ESERCIZI SCHEDA 5

ESERCIZIO 1

$$\lim_{x\to 0} \frac{e^{\sin x}-1}{\log(1+3x)} =$$

sviluppi di Mac Laurin: $\sin x = x + o(x)$ per $x \to 0$

$\log(1+x) = x + o(x)$ per $x\to 0$

cambiando la variabile in $3x$ si ha

$\log(1+3x) = 3x + o(3x) \overset{!}{=} 3x + o(x)$ per $x \to 0$

$e^x = 1 + x + o(x)$ per $x \to 0$

$\Rightarrow e^{\sin x} = e^{x+o(x)} = 1 + x + o(x) + o(x + o(x)) \overset{!}{=} 1 + x + o(x) + o(x) \overset{!}{=} 1 + x + o(x)$ per $x \to 0$

$$= \lim_{x\to 0} \frac{1+x+o(x)-1}{3x+o(x)} = \lim_{x\to 0}\frac{x+o(x)}{3x+o(x)} = \lim_{x\to 0} \frac{x\left(1+\frac{o(x)}{x}\right)}{3x\left(1+\frac{o(x)}{3x}\right)} = \frac{1}{3}$$

ESERCIZIO 2

ⓐ $$\lim_{x\to 0} \frac{\sin(x^3)}{x(1-\cos x)+x^4} =$$

sviluppi: $\sin x = x + o(x)$ per $x\to 0$ $\Rightarrow \sin(x^3) = x^3 + o(x^3)$

$\cos x = 1 - \frac{x^2}{2} + o(x^2)$ per $x \to 0$

$$= \lim_{x\to 0} \frac{x^3+o(x^3)}{x\left(1-1+\frac{x^2}{2}+o(x^2)\right)+x^4} = \lim_{x\to 0}\frac{x^3+o(x^3)}{\frac{x^3}{2}+o(x^3)+x^4} = \lim_{x\to 0}\frac{x^3+o(x^3)}{\frac{x^3}{2}+o(x^3)} = \lim_{x\to 0}\frac{x^3\left(1+\frac{o(x^3)}{x^3}\right)}{\frac{x^3}{2}\left(1+2\frac{o(x^3)}{x^3}\right)} = 2$$

ⓑ $$\lim_{x\to -\infty} x^3 \log\left(\frac{1-x^2}{2-x^2}\right)$$

cambio di variabile $t = \frac{1}{x}$ $\quad \lim_{x\to-\infty} t = \lim_{x\to -\infty}\frac{1}{x} = 0^- \Rightarrow t \to 0^-$ per $x \to -\infty$

$$\lim_{t\to 0^-} \frac{1}{t^3}\log\left(\frac{1-\frac{1}{t^2}}{2-\frac{1}{t^2}}\right) = \lim_{t\to 0^-}\frac{1}{t^3}\log\left(\frac{t^2-1}{2t^2-1}\right) = \lim_{t\to 0^-}\frac{1}{t^3}\log\left(\frac{2t^2-1-t^2}{2t^2-1}\right) = \lim_{t\to 0^-}\frac{1}{t^3}\log\left(1+\frac{-t^2}{2t^2-1}\right)$$

sviluppo: $\log\left(1+\frac{-t^2}{2t^2-1}\right) = -\frac{t^2}{2t^2-1} + o\left(-\frac{t^2}{2t^2-1}\right) = -\frac{t^2}{2t^2-1} + o(1) = \frac{-t^2+o(1)\cdot(2t^2-1)}{2t^2-1} =$

$= \frac{-t^2+o(2t^2-1)}{2t^2-1} = \frac{-t^2+o(t^2)}{2t^2-1}$

$$= \lim_{t\to 0^-}\frac{\frac{-t^2+o(t^2)}{2t^2-1}}{t^3} = \lim_{t\to 0^-}\frac{-t^2+o(t^2)}{2t^5-t^3} = \lim_{t\to 0^-}\frac{t^2\left(-1+\frac{o(t^2)}{t^2}\right)}{t^3(2t^2-1)} = -\infty$$

