar{I}}) \vec{n}_1 = \vec{0}$$

$$\begin{bmatrix} 0 & 0,02 & 0 \\ 0,02 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \vec{n}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow \text{normalizzato}$$

$$(\bar{\bar{\epsilon}} - \epsilon_2 \bar{\bar{I}}) \vec{n}_2 = \vec{0}$$

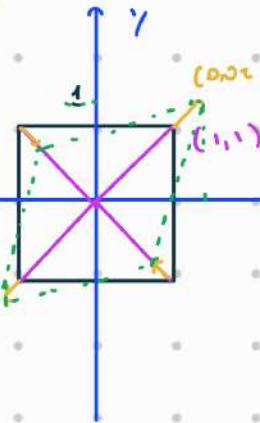
$$\begin{bmatrix} -0,02 & 0,02 & 0 \\ 0,02 & -0,02 & 0 \\ 0 & 0 & 0,02 \end{bmatrix} \rightsquigarrow \begin{cases} \alpha_1 + \alpha_2 = 0 \\ \alpha_1 - \alpha_2 = 0 \\ \alpha_3 = 0 \end{cases} \quad \begin{cases} \alpha_1 = \alpha^* \\ \alpha_2 = \alpha^* \\ \alpha_3 = 0 \end{cases} \quad \vec{n}_2 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}$$

$$(\bar{\bar{\epsilon}} - \epsilon_3 \bar{\bar{I}}) \vec{n}_3 = \vec{0}$$

$$\begin{bmatrix} 0,02 & 0,02 & 0 \\ 0,02 & 0,02 & 0 \\ 0 & 0 & 0,02 \end{bmatrix} \rightsquigarrow \begin{cases} \alpha_1 = -\alpha_2 \\ \alpha_3 = 0 \end{cases} \quad \vec{n}_3 = \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \\ 0 \end{bmatrix}$$

Oss: $\vec{n}_1 \quad \vec{n}_2 \quad \vec{n}_3$ sono ORTOGONALI

graficamente



(cos 45, sin 45)

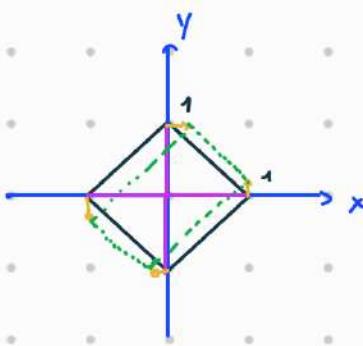
lavoriamo nel piano

$$\begin{bmatrix} 0 & 0,5\sqrt{2} \\ 0,5\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0,5\sqrt{2} \\ 0,5\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0,5\sqrt{2} \\ 0,5\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0,5\sqrt{2} \\ 0,5\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0,5\sqrt{2} \\ 0,5\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -0,5\sqrt{2} \\ 0,5\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0,5\sqrt{2} \\ 0,5\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0,5\sqrt{2} \\ -0,5\sqrt{2} \end{bmatrix}$$



$$\begin{bmatrix} 0 & 0,5\sqrt{2} \\ 0,5\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0,5\sqrt{2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0,5\sqrt{2} \\ 0,5\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0,5\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0,5\sqrt{2} \\ 0,5\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0,5\sqrt{2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0,5\sqrt{2} \\ 0,5\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -0,5\sqrt{2} \end{bmatrix}$$

estensimetrici

oss: prendendo

$$\vec{n} = [\cos \varphi, \sin \varphi, 0]$$

allora

$$\vec{n} \cdot \vec{e} \cdot \vec{n}$$

$$= [\cos \varphi, \sin \varphi, 0] \begin{bmatrix} \epsilon_x + \frac{1}{2} \gamma_{xy} & \frac{1}{2} \gamma_{xz} \\ \frac{1}{2} \gamma_{yx} & \epsilon_y + \frac{1}{2} \gamma_{yz} \\ \frac{1}{2} \gamma_{zx} & \frac{1}{2} \gamma_{zy} \end{bmatrix} \begin{bmatrix} \cos \varphi \\ \sin \varphi \\ 0 \end{bmatrix}$$

$$= [\cos \varphi, \sin \varphi, 0] \begin{bmatrix} \epsilon_x \cos \varphi + \frac{1}{2} \gamma_{xy} \sin \varphi \\ \frac{1}{2} \gamma_{yx} \cos \varphi + \epsilon_y \sin \varphi \\ \frac{1}{2} \gamma_{zx} \cos \varphi + \frac{1}{2} \gamma_{zy} \sin \varphi \end{bmatrix}$$

$$= \epsilon_x \cos^2 \varphi + \gamma_{xy} \cos \varphi \sin \varphi + \epsilon_y \sin^2 \varphi$$

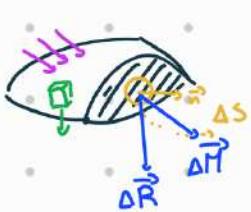
Teoria delle tensioni



assumo che sia
in EQUILIBRIO

- equilibrio globale
- equilibrio locale

prendo una sezione



considero

$$\lim_{\Delta S \rightarrow 0} \frac{\Delta R}{\Delta S} = \vec{t}$$

vettore di
TENSIONE

$$[\frac{F}{L^2}]$$

$$\lim_{\Delta S \rightarrow 0} \frac{\Delta \vec{n}}{\Delta S} = \vec{m}$$

CONTINUITÀ
CAUCHY

$$= \vec{0}$$

CONTINUITÀ
COSSERAT

molto complesso,
molto raro

Oss:

unità di misura

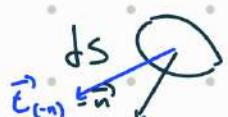
$$\begin{array}{l} \text{Lunghezza} \\ \text{Forza} \end{array} \quad \begin{array}{l} \text{mm} \\ \text{N} \end{array} \quad \rightarrow 1 \frac{\text{N}}{\text{mm}^2} = 1 \text{ MPa}$$

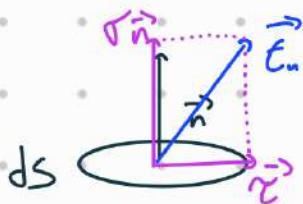
$$\begin{array}{l} \text{Lunghezza} \\ \text{Forza} \end{array} \quad \begin{array}{l} \mu\text{m} \\ \mu\text{N} \end{array} \quad \rightarrow 1 \frac{\mu\text{N}}{\mu\text{m}^2} = 1 \text{ MPa}$$

Lemma di Cauchy

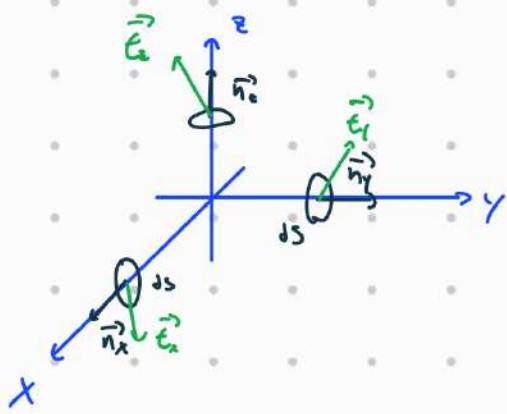


$$\vec{t}_n = -\vec{t}_{(n)}$$





$T_n \vec{n}$ componente normale del vettore \vec{E} : \vec{T}_n
 \vec{T}_t componente tangenziale del vettore \vec{E} : \vec{T}_t



: vettori: dati: tagli: sono:
 \vec{T}_x \vec{T}_y \vec{T}_z

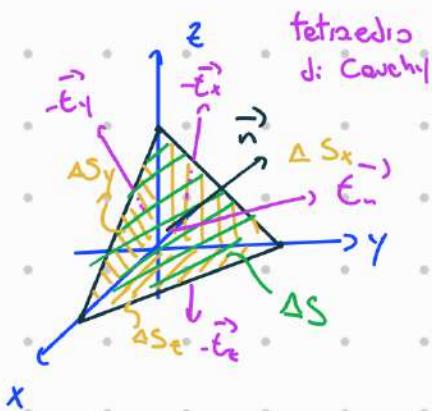
$$\begin{aligned} \vec{\sigma} &= \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \\ &= \begin{bmatrix} \tau_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_z \end{bmatrix} \end{aligned}$$

tensore $\vec{\sigma}$: tensione
di Cauchy

componenti
NORMALI

componenti
TANGENZIALI

Proprietà



$$\vec{n} = (\alpha, \beta, \gamma)$$

$$\Delta S_x = \Delta S \cdot \alpha$$

$$\Delta S_y = \Delta S \cdot \beta$$

$$\Delta S_z = \Delta S \cdot \gamma$$

forze di
volume

Calcolo l'equilibrio

$$\vec{E}_n \Delta S - \vec{E}_x \Delta S_x - \vec{E}_y \Delta S_y - \vec{E}_z \Delta S_z + \overbrace{\vec{b}}^{\uparrow} \Delta V = \vec{0}$$

$$\vec{E}_n - \vec{E}_x \alpha - \vec{E}_y \beta - \vec{E}_z \gamma + \overbrace{\frac{\vec{b}}{\Delta S}}^{\uparrow} \Delta V = \vec{0}$$

passando al limite $\Delta S \rightarrow 0$

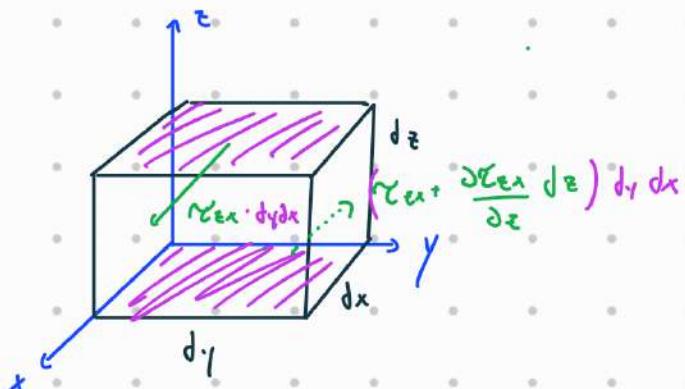
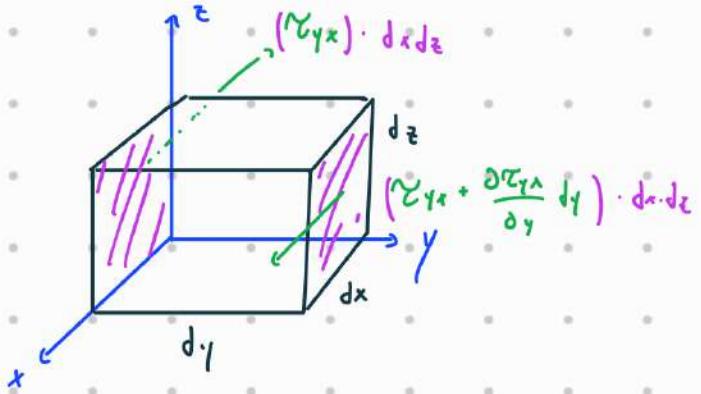
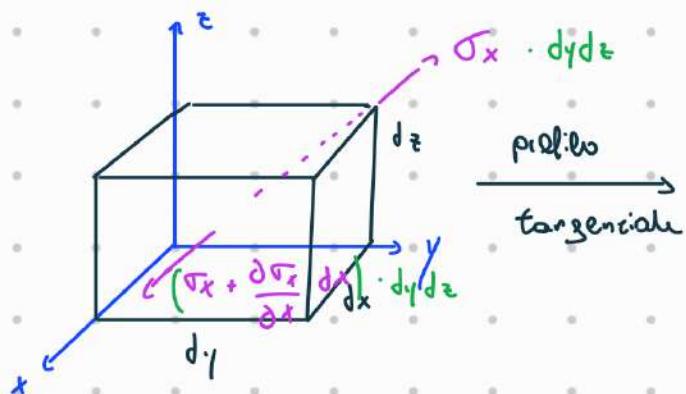
$$\vec{E}_n = \vec{E}_x \alpha + \vec{E}_y \beta + \vec{E}_z \gamma$$

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{xy} \\ \epsilon_{xz} \end{bmatrix} \alpha + \begin{bmatrix} \epsilon_{yy} \\ \epsilon_{yz} \\ \epsilon_{zx} \end{bmatrix} \beta + \begin{bmatrix} \epsilon_{zz} \\ \epsilon_{zy} \\ \epsilon_{xz} \end{bmatrix} \gamma$$

$$\vec{\epsilon}_n = \bar{\sigma} \cdot \vec{n}$$

formule di
Cauchy

Equazioni indeterminate dell'equilibrio



$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial z} dz - \sigma_x \right) dy dz + \left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y} dy - \sigma_{yx} \right) dx dy + \left(\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z} dz - \sigma_{zx} \right) dx dz + b_x dy dz = 0$$

diventa

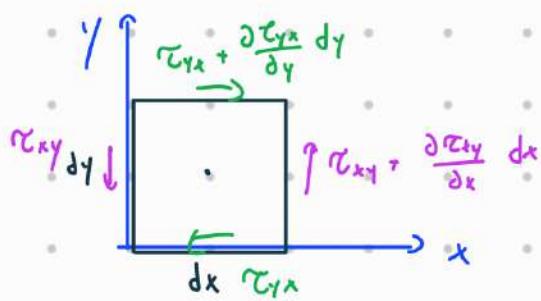
$$\frac{\partial \sigma_x}{\partial z} dz + \frac{\partial \sigma_{yx}}{\partial y} dy + \frac{\partial \sigma_{zx}}{\partial z} dz + b_x \cdot dz = 0$$

iterando per gli altri: otto: diventa

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + b_x &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + b_y &= 0 \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + b_z &= 0 \end{aligned}$$

$$\Rightarrow \operatorname{div}(\bar{\sigma}) + \vec{b} = \vec{0}$$

nel piano



$$\begin{aligned} & \tau_{xy} dy \quad \text{at top edge} \\ & \tau_{yx} dx \quad \text{at bottom edge} \\ & \tau_{xy} + \frac{\partial \tau_{yx}}{\partial y} dy \quad \text{at right edge} \\ & \tau_{xz} + \frac{\partial \tau_{xy}}{\partial x} dx \quad \text{at left edge} \end{aligned}$$

$$\begin{aligned} & \tau_{xy} dy dz \cdot dx + \frac{\partial \tau_{xy}}{\partial x} dx \cdot dy dz \cdot \frac{dx}{c} - \\ & - \tau_{yx} dx dz \cdot dy - \frac{\partial \tau_{yx}}{\partial y} dy \cdot dz \cdot \frac{dy}{c} = 0 \\ \Rightarrow \quad & \tau_{xy} = \tau_{yx} \end{aligned}$$

ordine superiore

$$\text{cioè } \tau_{ij} = \tau_{ji}$$

quindi:

Il tensore di tensione di Cauchy $\bar{\sigma}$ è simmetrico

Esiste \vec{n} t.c.

$$\bar{\sigma} \cdot \vec{n} = \sigma \vec{n}$$

$$(\bar{\sigma} - \sigma \vec{I}) \vec{n} = \vec{0}$$

scalare

} Diagonalizzare da
algebra lineare

$$\sigma^3 - I_1 \sigma^2 - I_2 \sigma - I_3 = 0 \quad \text{polinomio caratteristico}$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x - \sigma_x^2 - \sigma_y^2 - \sigma_z^2$$

$$I_3 = \det(\bar{\sigma})$$

I_1, I_2, I_3 sono gli INVARIANTI principali di $\bar{\sigma}$

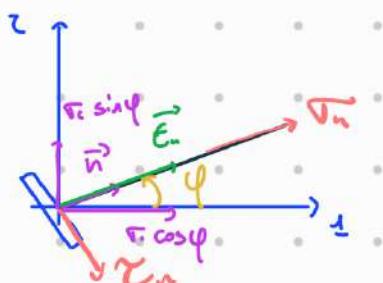
gli AUTVALORI vengono chiamati TENSIONI PRINCIPALI

oss: Principali stati di tensione

- Se $\sigma_1, \sigma_2, \sigma_3 \neq 0 \Rightarrow$ stato J: TENSIONE TRIASIALE
- Se $\sigma_1 = \sigma_2 = \sigma_3 = p \neq 0 \Rightarrow$ stato J: PRESSIONE IDROSTATICA
↳ esempio oggetto immerso
- Se $\sigma_1 \neq \sigma_2, \sigma_3 = 0 \Rightarrow$ stato J: TENSIONE BIASSIALE
- Se $\sigma_1 = \sigma_2 \neq 0, \sigma_3 = 0 \Rightarrow$ stato J: TENSIONE EQUI-BIASSIALE
- Se $\sigma_1 \neq 0, \sigma_2 = \sigma_3 = 0 \Rightarrow$ stato J: TENSIONE MONOASSIALE

Cerchio di Mohr interpretazione grafica delle tensioni

assumo stato di tensione biaxiale $\sigma_1, \sigma_2 \neq 0$



$$\vec{n} = (\cos\varphi, \sin\varphi, 0)$$

$$\vec{t}_n = \vec{\tau} \cdot \vec{n}$$

quindi

$$\vec{t}_n = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_1 \cos\varphi \\ \sigma_2 \sin\varphi \\ 0 \end{bmatrix}$$

nel sistema J: riferimento $\sigma_n - \tau_n$

$$\left\{ \begin{array}{l} \sigma_n = \sigma_1 \cos^2\varphi + \sigma_2 \sin^2\varphi \\ \tau_n = \sigma_1 \cos\varphi \sin\varphi - \sigma_2 \cos\varphi \sin\varphi \end{array} \right. \quad \left\{ \begin{array}{l} \sigma_n = \sigma_1 \cos^2\varphi + \sigma_2 \sin^2\varphi \\ \tau_n = \frac{\sigma_1 - \sigma_2}{2} \sin 2\varphi \end{array} \right.$$

cerco punti di massimi

$$\begin{aligned} \frac{\partial \sigma_n}{\partial \varphi} &= -2\sigma_1 \cos\varphi \sin\varphi + 2\sigma_2 \sin\varphi \cos\varphi \\ &= (\sigma_2 - \sigma_1) \sin 2\varphi = 0 \Rightarrow \varphi = 0, \frac{\pi}{2} \text{ ass: PRINCIPALI} \end{aligned}$$

$$\frac{\partial \tau_n}{\partial \varphi} = (\sigma_i - \sigma_c) \cos 2\varphi = 0 \Rightarrow \varphi = \frac{\pi}{4} \text{ rad/min. d: tensione tangenziale}$$

formule trigonometriche

$$\cos^2 \varphi = \cos^2 \varphi \cdot \sin^2 \varphi = 1 - \sin^2 \varphi = \cos^2 \varphi - 1$$

$$\left\{ \begin{array}{l} \sin^2 \varphi = \frac{1 - \cos 2\varphi}{2} \\ \cos^2 \varphi = \frac{1 + \cos 2\varphi}{2} \end{array} \right.$$

quindi:

$$\left\{ \begin{array}{l} \sigma_n = \sigma_i \frac{1 + \cos 2\varphi}{2} + \sigma_c \frac{1 - \cos 2\varphi}{2} \\ \tau_n = \frac{\sigma_i - \sigma_c}{2} \sin 2\varphi \end{array} \right.$$

$$\left\{ \begin{array}{l} \sigma_n = \frac{\sigma_i + \sigma_c}{2} + \frac{\sigma_i - \sigma_c}{2} \cos 2\varphi \\ \tau_n = \frac{\sigma_i - \sigma_c}{2} \sin 2\varphi \end{array} \right.$$

definisco

$$\sigma_i > \sigma_c$$

$$C = \frac{\sigma_i + \sigma_c}{2}$$

$$R = \frac{\sigma_i - \sigma_c}{2}$$

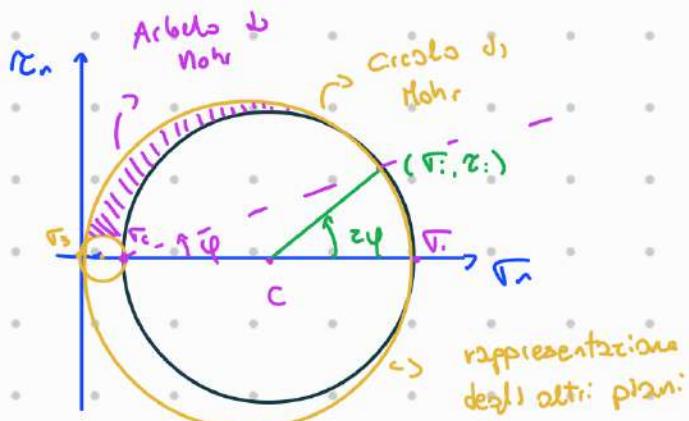
quindi:

$$\left\{ \begin{array}{l} \sigma_n = C + R \cos 2\varphi \\ \tau_n = R \sin 2\varphi \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \sigma_n - C = R \cos 2\varphi \\ \tau_n = R \sin 2\varphi \end{array} \right.$$

$$\Rightarrow (\sigma_n - C)^2 + \tau_n^2 = R^2$$

circconferenza:
 : centro C
 : raggio R

Rappresentazione grafica del cerchio di Mohr



$\sigma_n - C$ è detto

plano delle tensioni di Mohr

Calcolo delle tensioni: principali e costituzione del cerchio.

