ESERCIZI SCHEDA 12

ESERCIZIO 1

$y'(t) = t\,y(t)$ Moltiplico tutto per $e^{-\int t\,dt} = e^{-\frac{t^2}{2}}$

$e^{-\frac{t^2}{2}}y' - te^{-\frac{t^2}{2}}y = 0$

$\Rightarrow \frac{d}{dt}\left[e^{-\frac{t^2}{2}}y\right] = C \in \mathbb{R} \Rightarrow \boxed{y = Ce^{\frac{t^2}{2}}}$

ESERCIZIO 2

$$\begin{cases} y'(t) = 5y(t) + e^t \\ y(0) = 0 \end{cases}$$

$y'(t) - 5y(t) = e^t$ Moltiplico tutto per $e^{-A(t)}$ dove $A(t)$ è una primitiva di $5 \Rightarrow A(t) = 5t$

$e^{-5t}y'(t) - 5e^{-5t}y(t) = e^t \cdot e^{-5t}$

$\frac{d}{dt}\left[e^{-5t}y(t)\right] = e^{-4t} \Leftrightarrow e^{-5t}y(t) = \int e^{-4t}dt = -\frac{1}{4}\int -4e^{-4t}dt = -\frac{1}{4}e^{-4t} + c$

$\Leftrightarrow y(t) = e^{5t}\left(-\frac{1}{4}e^{-4t} + c\right)$

$y(0) = 0 \Leftrightarrow e^0\left(-\frac{1}{4}e^0 + c\right) = 0 \Leftrightarrow c - \frac{1}{4} = 0 \Leftrightarrow c = \frac{1}{4}$

$\Rightarrow y(t) = e^{5t}\left(-\frac{1}{4}e^{-4t} + \frac{1}{4}\right)$

$\Rightarrow \boxed{y(t) = \frac{e^{5t}}{4} - \frac{e^t}{4}}$

ESERCIZIO 3

$$\begin{cases} y'(t) = t\,y(t) + \alpha t \\ y(0) = 3 \end{cases}$$

$y'(t) - t\,y(t) = \alpha t$ Moltiplico ambo i membri per $e^{A(t)} = e^{-\int t\,dt} = e^{-\frac{t^2}{2}}$

$e^{-\frac{t^2}{2}}y'(t) - te^{-\frac{t^2}{2}}y(t) = \alpha te^{-\frac{t^2}{2}}$

$\frac{d}{dt}\left[e^{-\frac{t^2}{2}}y(t)\right] = \alpha te^{-\frac{t^2}{2}} \Leftrightarrow e^{-\frac{t^2}{2}}y(t) = \int \alpha te^{-\frac{t^2}{2}}dt = -\alpha\int -te^{-\frac{t^2}{2}}dt = -\alpha\left(e^{-\frac{t^2}{2}} + c\right)$

$\Leftrightarrow y(t) = e^{\frac{t^2}{2}}\left[-\alpha\left(e^{-\frac{t^2}{2}} + c\right)\right] = -\alpha\left(1 + ce^{\frac{t^2}{2}}\right)$

$$y(0)=3 \iff -\alpha\left(1+ce^{0}\right)=3 \iff -\alpha(1+c)=3 \iff -\alpha-c\alpha=3 \iff c=-\frac{\alpha+3}{\alpha}$$

$$\Rightarrow y(t)=-\alpha\left(1-\frac{\alpha+3}{\alpha}e^{\frac{t^2}{2}}\right) \iff \boxed{y(t)=-\alpha+(\alpha+3)e^{\frac{t^2}{2}}}$$