ⓒ $$\lim_{x\to 1}\frac{\sin(\sqrt{x}-1)}{x-1} =$$

cambio di variabile $t = \sqrt{x}-1 \Leftrightarrow \sqrt{x} = t+1 \Leftrightarrow x = t^2+2t+1 \Rightarrow t\to 0$ per $x \to 1$

$$= \lim_{t\to 0}\frac{\sin t}{t^2+2t+\cancel{1}-\cancel{1}} = \lim_{t\to 0}\frac{\sin t}{t^2+2t} =$$

sviluppo: $\sin t = t + o(t)$ per $t \to 0$

$$= \lim_{t\to 0} \frac{t+o(t)}{t^2+2t} = \lim_{t\to 0} \frac{t+o(t)}{2t+o(t)} = \frac{1}{2}$$

Esercizio 3

$f(x) = e^{-\frac{1}{x}}$ Considero un x^n con $n\in\mathbb{N}$: $\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x^n}$ cambio di variabile: $t=\frac{1}{x} \Rightarrow t\to+\infty$ per $x\to 0^+$

$$\lim_{t\to+\infty} t^n e^{-t} = \lim_{t\to+\infty} \frac{t^n}{e^t} = 0 \text{ per gerarchia } \forall n\in\mathbb{N}$$

$\Rightarrow e^{-\frac{1}{x}} = o(x^n)\ \forall n\in\mathbb{N} \Rightarrow$ $f(x)$ non ha ordine di infinitesimo

$g(x) = \log(1+x^3) = x^3 + o(x^3)$ per $x\to 0$ $\Rightarrow \lim_{x\to 0} \frac{\log(1+x^3)}{x^3} = 1 \Rightarrow$ $g(x)$ ha ordine di infinitesimo pari a 3.

$h(x) = x^{\frac{1}{\sqrt{x}}}$ $\quad \lim_{x\to 0^+} \frac{x^{\frac{1}{\sqrt{x}}}}{x^n} = \lim_{x\to 0^+} x^{\frac{1}{\sqrt{x}}-n} = \lim_{x\to 0^+} \exp\left\{\log\left(x^{\frac{1}{\sqrt{x}}-n}\right)\right\} = \lim_{x\to 0^+} \exp\left\{\underbrace{\underbrace{\left(\frac{1}{\sqrt{x}}-n\right)}_{\to+\infty}\underbrace{\log x}_{\to-\infty}}_{\to-\infty}\right\} = e^{-\infty} = 0\ \forall n\in\mathbb{N}$

$\Rightarrow$ $h(x)$ non ha ordine di infinitesimo.

$i(x) = \sqrt{1+x^2} - \cos x$: $\quad \lim_{x\to 0} \frac{\sqrt{1+x^2}-\cos x}{x^\alpha} = \lim_{x\to 0} \frac{\sqrt{1+x^2}-\cos x}{x^\alpha} \cdot \frac{\sqrt{1+x^2}+\cos x}{\sqrt{1+x^2}+\cos x} = \lim_{x\to 0} \frac{1+x^2-\cos^2 x}{x^\alpha} \cdot \frac{1}{\sqrt{1+x^2}+\cos x} =$

sviluppo $\cos x = 1 - \frac{x^2}{2} + o(x^2)$

$$= \lim_{x\to 0} \frac{1+x^2-\left(1-\frac{x^2}{2}+o(x^2)\right)}{x^\alpha} \cdot \lim_{x\to 0} \frac{1}{\underbrace{\sqrt{1+x^2}}_{\to 1}+\underbrace{\cos x}_{\to 1}} = \frac{1}{2}\lim_{x\to 0} \frac{\frac{3}{2}x^2+o(x^2)}{x^\alpha} = \ell\in\mathbb{R}\setminus\{0\} \Leftrightarrow \alpha=2$$

$i(x)$ ha ordine di infinitesimo pari a 2.

