Appello 3

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Analisi

1. No: $\partial_y(3x+5y) = 3 \neq \partial_x(3x+5y) = 3.$

2. $\nabla f(x,y) \parallel \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \Rightarrow \det \begin{vmatrix} 2x-2 & 1 \\ 2y-4 & 1 \end{vmatrix} = 0$

$\Rightarrow (2x-2) - (2y-4) = 0$

$\Rightarrow x - y + 1 = 0.\ (*)$

Serve poi $\nabla f(x,y) = \lambda\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\lambda \geq 0$: se vale $(*)$ è $y = x+1 \Rightarrow \nabla f(x,y) = (2x-2, 2x-2) = (2x-2)(1,1) = \frac{\lambda}{\sqrt{2}}(1,1)$ con $\lambda \geq 0 \Leftrightarrow \boxed{2x-2 \geq 0}$ cioè $x \geq 1$.

Si tratta quindi dei punti $\boxed{\{(x,y) : x \geq 1,\ y = x+1\}}.$

Alternativamente $\frac{\nabla f(x,y)}{\|\nabla f(x,y)\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$\Leftrightarrow \frac{(2x-2,\ 2y-4)}{\sqrt{(2x-4)^2 + (2y-4)^2}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \Leftrightarrow \begin{cases} 2x-2 = 2y-4 \\ \frac{2x-2}{\sqrt{2}\,|2x-2|} = \frac{1}{\sqrt{2}} \end{cases}$$

$$\Leftrightarrow \begin{cases} y = x+1 \\ \frac{x-1}{|x-1|} = 1 \end{cases} \Leftrightarrow \begin{cases} y = x+1 \\ x \geq 1 \end{cases}.$$

3. $$\int_{x^2+y^2 \leq 4} (x^2+y^2)\left(2 - \sqrt{x^2+y^2}\right) dx\, dy = \int_{[0,2]_\rho \times [0,2\pi]} \rho^2 (2-\rho)\rho\, d\rho$$

$$= 2\pi \int^2 \rho^3 (2-\rho)\, d\rho = \frac{16\pi}{5}$$

4. $\text{dist}((x,y,z),0) = x^2+y^2+z^2 = x^2+y^2+xy+1 := f(x,y)$

$\nabla f(x,y) = (2x+y,\ 2y+x) = 0 \iff x=y=0.$

5. $\underset{2x}{\underset{\shortparallel}{\partial_y(3+2xy)}} = \underset{2x}{\underset{\shortparallel}{\partial_x(x^2-3y^2)}}.$

$\vec{F}(x,y) = \nabla(3x+x^2y-y^3)$

$$\int_r \vec{F}\cdot dx = U(r(\pi)) - U(r(0))$$
$$= U(0,-e^{\pi}) - U(0,1) = e^{3\pi}+1.$$

6. $$\int_D x^2\sqrt{1+|\nabla f(x,y)|^2}\,dx\,dy$$

$(D = \{(x,y) : x^2+y^2 \le 1\},\quad f(x,y) = \sqrt{x^2+y^2})$

$$= \int_D x^2\sqrt{1+\frac{x^2+y^2}{x^2+y^2}}\,dx\,dy = \sqrt{2}\int_{x^2+y^2\le 1} x^2\,dx\,dy$$

$$= \frac{\sqrt{2}}{2}\int_{x^2+y^2} x^2+y^2\,dx\,dy = \frac{\sqrt{2}}{2}\left(\int_0^1 \rho^3\,d\rho\right)2\pi$$

$$= \sqrt{2}\pi/4$$

Probabilità

1. Anagrammi di SANSCRITO

= anagrammi di SSANCRITO : $\dfrac{9!}{2!}$

Anagrammi con 2S vicine: $\boxed{SS}$ANCRITO : $8!$

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$$\Rightarrow \quad \frac{9!}{2!} - 8!$$

2. $P(S\cap\bar{T}) = 0.1, \quad P(S\cap T) = 0.2, \quad P(S|\bar{T}) = 0.125$

a) $P(S) = P(S\cap\bar{T}) + P(S\cap T) = 0.3$

b) $P(S|\bar{T}) = \dfrac{P(S\cap\bar{T})}{P(\bar{T})} \Rightarrow$

$$0.125 = \frac{0.1}{1-P(T)}$$

$$\Rightarrow 1-P(T) = \frac{0.1}{0.125}$$

$$\Rightarrow P(T) = 1 - \frac{100}{125} = \frac{25}{125} = 1/5 = 0.2$$

c) $$P(T|\bar{S}) = \frac{P(\bar{S}|T)P(T)}{P(\bar{S})} = \frac{(1-P(S|T))P(T)}{P(\bar{S})}$$

$$= \frac{(P(T) - P(S\cap T))}{1-P(S)} = \frac{1/5 - 0.2}{0.7} = 0.$$

3. a) $X = 11 + \sigma Z \qquad \sigma = 0.5$

$$P(|X-11| \le 0.2) = P(\sigma|Z| \le 0.2) = P(|Z| \le \tfrac{2}{5} = 0.4)$$

$$= 2\Phi(0.4) - 1 \approx 0.31$$

b) $P(|X-11| \le 0.2) \ge 0.7 \Leftrightarrow P(|Z| \le \frac{0.2}{\sigma}) \ge 0.7$

