Appello 3 - prob

domenica 7 luglio 2019 15:57

Probabilità

1. (Bayes). I dati: $P(R|S)=0,9$; $P(S^c|R^c)=0.99$; $P(R)=0.8$

$$\overbrace{P(S|R)}^{x} = \frac{P(R|S)P(S)}{P(R)}$$

$$P(S) = \underbrace{P(S|R)}_{x}P(R) + \underbrace{P(S|R^c)}_{\substack{\| \\ (1-P(S^c|R^c))}}P(R^c)$$

$$x = \frac{P(R|S)\,(xP(R) + (1-P(S^c|R^c))P(R^c))}{P(R)}$$

$$\Rightarrow \quad xP(R)(1-P(R|S)) = (1-P(S^c|R^c))P(R^c)$$

$$\Rightarrow \quad x = \frac{(1-P(S^c|R^c))P(R^c)}{1-P(R|S)} = \frac{0.01\times 0.2}{0.1} = 2\%$$

In alternativa: $P(S) = \dfrac{P(S|R^c)P(R^c)}{P(R^c|S)} = \dfrac{0{,}01\times 0{,}2}{0{,}1} = \dfrac{2}{100}$

da cui $P(S|R) = \dfrac{P(R|S)P(S)}{P(R)} = \dfrac{0.9\times 0.02}{0.8} = \dfrac{9}{400}$

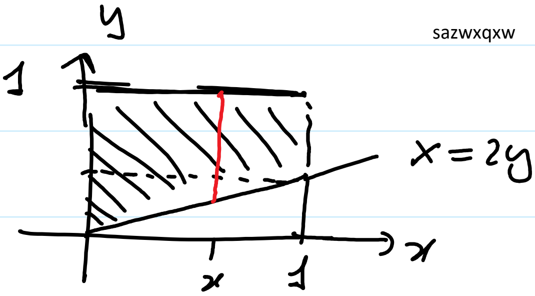
2. $f^a(x,y) = \begin{cases} ax^2(5-y) & x,y\in[0,1] \\ 0 & \text{altrimenti} \end{cases}$.

a) $f^a \geq 0$, $\int f_a(x,y) = \frac{1}{3}a\left[5y-\frac{1}{2}y^2\right]_0^1 = \frac{9}{6}a = \frac{3}{2}a = 1 \Leftrightarrow a = \frac{2}{3}$

b) $f_X^a(x) = \begin{cases} 0 & x\notin[0,1] \\ \frac{2}{3}x^2\left[5y-\frac{1}{2}y^2\right]_0^1 = \frac{2}{3}x^2\cdot\frac{9}{2} = 3x^2 & x\in[0,1] \end{cases}$

$$f_Y(y) = \begin{cases} 0 & y \notin [0,1] \\ \frac{2}{9}(5-y) & y \in [0,1]. \end{cases}$$

c) $$P(X \le 2Y) = \int_{\substack{0 \le x \le 2y \\ x \in [0,1] \\ y \in [0,1]}} \frac{2}{3} x^2 (5-y)\, dx\, dy$$



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$0 \le x \le 2y \quad y \in [0,1]$
$x \in [0,1]$

$$\Longleftrightarrow \quad x \in [0,1], \quad \frac{x}{2} \le y \le 1$$

$$P(X \le 2Y) = \frac{2}{3} \int_0^1 x^2 \int_{x/2}^1 (5-y)\, dy\, dx = \frac{3}{5}$$

3. T_i = tempo di vita batteria i

$$P(T_1 + \dots + T_{100} > 20260) > 0.4$$

$$\simeq P(\underbrace{100 \times 200 + \sqrt{100}\, \sigma Z}_{\parallel} > 20260) > 0.4$$

$$P(20.000 + 10\sigma Z > 20.260)$$

$$\parallel$$

$$P(\sigma Z > 26) = P\left(Z > \frac{26}{\sigma}\right) = 1 - \Phi\left(\frac{26}{\sigma}\right)$$

Si vuole $1 - \Phi\left(\frac{26}{\sigma}\right) > 0.4 \iff \Phi\left(\frac{26}{\sigma}\right) < 0.6$

Dato che $\Phi(0.25) < 0.6$, $\Phi(0.26) > 0.6$ è sufficiente che $\frac{26}{\sigma} < 0.25 \iff \sigma > \frac{26}{1/4} = 104$. #