Esercizio 4

$f(x) = o(x^2)$ per $x\to+\infty$

$$x^\alpha + x + 1 \begin{cases} \sim x^\alpha & \text{per } x\to+\infty & \text{se } \alpha>1 \\ \sim 2x & \text{per } x\to+\infty & \text{se } \alpha=1 \\ \sim x & \text{per } x\to+\infty & \text{se } \alpha<1 \end{cases}$$

Caso $\alpha>1$: $\quad \lim_{x\to+\infty} \frac{f(x)}{x^\alpha+x+1} = \lim_{x\to+\infty} \frac{o(x^2)}{x^\alpha} = \lim_{x\to+\infty} \underbrace{\frac{o(x^2)}{x^2}}_{\to 0} \cdot \frac{1}{x^{\alpha-2}} = \begin{cases} 0 & \text{se } \alpha\geq 2 \\ \text{non lo so se } \alpha\in(1,2) \end{cases}$

Caso $\alpha=1$: $\quad \lim_{x\to+\infty} \frac{f(x)}{x+x+1} = \lim_{x\to+\infty} \frac{o(x^2)}{2x} = \frac{1}{2}\lim_{x\to+\infty} \underbrace{\frac{o(x^2)}{x^2}}_{\to 0} \cdot \underbrace{x}_{\to+\infty}$ forma indeterminata

Caso $\alpha<1$ analogo al caso $\alpha=1$.

$$\Rightarrow \lim_{x\to+\infty} \frac{f(x)}{x^\alpha+x+1} = \begin{cases} 0 & \text{se } \alpha\geq 2 \\ \text{non so calcolarlo se } \alpha<2. \end{cases}$$

Esercizio 5

a) $\lim_{x\to 3} \frac{x^{10}-3^{10}}{x^{11}-3^{11}}$

cambio di variabile: $t=x-3 \Rightarrow t\to 0$ per $x\to 3$
$\Rightarrow x=t+3$

$\lim_{t\to 0} \frac{(t+3)^{10}-3^{10}}{(t+3)^{11}-3^{11}}$

Implemento il binomio di Newton ignorando tutti i coefficienti tranne gli ultimi due:

$$(t+3)^n = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} \cdot b^k$$

$$\Rightarrow (t+3)^{10} = t^{10} + \dots + 10\cdot 3^9 t + 3^{10}$$

$$\Rightarrow (t+3)^{11} = t^{11} + \dots + 11\cdot 3^{10} t + 3^{11}$$

$$= \lim_{t\to 0} \frac{t^{10}+\dots+10\cdot 3^9 t + 3^{10} - 3^{10}}{t^{11}+\dots+11\cdot 3^{10} t + 3^{11} - 3^{11}} = \lim_{t\to 0} \frac{\overbrace{(t^9+\dots+10\cdot 3^9)}^{\to 10\cdot 3^9} t}{\underbrace{(t^{10}+\dots+11\cdot 3^{10})}_{\to 11\cdot 3^{10}} t} = \frac{10\cdot 3^9}{11\cdot 3^{10}} = \boxed{\frac{10}{33}}$$

b) $\lim_{x\to 1} \frac{x-4\sqrt{x}+3}{x^2-1}$

Tratto il numeratore come se fosse un'equazione di 2° grado con $\sqrt{x}$ come variabile.

$$(\sqrt{x})_{1,2} = \frac{4\pm\sqrt{16-12}}{2} = \frac{4\pm 2}{2} = \begin{cases} \frac{6}{2} = 3 \\ \frac{2}{2} = 1 \end{cases}$$

$$\Rightarrow x-4\sqrt{x}+3 = (\sqrt{x}-1)(\sqrt{x}-3)$$

$$\lim_{x\to 1} \frac{(\sqrt{x}-1)(\sqrt{x}-3)}{(x+1)(x-1)} = \lim_{x\to 1} \frac{(\sqrt{x}-1)(\sqrt{x}-3)}{(x+1)(\sqrt{x}+1)(\sqrt{x}-1)} = \lim_{x\to 1} \frac{\sqrt{x}-3}{(x+1)(\sqrt{x}+1)} = \frac{-2}{2\cdot 2} = \boxed{-\frac{1}{2}}$$

c) Si può procedere come nel punto b o con un cambio di variabile come segue.