Stato di: tensione biaxiale

$$\bar{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\det(\bar{\sigma} - s\bar{I}) = 0 \rightarrow \det \begin{bmatrix} \sigma_x - s & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_y - s & 0 \\ 0 & 0 & -s \end{bmatrix}$$

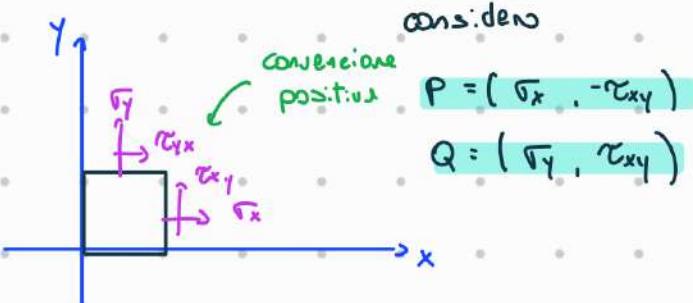
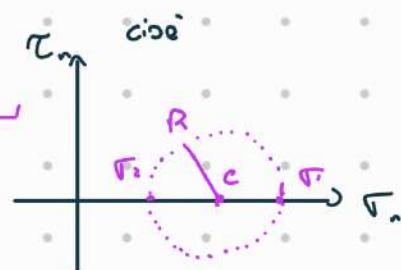
$$= -s((\sigma_x - s)(\sigma_y - s) - \tau_{xy}^2) = 0 \rightarrow \sigma_3 = 0$$

$$= s^2 - (\sigma_x + \sigma_y)s + \sigma_x\sigma_y - \tau_{xy}^2 = 0$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y \pm \sqrt{(\sigma_x + \sigma_y)^2 - 4\sigma_x\sigma_y + 4\tau_{xy}^2}}{2}$$

$$= \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

$$= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

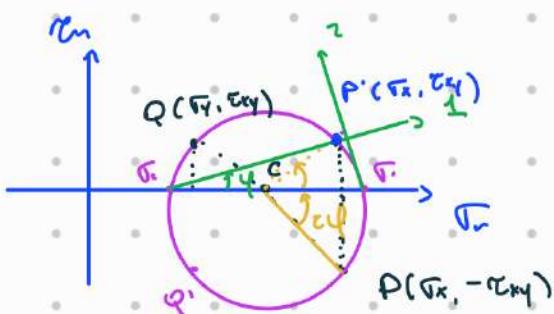


considen

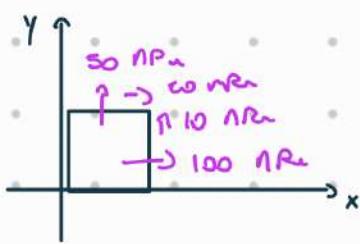
$$P = (\sigma_x, -\tau_{xy})$$

$$Q = (\sigma_y, \tau_{xy})$$

Costruendo il cerchio di Mohr diventa:



esercizio



$$\bar{\bar{F}} = \begin{bmatrix} 100 & \omega & 0 \\ \omega & 50 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\det \begin{bmatrix} 100 - s & \omega & 0 \\ \omega & 50 - s & 0 \\ 0 & 0 & -s \end{bmatrix} = -s((100 - s)(50 - s) - \omega^2)$$

$$= s(s^2 - 150s + 5000 - 400) = s(s^2 - 150s - 4600) = 0$$

gli autovettori sono

$$\tau_3 = 0$$

$$\tau_{1,2} = 75 \pm \sqrt{75^2 - 4600} = \begin{cases} \tau_1 = 107 \text{ NPa} \\ \tau_2 = 93 \text{ NPa} \end{cases}$$

le direzioni principali sono

$$\vec{n}_3 = (0, 0, 1)$$

$$(\bar{\bar{F}} - \tau_1 \bar{\bar{I}}) \vec{n}_1 = \vec{0}$$

$$\begin{bmatrix} 100 - 107 & \omega & 0 \\ \omega & 50 - 107 & 0 \\ 0 & 0 & -107 \end{bmatrix} = \begin{bmatrix} -7 & \omega & 0 \\ \omega & -57 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \vec{0}$$

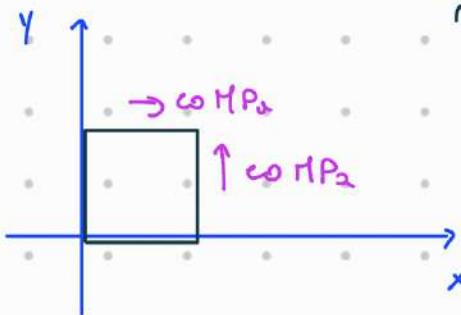
$$\begin{cases} -7x + 10y = 0 \\ \omega x - 57y = 0 \\ z = 0 \end{cases} \quad \begin{cases} x = \frac{10}{7}y \\ x = \frac{57}{\omega}y \\ z = 0 \end{cases} \quad \begin{cases} x = \frac{80}{7}y \\ y = y^* \\ z = 0 \end{cases}$$

$$\vec{n}_1 = \frac{1}{\sqrt{302}} \left(\frac{\omega}{7}, 1, 0 \right) = (0, 0.943, 0.331)$$

$$\varphi = \arctan \left(\frac{0.331}{0.943} \right) \approx 19^\circ, 3$$

ejercicio Solo sforci di taglio

$$\tau_{xy} = \text{co MP}_2$$



$$\bar{\sigma} = \begin{bmatrix} 0 & z_0 & 0 \\ z_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\det \begin{bmatrix} -s & z_0 & 0 \\ z_0 & -s & 0 \\ 0 & 0 & -s \end{bmatrix} = -s(s^2 - 400) = s(s+20)(s-20) = 0$$

$$\tau_1 = z_0 \quad \tau_2 = -z_0 \quad \tau_3 = 0$$

le direzioni principali sono:

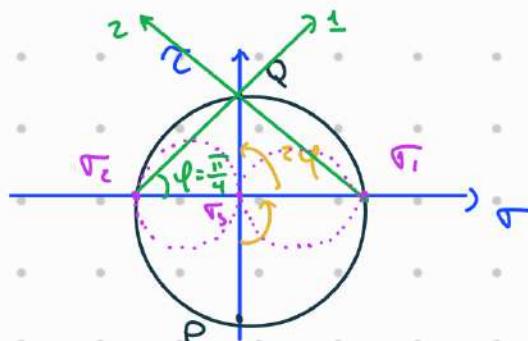
$$\tau_3: \vec{n}_3 = (0, 0, 1)$$

$$\tau_1: \text{Ker} \begin{bmatrix} -z_0 & z_0 & 0 \\ z_0 & -z_0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \text{Ker} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{n}_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right)$$

$$\tau_2: \text{Ker} \begin{bmatrix} z_0 & 0 & 0 \\ 0 & z_0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \text{Ker} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{n}_2 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0 \right)$$

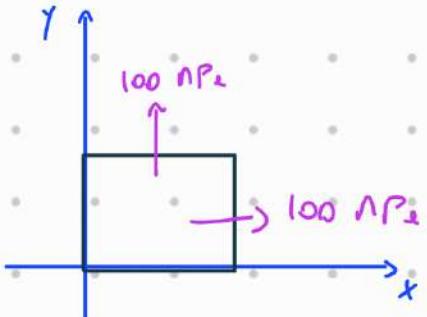
$$P = (\tau_{x1}, -\tau_{x1}) \rightarrow (0, -z_0)$$

$$Q = (\tau_1, \tau_{xy}) \rightarrow (0, z_0)$$



$$\text{oss: } \gamma_{\max} = \max \left\{ \frac{1}{2} |\tau_i - \tau_j| \right\} ; i, j = 1, 2, 3$$

esercizio



$$\bar{\bar{\tau}} = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

↪ osserviamo che è già diagonalizzato

Direzioni principali

$$\tau_3: \text{asse } z \quad \vec{n}_3 = (0, 0, 1)$$

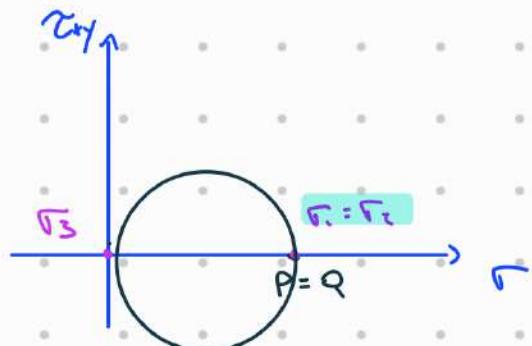
$\tau_1, \tau_2:$

$$K_{\tau\tau} = \begin{bmatrix} 100-100 & 0 & 0 \\ 0 & 100-100 & 0 \\ 0 & 0 & -100 \end{bmatrix}$$

$$\sim \begin{cases} \tau_1 = \tau_2 \\ \tau_2 = \tau_1 \\ \tau_3 = 0 \end{cases} \rightarrow \text{Tutto il piano } xy \text{ è disponibile}$$

$$P = (\tau_x, -\tau_{xy}) \rightarrow (100, 0)$$

$$Q = (\tau_y, \tau_{xy}) \rightarrow (100, 0)$$



def: Tensione media / Parte idrostatica delle tensioni

$$\bar{\bar{\tau}} = \begin{bmatrix} \tau_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_z \end{bmatrix}$$

$$I_1 = \tau_x + \tau_y + \tau_z = \tau_1 + \tau_2 + \tau_3$$

allora

$$\sqrt{\tau_m} = \frac{I_1}{3} = \frac{\tau_x + \tau_y + \tau_z}{3} = \frac{\tau_1 + \tau_2 + \tau_3}{3}$$

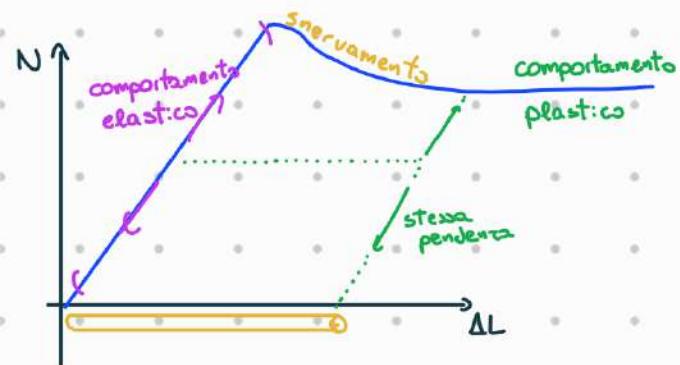
def: Parte deviatorica del tensore di tensione

$$\bar{\bar{\tau}}_{dev} = \bar{\bar{\tau}} - \sqrt{\tau_m} \bar{\bar{I}}$$

oss: $\bar{\bar{\tau}}_{dev}$ e $\bar{\bar{\epsilon}}_{dev}$ sono CORRELATI

Modelli costitutivi

prove di trazione



L'andamento del grafico dipende da:

- proprietà intrinseche del provino (tipologia materiale)
- geometria del provino

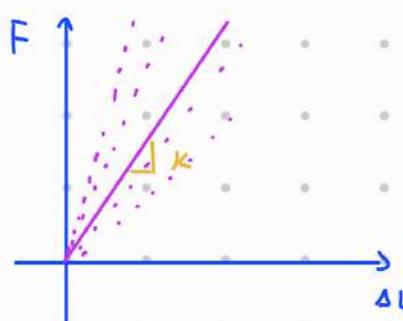
Tensione NORMATIVA

$$P = \frac{F}{A_0} \rightarrow \text{Area trasversale iniziale}$$

Deformazione NORMATIVA

$$\epsilon = \frac{\Delta l}{l_0}$$

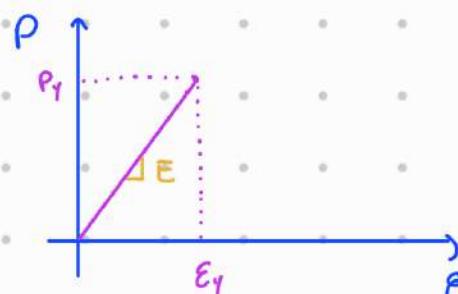
def:



$$k = \frac{F}{\Delta L}$$

dipende dalla geometria

RIGIDEZZA assiale del provino



$$E = \frac{P}{\epsilon}$$

MODULO DI YOUNG
modulo elastico longitudinale

$$\hookrightarrow P = E \epsilon \quad \text{Legge di Hooke}$$

caratteristica intrinseca del materiale

$$\bar{E} (1 P_A)$$

200 000

140 000

70 000

10 000 - 20 000

esempio

Leghe in ACCIAIO

Leghe in TITANIO

Leghe in ALLUMINIO

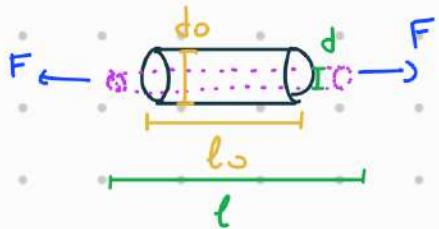
Legno

Elastomeri

PNEA

1

3000



tensione di Cauchy $\tau = \frac{F}{A}$ dove $A = \frac{\pi}{4} d^2$ area trasversale
se siamo in CAMPO ELASTICO vale $A \approx A_0$ $\tau \approx p$

$$\epsilon_t = \frac{\Delta l}{l_0} = \frac{l - l_0}{l_0}$$

deformazione LATERALE dovuta alla contrazione laterale

misure sperimentali

$$\epsilon_l = \frac{\Delta l}{l}$$

Sono DISCORDI

deformazione LONGITUDINALE

$$\nu = -\frac{\epsilon_t}{\epsilon_l}$$

coefficiente di Poisson

caratteristica intrinseca del materiale

esempi

Leghe in ACCIAIO 0,3

Leghe in TITANIO 0,33

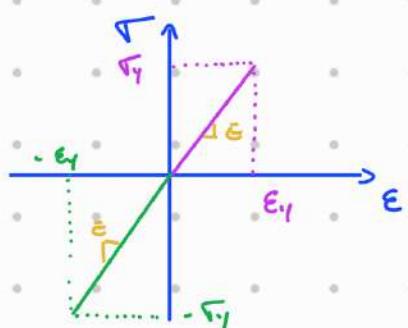
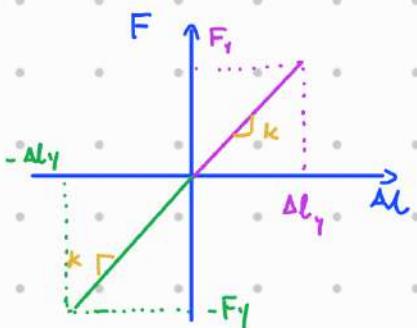
Leghe in ALLUMINIO 0,33

ELASTOMERI $\approx 0,5 \rightarrow$ valore CRITICO/massimo

ν

Compressione

è simmetrico



Energia

$$K = \frac{F}{\Delta l} \rightarrow F(\Delta l) = K \Delta l$$

energia
esterna

consider il lavoro $F(\Delta l) d(\Delta l)$ → $W = \int_0^{\Delta l} F(\Delta l) d(\Delta l) = \frac{1}{2} K \Delta l^2$

energia di
deformazione elastica

analogamente

$$\tau(\epsilon) \cdot d\epsilon \rightarrow w = \int_0^{\bar{\epsilon}} \tau(\epsilon) \cdot d\epsilon = \frac{1}{2} E \epsilon^2$$

lavoro di
pressione
interna

DENSITÀ di energia
di deformazione elastica

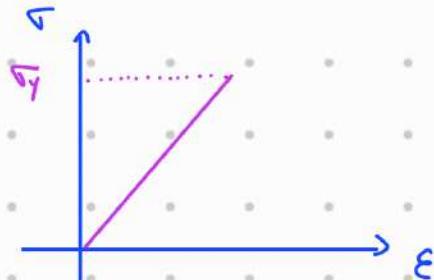
in campo
elastico

esempio T: 4 M 6 V

$$E = 110\,000 \text{ MPa}$$

$$\nu = 0,33$$

$$\sigma_y = 850 \text{ MPa}$$



$$\epsilon_y = \frac{\sigma_y}{E} = \frac{850}{110\,000} = 0,0077$$

$$\nu = -\frac{\epsilon_x}{\epsilon_l} \Rightarrow \nu \epsilon_{x,y} = -\nu \epsilon_y = -0,33 \cdot 0,0077 = -0,00254$$

$$\Delta = D_0 (1 + \epsilon_{x,y}) = 0,9975 D_0$$

$$\frac{A}{A_0} = \frac{\frac{\pi}{4} D^2}{\frac{\pi}{4} D_0^2} = \left(\frac{D}{D_0} \right)^2 = 0,9975^2 \approx 0,995$$

$$\Rightarrow D_0 = 5 \text{ mm} \quad F_1 = ?$$

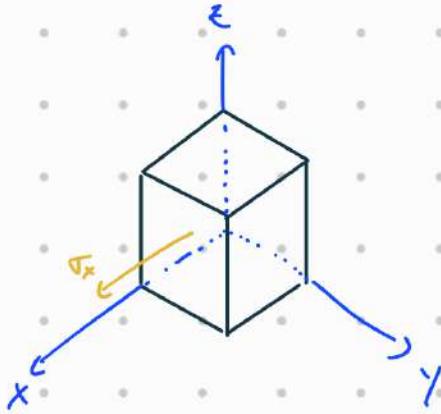
$$\pi = 3,1416$$

$$A_0 = \frac{\pi}{4} D_0^2 \approx 19,6 \text{ mm}^2$$

$$F_y = \sigma_y A_0 = 850 \cdot 19,6 = 16\,685,5 \text{ N} \approx 17\,000 \text{ kgf}$$

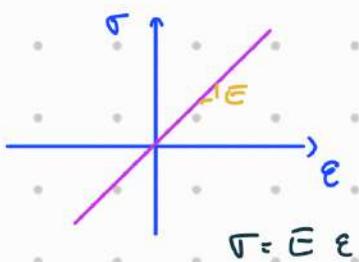
→ consider più gravità per la stabilità
fra statica e statica dinamica

L'equazione generale della legge di Hooke



$$\begin{aligned}\epsilon_x &= \frac{r_x}{E} - \frac{\nu}{E} r_y - \frac{\nu}{E} r_z \\ \epsilon_y &= -\frac{\nu}{E} r_x + \frac{r_y}{E} - \frac{\nu}{E} r_z \\ \epsilon_z &= -\frac{\nu}{E} r_x - \frac{\nu}{E} r_y + \frac{r_z}{E}\end{aligned}$$

altri considerazioni energetiche

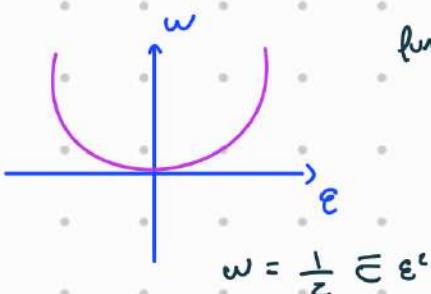


$$w(\epsilon) = \int_0^{\epsilon} \tau(\epsilon) d\epsilon \quad \text{cioè} \quad dw = \tau(\epsilon) d\epsilon$$

$$\tau(\epsilon) = \frac{dw}{d\epsilon}$$

funzione CONVessa

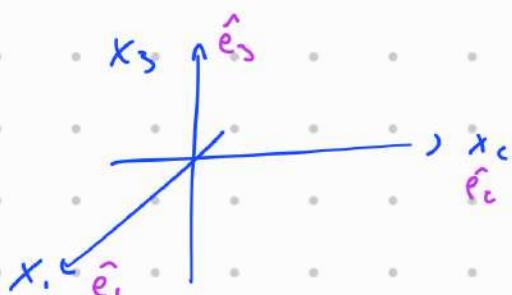
$$\begin{aligned}w(\epsilon) &\geq 0 \\ w(\epsilon) &= 0 \iff \epsilon = 0\end{aligned}$$



generalizzazione nelle altre dimensioni:

$$\begin{array}{l} r_x, r_y, r_z, r_{xy}, r_{yz}, r_{zx} \\ \epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx} \end{array} \quad \left. \begin{array}{l} \tau \\ \epsilon \end{array} \right\} \text{notazione molto pesante}$$

