Appello 4 -Soluzione

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# Analisi

1) $f(x,y) = \frac{x^2 y}{2} - x^2 + xy - 2x - y^2$

a) $\nabla f(x,y) = \left(xy - 2x + y - 2, \frac{x^2}{2} + x - 2y\right)$

$\nabla f(x,y) = 0 \iff \begin{cases} xy - 2x + y - 2 = 0 & (1) \\ x^2 + 2x - 4y = 0 & (2) \end{cases}$

(1): $y(x+1) = 2(x+1)$: se $x = -1$ le (2): $y = -1/4$

Se $x \neq -1$: $y = 2$ (2) $\Rightarrow$ $x^2 + 2x - 8 = 0$

$\Delta' = 9$, $x = -1 \pm 3 \begin{cases} -4 \\ 2 \end{cases}$

Pti critici: $(-4, 2)$, $(2, 2)$, $(-1, -\frac{1}{4})$.

$\text{Hess} f(x,y) = \begin{pmatrix} y-2 & x+1 \\ x+1 & -2 \end{pmatrix}$

$\text{Hess} f(2,2) = \begin{pmatrix} 0 & 3 \\ 3 & -2 \end{pmatrix}$ det $\text{Hess} f(2,2) = -9 < 0$: sella

$\text{Hess} f(-4,2) = \begin{pmatrix} -6 & 3 \\ 3 & -2 \end{pmatrix}$ det $\text{Hess} f(-4,2) = 12 - 9 < 0$: sella

$\text{Hess} f(-1, -\frac{1}{4}) = \begin{pmatrix} -11/4 & 3/4 \\ 3/4 & -2 \end{pmatrix}$ $\begin{cases} \det \text{Hess} f(-1, -\frac{3}{4}) = \frac{22}{4} - \frac{9}{16} > 0 \\ -\frac{11}{4} < 0: \quad (-1, -\frac{1}{4}) \text{ MAX locale stretto.} \end{cases}$

b) $\nabla f(0,1) = (-1, -2)$

$z = \nabla f(0,1) \cdot \begin{pmatrix} x-0 \\ y-1 \end{pmatrix} + f(0,1)$

$$= \begin{pmatrix}-1\\-2\end{pmatrix}\cdot\begin{pmatrix}x\\y-1\end{pmatrix} - 1 = -x-2(y-1)-1$$

$$\boxed{z = -x-2y+1}$$

c) $D_{\vec{u}} f(0,1) = \nabla f(0,1)\cdot\vec{u} = \begin{pmatrix}-1\\-2\end{pmatrix}\cdot\begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix} = -\frac{\sqrt{3}}{2} - 1$ .

2. $\begin{cases} y' + (1-x)y = xe^{-x} \\ y(1) = 0 \end{cases}$

$\int 1-x \, dx = x - \frac{x^2}{2}$ $\qquad [y' + (1-x)y]e^{x-\frac{x^2}{2}} = xe^{-\frac{x^2}{2}}$

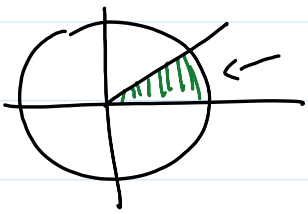
$\qquad (ye^{x-\frac{x^2}{2}})' = xe^{-\frac{x^2}{2}}$

$\Rightarrow \; ye^{x-\frac{x^2}{2}} = \int xe^{-\frac{x^2}{2}}dx = -e^{-\frac{x^2}{2}} + c$

$y(x) = -e^{-x} + ce^{-x+\frac{x^2}{2}}$ . È $y(1)=0 \iff -e^{-1} + ce^{-\frac{1}{2}} = 0$

$\iff c = e^{-\frac{1}{2}}$

$\Rightarrow \boxed{y(x) = -e^{-x} + e^{-x+\frac{x^2}{2}-\frac{1}{2}}}$

3.  $(\rho\cos t, \rho\sin t)$: $\rho\in[0,1]$, $t\in[0,\frac{\pi}{4}]$.

$$\int_D \frac{x+y}{1+x^2+y^2}dx\,dy = \int_{\substack{0\le\rho\le 1\\ t\in[0,\frac{\pi}{4}]}} \frac{\rho\cos t + \rho\sin t}{1+\rho^2}\rho\, d\rho\, dt$$

$$= \left(\int_0^1 \frac{\rho^2}{1+\rho^2}d\rho\right)\left(\int_0^{\frac{\pi}{4}} \cos t + \sin t \, dt\right)$$

$$\int_0^1 \frac{\rho^2}{1+\rho^2}d\rho = \int_0^1 1 - \frac{1}{1+\rho^2}d\rho = 1 - [\operatorname{arctg}\rho]_0^1 = 1 - \frac{\pi}{4}$$

$$\int_0 \frac{\ }{1+\rho^2}\, d\rho = \int_0 1 - \frac{\ }{1+\rho^2}\, d\rho = 1 - [\operatorname{arctg} \rho]_0 \ \cdots \ \frac{\ }{4}$$

$$\int_0^{\pi/4} \cos t + \sin t \, dt = \left[ \sin t - \cos t \right]_0^{\pi/4} = 1$$