$\lim_{x\to 4} \frac{\sqrt{x}-2}{x^2-5x+4}$

$t=\sqrt{x}-2 \Rightarrow t\to 0$ per $x\to 4$

$$\Rightarrow \sqrt{x}=t+2 \Leftrightarrow x=(t+2)^2 = t^2+4t+4$$
$$\Leftrightarrow x^2 = (t+2)^4 = (t^2+4t+4)^2 = t^4+16t^2+16+8t^2+8t^3+32t$$
$$\overset{!}{=} t^4+8t^3+24t^2+32t+16$$

$$\lim_{t\to 0} \frac{t}{t^4+8t^3+24t^2+32t+\cancel{16}-5t^2-20t-\cancel{20+4}} = \lim_{t\to 0} \frac{t}{t^4+8t^3+19t^2+12t} = \lim_{t\to 0} \frac{1}{t^3+8t^2+19t+12} = \boxed{\frac{1}{12}}$$

Esercizio 6

a)
$$\lim_{x\to 2^+} \frac{\log(5x^2-19)}{|x-2|^\alpha} = \lim_{x\to 2^+} \frac{\log(5x^2-20+1)}{(x-2)^\alpha} = \lim_{x\to 2^+} \frac{\log[\overbrace{5(x^2-4)}^{\to 0 \text{ per } x\to 2^+}+1]}{(x-2)^\alpha} = \lim_{x\to 2^+} \frac{5(x^2-4)+o(5(x^2-4))}{(x-2)^\alpha} = \lim_{x\to 2^+} \frac{5(x^2-4)(1+o(1))}{(x-2)^\alpha} =$$

$$= \lim_{x\to 2^+} \frac{5(x+2)(x-2)(1+o(1))}{(x-2)^\alpha} = \lim_{x\to 2^+} \underbrace{5(x+2)}_{\to 20}(x-2)^{1-\alpha}\underbrace{(1+o(1))}_{\to 1} \in \mathbb{R}\setminus\{0\} \Leftrightarrow \boxed{\alpha=1}$$

b)
$$\lim_{x\to 0^+} \frac{1-\sqrt[3]{\cos x}}{|x|^\alpha} = \lim_{x\to 0^+} \frac{1-\sqrt[3]{\cos x}}{x^\alpha} \cdot \frac{1+\sqrt[3]{\cos^2 x}+\sqrt[3]{\cos x}}{1+\sqrt[3]{\cos^2 x}+\sqrt[3]{\cos x}} = \lim_{x\to 0^+} \frac{1-\cos x}{x^\alpha\left(1+\sqrt[3]{\cos^2 x}+\sqrt[3]{\cos x}\right)} =$$

$$= \lim_{x\to0^+} \frac{1-\left(1-\frac{x^2}{2}+o(x^2)\right)}{x^\alpha\left(1+\sqrt[3]{\cos^2 x}+\sqrt[3]{\cos x}\right)} = \lim_{x\to0^+}\frac{\frac{x^2}{2}+o(x^2)}{x^\alpha\underbrace{\left(1+\sqrt[3]{\cos^2 x}+\sqrt[3]{\cos x}\right)}_{\to 3}} \in \mathbb{R}\setminus\{0\} \iff \alpha=2$$

ⓒ $$\lim_{x\to0^+}\frac{1}{|x|^\alpha\left[\exp(4\arcsin(2x))-1\right]}$$

sviluppo: $\arcsin(2x) = 2x+o(2x) = 2x+o(x)$ per $x\to0^+$

$e^{4\arcsin(2x)} = e^{4(2x+o(x))} = e^{8x+o(x)}$ per $x\to0^+$

$e^{8x+o(x)} = 1+(8x+o(x))+o\left(8x+o(x)\right) = 1+8x+o(x)$ per $x\to0^+$

$$\lim_{x\to0^+}\frac{1}{x^\alpha(\cancel{1}+8x+o(x)\cancel{-1})} = \lim_{x\to0^+}\frac{1}{8x^{\alpha+1}+o(x^{\alpha+1})} \iff \alpha=-1 \leadsto$$ La funzione non ha ordine di infinitesimo.