Esercizio 4

ⓐ $y'(t)-2y(t)=t-1+t\sin t$ Moltiplico ambo i membri per $e^{-A(t)}=e^{-\int 2dt}=e^{-2t}$

$$e^{-2t}y'(t)-2e^{-2t}y(t)=e^{-2t}(t-1+t\sin t)$$

$$\frac{d}{dt}\left[e^{-2t}y(t)\right]=e^{-2t}(t-1+t\sin t)$$

$$e^{-2t}y(t)=\int e^{-2t}(t-1+t\sin t)\,dt=\underbrace{\int te^{-2t}dt}_{\boxed{A}}-\underbrace{\int e^{-2t}dt}_{\boxed{B}}+\underbrace{\int te^{-2t}\sin t\,dt}_{\boxed{C}}$$

$\boxed{B}$ $-\int e^{-2t}dt=\int -e^{-2t}dt=\frac{1}{2}\int -2e^{-2t}dt=\frac{1}{2}e^{-2t}+c$

$\boxed{A}$ $\int te^{-2t}dt$ Integro per parti: $\begin{cases} f(t)=t \\ g'(t)=e^{-2t}\end{cases} \Rightarrow \begin{cases} f'(t)=1 \\ g(t)=-\frac{1}{2}e^{-2t}\end{cases}$

$$=-\frac{1}{2}te^{-2t}-\int -\frac{1}{2}e^{-2t}dt=-\frac{1}{2}te^{-2t}-\frac{1}{4}\int -2e^{-2t}dt=-\frac{1}{2}te^{-2t}-\frac{1}{4}e^{-2t}+c$$

$\boxed{C}$ **MOLTO LUNGO**

$\int te^{-2t}\sin t\,dt$ Integro per parti: $\begin{cases} f(t)=te^{-2t} \\ g'(t)=\sin t\end{cases} \Rightarrow \begin{cases} f'(t)=e^{-2t}-2te^{-2t} \\ g(t)=-\cos t\end{cases}$

$$\int te^{-2t}\sin t\,dt=-te^{-2t}\cos t+\int e^{-2t}\cos t\,dt\underline{-2\int te^{-2t}\cos t\,dt}$$

Integro per parti $\begin{cases} f(t)=te^{-2t} \\ g'(t)=\cos t\end{cases} \Rightarrow \begin{cases} f'(t)=e^{-2t}-2te^{-2t} \\ g(t)=\sin t\end{cases}$

$$=-2te^{-2t}\sin t+2\int e^{-2t}\sin t\,dt-4\int te^{-2t}\sin t\,dt$$

$$\int te^{-2t}\sin t\,dt=-te^{-2t}\cos t+\int e^{-2t}\cos t\,dt-2te^{-2t}\sin t+2\int e^{-2t}\sin t\,dt-4\int te^{-2t}\sin t\,dt$$

$$5\int te^{-2t}\sin t\,dt=-te^{-2t}\cos t+\int e^{-2t}\cos t\,dt-2te^{-2t}\sin t+2\int e^{-2t}\sin t\,dt$$

$$\int te^{-2t}\sin t\,dt=\frac{1}{5}\left(-te^{-2t}\cos t+\underline{\int e^{-2t}\cos t\,dt}-2te^{-2t}\sin t+2\int e^{-2t}\sin t\,dt\right)$$

Integro per parti: $\begin{cases} f(t)=e^{-2t} \\ g'(t)=\cos t\end{cases} \Rightarrow \begin{cases} f'(t)=-2e^{-2t} \\ g(t)=\sin t\end{cases}$

$$=e^{-2t}\sin t+2\int e^{-2t}\sin t\,dt$$

$$\int te^{-2t}\sin t\,dt = \frac{1}{5}\left(-te^{-2t}\cos t + e^{-2t}\sin t + 2\int e^{-2t}\sin t\,dt - 2te^{-2t}\sin t + 2\int e^{-2t}\sin t\,dt\right)$$

$$\int te^{-2t}\sin t\,dt = \frac{1}{5}\left(-te^{-2t}\cos t + e^{-2t}\sin t + 4\int e^{-2t}\sin t\,dt - 2te^{-2t}\sin t\right)$$