$\Rightarrow$ l'integrale vale $\boxed{1 - \pi/4}$

## Probabilità

1) a) A : allarme  B: contiene una bomba

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$
$$= 0.99 \cdot \frac{1}{5000} + 0.05 \cdot \frac{4999}{5000}$$

b) $$P(B|A) = \frac{P(A|B)P(B)}{P(A)} = \frac{0.99}{0.99 + 0.05 \cdot 4999}$$

2) a) $X \sim B\left(1000, \frac{18}{100}\right)$

b) $P(X \le 170) + P(X \ge 190)$

$$X \approx 1000 \times \frac{18}{100} + \sqrt{1000 \times \frac{18}{100} \times \frac{82}{100}}\ Z$$
$$= 180 + \underbrace{\sqrt{\frac{18 \times 82}{10}}}_{\sigma}\ Z \qquad Z \sim N(0,1)$$

$$P(X \le 170) \approx P(180 + \sigma Z \le 170) = P\left(Z \le \frac{-10}{\sigma}\right)$$
$$= \Phi(-10/\sigma) = 1 - \Phi\left(\frac{10}{\sigma}\right)$$

$$P(X \ge 190) \approx P(180 + \sigma Z \ge 190)$$
$$= P(\sigma Z \ge 10) = P\left(Z \ge \frac{10}{\sigma}\right) = 1 - \Phi\left(\frac{10}{\sigma}\right)$$

Quindi $P(X \le 170) + P(X \ge 190) \approx \boxed{2\left(1 - \Phi\left(\frac{10}{\sigma}\right)\right)}$  $\sigma = \sqrt{\frac{18 \times 82}{10}} = 6 \times \sqrt{\frac{41}{10}}$

$+\infty$  $, 0 \quad x \quad x < 0$

$- 6 \times \sqrt{10}$

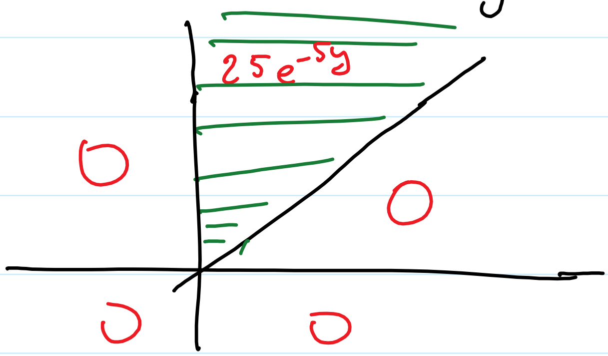
3. a) $f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \begin{cases} 0 & \text{se } x < 0 \\ \int_x^{+\infty} 25\, e^{-5y}\,dy & \text{se } x \geq 0 \end{cases}$

$$\int_x^{+\infty} 25\, e^{-5y}\,dy = -\frac{25}{5}\left[e^{-5y}\right]_x^{+\infty} = 5\, e^{-5x}$$

$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx = \begin{cases} 0 & \text{se } y < 0 \\ \int_0^{y} 25\, e^{-5y}\,dx & \text{se } y \geq 0 \end{cases}$

$$\int_0^{y} 25\, e^{-5y}\,dx = 25\, y\, e^{-5y}$$

(b) $f_{X,Y}(x,y)$:

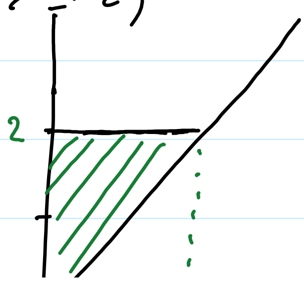



$X, Y$ non indipendenti perché
$f_X(x) f_Y(y) \neq f_{X,Y}(x,y)$.

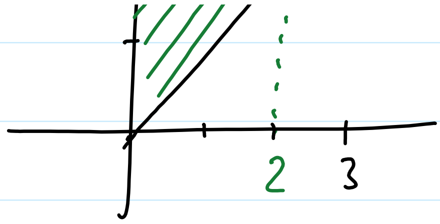
Ad esempio se $x > 0, y > 0$ è $f_X(x) f_Y(y) \neq 0$
mentre $f_{X,Y}(x,y) = 0$ se $y > x > 0$.

(c) $P(X \leq 3, Y \leq 2) = P(0 \leq X \leq 3,\ 0 \leq Y \leq 2)$

$= \int 25\, e^{-5y}\,dx\,dy =$



$$[0,3]\times[0,2]\cap\{0\le x<y\}$$




$$= \int_{[0,2]\times[0,2]\cap\{0\le x<y\}} 25\, e^{-5y}\,dx\,dy = \int_0^2 \left\{ \int_0^y 25\, e^{-5y}\,dx \right\} dy$$

$$= \int_0^2 25\, y\, e^{-5y}\,dy \underset{\text{p. parti}}{=} 25\left(\left[-\frac{1}{5}\, y e^{-5y}\right]_0^2 + \frac{1}{5}\int_0^2 e^{-5y}\,dy\right)$$

$$= 25\left(-\frac{2}{5}\, e^{-10} - \frac{1}{25}\left(e^{-10}-1\right)\right)$$

$$= -10e^{-10} - e^{-10} + 1 = \boxed{1-11\,e^{-10}}$$