ⓓ $$\lim_{x\to0^+}\frac{(\cos x)^{\frac{1}{x}}-1}{|x|^\alpha} = \lim_{x\to0^+}\frac{\exp\{\log((\cos x)^{\frac{1}{x}})\}-1}{x^\alpha} = \lim_{x\to0^+}\frac{\exp\{\frac{1}{x}\log(\cos x)\}-1}{x^\alpha}$$

sviluppo: $\cos x = 1-\frac{x^2}{2}+o(x^2)$ per $x\to0^+$

$\Rightarrow \log(\cos x) = \log\left(1-\frac{x^2}{2}+o(x^2)\right) = -\frac{x^2}{2}+o(x^2)+o\left(-\frac{x^2}{2}+o(x^2)\right) = -\frac{x^2}{2}+o(x^2)$ per $x\to0^+$

$\Rightarrow \frac{1}{x}\log(\cos x) = \frac{1}{x}\left(-\frac{x^2}{2}+o(x^2)\right) = -\frac{x}{2}+o(x)$ per $x\to0^+$

$\Rightarrow \exp\{\frac{1}{x}\log(\cos x)\} = e^{-\frac{x}{2}+o(x)} = 1-\frac{x}{2}+o(x)+o\left(-\frac{x}{2}+o(x)\right) = 1-\frac{x}{2}+o(x)$

$$= \lim_{x\to0^+}\frac{\cancel{1}-\frac{x}{2}+o(x)\cancel{-1}}{x^\alpha} = \lim_{x\to0^+}\frac{-\frac{x}{2}+o(x)}{x^\alpha} \in \mathbb{R}\setminus\{0\} \iff \alpha=1$$

Esercizio 7

ⓐ $$\lim_{x\to0} f(x) = \lim_{x\to0} \overbrace{\frac{f(x)}{x^2}}^{\to 10}\cdot \underbrace{x^2}_{\to0} = 0$$

ⓑ $$\lim_{x\to0}\frac{f(x)}{x} = \lim_{x\to0}\overbrace{\frac{f(x)}{x^2}}^{\to10}\cdot \underbrace{x}_{\to0} = 0$$

ⓒ $$\lim_{x\to0}\frac{f(x)\log x}{x} = \lim_{x\to0} x\,\frac{f(x)\log x}{x^2} = \overbrace{\lim_{x\to0}\frac{f(x)}{x^2}}^{\to10}\cdot\lim_{x\to0} x\log x$$

cambio di variabile $t=\frac{1}{x} \Rightarrow x=\frac{1}{t}$, $t\to+\infty$ per $x\to0^+$

$$= 10\lim_{t\to+\infty} t^{-1}\log(t^{-1}) = 10\lim_{t\to+\infty} -\overbrace{\frac{\log t}{t}}^{\to0} = 0$$

ⓓ $$\lim_{x\to0}\frac{\overbrace{f(x)}^{\to10}}{\underbrace{x^2\log x}_{\to-\infty}} = \frac{10}{-\infty} = 0$$

ⓔ $$\lim_{x\to0}\frac{f(x)}{\sin^2 x} = \lim_{x\to0}\frac{f(x)}{x^2}\cdot\frac{x^2}{\sin^2 x} = \lim_{x\to0}\overbrace{\frac{f(x)}{x^2}}^{\to10}\overbrace{\left(\frac{\sin x}{x}\right)}^{\to1}{}^{-2} = 10$$

ⓕ $$\lim_{x\to0}\frac{f(x)}{1-\cos x} = \lim_{x\to0}\frac{f(x)}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x} = \lim_{x\to0}\frac{f(x)}{1-\cos^2 x}(1+\cos x) = \lim_{x\to0}\underbrace{\frac{f(x)}{\sin^2 x}}_{\to10}\overbrace{(1+\cos x)}^{\to2} = 20$$

$\to 10$ dal punto ⓔ

Esercizio 8

ⓐ $$\lim_{n\to+\infty}\frac{9n^3+3n-1}{7n^2+4n} = (\text{per asintoticità}) = \lim_{n\to+\infty}\frac{9n^3}{7n^2} = \lim_{n\to+\infty}\frac{9}{7}n = +\infty$$