Integro per parti: $\begin{cases} f(t) = e^{-2t} \\ g'(t) = \sin t \end{cases} \Rightarrow \begin{cases} f'(t) = -2e^{-2t} \\ g(t) = -\cos t \end{cases}$

$$\int e^{-2t}\sin t\,dt = -e^{-2t}\cos t - 2\int e^{-2t}\cos t\,dt$$

Nuovamente per parti: $\begin{cases} f(t) = e^{-2t} \\ g'(t) = \cos t \end{cases} \Rightarrow \begin{cases} f'(t) = -2e^{-2t} \\ g(t) = \sin t \end{cases}$

$$\int e^{-2t}\cos t\,dt = e^{-2t}\sin t + 2\int e^{-2t}\sin t\,dt$$

$$\int e^{-2t}\sin t\,dt = -e^{-2t}\cos t - 2e^{-2t}\sin t - 4\int e^{-2t}\sin t\,dt$$

$$5\int e^{-2t}\sin t\,dt = -e^{-2t}\cos t - 2e^{-2t}\sin t$$

$$\int e^{-2t}\sin t\,dt = \frac{1}{5}\left(-e^{-2t}\cos t - 2e^{-2t}\sin t\right)$$

$$\int te^{-2t}\sin t\,dt = \frac{1}{5}\left(-te^{-2t}\cos t + e^{-2t}\sin t + \frac{4}{5}\left(-e^{-2t}\cos t - 2e^{-2t}\sin t\right) - 2te^{-2t}\sin t\right) + c$$

$$= -\frac{1}{5}te^{-2t}\cos t - \frac{2}{5}te^{-2t}\sin t - \frac{4}{25}e^{-2t}\cos t - \frac{3}{25}e^{-2t}\sin t + c$$

$$\Rightarrow e^{-2t}y(t) = -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + \frac{1}{2}e^{-2t} - \frac{1}{5}te^{-2t}\cos t - \frac{2}{5}te^{-2t}\sin t - \frac{4}{25}e^{-2t}\cos t - \frac{3}{25}e^{-2t}\sin t + c$$

$$\Rightarrow y(t) = -\frac{1}{2}t - \frac{1}{4} + \frac{1}{2} - \frac{1}{5}t\cos t - \frac{2}{5}t\sin t - \frac{4}{25}\cos t - \frac{3}{25}\sin t + ce^{2t}$$

$$\Rightarrow \boxed{y(t) = \frac{1}{4} - \frac{1}{2}t - \frac{4}{25}\cos t - \frac{3}{25}\sin t - \frac{1}{5}t\cos t - \frac{2}{5}t\sin t + ce^{2t}}$$

b

$$y(1) = 0 \iff \frac{1}{4} - \frac{1}{2} - \frac{4}{25}\cos(1) - \frac{3}{25}\sin(1) - \frac{1}{5}\cos(1) - \frac{2}{5}\sin(1) + ce^{2} = 0$$

$$\iff -\frac{1}{4} - \frac{9}{25}\cos(1) - \frac{13}{25}\sin(1) + ce^{2} = 0 \iff c = \frac{\frac{1}{4} + \frac{9}{25}\cos(1) + \frac{13}{25}\sin(1)}{e^2}$$

$$\Rightarrow y(t) = \frac{1}{4} - \frac{1}{2}t - \frac{4}{25}\cos t - \frac{3}{25}\sin t - \frac{1}{5}t\cos t - \frac{2}{5}t\sin t + e^{2t}\cdot\frac{\frac{1}{4} + \frac{9}{25}\cos(1) + \frac{13}{25}\sin(1)}{e^2}$$

$$\Rightarrow \boxed{y(t) = \frac{1}{4} - \frac{1}{2}t - \frac{4}{25}\cos t - \frac{3}{25}\sin t - \frac{1}{5}t\cos t - \frac{2}{5}t\sin t + \left(\frac{1}{4} + \frac{9}{25}\cos(1) + \frac{13}{25}\sin(1)\right)e^{t}}$$