Utilizzo notazione tensoriale e indic:



$$\vec{v} = v_i \vec{e}_i + v_j \vec{e}_j + v_k \vec{e}_k$$

$$\begin{aligned}\vec{v} \cdot \vec{v} &= [v_i, v_j, v_k] \begin{bmatrix} v_i \\ v_j \\ v_k \end{bmatrix} = v_i v_i + v_j v_j + v_k v_k \\ &= \sum_{i=1}^3 v_i v_i = v_i v_i \quad \text{convenzione} \\ &\quad \text{di Einstein}\end{aligned}$$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} A_{11}U_1 + A_{12}U_2 + A_{13}U_3 \\ A_{21}U_1 + A_{22}U_2 + A_{23}U_3 \\ A_{31}U_1 + A_{32}U_2 + A_{33}U_3 \end{bmatrix}$$

$$= \sum_{S=1}^3 A_{1S} U_S = A_{1S} U_S$$

$$A \cdot B = A_{1S} B_{S2}$$

quindi:

$$\bar{E}_{1S}$$

$$w(E_{1S}) = \int_0^{\bar{E}_{1S}} \tau_{1S}(E_{1S}) dE_{1S}$$

dei essere differenziabile esatto

$$dw = \tau_{1S} dE_{1S}$$

$$\hookrightarrow \tau_{1S} = \frac{dw}{dE_{1S}}$$

$$\frac{\partial \tau_{1S}}{\partial E_{2L}} = \frac{\partial w}{\partial E_{2L} \partial E_{1S}}$$

siano uguali

$$\frac{\partial \tau_{2L}}{\partial E_{1S}} = \frac{\partial w}{\partial E_{1S} \partial E_{2L}}$$

$$\Rightarrow \frac{\partial \tau_{1S}}{\partial E_{2L}} = \frac{\partial \tau_{2L}}{\partial E_{1S}}$$

materiale iperelastico

SIMETRIE RAGGIARIA

$$\tau_{1S} = \tau_{1S}(E_{2L})$$

materiale elastico

tensores costitutivo $\delta_1 \times \delta_1$



$$\tau_{1S} = D_{1S} E_{2L} \quad \text{cioè} \quad \tau_{11} = D_{1111} E_{11} + D_{1122} E_{12} + D_{1133} E_{13} \dots$$

però $\tau_{1S} = \tau_{2S}$ tensores di tensione simmetrico

simmetrie minori

$E_{1S} = E_{2S}$ tensore di deformazione

$$D_{1S} E_{2L} = D_{2S} E_{1L} = D_{1S} E_{1L}$$

Quindi
materiale elastico anisotropo

\hookrightarrow ci parametri indipendenti:

$$D_{1S} E_{1L} = D_{1S} E_{2L}$$

Materiale isotropo elastico

\hookrightarrow 2 parametri indipendenti:

$$E, \nu$$

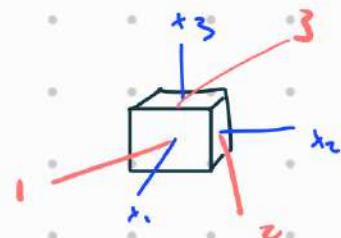
Densità di energia FORMA LINEARE

$$W = W(\varepsilon_{12})$$

nel caso di materiale ISOTROPO

$$W = W(\varepsilon_1, \varepsilon_2, \varepsilon_3) \rightarrow \varepsilon^3 - I_1 \varepsilon^2 + I_2 \varepsilon - I_3 = 0$$

$$\frac{!}{=} W(I_1, I_2, I_3)$$



le invarianti sono

$$\left\{ \begin{array}{l} W = \frac{1}{2} E \varepsilon^2 \\ \tau = \frac{\partial W}{\partial \varepsilon} \end{array} \right. \Rightarrow \tau = E \varepsilon$$

$$I_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z \rightarrow \text{lineare}$$

$$I_2 = \varepsilon_x \varepsilon_y + \varepsilon_y \varepsilon_z + \varepsilon_z \varepsilon_x - \frac{1}{3} (\delta_{xx}^2 + \delta_{yy}^2 + \delta_{zz}^2)$$

$$I_3 = \varepsilon_x \varepsilon_y \varepsilon_z$$

l'espressione più semplice è

$$W = a I_1^2 + b I_2$$

a, b costanti

\hookrightarrow alla fine a, b privo di significato fisico

quindi

$$\left\{ \begin{array}{l} Q = \frac{\lambda}{2} + \mu \\ b = -2\mu \end{array} \right.$$

λ, μ costanti di LANÉ

dove

$$\lambda = \frac{\nu E}{(1+\nu)(1-\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

quindi

$$W = \left(\frac{1}{2} + \mu \right) I_1^2 - \mu I_2$$

stesso τ_{ij}

$$\tau_x = \frac{\partial W}{\partial \varepsilon_x}, \quad \tau_y = \frac{\partial W}{\partial \varepsilon_y}, \quad \tau_z = \frac{\partial W}{\partial \varepsilon_z}$$

$$\tau_{xy} = \frac{\partial W}{\partial \delta_{xy}}, \quad \tau_{yz} = \frac{\partial W}{\partial \delta_{yz}}, \quad \tau_{xz} = \frac{\partial W}{\partial \delta_{xz}}$$

$$\begin{aligned}
 \Gamma_x &= \frac{\partial w}{\partial \epsilon_x} = \frac{\partial w}{\partial I_1} \cdot \frac{\partial I_1}{\partial \epsilon_x} + \frac{\partial w}{\partial I_c} \cdot \frac{\partial I_c}{\partial \epsilon_x} \\
 &= (1+z\mu) I_1 \cdot \frac{\partial I_1}{\partial \epsilon_x} - z\mu \frac{\partial I_c}{\partial \epsilon_x} \\
 &= (\lambda + z\mu) (\epsilon_x + \epsilon_y + \epsilon_z) \cdot 1 - z\mu \cdot (\epsilon_y + \epsilon_z) \\
 &= (\lambda + z\mu) \epsilon_x + \lambda (\epsilon_y + \epsilon_z)
 \end{aligned}$$

$$\begin{aligned}
 \gamma_{xy} &= \frac{\partial w}{\partial \gamma_{xy}} = \frac{\partial w}{\partial I_1} \cdot \frac{\partial I_1}{\partial \gamma_{xy}} + \frac{\partial w}{\partial I_c} \cdot \frac{\partial I_c}{\partial \gamma_{xy}} \\
 &= 0 \cdot z\mu \left(-\frac{1}{2} \gamma_{xy} \right) \\
 &= \mu \gamma_{xy}
 \end{aligned}$$

espliatoando le costanti di Lami

$$\lambda + z\mu = \frac{v \bar{E}}{(1+v)(1-zv)} + \frac{\bar{E}}{1+v} = \frac{(1-v) \bar{E}}{(1+v)(1-zv)}$$

in forma compatta

notazione di Voigt

$$\begin{bmatrix} \Gamma_x & \Gamma_{xy} & \Gamma_{xz} \\ \Gamma_{yx} & \Gamma_y & \Gamma_{yz} \\ \Gamma_{zx} & \Gamma_{zy} & \Gamma_z \end{bmatrix}$$

vettore delle componenti

$$\vec{\Gamma} = \begin{pmatrix} \Gamma_x \\ \Gamma_y \\ \Gamma_z \\ \Gamma_{xy} \\ \Gamma_{xz} \\ \Gamma_{yz} \end{pmatrix}$$

$$\begin{bmatrix} \epsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_z \end{bmatrix}$$

$$\vec{\epsilon} = \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{pmatrix}$$

quindi relazione costitutiva per materiale elastico lineare isotropo

$$\begin{bmatrix} \Gamma_x \\ \Gamma_y \\ \Gamma_z \\ \Gamma_{xy} \\ \Gamma_{xz} \\ \Gamma_{yz} \end{bmatrix} = \frac{\bar{E}}{(1+v)(1-zv)}$$

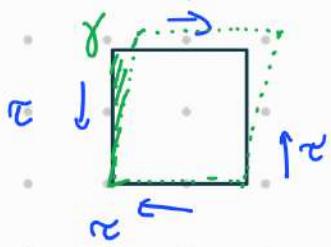
$$\begin{bmatrix} 1-v & v & v \\ v & 1-v & v \\ v & v & 1-v \end{bmatrix} \begin{bmatrix} \phi \\ \phi \\ O \end{bmatrix} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

in forme simboliche compatte

$$\vec{\sigma} = D \vec{\epsilon}$$

↳ matrice costitutiva /
matrice di RIGIDEZZA

oss: dato che $\tau_{ij} = \mu \gamma_{ij}$



Jelinsko costante $G = \mu = \frac{E}{2(1+\nu)}$

MODULO ELASTICO TANGENZIALE

Le costanti di Lame sono costanti ingegneristiche

se uso elementi speciali/principali

$$\begin{bmatrix} \sigma \\ \tau_c \\ \tau_s \\ 0 \\ 0 \\ 0 \end{bmatrix} = D \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

inoltre

$$\vec{\sigma} = D^{-1} \vec{\epsilon}$$

↳ matrice di DEFORMABILITÀ

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & -\nu & 1 & -\nu \\ -\nu & 1 & -\nu & -\nu & -\nu & 1 \\ -\nu & -\nu & 1 & -\nu & -\nu & -\nu \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi & & & & \phi & \\ & & & & & \phi \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}$$

Quali valori possono assumere E e ν ?

$$\omega = \frac{1}{c} \sqrt{\frac{E}{\rho}} \quad \text{e} \quad \omega = 0 \iff \vec{E} = \vec{0}$$

definita positiva

funzione policonvessa

per semplicità si utilizza

D' e sistema principale

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{bmatrix} \sigma_1' \\ \sigma_2' \\ \sigma_3' \end{bmatrix}$$

per essere definite positive

- $\det(D) > 0$
- $\det(D') > 0$
- $\det(D'') > 0$

quindi

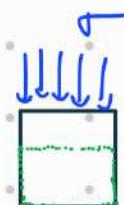
$$\left| \begin{array}{l} E > 0 \\ -1 < \nu < 1 \\ \nu < \frac{1}{c} \end{array} \right.$$

*restriction:
termo dinamico*

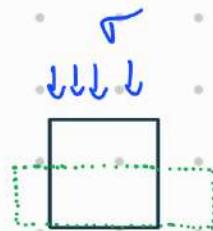
per materiali non troppo complessi vale:

$$0 \leq \nu < \frac{1}{c}$$

se $\nu = 0$ nessuna contrazione laterale
Schiisme



$$\text{se } 0 < \nu < \frac{1}{c}$$



se $-1 < \nu < 0$ materiali auxetic

$$se \quad v \rightarrow \frac{1}{2}$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{v}{E} \sigma_y - \frac{v}{E} \sigma_z$$

$$\varepsilon_y = -\frac{v}{E} \sigma_x + \frac{\sigma_y}{E} - \frac{v}{E} \sigma_z$$

$$\varepsilon_z = -\frac{v}{E} \sigma_x - \frac{v}{E} \sigma_y + \frac{\sigma_z}{E}$$

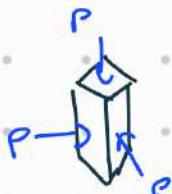
$$\Rightarrow \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1}{E} (\sigma_x + \sigma_y + \sigma_z) - \frac{2v}{E} (\sigma_x + \sigma_y + \sigma_z)$$

$$\frac{1}{E} I_1 - \frac{2v}{E} I_1 \quad \begin{matrix} \text{variazione} \\ \rightarrow \\ \text{d: volume} \end{matrix}$$

$$v \sim \frac{1}{2} \quad \begin{matrix} \text{materiali} \\ \text{incomprensibili} \end{matrix}$$

esempio

assumo stato di TENSIONE IDROSTATICA



$$\sigma_x = \sigma_y = \sigma_z = P$$

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \frac{P}{E} - \frac{v}{E} P$$

$$\frac{1}{E} P \frac{(1-v)}{E}$$

$$I_1 = 3 \frac{P(1-v)}{E}$$

$$\frac{P}{I_1} = \frac{P}{\Delta V / V_0} = \frac{E}{3(1-v)}$$

rigidezza
volumetrica (bulk modulus)

$$K_V = \frac{E}{3(1-v)}$$

$$\lim_{V \rightarrow \frac{1}{2}} K_V = +\infty$$

esercizio

Materiale elastico lineare isotropo. $E = 10.000 \text{ MPa}$. $\nu = 0,4$

Calcolare stato I: deformazione per $\sigma_x = 100 \text{ MPa}$ e
 stato I) Legge generalizzata di Hooke

$$\epsilon_x = \frac{\sigma_x}{E} = 0,01 \quad \epsilon_y = -\frac{\nu}{E} \sigma_x = -0,004 = \epsilon_z$$

$$\tau_{xz} = 0 \Rightarrow \gamma_{xz} = 0$$

$$\left. \begin{array}{l} \sigma_x = 100 \text{ MPa} \\ \sigma_y = 50 \text{ MPa} \end{array} \right.$$

stato II)

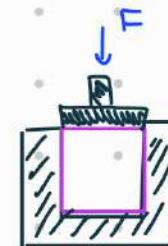
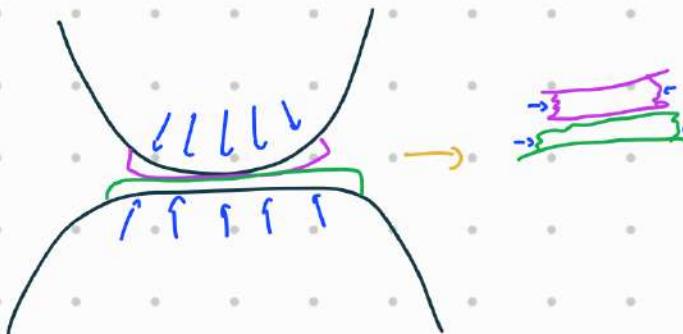
$$\gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0$$

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E} \sigma_y = \frac{100}{10000} - \frac{0,4}{10000} \cdot 50 = 0,0018$$

$$\epsilon_y = -\frac{\nu}{E} \sigma_x + \frac{\sigma_y}{E} = \frac{-0,4}{10000} \cdot 100 + \frac{50}{10000} = -0,001$$

$$\epsilon_z = -\frac{\nu}{E} \sigma_x - \frac{\nu}{E} \sigma_y = \frac{-0,4}{10000} (100 - 50) = -0,006$$

Stato di compressione confinata



lo stato deformativo:

$$\gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0$$

$$\left. \begin{array}{l} \epsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E} \sigma_y - \frac{\nu}{E} \sigma_z \\ \epsilon_y = -\frac{\nu}{E} \sigma_x + \frac{\sigma_y}{E} - \frac{\nu}{E} \sigma_z \\ \epsilon_z = -\frac{\nu}{E} \sigma_x - \frac{\nu}{E} \sigma_y + \frac{\sigma_z}{E} \end{array} \right.$$

$$\sigma_z = -\frac{F}{A}$$

$$\epsilon_x = \epsilon_y = 0$$

\rightarrow del disegno

$$\Rightarrow \sigma_x = \sigma_y = \sigma_z$$

quindi:

$$\left\{ \begin{array}{l} \frac{\sigma_x}{E} - \frac{\nu}{E} \sigma_y - \frac{\nu}{E} \sigma_z = 0 \\ -\frac{\nu}{E} \sigma_x + \frac{\sigma_y}{E} - \frac{\nu}{E} \sigma_z = 0 \\ \epsilon_x = -\frac{2\nu}{E} \sigma + \frac{\sigma_z}{E} \end{array} \right. \quad) \text{ uguali:} \quad \left\{ \begin{array}{l} \sigma (1-\nu) - \nu \sigma_z = 0 \\ \epsilon_z = -\frac{2\nu}{E} \sigma + \frac{\sigma_z}{E} \end{array} \right.$$

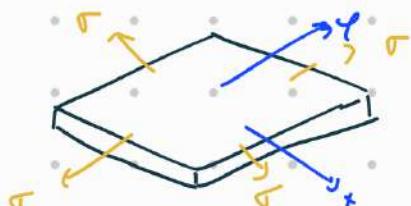
$$\left\{ \begin{array}{l} \sigma = \frac{\nu}{1-\nu} \sigma_z \\ \epsilon_z = -\frac{2\nu^2}{1-\nu} \sigma_z + \frac{1}{E} \sigma_z + \frac{\sigma_z}{E} \end{array} \right. \quad \left\{ \begin{array}{l} \sigma = \frac{\nu}{1-\nu} \sigma_z \\ \epsilon_z = \frac{\sigma_z}{E} + \frac{\nu}{1-\nu} \end{array} \right. (1-\nu)$$

$$\left\{ \begin{array}{l} \sigma = \frac{\nu}{1-\nu} \sigma_z \\ \left(\frac{\sigma_z}{\epsilon_z} \right) = \frac{1-\nu}{1+\nu} \frac{E}{1-2\nu} \\ \hookrightarrow \text{modulo J: rigidezza} \\ \text{a compressione confinata} \end{array} \right. H$$

Oss:

$$H = \frac{\sigma_z}{\epsilon_z} = \frac{1-\nu}{1+\nu} \frac{3\sqrt{E}}{3(\nu z \nu)} = \frac{1-\nu}{1+\nu} 3 K_v$$

Stato d: tensione equi-biassiale

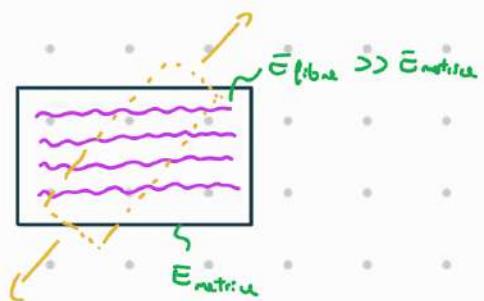


$$\frac{\sigma}{E} = \frac{\epsilon}{1-\nu}$$

$$\left\{ \begin{array}{l} \epsilon_x = \frac{\sigma}{E} - \frac{\nu}{E} \sigma \\ \epsilon_y = -\frac{\nu}{E} \sigma + \frac{\sigma}{E} \\ \epsilon_z = -\frac{2\nu}{E} \sigma \end{array} \right. \quad) \text{ uguali:} \quad \left\{ \begin{array}{l} \epsilon_x = \epsilon_y = \frac{1-\nu}{E} \sigma \\ \epsilon_z = -\frac{2\nu}{E} \sigma \end{array} \right.$$

Altri comportamenti dei materiali

Materiale elastico lineare ORTOINTR



$\bar{E}_x \gg \bar{E}_y \gg \bar{E}_z$

Si comporta in modo OLTREZZO alla TRAZIONE se si cambia direzione/angolo.
è SIMMETRICO A UN'asse

i parametri indipendenti sono 9:

- moduli elastici longitudinali E_x, E_y, E_z
- coefficienti di Poisson $\nu_{xy}, \nu_{xz}, \nu_{yz}$
- moduli elastici tangenziali G_{xy}, G_{xz}, G_{yz}

in particolare

$$E_x = \frac{\sigma_x}{\epsilon_x} = -\frac{\nu_{yx}}{E_y} \sigma_y - \frac{\nu_{zx}}{E_z} \sigma_z$$

$$E_y = -\frac{\nu_{xy}}{E_x} \sigma_x + \frac{\sigma_y}{\epsilon_y} - \frac{\nu_{zy}}{E_z} \sigma_z$$

$$\epsilon_z = -\frac{\nu_{xz}}{E_x} \sigma_x - \frac{\nu_{yz}}{E_y} \sigma_y + \frac{\sigma_z}{\epsilon_z}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}}$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G_{xz}}$$

Dato che il materiale è elastico

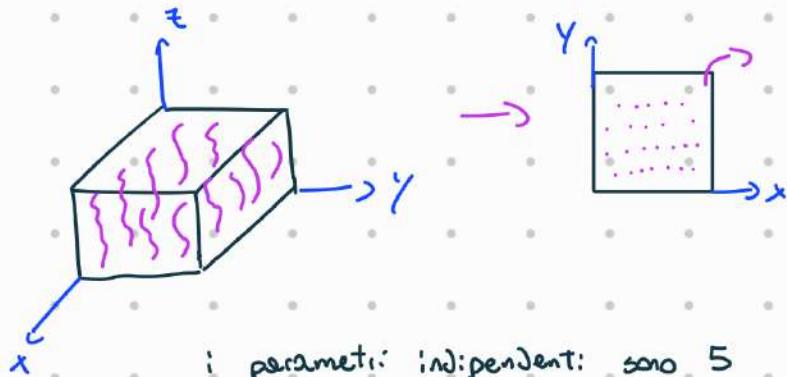
\hookrightarrow simmetrico

$$\frac{\nu_{yz}}{E_z} = \nu_{xy} = \frac{\nu_{xy}}{E_x}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G_{yz}}$$

\rightarrow parametri INDIPENDENTI!

