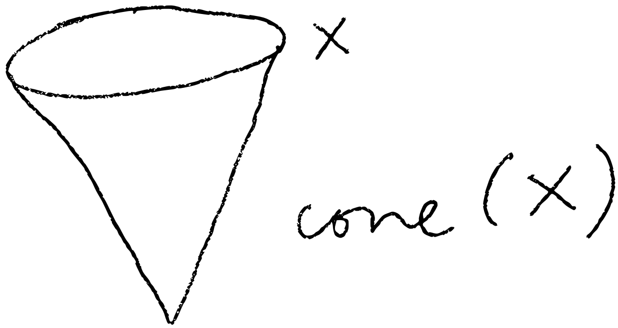
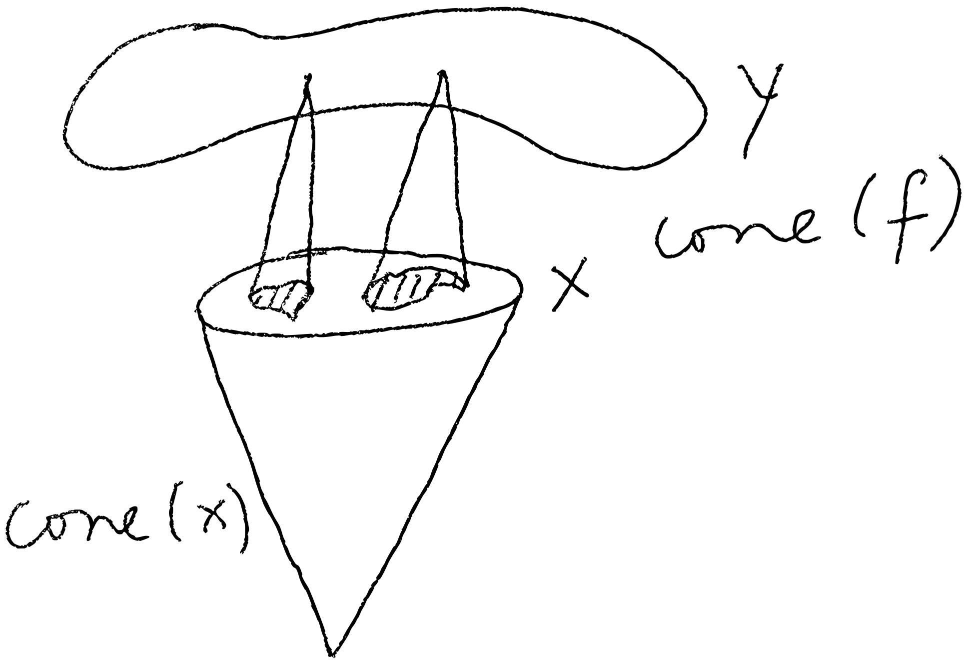


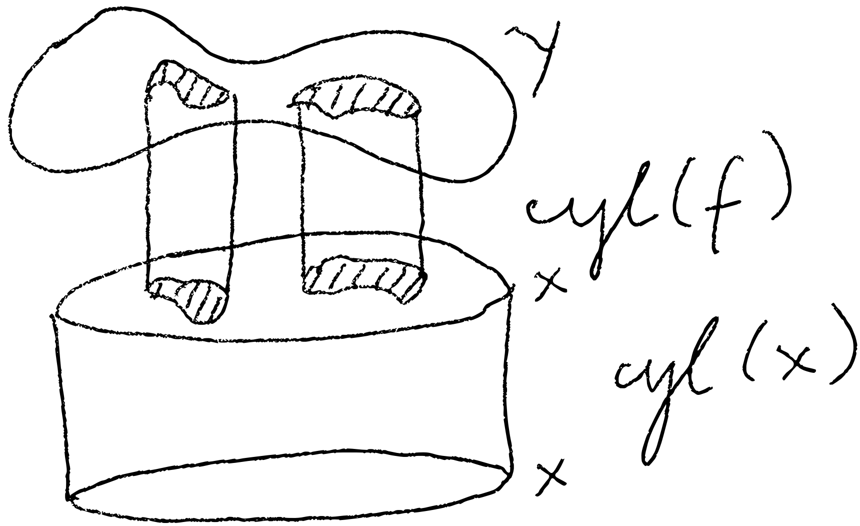
Now consider cone(f)
where $f : X \to Y$.

