CHAPTER B

Vector differential operators in frequently used reference systems

Rectangular cartesian coordinates (x_1, x_2, x_3)

$$\nabla f = \sum_{i=1}^{3} \hat{a}_i \frac{\partial f}{\partial x_i} \quad , \tag{B.1}$$

$$\nabla \cdot \mathbf{A} = \sum_{i=1}^{3} \frac{\partial A_i}{\partial x_i} \quad , \tag{B.2}$$

$$\begin{aligned}\nabla \times \mathbf{A} &= \hat{a}_1 \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \right) + \hat{a}_2 \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \right) + \hat{a}_3 \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right) \\ &= \begin{vmatrix} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ A_1 & A_2 & A_3 \end{vmatrix} \quad ,\end{aligned} \tag{B.3}$$

$$\nabla^2 f = \sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i^2} \quad , \tag{B.4}$$

$$\nabla^2 \mathbf{A} = \sum_{i=1}^{3} \hat{a}_i \, \nabla^2 A_i \quad . \tag{B.5}$$

Note that sometimes, in the text, the base unit vectors are denoted as $\hat{x}_1, \hat{x}_2, \hat{x}_3$.

Cylindrical coordinates (r, φ, z)

$$\nabla f = \hat{a}_r \frac{\partial f}{\partial r} + \hat{a}_\varphi \frac{1}{r} \frac{\partial f}{\partial \varphi} + \hat{a}_z \frac{\partial f}{\partial z} \quad , \tag{B.6}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z} \quad , \tag{B.7}$$

$$\nabla \times \mathbf{A} = \hat{a}_r \left(\frac{1}{r} \frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} \right) + \hat{a}_\varphi \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) +$$

$$+\,\hat{a}_z\left(\frac{1}{r}\,\frac{\partial(rA_\varphi)}{\partial r}-\frac{1}{r}\,\frac{\partial A_r}{\partial\varphi}\right)\quad, \tag{B.8}$$

$$\nabla^2 f \;=\; \frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\,\frac{\partial^2 f}{\partial\varphi^2}+\frac{\partial^2 f}{\partial z^2}\quad, \tag{B.9}$$

$$\begin{aligned}\nabla^2\mathbf{A} \;=\;& \hat{a}_r\left(\nabla^2 A_r-\frac{2}{r}\,\frac{\partial A_\varphi}{\partial\varphi}-\frac{A_r}{r^2}\right)+\\ &+\hat{a}_\varphi\left(\nabla^2 A_\varphi+\frac{2}{r^2}\,\frac{\partial A_r}{\partial\varphi}-\frac{A_\varphi}{r^2}\right)+\hat{a}_z(\nabla^2 A_z)\quad.\end{aligned} \tag{B.10}$$

Note that sometimes, in the text, the base unit vectors are denoted as $\hat{r},\hat{\varphi},\hat{z}$.

Spherical coordinates (r,ϑ,φ)

$$\nabla f \;=\; \hat{a}_r\,\frac{\partial f}{\partial r}+\hat{a}_\vartheta\,\frac{1}{r}\,\frac{\partial f}{\partial\vartheta}+\hat{a}_\varphi\,\frac{1}{r\,\sin\vartheta}\,\frac{\partial f}{\partial\varphi}\quad, \tag{B.11}$$

$$\nabla\cdot\mathbf{A} \;=\; \frac{1}{r^2}\,\frac{\partial}{\partial r}\,(r^2A_r)+\frac{1}{r\,\sin\vartheta}\,\frac{\partial}{\partial\vartheta}\,(\sin\vartheta\,A_\vartheta)+\frac{1}{r\,\sin\vartheta}\,\frac{\partial A_\varphi}{\partial\varphi}\,, \tag{B.12}$$

$$\begin{aligned}\nabla\times\mathbf{A} \;=\;& \frac{\hat{a}_r}{r\,\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}\,(A_\varphi\,\sin\vartheta)-\frac{\partial A_\vartheta}{\partial\varphi}\right]+\\ &+\frac{\hat{a}_\vartheta}{r}\left[\frac{1}{\sin\vartheta}\,\frac{\partial A_r}{\partial\varphi}-\frac{\partial}{\partial r}\,(rA_\varphi)\right]+\frac{\hat{a}_\varphi}{r}\left[\frac{\partial}{\partial r}\,(rA_\vartheta)-\frac{\partial A_r}{\partial\vartheta}\right],\end{aligned} \tag{B.13}$$

$$\nabla^2 f \;=\; \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\,\sin\vartheta}\,\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial f}{\partial\vartheta}\right)+\frac{1}{r^2\,\sin^2\vartheta}\,\frac{\partial^2 f}{\partial\varphi^2}\,, \tag{B.14}$$

$$\begin{aligned}\nabla^2\mathbf{A} \;=\;& \hat{a}_r\left[\nabla^2 A_r-\frac{2}{r^2}\left(A_r+\frac{1}{\operatorname{tg}\vartheta}\,A_\vartheta+\frac{1}{\sin\vartheta}\,\frac{\partial A_\varphi}{\partial\varphi}+\frac{\partial A_\vartheta}{\partial\vartheta}\right)\right]+\\ &+\hat{a}_\vartheta\left[\nabla^2 A_\vartheta-\frac{1}{r^2}\left(\frac{1}{\sin^2\vartheta}\,A_\vartheta-2\,\frac{\partial A_r}{\partial\vartheta}+\frac{2}{\operatorname{tg}\vartheta\,\sin\vartheta}\,\frac{\partial A_\varphi}{\partial\varphi}\right)\right]+\\ &+\hat{a}_\varphi\left[\nabla^2 A_\varphi-\frac{1}{r^2}\left(\frac{1}{\sin^2\vartheta}\,A_\varphi-\frac{2}{\sin\vartheta}\,\frac{\partial A_r}{\partial\varphi}-\frac{2}{\operatorname{tg}\vartheta\,\sin\vartheta}\,\frac{\partial A_\vartheta}{\partial\varphi}\right)\right].\end{aligned} \tag{B.15}$$

Note that sometimes, in the text, the base unit vectors are denoted as $\hat{r},\hat{\vartheta},\hat{\varphi}$.