Materiale elastico trasversalmente isotropo



i parametri indipendenti sono 5

$$E_x = E_y = \bar{E}$$

$$G_{xy} = G_z = \frac{\bar{E}}{2(1+\nu)}$$

$$\nu_{xy} = \nu_{yz} = \nu$$

$$\bar{E}_z \neq \bar{E}$$

$$\nu_{zx} = \nu_{ty}$$

$$G_{zx} = G_{ty}$$

Criteri di sicurezza

Per stimare la sicurezza di uno stato deformatorio si utilizza:

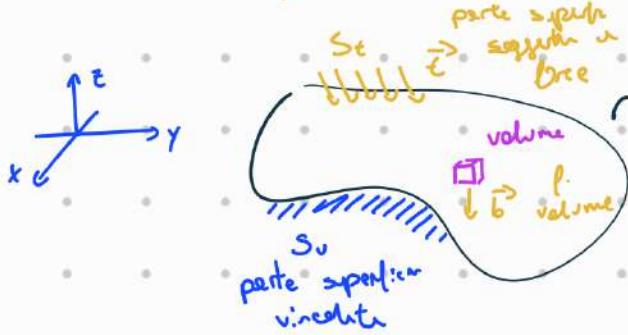
TENSIONE EQUIVALENTE DI VON MISES

$$\sqrt{\sigma_n} = \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - (\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x)} + 3(\nu_{xy}^2 + \nu_{yz}^2 + \nu_{xz}^2)$$

Se si è in campo elastico, per la sicurezza bisogna

$$\sigma_n \leq \frac{\sigma_{yield}}{FS} \rightarrow \text{fattore di sicurezza (moltiplicatore)}$$

Problema dell'equilibrio elastico



S superficie laterale

$$S_v \cap S_t = \emptyset$$

$$S_v \cup S_t = S$$

$$\vec{U} (v, v_r, w)$$

\hookrightarrow tensore deformazioni infinitesime

$$\bar{\epsilon} = \frac{1}{2} (\nabla \vec{U} + \nabla^T \vec{U})$$

tensori tensioni di Chechij

eq. indennite di equilibrio

$$\operatorname{div}(\bar{\tau}) + \vec{b} = \vec{0} \quad \text{su } V$$

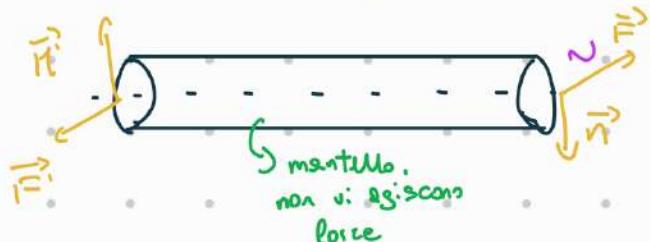
$$\bar{\tau} \cdot \vec{n} = \vec{\epsilon} \quad \text{su } S_t$$

legame costitutivo

$$\vec{\tau} = \mathbb{D} \vec{\epsilon}$$

dove \mathbb{D} matrice costitutiva

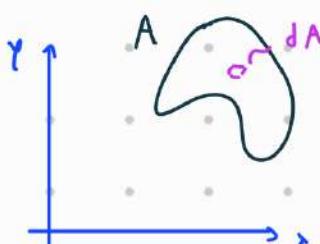
Problema di de Saint-Venant



proprietà dell'area delle sezioni

- prisma RETTO di sezione generica
- materiale elastico, lineare, isotropo e omogeneo
- sistema di forze/momenti: applicata alle sezioni di ESTREMITÀ e si AUTOEQUILIBRANO
- superficie LATERALE priva di forze

\hookrightarrow geometrie delle sezioni



$$N=1$$

Densità di superficie unitaria

momento statico di primo ordine rispetto all'asse x

$$S_x = \int_A y \, dA$$

$$(S_y = \int_A x \, dA)$$

il BARICENTRO rispetto al sistema è

$$x_G = \frac{S_y}{A} = \frac{\int_A x \, dA}{\int_A dA}$$

$$y_G = \frac{S_x}{A} = \frac{\int_A y \, dA}{\int_A dA}$$

OSS: questo concetto si applica anche su aree PUNTALE

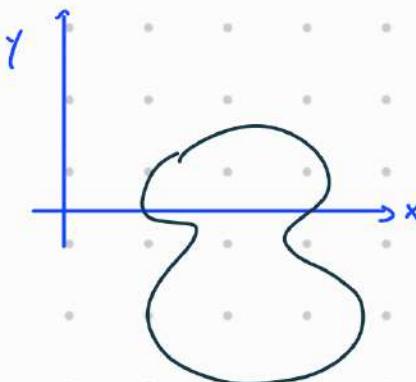


$$S_x = \sum_{i=1}^n A_i y_i$$

$$S_y = \sum_{i=1}^n A_i x_i$$

$$y_G = \frac{S_x}{A}, \quad x_G = \frac{S_y}{A}$$

Momento d'inerzia rispetto ad un'asse (rispetto all'asse x)



$$J_x = \int_A y^2 \, dA$$

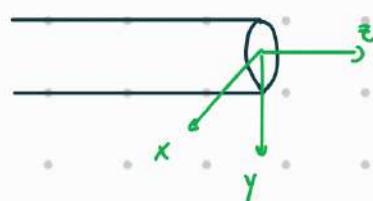
$$J_y = \int_A x^2 \, dA$$

Momento d'inerzia misto

$$J_{xy} = \int_A xy \, dA$$

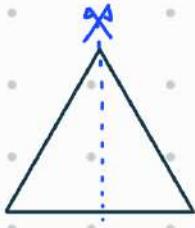
$$\begin{bmatrix} J_x & J_{xy} \\ J_{xy} & J_y \end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix} J_x & 0 \\ 0 & J_y \end{bmatrix}$$



esistono sempre
assi PRINCIPALI

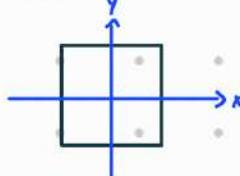
Oss:



l'asse di simmetria rappresenta un'asse principale

esempio

sezione quadrata

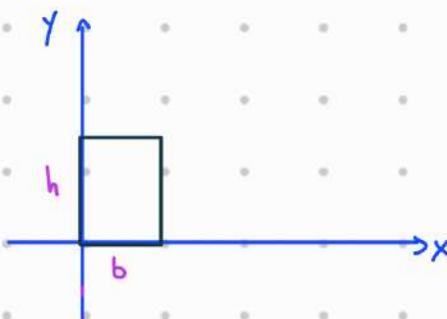


$$J_x = J_y$$

$$J_{x_f} = 0$$

autovettori uguali

sezione rettangolare



$$S_x = \int_A y \, dA = \int_0^h b y \, dy = \frac{1}{2} b h^2$$

$$A = b h$$

$$y_G = \frac{S_x}{A} = \frac{\frac{1}{2} b h^2}{b h} = \frac{h}{2}$$

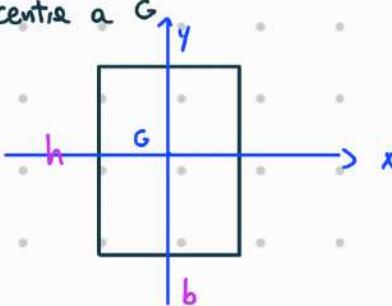
$$S_y = \int_A x \, dA = \int_0^b h x \, dx = \frac{1}{2} h b^2$$

$$x_G = \frac{S_y}{A} = \frac{\frac{1}{2} h b^2}{b h} = \frac{b}{2}$$

$$J_x = \int_A y^2 \, dA = \int_0^h b y^2 \, dy = \frac{1}{3} b h^3$$

$$J_y = \int_A x^2 \, dA = \int_0^b h x^2 \, dx = \frac{1}{3} h b^3$$

se si centra a G

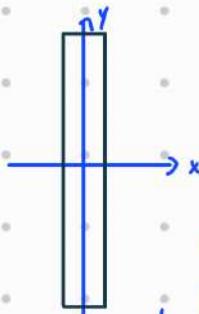


momenti d'inerzia

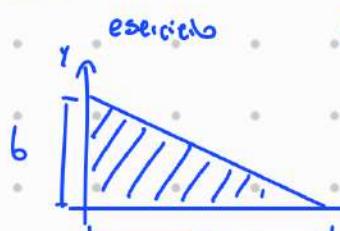
$$J_x = \int_P y^2 \, dA = \int_{-h/2}^{h/2} b y^2 \, dy = \frac{1}{12} b h^3$$

$$J_y = \int_A x^2 \, dA = \int_{-b/2}^{b/2} h x^2 \, dx = \frac{1}{12} h b^3$$

Oss:



$J_x >> J_y$ è più difficile scrivere su x



$$J_x = \int y^2 dA = \int_0^a \left[-\frac{b^2}{a^2} x^3 + \frac{3b^3}{a^2} x^2 - \frac{3b^3}{a^2} x + b^2 \right] dx \\ = -\frac{b^3}{9} a + \frac{b^3}{2} a - \frac{3}{2} \cdot \frac{b^3}{2} a + b^3 a \\ = -\frac{1}{3} a + \frac{1}{2} a = -\frac{1}{6} a = \frac{1}{6}$$

$$S_x = \int y(x) dA \quad y(x) = -\frac{b}{a}x + b \Rightarrow S_x = \int y^2 dA$$

$$\frac{1}{3} b^2 a - b^2 a + b^2 a = \frac{b^2 a}{3}$$

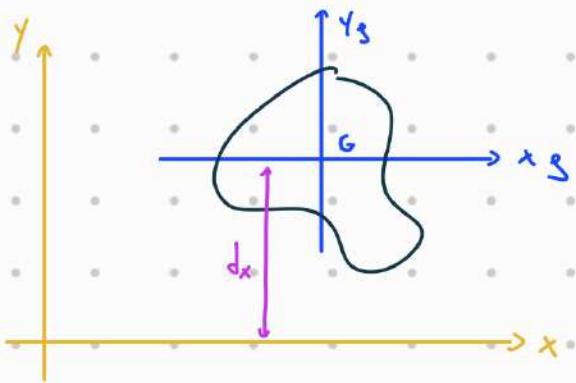
$$\int_0^a \frac{b^2}{a^2} x^2 - \frac{2b^2}{a} x + b^2 dx$$

d. conseguenza

$$S_y = \frac{a^2 b}{3}$$

$$G = \begin{cases} x_G = a/6 \\ y_G = b/6 \end{cases}$$

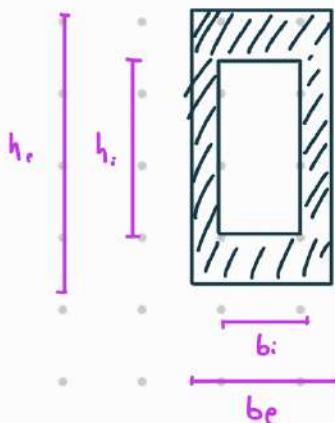
Oss: Teorema di Steiner



$$J_{xg} = \int y_g^2 dA$$

$$J_x = \int_A y^2 dA = \int_A (y_g + d)^2 dA = \int_A (y_g^2 + 2y_g d + d^2) dA \\ = \int_A y_g^2 dA + \int_A d^2 dA + 2d \int_A y_g dA \\ = J_{xg} + A d_x^2$$

Esempio Rettangolo canale

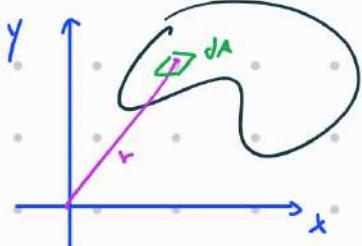


$$J_x = \frac{1}{12} (b_e h_e^3 - b_i h_i^3)$$

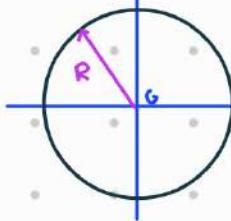
$$J_y = \frac{1}{12} (h_e b_e^3 - h_i b_i^3)$$

Momento di inerzia polare

$$J_p = \int r^2 dA = \int (x^2 + y^2) dA = J_x + J_y$$



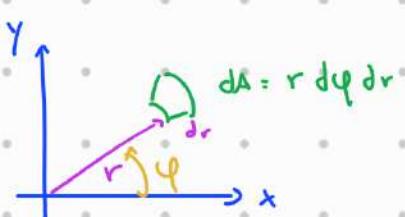
esempio Circolare



$$S_p = J_x + J_y \quad \wedge \quad J_x = J_y$$

$$\Rightarrow J_x = J_y = \frac{S_p}{c}$$

uso riferimento polare



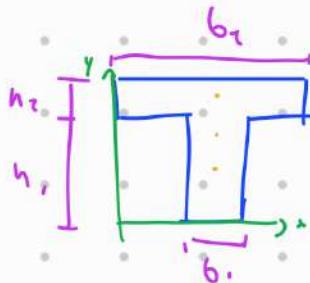
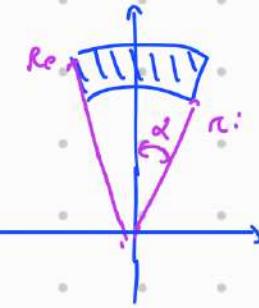
$$S_p = \int_A r^2 dA = \int_0^R \int_0^{2\pi} r^2 \cdot r d\varphi dr = c\pi \int_0^R r^3 dr = \frac{\pi R^4}{4}$$

punto

$$J_x = J_y = \frac{S_p}{c} = \frac{\pi R^4}{4} = \frac{\pi}{64} D^4$$

esercizio

$$J_p = \int_A r^2 dA = c\alpha \int_{R_i}^{R_o} r^2 dr = \alpha \left(\frac{R_o^4}{2} - \frac{R_i^4}{2} \right) = \alpha \left(\frac{D_o^4}{64} - \frac{D_i^4}{64} \right)$$



$$x_G = \frac{b_c}{2}$$

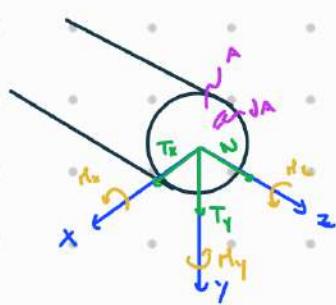
$$y_G = \frac{\int_A x dA}{\int_A dA} \rightarrow h_c \int_0^{b_c} x dx + h_i \int_0^{b_i} x dx = \frac{h_c \frac{b_c^2}{2} + h_i \frac{b_i^2}{2}}{h_c b_c + h_i b_i}$$

$$= \frac{h_c b_c \cdot \frac{b_c}{2} + h_i b_i \cdot \frac{b_i}{2}}{h_c b_c + h_i b_i}$$

media ponderata

Distribuzione di carico delle sezioni

s. prendiamo in considerazione il solido di Saint-Venant



s. ha:

$$N = \int_A \sigma_z \, dA \quad T_x = \int_A \tau_{xz} \, dA \quad T_y = \int_A \tau_{yz} \, dA$$

$$R_x = \int_A y \sigma_z \, dA \quad R_y = - \int_A x \sigma_z \, dA \quad R_z = \int_A (\tau_{yz} x - \tau_{xz} y) \, dA$$

Sollecitazione axiale

$$N \neq 0 ; \quad T_x = T_y = R_x = R_y = R_z = 0$$

prop: Teorema di Kirchhoff

se esiste la soluzione al problema elastico, allora è UNICA

s. suppone

$$\begin{cases} \sigma_z = k \text{ costante} \\ \tau_{xz} = \tau_{yz} = 0 \end{cases} \Rightarrow N = \int_A \sigma_z \, dA = k A \rightarrow \sigma_z = k = \frac{N}{A}$$

s. osserva:

$$R_x = \int_A y \sigma_z \, dA = k \int_A y \, dA = 0 \Leftrightarrow \text{sistema baricentrico}$$

$$R_y = - \int_A x \sigma_z \, dA = -k \int_A x \, dA = 0 \Leftrightarrow \text{sistema baricentrico}$$

Dato che

$$\epsilon_z = \frac{\sigma_z}{E} \Rightarrow \epsilon_z = \frac{N}{EA} \rightarrow \text{rigidezza axiale della sezione} [F]$$

$$\epsilon_x = \epsilon_y = -\frac{y}{E} \sigma_z$$

esempio



Δl ?

$$\Delta l = \int_0^l \epsilon_z \, dz = \int_0^l \frac{N}{EA} \, dz = \frac{Nl}{EA}$$

definisco

$$k_a = \frac{EA}{l}$$

rigidezza
axiale
della trave
[F/L]

$$\Rightarrow \Delta l = \frac{N}{k_a}$$

$$W = \frac{1}{2} N \Delta l = \frac{1}{2} k_a \Delta l^2 = \int_V w \, dV$$

$$V \hookrightarrow \frac{1}{2} E \epsilon_z^2 = \frac{1}{2} \sigma_z \epsilon_z$$

Sollecitazione di momento flettente retto

$$M_x \neq 0$$

$$\begin{cases} N = T_x = T_y = 0 \\ M_y = M_z = 0 \end{cases}$$

Suppongo

$$\begin{cases} \sigma_\epsilon = K_y \text{ costante} \\ T_{xx} = T_{yy} = 0 \\ \hookrightarrow T_x = T_y = \tau_z = 0 \end{cases}$$

\Rightarrow

$$N = \int_A \sigma_\epsilon dA = K \int_A y dA = 0 \quad \text{sistema biaxiale}$$

$$M_y = - \int_A x \sigma_\epsilon dA = -K \int_A xy dA = 0 \quad \text{sistema biaxiale}$$

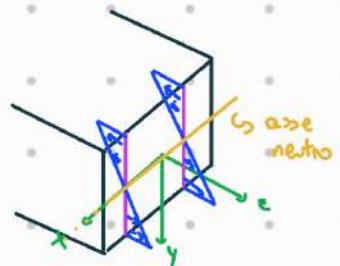
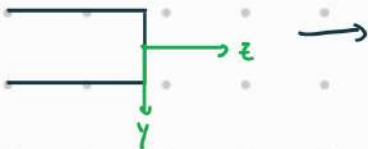
A

secondo ordine
misto

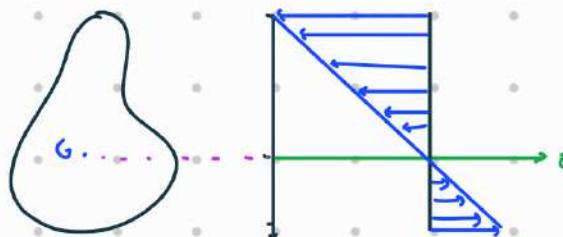
$$M_x = \int_A y \sigma_\epsilon dA = K \int_A y^2 dA = K J_x \quad \approx K = \frac{M_x}{J_x}$$

$$\sigma_\epsilon = \frac{M_x}{J_x} y \quad \text{formule J.J. Navier}$$

RAPPRESENTAZIONI GRAFICHE



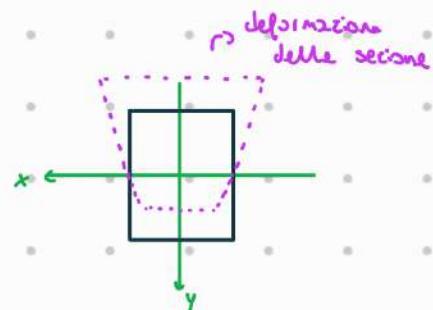
oss.: proprietà del biaxiale e asse neutro



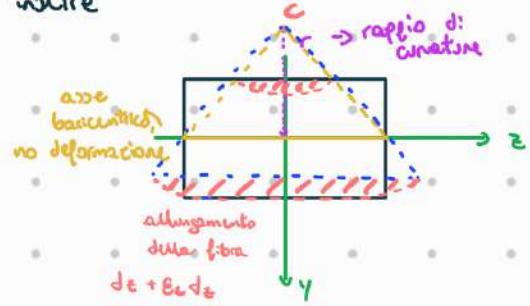
Dato che

$$\sigma_\epsilon = \frac{\tau_\epsilon}{E} = \frac{M_x}{E \cdot J_x} y \quad \text{graficamente}$$

$$\epsilon_x = \epsilon_y = -\frac{1}{E} \tau_\epsilon = -\frac{1}{E} \frac{M_x}{J_x} y$$



oltre



più mi allontano dall'asse neutro
più forte è la "sollecitazione"?

$$\epsilon_z = \frac{\gamma}{E}$$

considero i triangoli simili

$$\frac{dz}{r} = \frac{(1 + \epsilon_z) dz}{r + y}$$

angolo di curvatura

$$\frac{r+y}{r} = 1 + \epsilon_z \Rightarrow 1 + \frac{y}{r} = 1 + \epsilon_z$$

$$\hookrightarrow \frac{1}{r} = \frac{\epsilon_z}{y} = \frac{M_x}{E J_x} \frac{y}{y}$$