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & \mathrm{cone}(X) \arrow[d] \\
Y \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \mathrm{cone}(f)
\end{tikzcd}
$$



$Y \hookrightarrow \mathrm{cone}(f)$ is an embedding, but $X \longrightarrow \mathrm{cone}(f)$ is not.

We ask: when is $X \longrightarrow \mathrm{cone}(f)$ an embedding?



When we use cyl instead of cone. Let us formalize this cylinder-like property.

$$\begin{tikzcd} X \arrow[r, tail] \arrow[rr, bend left, "f"] & \mathrm{cyl}(f) \arrow[r, "\sim"] & Y \end{tikzcd}$$

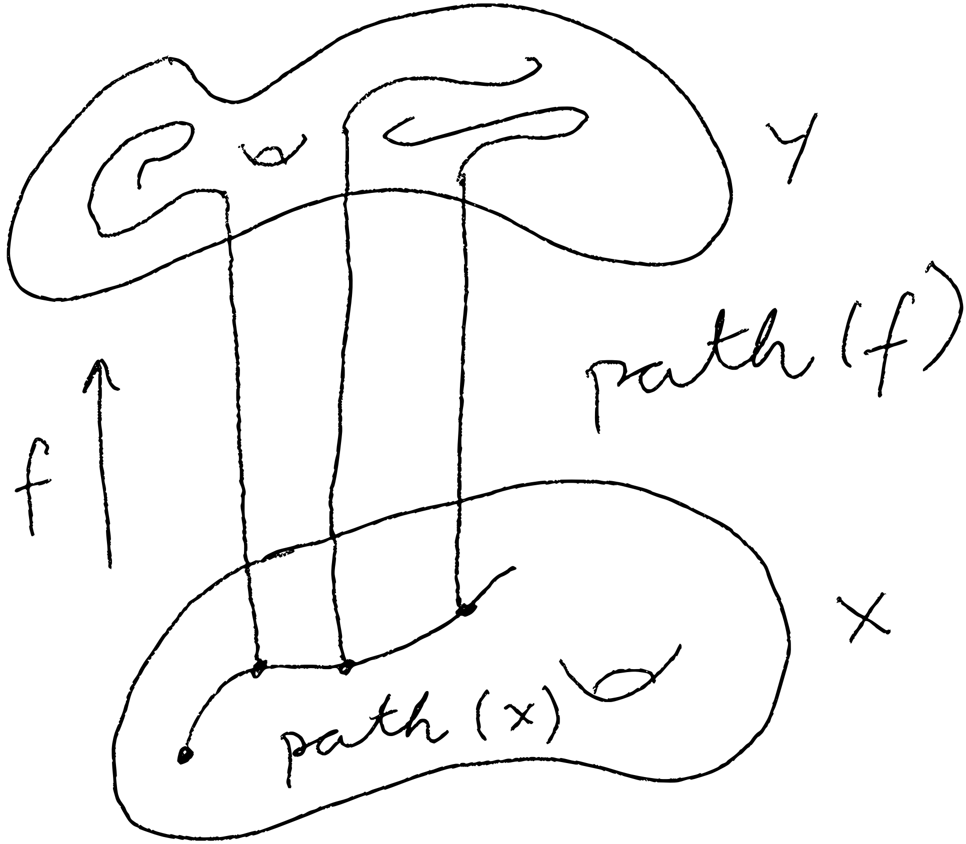
Now, investigate the dual question

$$\mathrm{cyl} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{path}$$

$$X \times I \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} X^I$$

$$\begin{tikzcd}
\mathrm{path}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
\mathrm{path}(Y) \arrow[r] & Y
\end{tikzcd}$$


path(x)
X


Y
path(f)
f
X
path(x)

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[rr, bend right, "f"'] & \mathrm{path}(f) \arrow[r, two heads] & Y \end{tikzcd}$$

$X \xrightarrow{\sim} \mathrm{path}(f)$ is a strong def. retract.