Esercizio 5

$$\begin{cases} y'(t) = \dfrac{t}{[y(t)]^4} \\ y(0) = 1 \end{cases}$$

Applico il metodo delle variabili separabili:

$$y'(t) = \frac{dy}{dt} = \frac{t}{[y(t)]^4} \iff [y(t)]^4 dy = t\,dt$$

$$\iff \int [y(t)]^4 dy = \int t\,dt$$

$$\iff \frac{1}{5}[y(t)]^5 = \frac{t^2}{2} + c$$

$$\Leftrightarrow \quad y(t)=\sqrt[5]{\frac{5}{2}t^2+5c}$$

$$y(0)=1 \Leftrightarrow \sqrt[5]{5c}=1 \Leftrightarrow 5c=1 \Leftrightarrow c=\frac{1}{5} \quad \Rightarrow \quad y(t)=\sqrt[5]{\frac{5}{2}t^2+1}$$

Esercizio 6

$$\begin{cases} y'(t)+\dfrac{6t+3}{t^2+t+1}\left(y(t)-1\right)^2=0 \\ y(0)=1 \end{cases}$$

Una soluzione è la funzione costante $y(t)=1$ e soddisfa $y(0)=1$

Esercizio 7

pendenza $=-\frac{y}{x} \Rightarrow y'(x)=-\frac{y}{x}$

passante per il punto $(1,1) \Rightarrow y(1)=1$

$$y'=-\frac{y}{x} \Leftrightarrow \frac{dy}{dx}=-\frac{y}{x} \Leftrightarrow \frac{1}{y}dy=-\frac{1}{x}dx \Leftrightarrow \int\frac{1}{y}dy=-\int\frac{1}{x}dx \Leftrightarrow \log|y|=-\log|x|+c=\log\left(\frac{1}{|x|}\right)+c$$

$$\Leftrightarrow \quad |y|=e^{\log\left(\frac{1}{|x|}\right)+c}=e^c\cdot e^{\log\left(\frac{1}{|x|}\right)}=\frac{e^c}{|x|}$$

$$\Rightarrow \quad y=\pm\frac{e^c}{|x|}$$

Dato che nel punto $(1,1)$ $y(x)$ è positiva e il termine $\frac{e^c}{|x|}$ è positivo, allora scelgo: $y=\frac{e^c}{|x|}$

$$y(1)=1 \Leftrightarrow \frac{e^c}{|1|}=1 \Leftrightarrow e^c=1 \quad \Rightarrow \quad y=\frac{1}{|x|}$$

Esercizio 8

$$\begin{cases} y'+\dfrac{y}{t^2}=-\dfrac{1}{t^2} \Leftrightarrow y'=-\dfrac{y+1}{t^2} \\ y(1)=e \end{cases}$$

Applico il metodo delle variabili separabili:

$$\frac{dy}{dt}=-\frac{y+1}{t^2} \Leftrightarrow \frac{1}{y+1}dy=-\frac{1}{t^2}dt \Leftrightarrow \int\frac{1}{y+1}dy=\int-\frac{1}{t^2}dt$$

$$\Leftrightarrow \log|y+1|=\frac{1}{t}+c$$

$$\Leftrightarrow |y+1|=e^{\frac{1}{t}+c} \Leftrightarrow y=-1\pm e^{\frac{1}{t}+c}$$

Analogamente all'esercizio precedente, scelgo $y=-1+e^{\frac{1}{t}+c}=-1+e^c\cdot e^{\frac{1}{t}}$

$y(1)=e \iff -1+e^c\cdot e = e \iff e^c = \frac{e+1}{e}$

$$\Rightarrow y=-1+\frac{e+1}{e}\cdot e^{\frac{1}{t}} \Rightarrow \boxed{y=-1+(e+1)e^{\frac{1}{t}-1}}$$