\hookrightarrow inverso del rapporto di curvatura \rightarrow curvatura % (estiranea)

definisco

$$K = \frac{1}{r} = \frac{|y''(z)|}{[1 + (y'(z))^2]^{3/2}}$$

espressione esatta
della curvatura

$y'(z) \ll 1$
deformazione
piccola

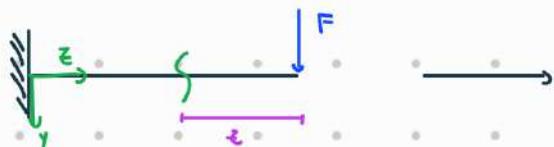
$$|y''(z)| \Rightarrow \pm y''(z) = \frac{M_x}{E J_x}$$

convenzione segno

$$y''(z) = - \frac{M_x}{E J_x}$$

eq. differenziale
della linea elastica

esercizio



$$M(z) + Fz = 0$$

$$\hookrightarrow M(z) = -Fz$$

Dato che

$$M(x) = q \quad \text{carico J. distribuito}$$

$$\hookrightarrow y''(x) = \frac{q}{E J_x}$$

esercizio

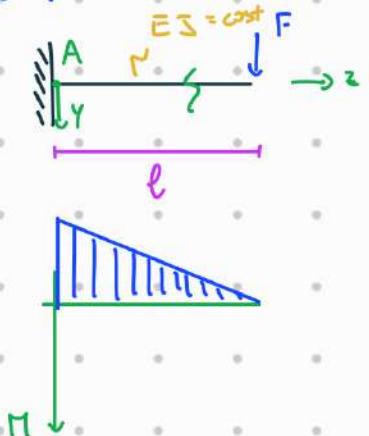
nuovo tubo

stesso materiale, diametro differente

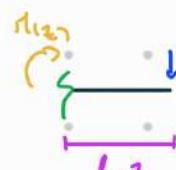
$$J_x = \frac{\pi}{64} D^4 \Rightarrow \frac{J_{x,17}}{J_{x,15}} = \left(\frac{17}{15}\right)^4 \approx 1,65$$

$$A = \frac{\pi}{4} D^2 \Rightarrow \frac{A_{12}}{A_{15}} = \left(\frac{17}{15}\right)^2 \approx 1,29$$

esempio Calcolo della deformazione



$$\gamma''(z) = -\frac{M(z)}{EI}$$



$$M(z) = -F(l-z)$$

quindi:

$$EI \gamma''(z) = F(l-z)$$

$$EI \gamma'(z) = Fz - \frac{1}{2}Fz^2 + C_1 \quad \text{rotazione imposta}$$

so che

$$EI \gamma'(0) = 0 \rightarrow C_1 = 0$$

$$EI \gamma(z) = \frac{1}{2}Fz^2 - \frac{1}{6}Fz^3 + C_2$$

so che

$$EI \gamma(l) = 0 \rightarrow C_2 = 0$$

$$\gamma'(z) = \frac{1}{EI} \left(Flz - \frac{1}{2}Fz^2 \right)$$

$$\gamma(z) = \frac{1}{EI} \left(\frac{1}{2}Flz^2 - \frac{1}{6}Fz^3 \right)$$



$$\gamma'(l) = \frac{1}{EI} \left(Fl^2 - \frac{1}{2}Fl^2 \right) = \frac{Fl^2}{2EI}$$

$$\gamma(l) = \frac{1}{EI} \left(\frac{1}{2}Fl^3 - \frac{1}{6}Fl^3 \right) = \frac{Fl^3}{3EI}$$

oss:

$$\eta = \frac{F l^3}{3EI} \Rightarrow \frac{F}{\eta} = \frac{3EI}{l^3}$$

$\frac{[F]}{[\eta]}$ rigidezza
flessionale

Lavoro delle forze esterne

$$L = \frac{1}{2} \eta F^2 = \frac{1}{2} F^2 \frac{l^3}{3EI}$$

$$W = \int_V w dV$$

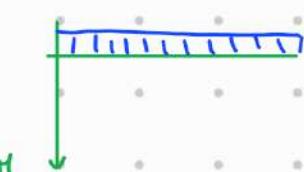
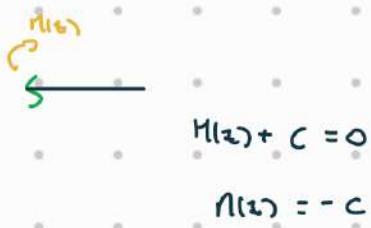
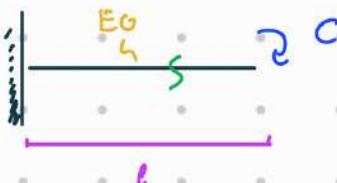
densità di
energia

energia di deformazione elastica

$$w = \frac{1}{2} \sigma_z E_z \quad \text{dove } \sigma_z = \frac{\pi}{l} \cdot \gamma$$

$$\epsilon_z = \frac{\pi}{EI} \cdot \gamma$$

esempio



$$\gamma''(z) = -\frac{M(z)}{EI} = \frac{C}{EI}$$

$$\gamma'(z) = \frac{C}{EI} z + \varphi_1$$

$$\gamma'(0) = 0 \rightarrow \varphi_1 = 0$$

$$\gamma(z) = \frac{1}{2} \frac{C}{EI} z^2 + \varphi_2$$

$$\gamma(l) = 0 \rightarrow \varphi_2 = 0$$

$$\gamma'(z) = \frac{C}{EI} z$$

$$\gamma(z) = \frac{1}{2} \frac{C}{EI} z^2$$



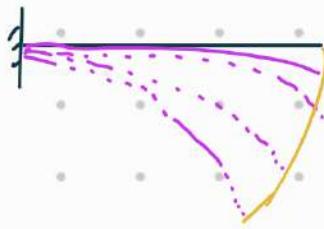
$$\gamma(l) = \frac{Cl}{EI}$$

$$\gamma'(l) = \frac{1}{2} \frac{Cl}{EI}$$

oss:

$$L_u = \frac{1}{2} C \cdot \varphi = \frac{1}{2} C \cdot \gamma'(l) = \frac{1}{2} \frac{Cl}{EI}$$

N.B. Questo modella funzione per strutture che non si deformano eccessivamente e non considerano quindi spostamenti lungo l'asse z



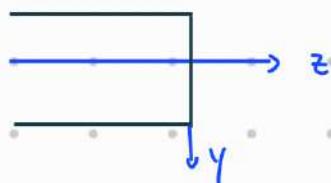
PRESSO(tenso) - flessione retta combinazione delle precedenti:

$$N \neq 0, M_x \neq 0, N_y = N_z = T_y = T_x = 0$$

PRESSO flessione $N < 0, M_x \neq 0$

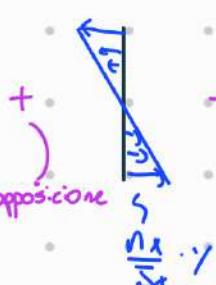
TENSO flessione $N > 0, M_x \neq 0$

$$\sigma_z = \frac{N}{A} + \frac{M_x}{J_x} \cdot y$$



$$\epsilon_z = \frac{\sigma_z}{E}$$

$$\epsilon_x = \epsilon_y = -V \epsilon_z$$



$$\gamma_{zx} = \gamma_{xy} = 0$$

$$\sigma_z > 0$$

$$\sigma_z(y) = 0 \rightarrow \frac{N}{A} + \frac{M_x}{J_x} \cdot y \quad \text{eq. dell'asse neutro}$$

Flessione deviata

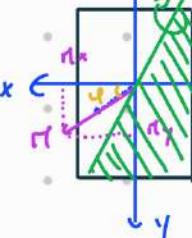
$$M_x \neq 0, N_y \neq 0, N = T_x = T_y = N_z = 0, J_x \neq J_y$$

$$\sigma_z = \frac{M_x}{J_x} y - \frac{N_y}{J_y} \cdot x$$

$$\begin{cases} N_x = N \cos \varphi \\ N_y = N \sin \varphi \end{cases}$$

$$\Rightarrow \sigma_z = 0 \rightarrow \frac{M_x}{J_x} \cdot y - \frac{N_y}{J_y} \cdot x = 0$$

dentro il piano $\sigma_z = 0$
 $M_x < 0$ regole mano su t



$$\frac{y}{x} = \frac{N_y}{N_x} \cdot \frac{J_x}{J_y} = \frac{J_x}{J_y} \tan \varphi \geq 1$$

Oss: se $J_x = J_y$, flessione sarà semplice PERPENDICOLARE

Pièce (tense) - flessione deviata combinazione precedente:

$$\sigma_z = \frac{N}{A} + \frac{N_x}{J_x} \cdot y - \frac{N_y}{J_y} \cdot x$$

Flessione composta

(taglio costante)



$$M(z) + T_y(l-z) = 0$$

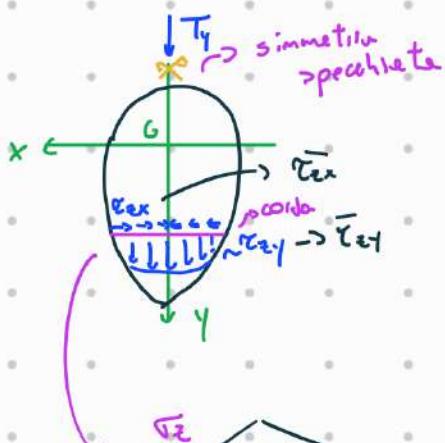
$$\Rightarrow N(z) = T_y(z-l)$$

$$\frac{dN_x(z)}{dz} = T_y$$

→ andamento degli spicci: Jy carico?

soluzione approssimata

di Jourawsky

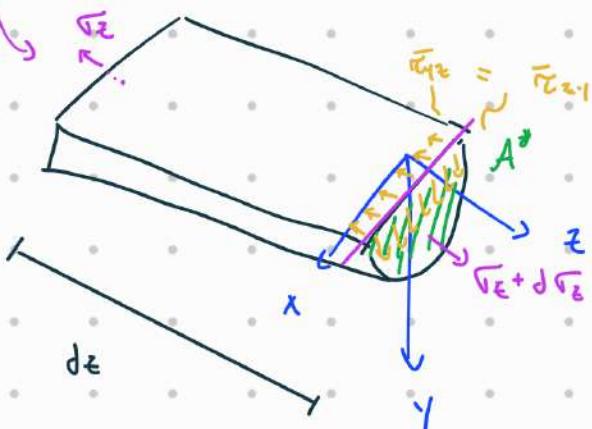


→ non tengo conto della struttura per intenz

$$\bar{\tau}_{xy} = \frac{1}{b} \int \tau_{xy} db$$

$$\bar{\tau}_{xz} = 0$$

→ trascurare corde



Studio l'equilibrio
del corpo delimitato dalle corde

$$\int_{A^*} (\tau_z + d\tau_z - \bar{\tau}_z) dA^* - \bar{\tau}_{xy} b dz = 0$$

Navier

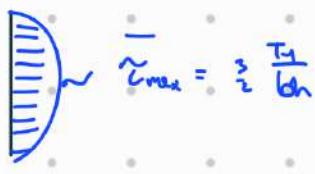
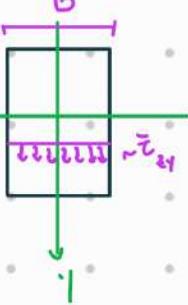
$$\bar{\tau}_{xy} b dz = \int_{A^*} d\tau_z dA^* \Rightarrow \bar{\tau}_{xy} = \frac{1}{b} \int_{A^*} \frac{d\tau_z}{dz} dA^* = \frac{1}{b} \int_{A^*} \frac{1}{J_x} \cdot \frac{dy}{dz} dA^* = \frac{1}{b \cdot J_x} \int_{A^*} y dA^*$$

cose

$$\bar{\tau}_{xy} = \frac{1}{b} \int_{A^*} \frac{dN_x}{dz} \cdot \frac{y}{J_x} dA^* = \frac{1}{b} \int_{A^*} T_y \cdot \frac{y}{J_x} dA^* = \frac{T_y}{b \cdot J_x} \int_{A^*} y dA^*$$

è un valore medio

esempio $\downarrow T_y$



$$\bar{C}_{z1} = \frac{T_y S_x^*}{J_x b}$$

$$S_x^* = \frac{1}{12} b h^3$$

dove $S_x^* = A^* \cdot y_G^*$

$$y_G^* = \left(\frac{h}{2} + y\right) \cdot \frac{1}{2}$$

$$A^* = \left(\frac{h}{2} - y\right) b$$

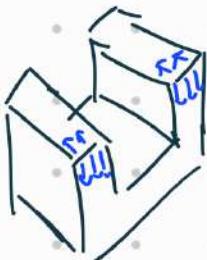
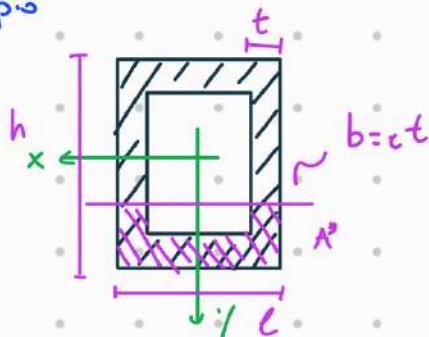
quindi:

$$S_x^* = \left(\frac{h}{2} - y\right) b \cdot \left(\frac{h}{2} + y\right) \cdot \frac{1}{2} = \frac{b}{2} \left[\left(\frac{h}{2}\right)^2 - y^2 \right]$$

$$\bar{C}_{z1} = \frac{T_y \cdot \frac{b}{2} \left[\left(\frac{h}{2}\right)^2 - y^2 \right]}{\frac{1}{12} b h^3 \cdot b} = \frac{6 T_y \left[\left(\frac{h}{2}\right)^2 - y^2 \right]}{6 h^3}$$

$$\bar{C}_{z1} \left(\pm \frac{h}{2} \right) = 0 \quad \bar{C}_{z1} (0) = \frac{6 T_y \left(\frac{h}{2} \right)^2}{6 h^3} = \frac{3}{2} \frac{T_y}{b h} = \bar{C}_{max}$$

esempio

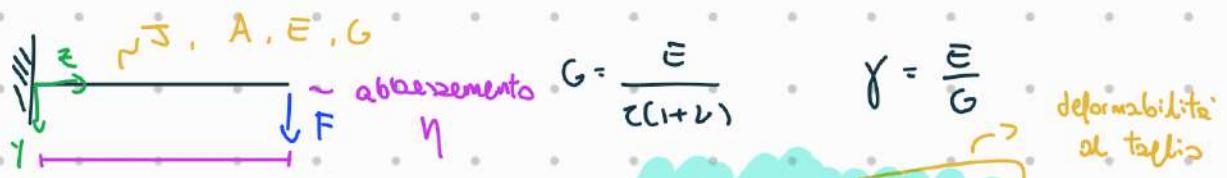


$$J_x = \frac{1}{12} (l h^3 - (l-b)(h-b)^3) = \frac{1}{12} (l h^3 - (l-b)(h^3 - b^3 - 3h^2b + 3hb^2))$$

$$= \frac{1}{12} (l h^3 - l h^3 + b^3 l + 3h^2 b l - 3hb^2 l + 6h^3 - 6h^2 + 3h^2 b^2 - 3hb^2)$$

$$= \frac{1}{12} b (-b^3 + (l-3h)b^2 + 3h(h-l)b + (3h^2 l + h^3))$$

esempio



$$G = \frac{E}{2(1+\nu)}$$

$$\gamma = \frac{E}{G}$$

$$\eta = \frac{Fl^3}{3EI} + \chi \frac{F \cdot l}{GA}$$

deformabilità
alle flessioni

azione di:
taglio \rightarrow dipende dalla
geometria
delle sezioni

deformabilità
di taglio

oss: se la trave è sufficientemente LUNGA, la FLESSIONE ha carattere DOMINANTE

: se la trave è TORZA, il contributo al taglio è molto più maggiore

def:

VALORI DI χ \rightarrow alcune formule lo mette al denominatore, attenzione

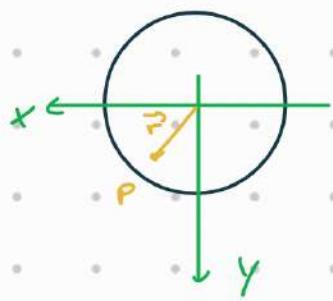
sezione	valore χ
rettangolare	b/s
circolare	$10/g$
circolare a pareti sottili	ζ

Torsione uniforme

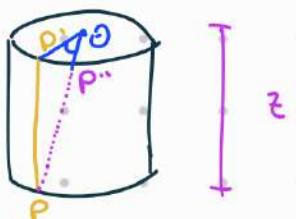
formula di Bredt



∞ : sezione circolare



$$\vec{r} = x\hat{i} + y\hat{j}$$



$$\vec{\theta} = \Theta \cdot \hat{z} \cdot \hat{k}$$

↳ rotazione unitaria

$$\begin{aligned}\vec{J} &= \vec{\theta} \times \vec{r} = \Theta_z \hat{k} \times (x\hat{i} + y\hat{j}) \\ &= \Theta_z x \hat{k} \times \hat{i} + \Theta_z y \hat{k} \times \hat{j} \\ &= \Theta_z x \hat{j} - \Theta_z y \hat{i}\end{aligned}$$

quindi:

$$\vec{v} = (u, v, w) \quad \left\{ \begin{array}{l} u = -\Theta_z y \\ v = \Theta_z x \\ w = 0 \end{array} \right.$$

componenti della deformazione

$$\epsilon_x = \frac{\partial u}{\partial x} = 0 \quad \epsilon_y = \frac{\partial v}{\partial y} = 0 \quad \epsilon_z = \frac{\partial w}{\partial z} = 0$$

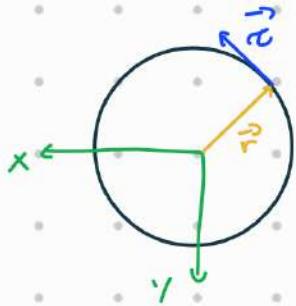
$$\gamma_{xx} = \gamma_{yy} = \gamma_{zz} = 0 \rightarrow N = 0, \tau_x = 0, \tau_y = 0 \quad \text{compatibile con} \quad \text{esistenza uniforme}$$

$$\left\{ \begin{array}{l} \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -\Theta_z + \Theta_z = 0 \\ \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0 - \Theta_y = -\Theta_y \\ \gamma_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0 + \Theta_x = \Theta_x \\ \tau_{zx} = -\Theta_y \gamma_y \Rightarrow \vec{\tau} = \tau_{zx} \hat{i} + \tau_{zy} \hat{j} \\ \tau_{zy} = \Theta_x \gamma_y \end{array} \right.$$

quindi:

$$\vec{\tau} \cdot \vec{r} = \tau_{zx} \cdot x + \tau_{zy} \cdot y = 0$$

cioè $\vec{r} \perp \vec{\tau}$



$$\begin{aligned}\tau_t &= \int_A (\tau_{zx} \cdot x - \tau_{zy} \cdot y) dA \\ &= \int_A (\Theta G x^2 + \Theta G y^2) dA \\ &= \Theta G \int_A (x^2 + y^2) dA = \Theta G \int_A r^2 dA \\ &= \Theta G J_p\end{aligned}$$

$$\tau_x = \int_A \tau_{zx} dA = \int_A -\Theta G y dA = -\Theta G \int_A y dA = 0$$

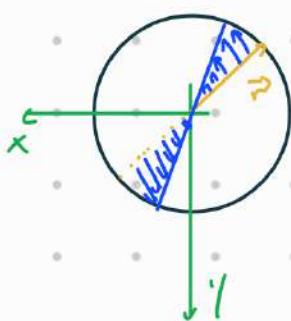
$$\tau_y = \int_A \tau_{zy} dA = \int_A \Theta G x dA = \Theta G \int_A x dA = 0$$

$$\tau = |\vec{\tau}| = \sqrt{\tau_{zx}^2 + \tau_{zy}^2} = \sqrt{\Theta G^2 y^2 + \Theta G^2 x^2} = \Theta G r$$

$$\tau = \frac{\tau_t}{G J_p} \cdot G r \rightarrow \tau = \frac{\tau_t}{J_p} \cdot r$$

$$\hookrightarrow r=0 \rightarrow \tau=0$$

$$r=R \rightarrow \tau_{max} = \frac{\tau_t}{J_p} \cdot R$$



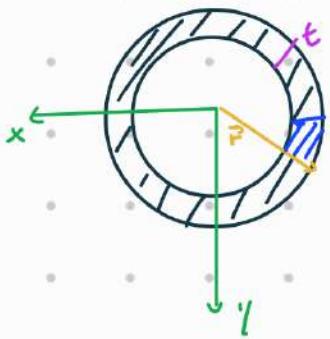
valori numerici

$$J_x = J_y = \frac{\pi}{64} D^4$$

$$J_p = J_x + J_y = \frac{\pi}{32} D^4 = \frac{\pi}{2} R^4$$

$$\tau_{max} = \frac{\tau_t}{\frac{\pi}{2} R^4} \cdot R = \frac{2}{\pi} \frac{\tau_t}{R^3}$$

caso: sezione circolare conca



$$t = R_e - R_i$$

$$I_p = \frac{\pi}{2} (R_e^4 - R_i^4)$$

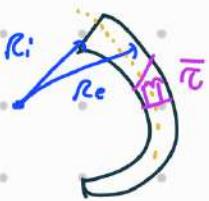
$$\gamma = \frac{It}{I_p} r \quad \text{con} \quad R_i \leq r \leq R_e$$

$$\text{fattur} \quad r_i = \frac{It}{I_p} R_i$$

$$r_e = \frac{n_e}{I_p} R_e$$

caso: sezione circolare a PARETE SOTTILE

$$R_e, R_i \gg t \quad \text{e} \quad R_e, R_i > 10t \quad \text{dove} \quad t = R_e - R_i$$



$$R_m = \frac{R_i + R_e}{2} \Rightarrow \begin{cases} R_e = R_m + \frac{t}{2} \\ R_i = R_m - \frac{t}{2} \end{cases}$$

$$\bar{c} = \frac{\pi t}{\frac{\pi}{2} (R_e^2 - R_i^2)} \cdot R_m$$

dove

$$\begin{aligned} (R_e^2 - R_i^2) &= (R_e^2 + R_i^2)(R_e^2 - R_i^2) \\ &\stackrel{!}{=} (R_e^2 + R_i^2)(R_e - R_i)(R_e + R_i) \\ &\stackrel{!}{=} (R_e^2 + R_i^2) t \cdot 2R_m \\ &\stackrel{!}{=} ((R_m + \frac{t}{2})^2 + (R_m - \frac{t}{2})^2) t \cdot 2R_m \quad \text{piccolo} \\ &\stackrel{!}{=} (R_m^2 + R_m t + \frac{t^2}{4} + R_m^2 - R_m t + \frac{t^2}{4}) \cdot 2R_m \cdot t \\ &= 4R_m^3 t \end{aligned}$$

cose:

$$\bar{c} = \frac{\pi t}{\frac{\pi}{2} \cdot 4R_m^3 t} R_m = \frac{\pi t}{2R_m^2 t} = \frac{\pi t}{2R_m^2 t}$$

oss:

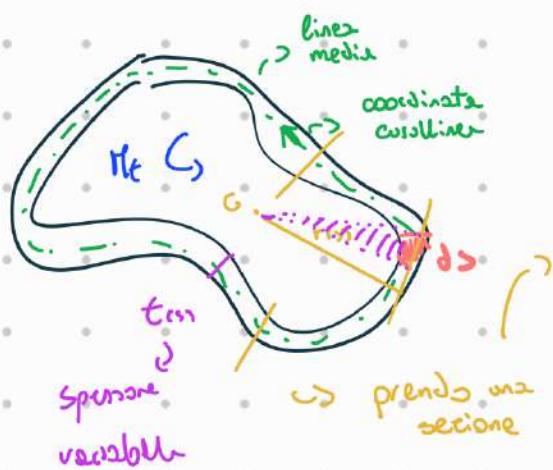
nelle sezioni CHIUSE a PARETE SOTTILE si può generalizzare e riunire le formule J. Bernoulli, cioè

$$\bar{c} = \frac{M_e}{c \pi t}$$

$\bar{c} \sim$ Area racchiusa
dal raggio medio

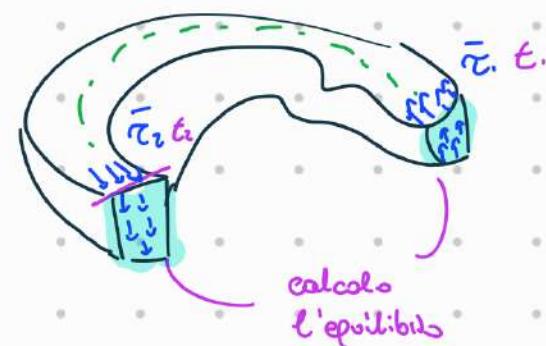
generalizzazione

Formula di Bernoulli



sezioni generate chiuse a parete sottile

(sezioni alla Bernoulli)



q. se ne è

$$\bar{t}_{tot} \cdot A_e = \bar{t}_m \cdot t_m \cdot \Delta e \Rightarrow \bar{t}_e \cdot t_e = \bar{t}_m \cdot t_m$$

cose

$$\bar{t}_{tot} \cdot t_{tot} = \text{cost}$$

$$\begin{aligned} \bar{t}_e &= \oint \bar{t}_m t(s) r(s) ds = \bar{t}_m t_{tot} \oint r(s) ds = \bar{t}_m t_{tot} \cdot \text{Area media} \\ &= \bar{t}_m t_{tot} \end{aligned}$$

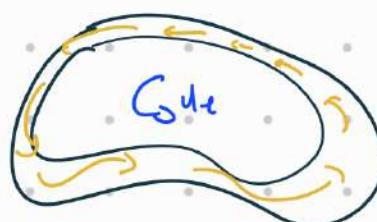
cioè

$$\bar{t}_m = \frac{\bar{t}_e}{t_{tot}}$$

oss: se prendessimo un fluido IDEALE NEWTONIANO in moto laminare.

c'è ANALOGIA IDRODINAMICA

\hookrightarrow t si comporterebbe come un fluido



cosa succederebbe
se tagliassi?



le geometrie complesse si
possono approssimare a rettangoli

sezione rettangolare



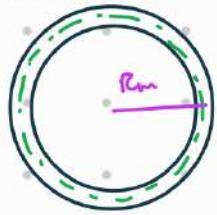
$$J_t = \frac{1}{3} a t^3$$

$$T_{max} = \frac{\bar{t}_e}{J_t} \cdot t$$

momento di
inercia torsionale

esempio

sezione chiusa

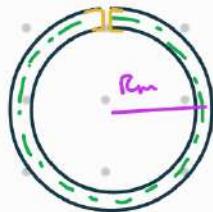


$$C_{max} = \bar{E} = \frac{\pi t}{2\pi t} = \frac{\pi t}{2}$$

$$S_t = \pi R_m^2 t$$

$$C_{max} = \frac{\pi t}{2\pi R_m t} = \frac{\pi}{2R_m}$$

sezione aperta



$$C_{max} = \frac{\pi t}{S_t} = \frac{\pi t}{\frac{1}{3}\pi R_m t} = \frac{3}{R_m}$$

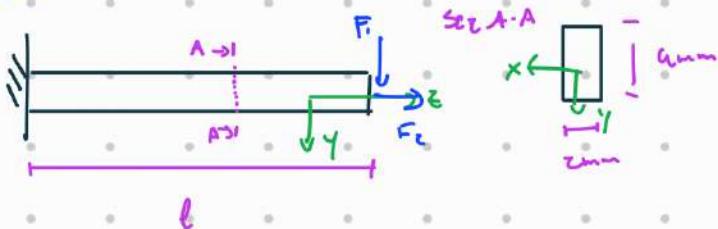
$$\bar{J}_e = \sum_i \frac{1}{3} a_i t_i^3 = \frac{1}{3} \pi R_m t^3$$

$$C_{max} = \frac{\pi t}{\frac{1}{3}\pi R_m t} = \frac{3}{R_m} = \frac{3}{2\pi R_m t}$$

$$\frac{C_{max}}{C_{max}^c} = \frac{3}{2} \frac{\pi t}{\pi R_m t} = \frac{3}{2} \frac{R_m}{t} \quad R_m \gg t$$

la sezione chiusa è molto più rigida a parità di \bar{E}

esercizio



$$l = 100 \text{ mm}$$

lega: titano

$$E = 110 \text{ GPa}$$

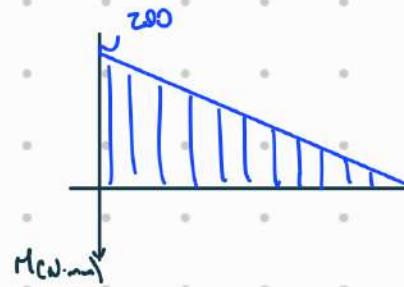
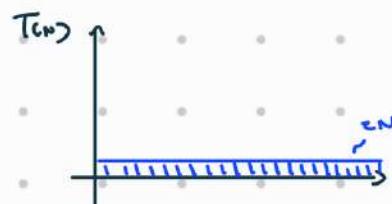
$$V = 0,33$$

$$\sigma_y = 660 \text{ MPa}$$

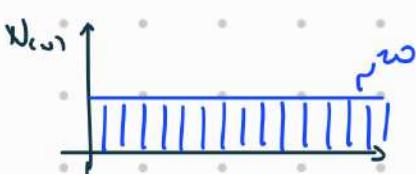
$$F_U = 2 \text{ N}$$

$$F_L = 20 \text{ N}$$

> Calcolo delle sollecitazioni e delle tensioni:

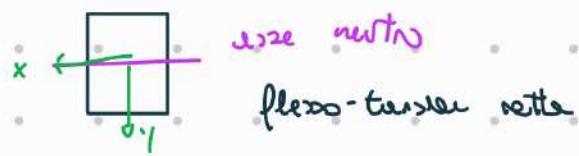


momento
asse x



Andamento tensioni normali

sono in un problema piano



asse neutro
flesso-tensioni rette

$$\sigma_z = \frac{N}{A} + \frac{M_x}{J_x} \cdot y$$

$$A = c \cdot a = 8 \text{ mm}^2$$

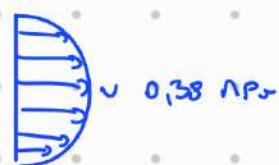
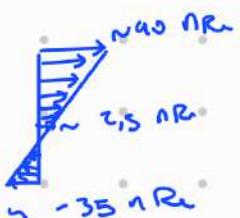
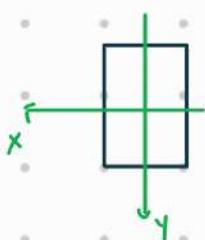
$$J_x = \frac{1}{12} \cdot c \cdot a^3 = 10,66 \text{ mm}^3$$

$$\Rightarrow \sigma_z = \frac{c\sigma}{8} + \frac{(-c\tau)}{10,66} \cdot y$$

$$\sigma_z (-c \text{ mm}) = \frac{c\sigma}{8} + \frac{(-c\tau)}{10,66} (-c) = 40 \text{ MPa}$$

$$\sigma_z (c \text{ mm}) = \frac{c\sigma}{8} + \frac{(-c\tau)}{10,66} (c) = -35 \text{ MPa}$$

$$\sigma_z (0 \text{ mm}) = c_1,5 \text{ MPa}$$



Tensioni tangenziali (τ_{xy})

$$\tau_{xy} = \frac{T s}{J_b} \quad \rightarrow \quad \tau_{xy, \max} = \frac{3}{2} \frac{T}{A} = \frac{3}{2} \frac{c}{c} \frac{c}{8} = 0,38 \text{ MPa}$$

Tensione equivalente J: Vda file

$$\sigma_{eq} = \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - (\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x) + 3(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}$$

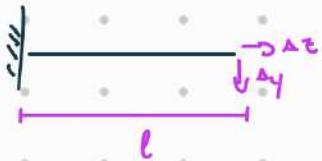
$\sigma_x = \sigma_y = \tau_{xy} = 0$ problema J de Saint-Venant

$$\sigma_{eq} = \sqrt{\sigma_z^2 + 3(\tau_{xy}^2 + \tau_{zx}^2)} = \sqrt{\sigma_z^2 + 3\tau_{xy}^2}$$

$$\text{Per } \gamma_{\text{c-conv}} \Rightarrow \sigma_{\text{m}} = \frac{\sigma}{\gamma_{\text{c-conv}}} = 40 \text{ MPa} < \sigma_{\text{yield}} \text{ GCU NR}$$

fattore di sicurezza $FS = \frac{\sigma_y}{\sigma_m} = \frac{160}{40} = 16,5$

5) calcolo dello spostamento verticale



$E \cdot A$ = rigidezza ASSIALE della sezione

$E \cdot J$ = rigidezza FLESSIONALE della sezione

$$\Delta_x = \frac{N}{E \cdot A} \cdot l = \frac{20 \cdot 100}{110 \cdot 10^3 \cdot 8} = 0,002 \text{ mm}$$

$$\begin{aligned} \Delta_y &= \frac{F \cdot l^3}{3E \cdot J} + \alpha \frac{F \cdot l}{G \cdot A} \quad \rightarrow G = \frac{E}{2(1+\nu)} = \frac{110 \cdot 10^3}{2(1+0,33)} \\ &= \frac{2 \cdot 100^3}{3 \cdot 110 \cdot 10^3 \cdot 10 \cdot 10^6} + \frac{6}{5} \frac{2 \cdot 100}{41353 \cdot 8} \\ &\approx 0,57 + 0,007 = 0,5707 \text{ mm} \end{aligned}$$

OSS: sostituti di materiali siliconici PDMS

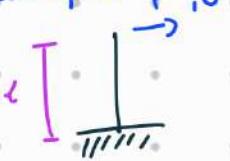
testi: ingegneriali



è importante tenere sotto delle deformabilità

=specificamente se è geometrica TOZZA

esempio:



$$l = 30 \mu\text{m}$$

$$\text{sezion circular } \phi = 10 \mu\text{m}$$

$$\text{PDMS: modulo di Young } E = 1,4 \text{ N/m}$$

$$\text{coefficiente di Poisson } \nu \approx 0,5$$

10%

$$F \approx \mu \text{N}$$

$$U = \frac{Fl^3}{3EI} + \chi \cdot \frac{Fl}{GA} = F \left(\frac{l^3}{3EI} + \chi \cdot \frac{l}{GA} \right)$$

$$\text{considero } k = \frac{U}{F} = \frac{l^3}{3EI} + \chi \cdot \frac{l}{GA}$$

deformabilità

$$A = \frac{\pi}{4} \phi^2 = \frac{\pi}{4} \cdot 10^4 = 78,54 \text{ } \mu\text{m}^2$$

$$I = \frac{\pi}{64} \phi^4 = \frac{\pi}{64} \cdot 10^8 = 490,87 \text{ } \mu\text{m}^4$$

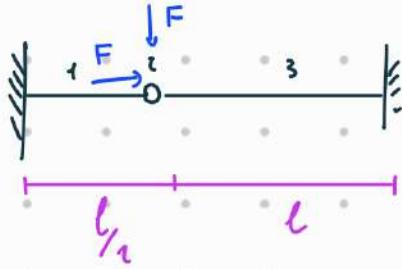
$$G = \frac{E}{2(1+\nu)} = \frac{1,4}{2(1+0,5)} = 0,667$$

perindi

$$k' = \frac{30^3}{3 \cdot 1,4 \cdot 490,87} + \frac{10}{9} \frac{30}{0,667 \cdot 78,54} \\ = 13,1 + 0,91 = 14,01 \text{ } \mu\text{m/N}$$

contribuisce
al 7%

Esercizio



area trasversale A

momento di incis J

molla elastica longitudinale E

Determinare componenti & sollecitazioni

metodo degli spostamenti

stretture per stretture

(metodo delle forze) $\rightarrow 8-6=2$
2 incognite per stretture

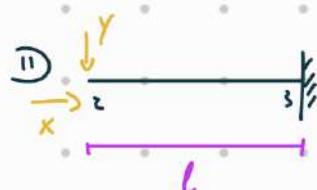
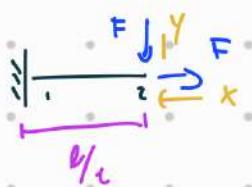
gradi di libertà = 6

condizioni di vincolo = 8

stretture fisse

Sistema principale

D



rigidezza assiale ΣA

rigidezza flessionale ΣJ

$$\text{so che } U_e^{(D)} = U_e^{(II)} \quad \Rightarrow \quad \text{equaz. di congruenza}$$

$$U_e^{(D)} = \frac{N}{EA} \frac{l}{2} = \frac{F \cdot x}{EA} \frac{l}{2}$$

sistema D

$$\text{sforzo assiale } N = F - x$$

$$U_e^{(D)} = \frac{N}{EA} \frac{l}{2} = \frac{F-x}{EA} \frac{l}{2}$$

sistema II

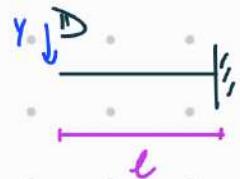
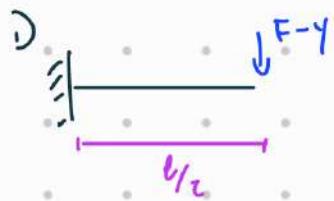
$$\text{sforzo assiale } N = -x$$

$$U_e^{(II)} = \frac{x}{EA} l \quad \rightarrow \quad \text{segno deve essere coerente}$$

equazione di congruenza

$$U_e^{(D)} = U_e^{(II)} \rightarrow \frac{F-x}{EA} \frac{l}{2} = \frac{x}{EA} l \rightarrow x = \frac{F}{3}$$

l'elemento rigido
assorbe più forza



$$v_c^D = \frac{(F-\gamma) (\frac{l}{c})^3}{3EI}$$

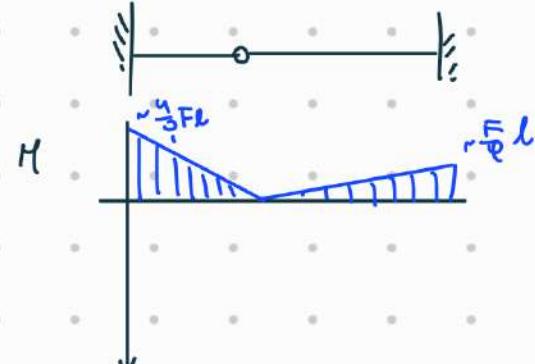
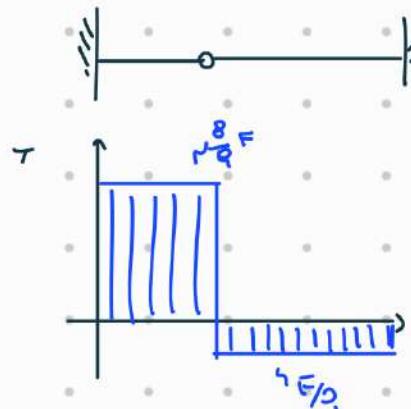
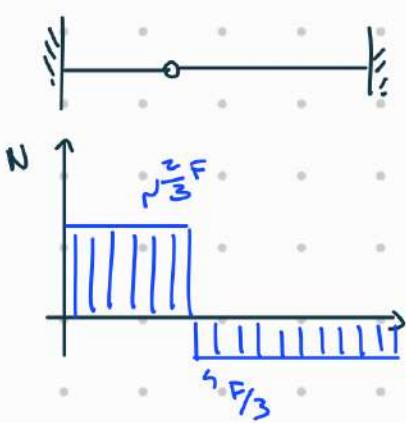
$$v_c^D = \frac{\gamma l^3}{3EI}$$

equazione di congruenza

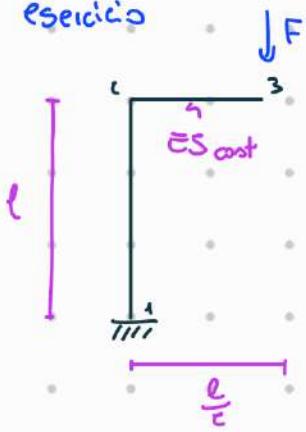
$$v_i^D = v_c^D \rightarrow \frac{(F-\gamma) l^3}{3EI} = \frac{\gamma l^3}{3EI}$$

$\gamma = F/q$ → per avere la stessa deformazione flessionale, le strutture p.i. lungo ricevono di meno forza.

