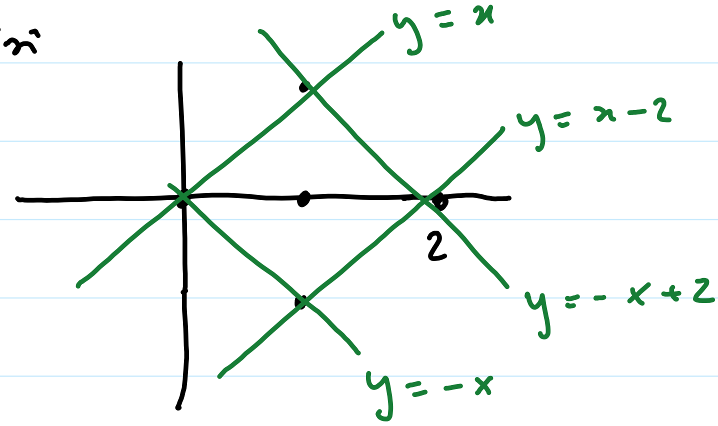
Appello 3

martedì 19 luglio 2016 11:37

Analisi

3.



$$D = \{(x,y) : \quad x-y \in [0,2], \quad x+y \in [0,2]\}$$

Posto $u = x-y$, $v = x+y$ è $\begin{cases} x = \frac{u+v}{2} \\ y = \frac{v-u}{2} \end{cases} = \phi(u,v)$

$$\text{Jac}\,\phi(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

$$\frac{x-y}{x+y+1} = \frac{u}{v+1} \quad e$$

$$\int_D \sqrt{\frac{x-y}{x+y+1}}\, dx\, dy = \int_{[0,2]\times[0,2]} \frac{1}{2} \sqrt{\frac{u}{v+1}}\, du\, dv$$

$$= \frac{1}{2} \int_0^2 \sqrt{u}\, du \int_0^2 \frac{1}{\sqrt{v+1}}\, dv$$

$$= \frac{1}{2} \left[\frac{2}{3} u^{\frac{3}{2}}\right]_0^2 \left[2(v+1)^{\frac{1}{2}}\right]_0^2 = \frac{1}{3} \cdot 2^{3/2} \cdot 2(\sqrt{3}-1)$$

$$= \frac{4}{3}\sqrt{2}\,(\sqrt{3}-1)$$

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1. $\gamma(0)=(1,1,0)$, $\quad \gamma(\pi)=(e^{\pi},-1,0)$

$$\int_\gamma \nabla(1+x^2yz)\cdot d\gamma = \left[1+x^2yz\right]_{\gamma(0)}^{\gamma(\pi)} = 0$$

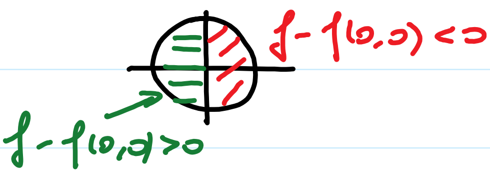
4. $\lim\limits_{(0,0)} \dfrac{y^4-2x^2}{y^4+x^2}$ non esiste. Infatti

su $x=my^2$ è $\dfrac{y^4-2x^2}{y^4+x^2}=\dfrac{1-2m^2}{1+m^2}$.

2. $f(x,y)=2-xy^2$. $\quad \partial_x f(x,y)=-y^2 \quad \partial_y f(x,y)=-2xy$

$$\mathrm{Hess} f(x,y)=\begin{pmatrix}0 & -2y\\ -2y & -2x\end{pmatrix} \Rightarrow \mathrm{Hess} f(0,0)=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

L'Hessiano non permette di concludere.
Si ha $f(x,y)-f(0,0)=(2-xy^2)-2=-xy^2$ ed è
$xy^2<0$ se $x<0$, $xy^2>0$ se $x>0$: in ogni intorno
di $(0,0)$ $f(x,y)-f(0,0)$ assume valori di ambo i
segni e quindi $(0,0)$ è di SELLA.



5. $\begin{cases} y'=y/10-40\\ y(0)=500\end{cases}$

$$y'=y/10-40 \Rightarrow y'-\frac{y}{10}=-40$$
$$\Rightarrow \left(y e^{-t/10}\right)'=-40e^{-t/10}$$
$$\Rightarrow y(t)e^{-t/10}-y(0)=-40\int_0^t e^{-s/10}\,ds$$

$$\Rightarrow \quad y(t)e^{-t/10} - y(0) = -40\int_0^t e^{-s/10}\,ds$$
$$= -40\left(-10e^{-t/10}+10\right)$$
$$= 400e^{-t/10} - 400$$

$$\Rightarrow \quad y(t) = 500e^{t/10} + 400 - 400e^{t/10}$$
$$= 100e^{t/10} + 400.$$

Si ha $\quad y(t) = 600 \iff 100e^{t/10} = 200$
$$\iff e^{t/10} = 2 \iff t/10 = \lg 2$$
$$\iff t = 10\lg 2.$$