(b) $$\lim_n \frac{-2n^2+4n+3}{3n^5-4n^4} = (\text{per asintoticità}) = \lim_n \frac{-2n^2}{3n^5} = 0$$

(c) $$\lim_n \frac{(-1)^n n^2+n}{n^3+1} = (\text{per asintoticità}) = \lim_n \frac{(-1)^n n^2}{n^3} = 0$$

(d) $$\lim_n \left[\log_{20}(n+3) - \log_{20}\sqrt[3]{n^7+n^5}\right] = \lim_n \log_{20}\frac{n+3}{\sqrt[3]{n^7+n^5}} = \lim_n \log_{20}\frac{n\overbrace{\left(1+\frac{3}{n}\right)}^{\to 1}}{n^{\frac{7}{3}}\underbrace{\sqrt[3]{1+\frac{1}{n^2}}}_{\to 1}} = \log_{20} 1 = 0$$

ESERCIZIO 9

(a) $$\lim_n n\log\left(1+\sin\left(\tfrac{1}{n}\right)\right) =$$

sviluppo: $\sin\left(\frac{1}{n}\right) = \frac{1}{n}+o\left(\frac{1}{n}\right)$ per $n\to+\infty$

$$\Rightarrow \log\left(1+\sin\left(\tfrac{1}{n}\right)\right) = \log\left(1+\tfrac{1}{n}+o\left(\tfrac{1}{n}\right)\right) = \tfrac{1}{n}+o\left(\tfrac{1}{n}\right)+o\left(\tfrac{1}{n}+o\left(\tfrac{1}{n}\right)\right)$$

$$= \tfrac{1}{n}+o\left(\tfrac{1}{n}\right) \quad \text{per } n\to+\infty$$

$$= \lim_n n\left(\tfrac{1}{n}+o\left(\tfrac{1}{n}\right)\right) = \lim_n 1+o(1) = 1$$

(b) $$\lim_n \frac{\log(\log n)}{1+\log^2 n} =$$

cambio di variabile $k=\log n \Rightarrow k\to+\infty$ per $n\to+\infty$

$$= \lim_k \frac{\log k}{1+k^2} = 0 \quad \text{per gerarchia}$$

(c) $$\lim_n \frac{\arcsin\left(\frac{1}{n!}\right)}{\arccos\left(\frac{1}{n!}\right)}\cdot n! =$$

cambio di variabile $k=\frac{1}{n!} \Rightarrow k\to 0^+$ per $n\to+\infty$

$$= \lim_{k\to 0} \overbrace{\frac{\arcsin k}{k}}^{\to 1}\cdot\frac{1}{\underbrace{\arccos k}_{\to \frac{\pi}{2}}} = \frac{2}{\pi}$$

ESERCIZIO 10

(a) $$\lim_n \frac{5n^4+n^3+1}{3^{2n}+5^n} = \lim_n \frac{5n^4+n^3+1}{9^n+5^n} = \lim_n \underbrace{\frac{5n^4}{9^n}}_{\to 0 \text{ per gerarchia}}\cdot\frac{\overbrace{1+\frac{1}{5n}+\frac{1}{5n^4}}^{\to 1}}{\underbrace{1+\left(\frac{5}{9}\right)^n}_{\to 1}} = 0$$

(b) $$\lim_n \frac{\log^7(n^5)+2n}{n+\log(e^n+1)} = \lim_n \frac{5^7\log^7 n+2n}{n+\log\left[e^n\left(1+\frac{1}{e^n}\right)\right]} = \lim_n \frac{5^7\log^7 n+2n}{2n+\log(1+e^{-n})} = \lim_n \overbrace{\frac{2n}{2n}}^{1}\cdot\frac{1+5^7\overbrace{\frac{\log^7 n}{2n}}^{\to 0 \text{ per gerarchia}}}{1+\underbrace{\frac{\log(1+e^{-n})}{2n}}_{\to 0}} = 1$$

$$= \log e^n+\log(1+e^{-n}) = n+\log(1+e^{-n})$$

(c) $$\lim_n \frac{2^{n+1}+\sin n}{2^n+n!} = \lim_n \frac{2\cdot 2^n+\sin n}{2^n+n!} = \lim_n \underbrace{\frac{2^n}{n!}}_{\to 0 \text{ per gerarchia}}\cdot\frac{2+\overbrace{\frac{\sin n}{2^n}}^{\to 0}}{1+\underbrace{\frac{2^n}{n!}}_{\to 0 \text{ per gerarchia}}} = 0\cdot 2 = 0$$