Soluz. tensioni:

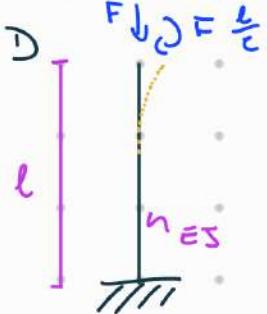


esercizio



calcolare trascinazione e rotazione delle sezione 3

applica sovrapposizione dell'effetti



ignora EA

$$v_3^D = \frac{Fl}{c} \cdot \frac{l^2}{2EJ} = \frac{Fl^3}{4EJ}$$

$$\varphi_3^D = \frac{Fl}{c} \cdot \frac{l}{EJ} = \frac{Fl^2}{2EJ} \text{ oraria} \rightarrow \text{negat.}$$

$$v_3^D = v_3 \quad \text{trascinazione verso destra}$$

$$\varphi_3^D = \varphi_3 \quad \text{rotazione oraria}$$

$$\tau_3^D = \varphi_3 \frac{l}{c} = \frac{Fl^3}{4EJ} \quad \text{trascinazione verso il basso}$$

D



$$v_3^D = 0$$

$$\varphi_3^D = \frac{F(v_3)^2}{2EJ} = \frac{Fl^2}{8EJ} \quad \text{oraria}$$

$$\tau_3^D = \frac{F(l/2)^2}{2EJ} = \frac{Fl^2}{8EJ} \quad \text{trascinazione verso il basso}$$

Spostamento complessivo

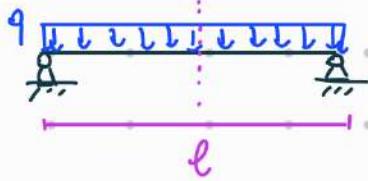
$$v_3^{\text{tot}} = v_3^D + v_3^D = \frac{Fl^3}{4EJ} \quad \text{verso destra}$$

$$v_3^{\text{tot}} = v_3^D + v_3^D = \frac{Fl^3}{4EJ} + \frac{Fl^3}{8EJ} = \frac{7}{24} \frac{Fl^3}{EJ}$$

$$\varphi_3^{\text{tot}} = \varphi_3^D + \varphi_3^D = \frac{Fl^2}{2EJ} + \frac{Fl^2}{8EJ} = \frac{5}{8} \frac{Fl^2}{EJ}$$

Simmetrie

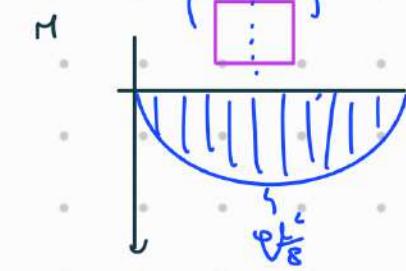
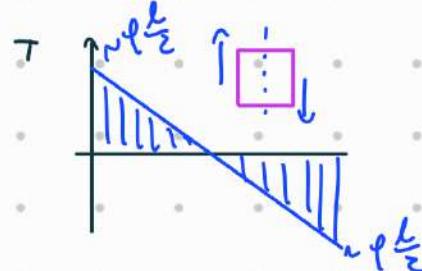
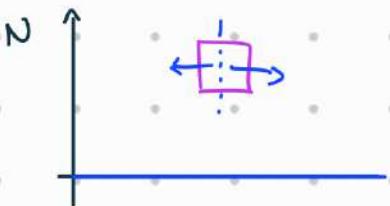
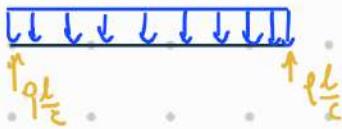
esercizio



struttura simmetrica caricata simmetricamente

proprietà omogenee: A, J, E, V

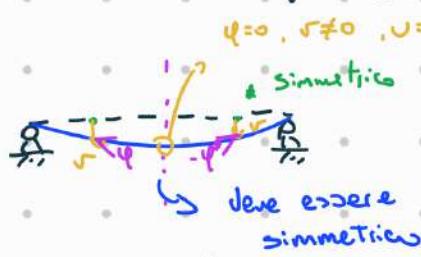
d) geometrica e materiale



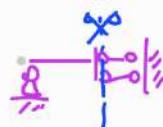
oss: • M è simmetrico

• T è antisimmetrico

considero la struttura deformata

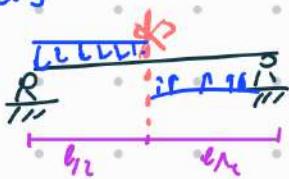


$\varphi=0, r \neq 0, u=0 \rightarrow$ ricorda vincolo doppia bille

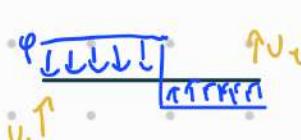


cioè posso studiare solo una metà della struttura

esercizio struttura simmetrica caricata antisimmetricamente



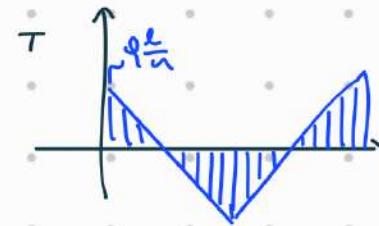
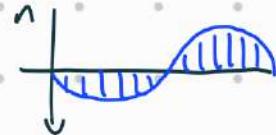
sistema più rapido

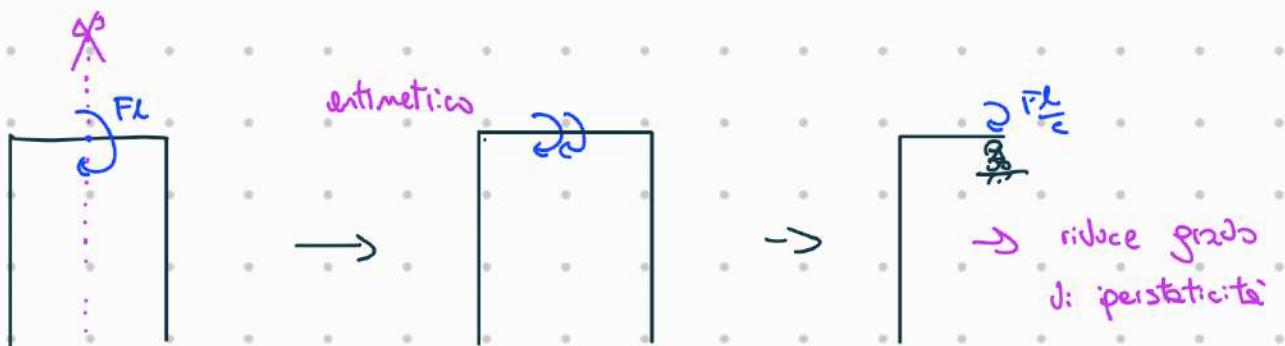
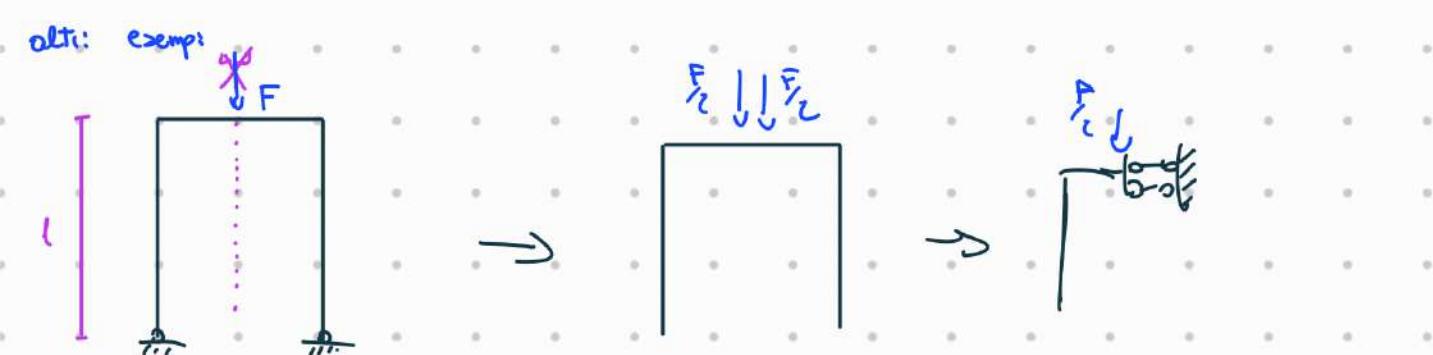


momento z:

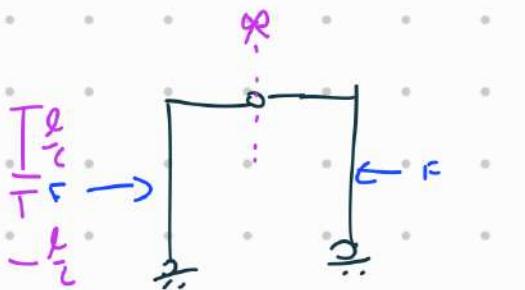
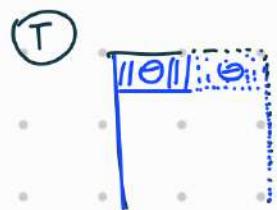
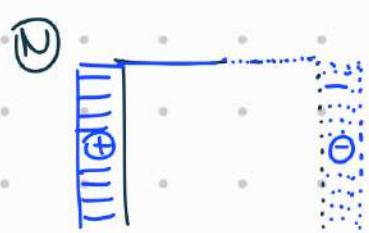
$$\frac{qL}{c} \cdot \frac{3}{8}l - \frac{qL}{2} \cdot \frac{l}{4} - V \cdot l = 0$$

$$V_1 = \frac{1}{4}qL \Rightarrow V_c = -\frac{1}{4}qL$$

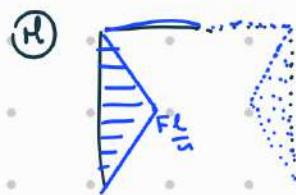
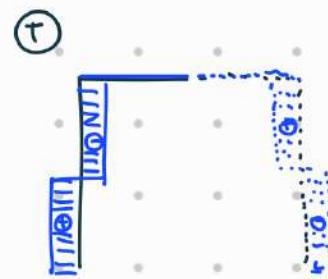
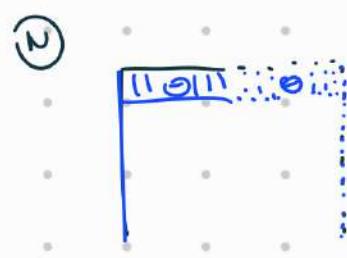
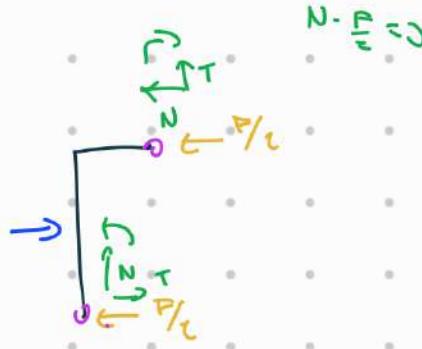




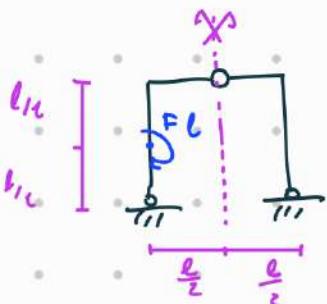
Diagrammi: J: sollecitazione



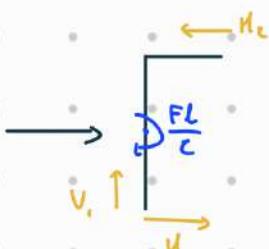
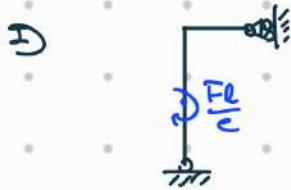
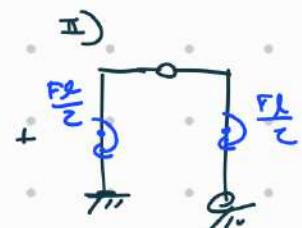
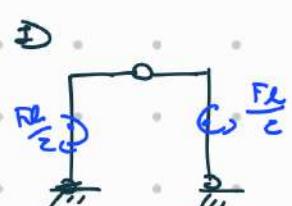
N.B.
D. BICELLA
la centrale interna è importante



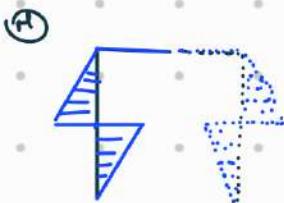
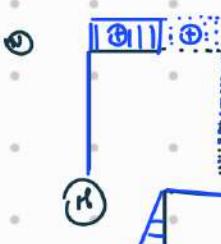
esempi simmetrie con vincoli



$\xrightarrow{\text{supposte effett.}}$



$$\left\{ \begin{array}{l} h_0 l - \frac{F_L}{2} = 0 \Rightarrow h_0 = F_L / l \\ v_0 = 0 \quad M_0 + h_0 z - \frac{F_L}{2} = 0 \Rightarrow \end{array} \right.$$

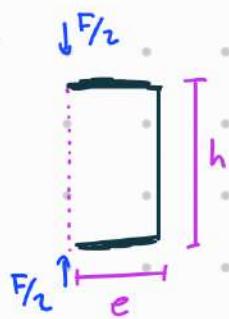


esempi fissatori spirali

ASTM vs ISO



ASTM



$e = 62 \text{ mm}$

$h = 76 \text{ mm}$

Titanio

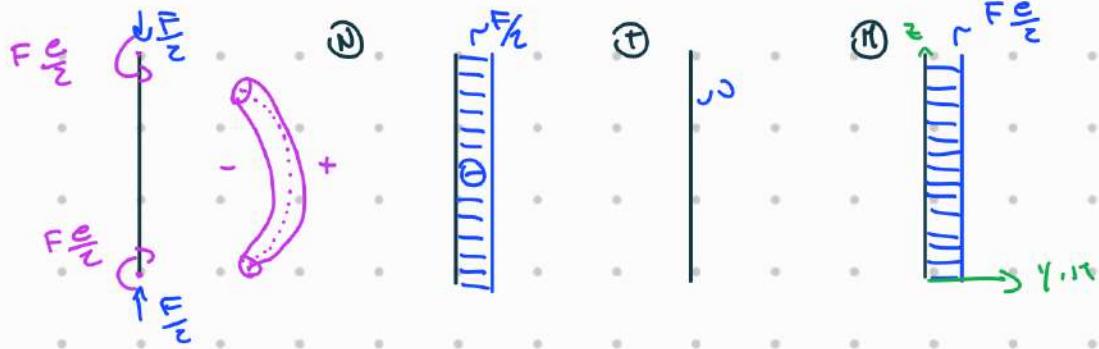
$\sigma_y = 850 \text{ N/mm}^2$

$\phi = 7 \text{ mm}$

Calcolare carico limite F_L corrispondente

al limite elastico della trave

Sollecitazioni sulla barra



Schema Tensio-flessione

$$\sigma_z = \frac{N}{A} + \frac{\tau}{J} y$$

compressione

$$! \quad \frac{F_e}{z} + \frac{F_e z}{J} \cdot (-\phi/z)$$

tensione equivalente di von Mises

$$\sigma_{vn} = \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - (\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x) + 3(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2)}$$

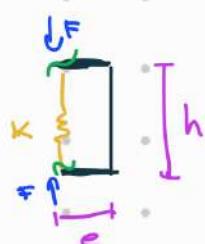
sen
jerk

$$\sqrt{\sigma_z^2 + 3(\tau_{xz}^2 + \tau_{yz}^2)} \rightarrow \sigma_{vn} = 1/\sigma_z$$

$$\sigma_{vn} = \frac{F_e}{zA} + \frac{F_e c}{4J} \phi = \sigma_p$$

$$F_e = \frac{\sigma_p}{\frac{1}{zA} + \frac{c\phi}{4J}} = 1400 \text{ N} \quad 1 \text{ m} \quad 1 \text{ kg} \quad F$$

150



$$h = 76 \text{ mm}$$

$$e = 40 \text{ mm}$$

$$K = 190 \text{ N/mm}$$

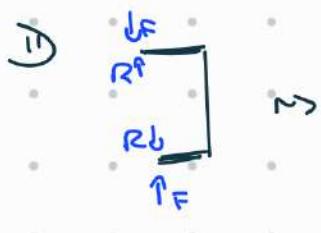
$$F = 200 \text{ N}$$

Tensione

$$\sigma_y = 890 \text{ N/mm}^2$$

$$E = 110000 \text{ N/mm}^2$$

$$\textcircled{1} \quad \begin{array}{l} \downarrow R \\ K \\ \uparrow R \end{array}$$



$$\textcircled{3} \quad \begin{array}{c} \downarrow (F-R)/2 \\ I \\ \uparrow I \\ (F-R)/2 \end{array}$$

$\frac{R}{K}$ accorciamento delle molle

$$\frac{R}{K} = c\eta \quad \text{equazione 1, congruenza}$$

$$\frac{\downarrow F_E}{E\delta} \sim \left[\frac{h}{c} \right]$$

$$\frac{R \cdot c \cdot \epsilon_S + F_E}{c} \sim \frac{h}{c}$$

$$\eta = \varphi \cdot e$$

$$\varphi = \frac{\left(\frac{F-R}{c} \cdot e \right) h_c}{E \cdot J}$$

$$\eta = \varphi \cdot e = \frac{(F-R) e^2 h}{4 E \delta}$$

$$\frac{R}{K} = c\eta = \frac{(F-R) e^2 h}{c E J} \cdot \frac{1}{k_p}$$

$$R \left(\frac{1}{K} + \frac{1}{k_p} \right) = \frac{F}{k_p}$$

$$R = \frac{K}{K+k_p} F$$

grafici sollecitazioni

N



T



H



$$|\nabla \epsilon| : \frac{|N|}{A} + \frac{|M \phi/c|}{J}$$

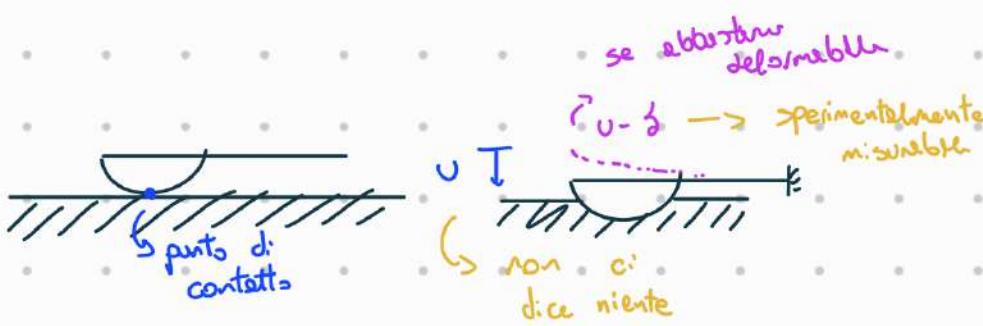
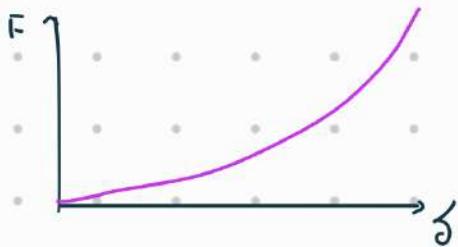
Problema del contatto elastico

δF forza di indentazione I problemi di indentazione

materiali isotropi

$$F = \frac{4}{3} \frac{\epsilon}{1-\nu^2} \sqrt{R} \delta^{3/2}$$

↳ specifico per INDENTAZIONI SPHERICHE

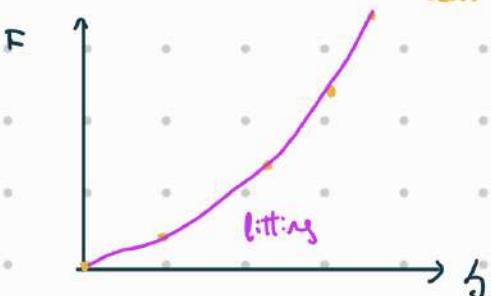


(este) rigidezza del cantilever

$$F = K(\nu - \delta)$$

$$= \frac{4}{3} \frac{E}{1-\nu^2} \sqrt{R} \delta^{3/2}$$

↳ solitamente ignota



OSS: in quest' problema la scala è
 $K \approx 0.1 \text{ N/m}$

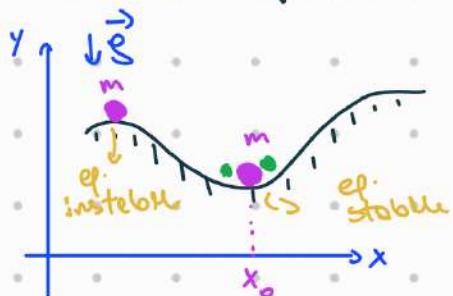
OSS: estensimetro GANGE

calcolo ϵ_e esempio fissato spazio

$$\sigma_e = \frac{N}{A} + \frac{F}{S_e} \cdot l \rightarrow \epsilon_e E = \sigma_e \quad \text{per verificare, basta l'estensione GANGE}$$

Instabilità dell'equilibrio elastico

Cos'è un equilibrio stabile?



$$\Pi = \text{energia potenziale totale}$$

$$= mgx + \text{cost}$$

$$\Pi(x_0 + dx) = \Pi(x_0) + \frac{\partial \Pi}{\partial x} \Big|_{x=x_0} dx$$

Definiamo $\delta_x = \frac{\partial \Pi}{\partial x} \Big|_{x=x_0} dx$

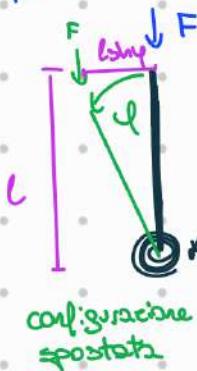
$$= 0 \rightarrow \text{configurazione di equilibrio}$$

Come definire tipologie di stabilità?

$$\Pi(x_0 + dx) = \Pi(x_0) + \frac{\partial \Pi}{\partial x} \Big|_{x=x_0} dx + \frac{1}{2} \frac{\partial^2 \Pi}{\partial x^2} \Big|_{x=x_0} dx^2$$

se $\delta_x^2 > 0$ stabile se vario via del punto di equilibrio, si tende a ritornare

Esempio



k rigidezza rotazionale

F forza INVARIANTE per direzione

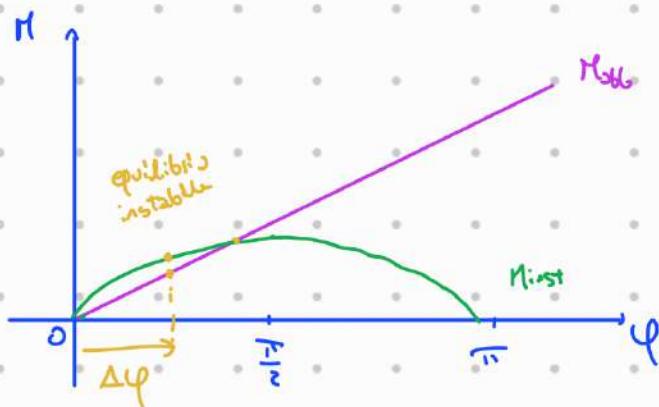
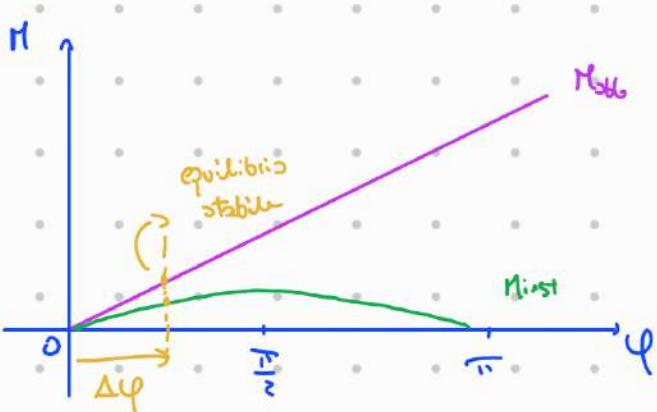
$M_{\text{inst}} = Fl \sin \varphi$ momento instabilizzante

$$M_{\text{stab}} = K\varphi$$

momento stabilitizzante

per el

$$M_{\text{inst}} = M_{\text{stab}}$$



$$h_{\text{stab}} = k\varphi \rightarrow \frac{dh_{\text{stab}}}{d\varphi} = k \xrightarrow{\varphi=0} k \Rightarrow F_c = \frac{k}{2}$$