Probabilità

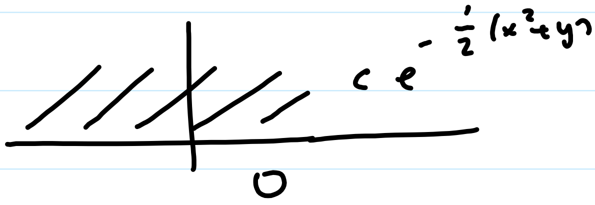
1. $$P(\text{Rosse} \mid \text{Stesso Colore}) = \frac{P(\text{Stesso Colore} \mid \text{Rosse})P(\text{Rosse})}{P(\text{Stesso Colore})}$$

$$\underbrace{P(\text{Stesso Colore})}_{S} = \underbrace{P(S|R)P(R)}_{\text{2 Rosse}} + \underbrace{P(S|N)P(N)}_{\text{2 Nere}} + \underbrace{P(S|V)P(V)}_{\text{2 Verdi}}$$
$$= P(R) + P(N) + P(V)$$
$$= \frac{5\times4 + 3\times2 + 4\times3}{12\times11} = \frac{38}{12\times11}$$

$$P(R) = \frac{5\times4}{12\times11}$$

$$\Rightarrow \quad P(\text{Rosse} \mid \text{St. Colore}) = \frac{5\times4}{38} = \frac{10}{19}.$$

2.



a) Deve essere $$\int_{\mathbb{R}\times[0,+\infty[} c e^{-\frac{1}{2}(x^2+y)}\,dx\,dy = 1$$

$$\Leftrightarrow c \int_{\mathbb{R}} e^{-x^2/2} \int_0^{+\infty} e^{-y/2} dy = 1$$

$$\left.\begin{array}{l} \displaystyle\int_{\mathbb{R}} e^{-x^2/2} dx \underset{(t=\frac{x}{\sqrt{2}})}{=} \sqrt{2} \int_{\mathbb{R}} e^{-t^2} dt = \sqrt{2\pi} \\ \displaystyle\int_0^{+\infty} e^{-y/2} dy = -\left[2e^{-y/2}\right]_0^{\infty} = 2 \end{array}\right\} \begin{array}{l} \Rightarrow 2c\sqrt{2\pi} = 1 \\ \Rightarrow c = \dfrac{1}{\sqrt{8\pi}} \end{array}$$

(b) • $\forall x \quad f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)\, dy$

$$= c \int_0^{+\infty} e^{-\frac{1}{2}x^2} e^{-y/2} dy = c e^{-\frac{1}{2}x^2} \int_0^{+\infty} e^{-y/2} dy$$

$$= 2c e^{-\frac{1}{2}x^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

• Se $y < 0$ $\quad f_Y(y) = 0$

Se $y \geq 0$ $\quad f_Y(y) = c \int_{\mathbb{R}} e^{-\frac{1}{2}x^2} e^{-y/2} dx = c e^{-y/2} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2}$

$$= \sqrt{2\pi}\, c\, e^{-y/2}$$

$$= \frac{1}{2} e^{-y/2}.$$

(c) $f_X \neq f_Y$: X e Y non sono ident. distribuite.

Si ha poi $f_X(x) f_Y(y) = \begin{cases} \frac{1}{\sqrt{8\pi}} e^{-\frac{1}{2}x^2 - \frac{1}{2}y} & \text{se } y > 0 \\ 0 & \text{altrim.} \end{cases} = f_{X,Y}(x,y)$

$\Rightarrow$ X e Y sono indipendenti.

(d) X, Y indip $\Rightarrow E(XY) = E(X)E(Y) = 0$ dato che $E(X) = 0$.

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(e) $$P(Y > X^2) = \int_{x^2 < y} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{\mathbb{R}} \left\{ \int_{x^2}^{+\infty} c\, e^{-\frac{1}{2}x^2 - \frac{1}{2}y} dy \right\} dx$$

$$= \int_{\mathbb{R}} c\, e^{-\frac{1}{2}x^2} \left[-2 e^{-y/2} \right]_{y=x^2}^{+\infty} dx = \int_{\mathbb{R}} c\, e^{-\frac{1}{2}x^2} \cdot 2 e^{-x^2/2} dx$$

$$= 2c \int_{\mathbb{R}} e^{-x^2} dx = 2c\sqrt{\pi} = \frac{1}{\sqrt{2\pi}} \sqrt{\pi} = \frac{1}{\sqrt{2}}.$$

3. $X_1 + \dots + X_n \sim Po(4n) \Rightarrow P(S_n = k) = (4n)^k \dfrac{e^{-4n}}{k!} \quad \forall k$

$Var(S_n) = E(S_n) = 4n$

T. C. Limite $\Rightarrow P(S_n > 390) \sim P(4n + \sqrt{4n}\, Z > 390)$

con $Z \sim N(0,1)$

$$= P\left(Z > \frac{390 - 4n}{2\sqrt{n}}\right) = 1 - \Phi\left(\frac{390-4n}{2\sqrt{n}}\right)$$

cioè $P(S_n > 390) > 0.5$

$$1 - \Phi\left(\frac{390-4n}{2\sqrt{n}}\right) > 0.5 \Leftrightarrow \Phi\left(\frac{390-4n}{2\sqrt{n}}\right) < 0.5 = \Phi(0)$$

$$\Leftrightarrow \frac{390-4n}{2\sqrt{n}} < 0 \Leftrightarrow 390 < 4n$$

$$\Leftrightarrow n > \frac{390}{4} = \frac{195}{2}.$$