Esercizio 9

$(t-3)^2 y' = t(y-1)$

Applico il metodo delle variabili separabili:

$$y' = \frac{t}{(t-3)^2}(y-1) \iff \frac{dy}{dt} = \frac{t}{(t-3)^2}(y-1)$$

$$\iff \frac{1}{y-1}dy = \frac{t}{(t-3)^2}dt \iff \int \frac{1}{y-1}dy = \int \frac{t}{(t-3)^2}dt$$

$$\int \frac{1}{y-1}dy = \log(y-1)+c$$

Secondo integrale: cerco A e B tali che $\frac{A}{t-3}+\frac{B}{(t-3)^2} = \frac{At-3A+B}{(t-3)^2} = \frac{t}{(t-3)^2} \Rightarrow \begin{cases} At=t \iff A=1 \\ -3A+B=0 \iff B=3A=3 \end{cases}$

$$\int \frac{t}{(t-3)^2}dt = \int\frac{1}{t-3}dt + \int\frac{3}{(t-3)^2}dt = \log|t-3| - \frac{3}{t-3}+c$$

$$\Rightarrow \log(y-1) = \log|t-3| - \frac{3}{t-3}+c$$

$$\iff y-1 = e^{\log|t-3|-\frac{3}{t-3}+c}$$

$$\iff y = 1+|t-3|e^c e^{-\frac{3}{t-3}}$$

Mi accorgo anche che $(t-3)^2y' = t(y-1)$ è risolvibile dalla funzione costante $y(t)=1$

$\Rightarrow$ Soluzioni: $\boxed{y_1(t)=1, \quad y_2(t)=1+C|t-3|e^{\frac{3}{3-t}}}$

Esercizio 10 (Le costanti A, B e C utilizzate sono in $\mathbb{R}$)

ⓐ $y''-y'-2y=0$

Polinomio caratteristico associato: $\lambda^2-\lambda-2=0$ $\quad \lambda_{1,2} = \frac{1\pm\sqrt{1+8}}{2}$ $\quad$ quindi $\lambda_1=-1$, $\lambda_2=2$

$$\boxed{y=Ae^{-t}+Be^{2t}}$$

ⓑ $y''-2y'-8y=0$

Polinomio caratteristico associato: $\lambda^2-2\lambda-8=0$ $\quad \lambda_{1,2}=\frac{2\pm\sqrt{4+32}}{2}$ $\quad$ quindi $\lambda_1=-2$, $\lambda_2=4$

$$\boxed{y=Ae^{-2t}+Be^{4t}}$$

ⓒ $y''-4y'+4y=0$

Polinomio caratteristico associato: $\lambda^2-4\lambda+4=0 \iff (\lambda-2)^2=0 \iff \lambda=2$ $\quad \boxed{y=(A+Bt)e^{2t}}$

d) $y''+12y'+36y=0$

Polinomio caratteristico associato: $\lambda^2+12\lambda+36=0 \Leftrightarrow (\lambda+6)^2=0 \Leftrightarrow \lambda=-6$

$$\boxed{y=(A+Bt)e^{-6t}}$$

e) $y''+16y=0$

Polinomio caratteristico associato: $\lambda^2+16=0 \Leftrightarrow \lambda_{1,2}=\frac{0\pm\sqrt{0-64}}{2}=\frac{\pm\sqrt{-64}}{2}=\frac{\pm 8\cdot\sqrt{-1}}{2}=\pm 4\cdot\sqrt{-1}$

$$\boxed{y=A\cos(4t)+B\sin(4t)}$$

f) $y''+2y'+5y=0$

Polinomio caratteristico associato: $\lambda^2+2\lambda+5=0 \Leftrightarrow \lambda_{1,2}=\frac{-2\pm\sqrt{4-20}}{2}=\frac{-2\pm\sqrt{-16}}{2}=\frac{-2\pm 4\sqrt{-1}}{2}=-1\pm 2\sqrt{-1}$

$$\boxed{y=e^{-t}(A\cos(2t)+B\sin(2t))}$$