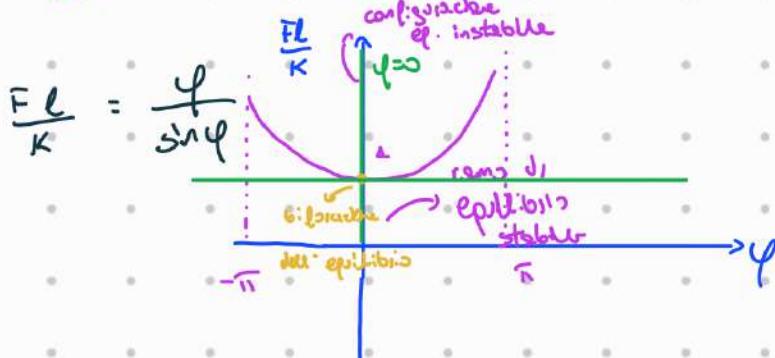
carico critico
J: instabilità

$$h_{\text{inst}} = Fl \sin\varphi \rightarrow \frac{dh_{\text{inst}}}{d\varphi} = Fl \cos\varphi \xrightarrow{\varphi=0} Fl$$

oppure

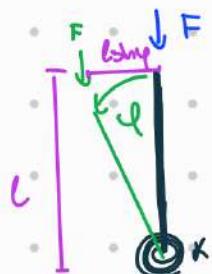
$$Fl \sin\varphi = k\varphi \quad \text{ha soluzioni:}$$

- $\varphi=0$ configurazione \downarrow cl. iniziale
- $\varphi \neq 0$ configurazione eq. instabile



se rotazioni piccole $\sin\varphi \approx \varphi$

$$\text{quindi } Fl \sin\varphi = k\varphi \rightarrow \frac{Fl}{K} = 1$$



$$W = \frac{1}{2} K \dot{\varphi}^2$$

U = potenziale delle forze

$$= Fl\varphi \rightarrow \frac{\partial U}{\partial \varphi} : F$$

$$U = F(l - \cos\varphi) \rightarrow \ddot{\varphi} = \frac{1}{2} k \dot{\varphi}^2 - Fl(l - \cos\varphi)$$

$$\frac{1}{2} m \dot{\varphi}^2 + F U - (l - \cos\varphi)$$

velocizzi zero

dejto $\ddot{\varphi} = 0$

Trave di Euler

esercizio

deformabile flessivamente



EJ rigidezza flessionale

sistema principale



$$\begin{cases} H_1 = F \\ V_1 = V_2 = 0 \end{cases}$$

non dovrebbe flessionarsi
se F compire abbastanza
grande instabilità elastica

configurazione deformata



$$F(y) - F\gamma(z) = 0$$

Q. diff. delle linee elastiche

$EJ\gamma'(z) = -F\gamma(z)$ → piccole deformazioni, però va bene

$$EJ\gamma''(z) + F\gamma(z) = 0 \quad \text{diventa}$$

$$\gamma'' + \frac{Fy}{EJ} = 0 \quad \text{con} \quad \alpha^2 = \frac{F}{EJ}$$

$$\gamma'' + \alpha^2 \gamma = 0$$

$$\gamma = A \sin(\alpha z) + B \cos(\alpha z)$$

$$\gamma' = A \alpha \cos(\alpha z) - B \alpha \sin(\alpha z)$$

$$\gamma'' = -A \alpha^2 \sin(\alpha z) - B \alpha^2 \cos(\alpha z) = -\alpha^2 \gamma$$

condizioni al contorno

$$\begin{cases} \gamma(0) = 0 \\ \gamma(z=l) = 0 \end{cases}$$

$$B=0$$

$$A \sin(\alpha l) + B \cos(\alpha l) = 0$$

$$B=0$$

$$A \sin(\alpha l) = 0$$

$A=0$ configurazione iniziale

$$A \sin(\alpha l) = 0$$

$A \neq 0 \quad \alpha l = 0 \quad \text{carico nullo}$

$$\alpha l = \pm \pi, \pm 2\pi \dots$$

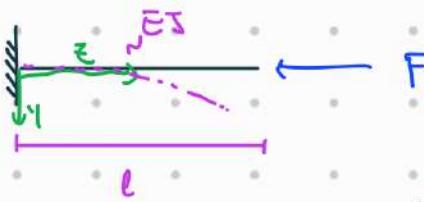
prendiamo $\alpha l = \pm \pi$

$$(\alpha l)^2 = \pi^2 \rightarrow \frac{F}{EJ} l^2 = \pi^2 \rightarrow F_{cr} = \frac{\pi^2 EJ}{l^2}$$

carico critico
di instabilità
Euleriano

OSS: se si supera, la trave può rompersi anche se è in regime elastico

esercizio



$$M(z) + F(e - y(l)) = 0 \quad M(z) = -EJ y''(z)$$

$$-EJ y''(z) + Fe - Fy(l) = 0$$

$$EJ y'' + Fy = Fe$$

$$y'' + \frac{F}{EJ} y = \frac{F}{EJ} e \quad \alpha^2 = \frac{E}{EJ}$$

$$y'' + \alpha^2 y = \alpha^2 e \quad \text{non omogenea}$$

soluzione generale

$$y = y_0 + \bar{y}$$

y_0 soluzione generale dell'omogenea associata

\bar{y} soluzione particolare

Casi:

$$y = A\sin(\alpha z) + B\cos(\alpha z) + e$$

$$y' = A\alpha \cos(\alpha z) - B\alpha \sin(\alpha z)$$

Condizione J1: contorno

$$\begin{cases} y(0) = 0 \\ y(l) = 0 \end{cases}$$

$$\begin{cases} B + e = 0 \\ Al = 0 \end{cases}$$

$$\begin{cases} B = -e \\ A = 0 \end{cases}$$

$$y(z) = -e \cos(\alpha z) + e$$

$$y(l) = -e \cos(\alpha l) + e \neq 0$$

$$\Leftrightarrow \cos(\alpha l) = 0$$

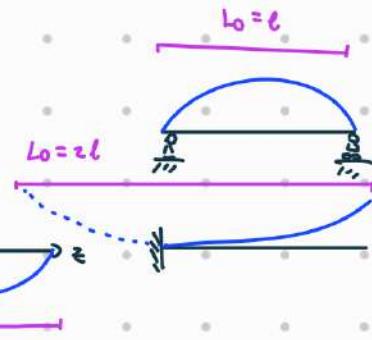
$$\alpha l = \pm \frac{\pi}{2}$$

$$\alpha^2 l^2 = \frac{\pi^2}{4} \rightarrow \frac{F}{EJ} l^2 = \frac{\pi^2}{4} \rightarrow F_{cr} = \frac{\pi^2 EJ}{4l^2}$$

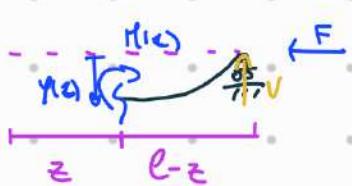
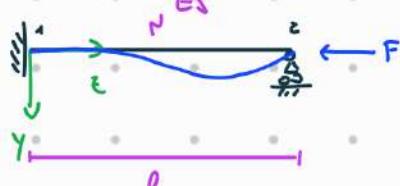
oss: I due esempi possono essere assimilati in una unica formula,

$$F_{cr} = \frac{\pi^2 EI}{L_0^2} \quad \text{dove } L_0 \text{ lunghezza libera J: Miseurone}$$

L_0 è la semilunghezza della sinusoida nella quale si ha la deformazione



esercizio



$$M(z) - F_l(z) - V(l-z) = 0$$

$$\text{dove } M(z) = EIy''$$

$$-EIy'' - Fy = V(l-z)$$

$$EIy'' + Fy = V(l-z)$$

$$y'' + \frac{F}{EI}y = \frac{V(l-z)}{EI} \quad \alpha = \frac{F}{EI}$$

$$y'' + \alpha^2 y = \frac{V}{F} \alpha^2 (z-l)$$

soltuzione generale

$$y = A \sin(\alpha z) + B \cos(\alpha z) + \frac{V}{F} (z-l)$$

$$y'(z) = A \alpha \cos(\alpha z) - B \alpha \sin(\alpha z) + \frac{V}{F}$$

Condizioni al contorno

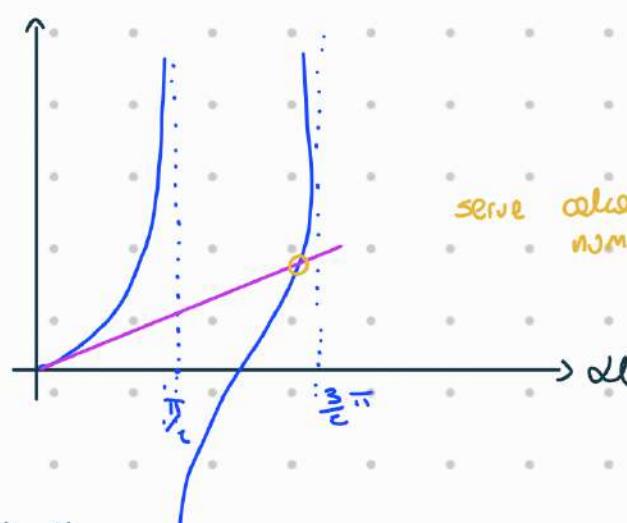
$$\begin{cases} y(0) = 0 \\ y(l) = 0 \\ y'(0) = 0 \end{cases} \quad \begin{cases} B - \frac{V}{F} l = 0 \\ A \sin(\alpha l) + B \cos(\alpha l) = 0 \\ A \alpha + \frac{V}{F} = 0 \end{cases}$$

$$\begin{cases} B = \frac{V}{F} l \\ \frac{\sin(\alpha l)}{\cos(\alpha l)} = -\frac{B}{A} \\ A = -\frac{V}{\alpha F} \end{cases}$$

$$\begin{cases} A = -\frac{V}{\alpha F} \\ B = \frac{V}{F} l \\ \tan(\alpha l) = \alpha l \end{cases}$$

$\tan(\alpha l)$

αl



serve calcoli numerici

$$\alpha l = 4,15 \quad \tan(\alpha l) = 4,15$$

$$F_{cr} = \frac{\pi^2 EI}{(0,60l/l)^2}$$

def: • Dato che

$$F_{cr} = \frac{\pi^2 E \Sigma}{l_0^2} \Rightarrow \sigma_{cr} = \frac{F_{cr}}{A}$$

tensione
costante

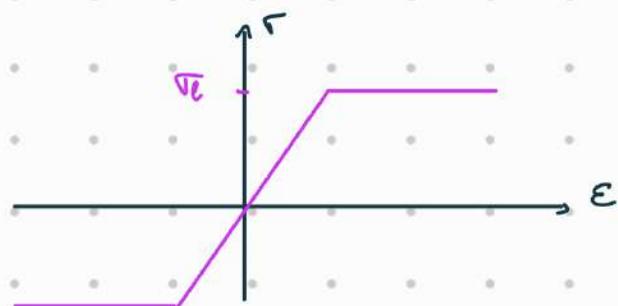
area trasversale
costante

$$r^2 = \frac{\Sigma}{A} \Rightarrow \sigma_{cr} = \frac{\pi^2 E}{(l_0/r)^2}$$

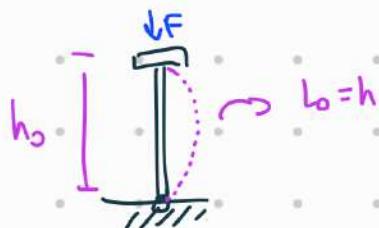
rapporto
di incidenza

$$\lambda = \frac{l_0}{r}$$

s'elenco



esempio per bastone J: Charlie Chaplin



$$h = 80 \text{ cm}$$

$$E = 11000 \text{ MPa}$$

longitudinale

$$\sigma_c = 54 \text{ MPa}$$

$$\phi = 40 \text{ mm}$$

Cerco limite corrispondente σ_c

$$\sigma_c = A \cdot \sigma_c \quad \text{dove } A = \pi \frac{\phi^2}{4} = 314,1 \text{ mm}^2$$
$$= 314,1 \cdot 54 = 16965 \text{ N/mm}^2 \approx 17 \text{ MPa}$$

$$F_{cr} = \frac{\pi^2 E \Sigma}{l_0^2}$$

$$\text{dove } \Sigma = \frac{\pi}{64} \phi^4 = 7854 \text{ mm}^2$$

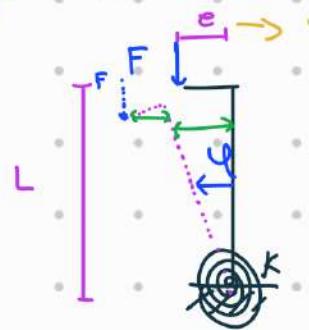
$$= \frac{\pi^2 \cdot 11000 \cdot 7854}{800^2}$$

$$= 1372 \text{ N} \approx 135,3 \text{ kN}$$

quindi cerco limite delle stirature è

$$\min \{ \sigma_c, F_{cr} \}$$

esempio Effetto delle imperfezioni



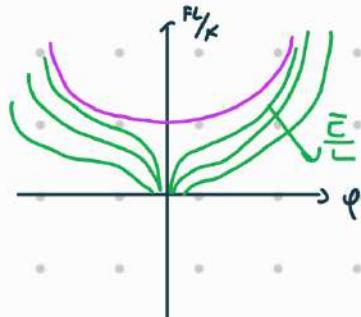
$$T_{\text{inst}} = F(L \sin \varphi + e \cos \varphi)$$

$$M_{\text{inst}} = K\varphi$$

$$\hookrightarrow F(L \sin \varphi + e \cos \varphi) = K\varphi$$

$$FL(\sin \varphi + \frac{e}{L} \cos \varphi) = K\varphi$$

$$\frac{FL}{K} = \frac{\varphi}{\sin \varphi + \frac{e}{L} \cos \varphi}$$



ideale

$$\frac{FL}{K} = \frac{\varphi}{\sin \varphi} + \frac{e}{L} \cos \varphi$$

esempio Trave di Euler

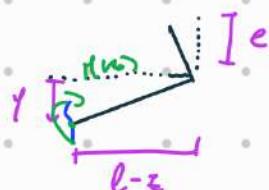


$$F_{\text{cr}} = \frac{\pi^2 E S}{l^2}$$

↓ imperf.



sistema
principio



$$\begin{cases} M(z) - F(e + y(z)) = 0 \\ M(z) = -E S y''(z) = 0 \end{cases}$$

$$\hookrightarrow -E S y'' - F(e + y) = 0$$

$$E S y'' + Fy = -Fe$$

$$y'' + \alpha^2 y = -\alpha^2 e$$

$$y = A \sin(\alpha t) + B \cos(\alpha t) - e^{\alpha t}$$

Condizioni di contorno

$$\begin{cases} y(0) = 0 \\ y(l) = 0 \end{cases} \quad \begin{cases} B - e = 0 \\ A \sin(\alpha l) + B \cos(\alpha l) - e = 0 \end{cases} \quad \begin{cases} B = e \\ A \sin(\alpha l) = e - e \cos(\alpha l) \end{cases}$$

$$\begin{cases} B = e \\ A = \frac{e(1 - \cos(\alpha l))}{\sin(\alpha l)} \end{cases}$$

$$y = \frac{e(1 - \cos(\alpha l))}{\sin(\alpha l)} \sin(\alpha t) + e \cos(\alpha t) - e$$

$$\Rightarrow y\left(\frac{l}{c}\right)$$

$$y\left(\frac{l}{c}\right) = \text{cont:}$$

che

$$\sin(\alpha l) = e \sin\left(\frac{\alpha l}{c}\right) \cos\left(\frac{\alpha l}{c}\right)$$

$$\cos(\alpha l) = \cos^2\left(\frac{\alpha l}{c}\right) - \sin^2\left(\frac{\alpha l}{c}\right)$$

notazione molto elegante

$$\sin = \sin\left(\frac{\alpha l}{c}\right)$$

$$\cos = \cos\left(\frac{\alpha l}{c}\right)$$

$$y\left(\frac{l}{c}\right) = \frac{e}{c \sin \cos} \left(1 - \cos^2 + \sin^2 \right) \sin + \cos - e$$

$$= \frac{e \sin^2}{\cos} + \cos - e$$

$$= \frac{e \sin^2 + e \cos^2 - e \cos}{\cos}$$

$$= \frac{e(1 - \cos)}{\cos} = \frac{e(1 - \cos(\alpha \frac{l}{c}))}{\cos(\alpha \frac{l}{c})}$$

ricordando che

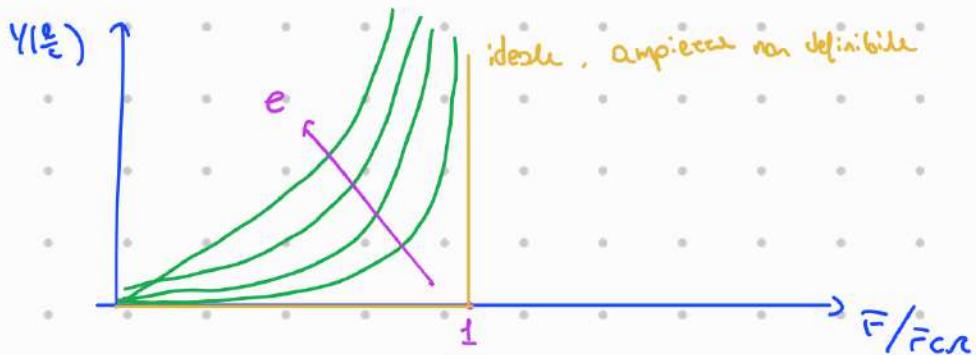
$$\alpha = \sqrt{\frac{F_{\text{es}} \cdot F_{\text{cr}}}{EJ}} = \sqrt{\frac{F}{F_{\text{cr}}} \cdot \frac{F_{\text{cr}}}{EJ}}$$

$$F_{\text{cr}} = \frac{\pi^2 EJ}{l^2}$$

$$= \sqrt{\frac{F}{F_{\text{cr}}} \cdot \frac{\pi^2}{l^2}} = \frac{\pi}{l} \sqrt{\frac{F}{F_{\text{cr}}}}$$

Jr. Jefinition

$$\gamma(\frac{F}{F_{cr}}) = \frac{e \left(1 - \cos \left(\frac{\pi}{c} \sqrt{\frac{F}{F_{cr}}} \right) \right)}{\cos \left(\frac{\pi}{c} \sqrt{\frac{F}{F_{cr}}} \right)}$$



esempio



condizioni al contorno

$$\gamma(0) = 0 \Rightarrow C_0 = 0$$

$$\gamma(L) = 0 \Rightarrow$$

$$\frac{Fe}{6EI} L^3 - \frac{Fe}{2EI} L^2 + C_1 L = 0 \Rightarrow C_1 = \frac{Fe}{2EI} L$$



$$M_r = F$$

$$M_r: Fe - V_r L = 0 \\ \hookrightarrow V_r = F \frac{e}{L} = V_r$$

$$M(x) + F \frac{e}{L} x - Fe = 0 \Rightarrow M(x) = Fe \left(1 - \frac{x}{L} \right)$$

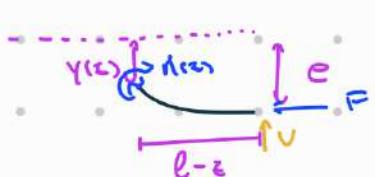
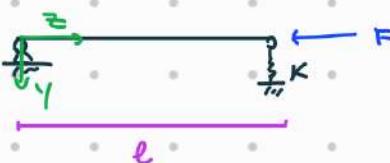
$$\gamma'' = -\frac{M}{EI} = -\frac{Fe}{EI} + \frac{Fe}{EIL} x$$

$$\gamma' = -\frac{Fe}{EI} x + \frac{Fe}{EIL} x^2 + C_1$$

$$\gamma = \frac{Fe}{6EIL} x^3 - \frac{Fe}{2EIL} x^2 + C_1 x + C_2$$

esempio

centrato + flessione



$$\begin{cases} M(z) + F(e - \gamma(z)) - V(l - z) = 0 \\ M(z) = -EI \gamma''(z) \\ \hookrightarrow -EI \gamma''(z) + Fe + F\gamma(z) - V(l - z) = 0 \\ EI \gamma''(z) + F\gamma(z) = Fe + V(l - z) \\ \gamma''(z) + \frac{F}{EI} \gamma(z) = \frac{Fe}{EI} + \frac{V}{EI}(l - z) \end{cases}$$