(d) $$\lim_n \frac{n!-(n-1)!}{(n-2)!} = \lim_n \frac{n(n-1)(n-2)!-(n-1)(n-2)!}{(n-2)!} = \lim_n (n^2-n-n+1) = \lim_n (n^2-2n+1) = \lim_n (n-1)^2 = +\infty$$

(e) $\lim_n n^{\frac{n\log^2 n}{n!}} = \lim_n \exp\left\{\frac{n\log^2 n}{n!}\log n\right\} = \exp\left\{\frac{n\log^3 n}{n(n-1)!}\right\} = \exp\left\{\underbrace{\frac{\log^3 n}{(n-1)!}}_{\to 0 \text{ per gerarchia}}\right\} = e^0 = 1$

Esercizio 11

(a) $\lim_n \frac{\log\left(\frac{1}{n^5}+\frac{1}{n^2}\right)}{4n^4} = \lim_n \frac{\log\left(\frac{n^3+1}{n^5}\right)}{4n^4} = \lim_n \frac{\log(n^3+1)-\log(n^5)}{4n^4} = \lim_n \frac{\log(n^3+1)}{4n^4} - \lim_n \underbrace{5\frac{\log n}{4n^4}}_{\to 0 \text{ per gerarchia}} =$

$= \lim_n \frac{\log\left[n^3\left(1+\frac{1}{n^3}\right)\right]}{4n^4} = \lim_n \left(\underbrace{\frac{\log(n^3)}{4n^4}}_{\to 0 \text{ per gerarchia}} + \underbrace{\frac{\log\left(1+\frac{1}{n^3}\right)}{4n^4}}_{\to 0}\right) = 0$

(b) $\lim_n (2n+3)^2\left(1-\cos\left(\frac{1}{n}\right)\right) =$

sviluppo: sottointendendo il cambio di variabile

$\cos\left(\frac{1}{n}\right) = 1 - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)$ per $n\to+\infty$

$= \lim_n (2n+3)^2\left(\cancel{1}-\cancel{1}+\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\right)$

$= \lim_n (4n^2+12n+9)\left(\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\right) = \lim_n \left(\underbrace{\frac{4n^2+12n+9}{2n^2}}_{\to 2} + \underbrace{o(1)}_{\to 0}\right) = 2$

(c) $\lim_n \frac{n^{\frac{1}{\sin(1/n)}}}{\exp\left(\frac{\arctan(2n)}{\sqrt{\log n}}\right)} =$ (divido numeratore e denominatore secondo il teorema sull'algebra dei limiti) $=$

$= \frac{\lim_n\left(n^{\frac{1}{\sin(1/n)}}\right)}{\lim_n\left[\exp\left(\frac{\arctan(2n)}{\sqrt{\log n}}\right)\right]}$

denominatore: $\arctan(2n) \sim \frac{\pi}{2}$ per $n\to+\infty$

$\Rightarrow \lim_n\left[\exp\left(\frac{\arctan(2n)}{\sqrt{\log n}}\right)\right] = \lim_n\left[\exp\left(\underbrace{\frac{\pi/2}{\sqrt{\log n}}}_{\to 0}\right)\right] = e^0 = 1$

numeratore: $\lim_n n^{\frac{1}{\sin(1/n)}} = \lim_n \exp\left\{\log n^{\frac{1}{\sin(1/n)}}\right\} =$