$$\Leftrightarrow 2\Phi\left(\frac{0.2}{\sigma}\right) - 1 \ge 0.7 \Leftrightarrow \Phi\left(\frac{0.2}{\sigma}\right) \ge 0.85$$

$$\Leftrightarrow \frac{0.2}{\sigma} \ge 1. \qquad \Leftrightarrow \sigma \le \frac{0.2}{1.04} = \frac{5}{26}$$

$$\Leftrightarrow \mathrm{Var}\, X \le \left(\frac{0.2}{1.04}\right)^2 = \left(5/26\right)^2$$

4. $f_c(x,y) = \begin{cases} c(x^2+y) & 0\le x\le y\le 2 \\ 0 \text{ altrimenti} & \end{cases}$

$$\int_{\mathbb{R}^2} f_c(x,y)\,dx\,dy = 1 \iff c\int_{0\le x\le y\le 2} x^2+y \quad dx\,dy = 1$$

$$\iff c\int_0^2\int_0^y (x^2+y)\,dx\,dy = 1 \iff c\int_0^2 \left[\frac13 x^3+yx\right]_{x=0}^{x=y} dy = 1$$

$$\iff c\int_0^2 \left(\frac13 y^3 + y^2\right) dy = 1 \iff c = \frac{1}{\frac1{12}2^4+\frac13 2^3} = \frac{1}{\frac43+\frac83}$$
$$= \frac{3}{12} = \boxed{\frac14}$$

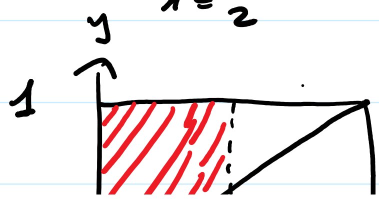
b) $f_Y(y) = \begin{cases} 0 & \text{se} \quad y\notin[0,2] \\ \int_{\mathbb{R}} f_{\frac14}(x,y)\,dx & y\in[0,2]\end{cases}$

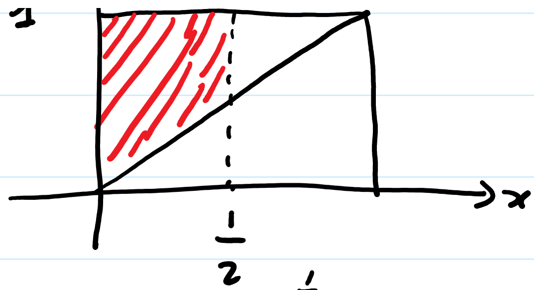
Se $y\in[0,2]$ $\quad \int f_{\frac14}(x,y)\,dx = \frac14\int_0^y (x^2+y)\,dx = \frac13 y^3+y^2$.

$$\Rightarrow f_Y(y) = \frac14\begin{cases} 0 & y\notin[0,2] \\ \frac13 y^3+y^2 & y\in[0,2]\end{cases}$$

c) $P(X\le \frac12,\ Y\le 1) = \int_{]-\infty,\frac12]\times]-\infty,1]} f_{\frac14}(x,y)\,dx\,dy$

$$= \int_{\substack{0\le x\le y\le 2\\ x\le \frac12\\ y\le 1}} \frac14 (x^2+y)\,dx\,dy = \frac14 \int_{\substack{0\le x\le y\\ y\le 1\\ x\le\frac12}} x^2+y \quad dx\,dy$$





$$= \frac{1}{4}\int_0^{\frac{1}{2}}\int_x^1 x^2+y \quad dy \quad dx = \frac{1}{4}\int_0^{\frac{1}{2}} \left[x^2y+\frac{1}{2}y^2\right]_{y=x}^{y=1} dx$$

$$= \frac{1}{4}\int_0^{\frac{1}{2}} \left(x^2+\frac{1}{2}\right)-\left(x^3+\frac{1}{2}x^2\right) dx = \frac{1}{4}\int_0^{\frac{1}{2}} \frac{1}{2}x^2-x^3+\frac{1}{2}\, dx$$

$$= \frac{1}{4}\left[\frac{1}{6}x^3-\frac{1}{4}x^4+\frac{1}{2}x\right]_0^{\frac{1}{2}} = \frac{1}{4}\left[\frac{1}{6}\left(\frac{1}{2}\right)^3-\frac{1}{4}\left(\frac{1}{2}\right)^4+\frac{1}{2^2}\right] = \frac{49}{768}$$

$$P\left(X\le\frac{1}{2} \mid Y\le 1\right) = \frac{P\left(X\le\frac{1}{2}, Y\le 1\right)}{P(Y\le 1)} = \frac{49/768}{\int_0^1 \frac{1}{12}(y^3+3y^2)\, dy}$$

$$= \frac{49/768}{\frac{1}{12}\left[\frac{1}{4}+1\right]} = \frac{49/768}{\frac{1}{12}(5/4)} = \frac{49}{80}.$$