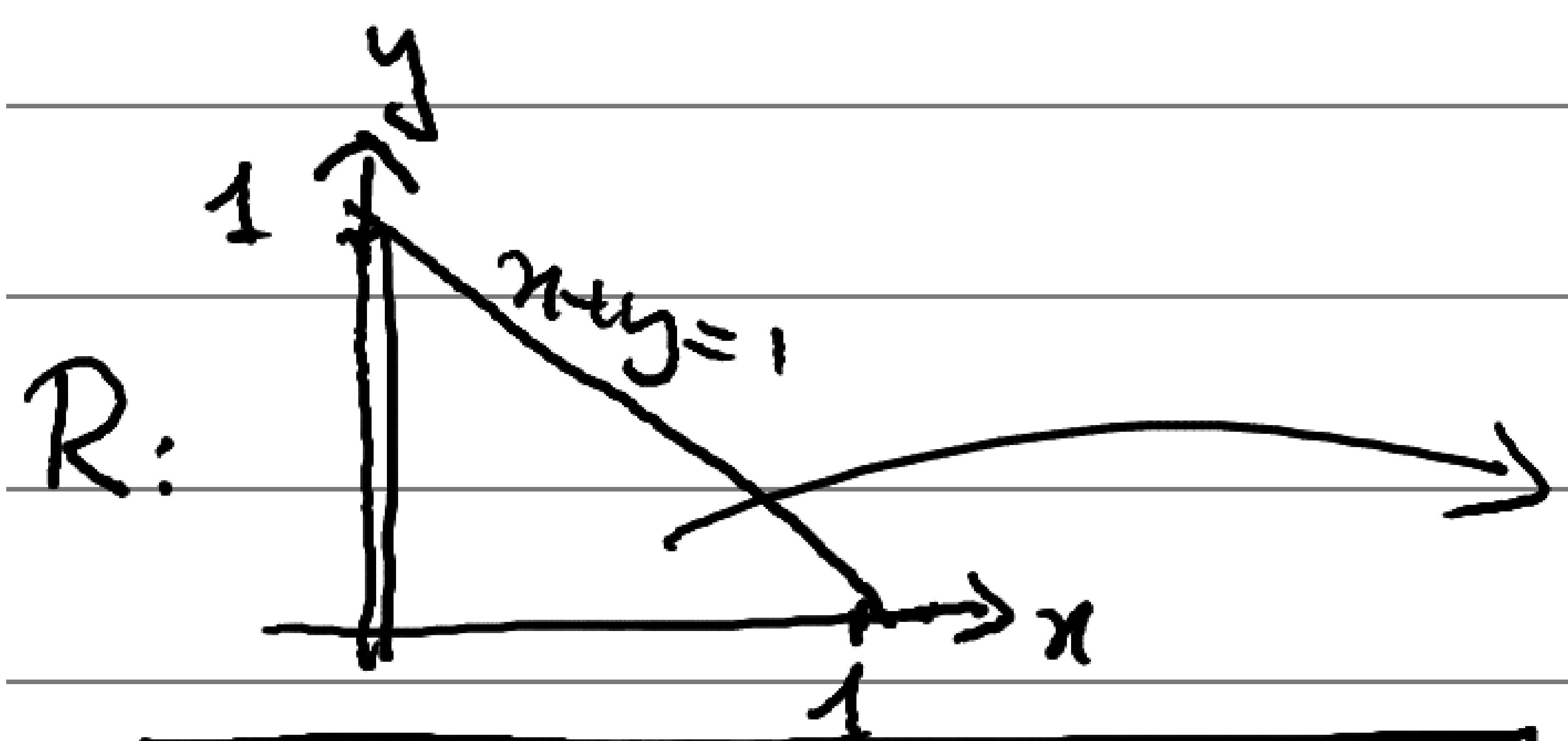
Analisi

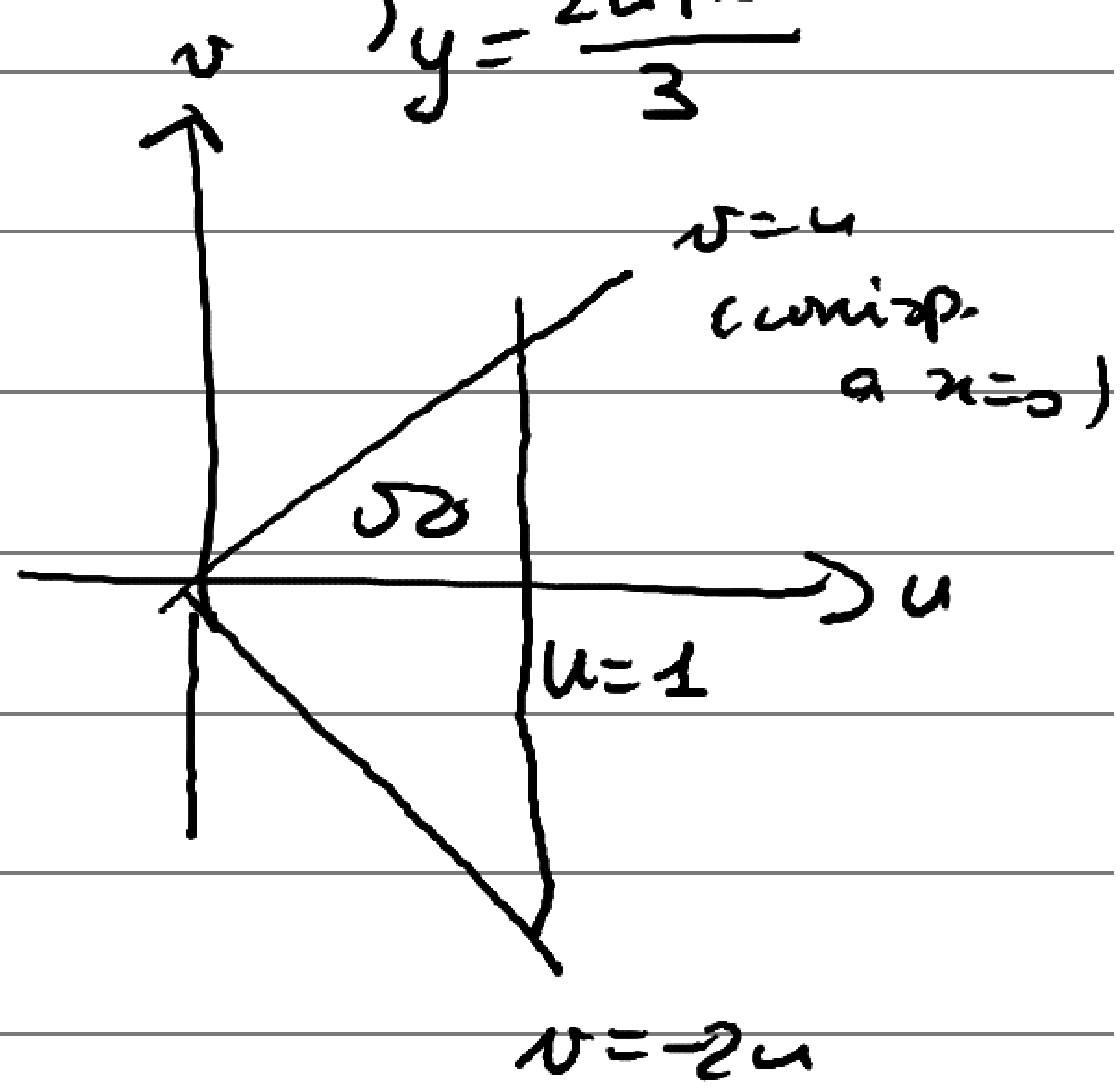
A1) $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$

$$= \int_R \underbrace{\sqrt{x+y}\,(y-2x)^2}_{f(x,y)}\,dy\,dx \qquad R=\{x\in[0,1],\ 0\le y\le 1-x\}$$

Poniamo $u=x+y$, $v=y-2x$. La funzione $(x,y)\longmapsto(u(x,y),v(x,y))$ è biiettiva (lineare e $\det\begin{pmatrix}1&1\\-2&1\end{pmatrix}=3\neq0$)

Si ha $(u,v)=(x+y,-2x+y) \Leftrightarrow \begin{cases} x=\dfrac{u-v}{3}\\[4pt] y=\dfrac{2u+v}{3}\end{cases}$





$\boxed{\Omega=\{0\le u\le 1,\ -2u\le v\le u\}}$

$$\int_R f(x,y)\,dx\,dy$$

$$= \int_\Omega \sqrt{u}\,v^2\cdot\underbrace{\left|\det\frac{\partial(x,y)}{\partial(u,v)}\right|}_{1/3}\,du\,dv$$

$$= \frac{1}{3} \int_{\substack{u \in [0,1] \\ -2u \le v \le u}} \sqrt{u}\, v^2 = \frac{1}{3} \int_0^1 \sqrt{u} \int_{-2u}^{u} v^2 \, dv \, du$$

$$= \frac{1}{9} \int_0^1 \sqrt{u} \left[v^3 \right]_{v=-2u}^{v=u} du = \frac{1}{9} \int_0^1 \sqrt{u}\, 9u^3$$

$$= \int_0^1 u^{7/2} du = \boxed{\frac{2}{9}}$$

(A2) 1. $\vec{F}$ è radiale, cioè del tipo $\varphi(|X|)\frac{X}{|X|}$

con $X = (x,y)$. Una primitiva è data da

$$U(X) = \int \varphi(r)\, dr \quad (r = |X|).$$

Qui $\varphi(r) = r^6$ $\int \varphi(r)\, dr = \frac{r^9}{9} + C$

$\Rightarrow$ Una primitiva è $U(x,y) = \frac{1}{9}\left(\sqrt{x^2+y^2}\right)^9.$