$= \lim_n \exp\left\{\frac{1}{\sin(1/n)}\log n\right\}$

sviluppo: $\sin\left(\frac{1}{n}\right) = \frac{1}{n} + o\left(\frac{1}{n}\right)$ per $n\to+\infty$

$= \lim_n \exp\left\{\frac{\overbrace{\log n}^{\to+\infty}}{\underbrace{\frac{1}{n}+o\left(\frac{1}{n}\right)}_{\to 0}}\right\} = e^{+\infty} = +\infty$

$= \frac{+\infty}{0} = +\infty$

(d) $\lim_n \frac{n^n}{(n+1)^{n+1}} =$ cambio di variabile $k=n+1 \Leftrightarrow n=k-1 \Rightarrow k\to+\infty$ per $n\to+\infty$

$= \lim_k \frac{(k-1)^{k-1}}{k^k} = \lim_k \frac{1}{k-1}\cdot\frac{(k-1)^k}{k^k} = \lim_k \frac{1}{k-1}\cdot\left(\frac{k-1}{k}\right)^k = \lim_k \underbrace{\frac{1}{k-1}}_{\to 0}\underbrace{\left(1+\frac{-1}{k}\right)^k}_{\to e^{-1}} = 0$

(e) $\lim_n \frac{n^{n+1}}{(n+1)^n} = \lim_n n\cdot\frac{n^n}{(n+1)^n} = \lim_n n\cdot\left(\frac{n}{n+1}\right)^n = \lim_n n\cdot\left[\left(\frac{n+1}{n}\right)^n\right]^{-1} = \lim_n \underbrace{n}_{\to+\infty}\cdot\underbrace{\left[\underbrace{\left(1+\frac{1}{n}\right)^n}_{\to e}\right]^{-1}}_{\to\frac{1}{e}} = +\infty$

(f) $$\lim_n \frac{\log[(n+4)!]-\log(n!)}{n\left(\sqrt[n]{n^5}-1\right)} = \frac{\lim_n\{\log[(n+4)!]-\log(n!)\}}{\lim_n\left[n\left(\sqrt[n]{n^5}-1\right)\right]}$$

numeratore: $$\lim_n\{\log[(n+4)!]-\log(n!)\} = \lim_n \log\left(\frac{(n+4)!}{n!}\right) = \lim_n\left(\frac{(n+4)(n+3)(n+2)(n+1)\cancel{n!}}{\cancel{n!}}\right) =$$

$$= \lim_n \log[(n+4)(n+3)(n+2)(n+1)] = \lim_n \log(n^4) \text{ per asintoticità}$$

denominatore: $$\lim_n\left[n\left(\sqrt[n]{n^5}-1\right)\right] = \lim_n\left[n\left(n^{\frac{5}{n}}-1\right)\right] = \lim_n\left[n\left(\exp\{\log n^{\frac{5}{n}}\}-1\right)\right] =$$

$$= \lim_n\left[n\left(\exp\left\{5\frac{\log n}{n}\right\}-1\right)\right]$$

mi accorgo che per $n\to+\infty$, $\frac{\log n}{n}\to 0$ per gerarchia

$\Rightarrow$ sviluppo: $\exp\left(5\frac{\log n}{n}\right) = 1+5\frac{\log n}{n}+o\left(\frac{\log n}{n}\right)$ per $n\to+\infty$

$$= \lim_n\left[n\left(\cancel{1}+\frac{5\log n}{n}+o\left(\frac{\log n}{n}\right)-\cancel{1}\right)\right] = \lim_n\left[\cancel{n}\left(5\frac{\log n}{\cancel{n}}+o\left(\frac{\log n}{\cancel{n}}\right)\right)\right] =$$

$$= \lim_n\left(5\log n + o(\log n)\right)$$

$$= \frac{\lim_n \log(n^4)}{\lim_n\left(5\log n+o(\log n)\right)} = \lim_n \frac{\log(n^4)}{5\log n+o(\log n)} = \lim_n \frac{4\log n}{5\log n+o(\log n)} = \lim_n \frac{4\cancel{\log n}}{\cancel{\log n}(5+o(1))} = \lim_n \frac{4}{5+o(1)} = \frac{4}{5}$$