2. $\gamma(0)=(1,3)$, $\gamma(1)=(2,4)$. Per il T. fond. del calcolo per i campi conservativi:

$$\int_\gamma \vec{F}\cdot d\gamma = U(\gamma(1)) - U(\gamma(0))$$

$$= U(2,4) - U(1,3)$$

$$= \frac{1}{9}\left[\left(\sqrt{2^2+4^2}\right)^9 - \left(\sqrt{1+9}\right)^9\right]$$

$$= \frac{1}{9}\left(20^{9/2} - 10^{9/2}\right).$$

A3) $B=\{(x,y): x^2+y^2\le 1\}$ è compatto (chiuso e limitato) e T è continua $\Rightarrow$ $\exists$ min/max assoluto.

a) Punti critici interni.

$$\nabla T(x,y) = (2x-1, 4y) = (0,0) \Leftrightarrow \begin{cases} x=\frac{1}{2} \\ y=0 \end{cases} :$$

unico pto critico interno è $\left(\frac{1}{2},0\right)$, dove T vale $T\left(\frac{1}{2},0\right) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

b) Studiamo T sul bordo di B:

Posto $x=\cos t$, $y=\sin t$, $t\in[0,2\pi]$ è

$$T(\cos t, \sin t) = \cos^2 t + 2\sin^2 t - \cos t$$
$$= \cos^2 t + 2(1-\cos^2 t) - \cos t$$
$$= -\cos^2 t - \cos t + 2$$
$$= -z^2 - z + 2, \quad z=\cos t \in [-1,1]$$

I MAX/MIN di T su ∂B sono quelli di

$\varphi(z) = -z^2 - z + 2$ su $[-1,1]$.

È $\varphi'(z) = -2z - 1$

	-1		$-1/2$		1
φ'		$+$	0	$-$	
φ	2	$\nearrow$	$2+\frac{1}{4} = 9/4$	$\searrow$	0

$\Rightarrow$ MAX T = MAX$\{9/4, -1/4\} = 9/4$ (assunto sul bordo)

MIN T = MIN$\{2, 0, -1/4\} = -1/4$ (assunto all'interno)

Domande Teoriche

(T1) $y' - 2y \le 1 \Rightarrow (y e^{-2t})' \le e^{-2t}$

$\Rightarrow y e^{-2t} - y(0) \le \int_0^t e^{-2s}\, ds$

$\Rightarrow y e^{-2t} \le 5 - \frac{1}{2}(e^{-2t} - 1) = \frac{11}{2} - \frac{e^{-2t}}{2}$

$\Rightarrow y(t) \le \frac{11}{2} e^{2t} - \frac{1}{2}$

Per $t = \lg 3$ viene $y(\lg 3) \le \frac{11}{2} e^{\lg 9} - \frac{1}{2}$

$$\begin{aligned} &\;\; \| \\ &\frac{9 \times 11}{2} - \frac{1}{2} \\ &= \frac{1}{2}\, 98 = \boxed{49}. \end{aligned}$$

Valore raggiunto quando $y(t) = \frac{11}{2} e^{2t} - \frac{1}{2}$.

(T2) $D_{\vec{v}} f(0,0) = 3u_1 - 4u_2 = 0 \Leftrightarrow u_2 = \frac{3}{4} u_1$:

$\vec{v} = \pm (1, \frac{3}{4}) / \sqrt{1 + 9/16} = \pm \frac{(1, \frac{3}{4})}{5} 4 = \pm (\frac{4}{5}, \frac{3}{5})$.

2. f differenziabile $\Rightarrow$

$$f(x,y) - f(0,0) = \partial_x f(0,0)x + \partial_y f(0,0)y$$
$$+ R(x,y)\sqrt{x^2+y^2}, \quad \underset{(x,y)\to(0,0)}{R(x,y)\to 0}$$

$$\Rightarrow f(x,y) = 3x - 4y + R(x,y)\sqrt{x^2+y^2}$$

$$\Rightarrow \frac{f(x,y) - 3x + 4y}{(x^2+y^2)^{1/4}} = R(x,y)(x^2+y^2)^{1/4} \to 0$$

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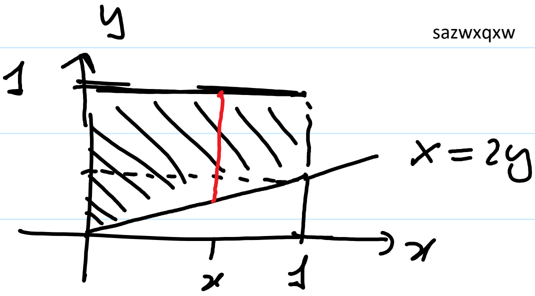

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