Investigations into syntactic iterated parametricity and cubical type theory

(report on work in progress)

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Investigations into *syntactic* iterated parametricity and cubical type theory, our approach

We look at *primitive-recursive translations* from type theory to type theory, as in:

Barras-Coquand-Huber 2014 (truncated semi-simplicial sets), Sozeau-Tabareau 2014 (1-groupoid), Boulier-Pédrot-Tabareau 2017, Tabareau-Tanter-Sozeau 2018 and 2019 (univalent parametricity), Altenkirch-Boulier-Kaposi-Tabareau 2019 (setoid), ...

rather than to models expressed in a *metalanguage* (often set theory):

Cohen-Coquand-Huber-Mörtberg 2015, Bernardy-Coquand-Moulin 2015, Nuyts 2017, Cavallo-Harper 2019 and 2020, ...

In particular, we expect to escape metalanguage issues such as the impact of a classical set-theoretic equality on the ability to equip paths with a J-style reduction rule (Swan 2018, if we understood correctly).

Investigations into *iterated parametricity* and cubical type theory, our approach

We look at (semi-)cubical sets from the *indexed* side of the Grothendieck's correspondence between fibered and indexed categories (type-theoretically: $\Sigma E : \mathsf{U}_l.\,(E \to B) \;\simeq\; B \to \mathsf{U}_l$), iteratedly.

This means defining cubical sets from the very same *dependently-typed* higher-dimensional relations that characterise iterated parametricity:

$$
\begin{array}{lcll}
A_0 & \triangleq & \mathsf{U}_l & \text{(points)} \\
A_1 & \triangleq & A_0 \times A_0 \to \mathsf{U}_l & \text{(segments)} \\
A_2 & \triangleq & \Pi a_L : (A_0 \times A_0).\, \Pi a_{L*} : A_1 a_L. & \\
 & & \Pi a_R : (A_0 \times A_0).\, \Pi a_{R*} : A_1 a_R. & \\
 & & A_1((a_L)_L, (A_R)_L) \times A_1((a_L)_R, (A_R)_R) \to \mathsf{U}_l & \text{(squares)} \\
 & \ldots & &
\end{array}
$$

Investigations into iterated parametricity and *cubical type theory*, our approach

We adopt the view that *parametric type theories* and *cubical type theories* are incarnations of a same type theory with *cubical equality*, differing in the kind of extra structure it comes with:

- bridge-based, as in parametric type theory (e.g. Bernardy-Coquand-Moulin 2015)

$$A =_{\mathsf{U}_l} B \ \triangleq\ A \times B \to \mathsf{U}_l$$

- equivalence-based, as in cubical type theory (Cohen-Coquand-Huber-Mörtberg 2014, ...)

$$A =_{\mathsf{U}_l} B \ \triangleq\ \left\{ \begin{array}{l} =_* : A \times B \to \mathsf{U}_l \\ \mathsf{contr}_L : \Pi a : A.\, \mathsf{iscontr}(\Sigma b : B.\, a =_* b) \\ \mathsf{contr}_R : \Pi b : B.\, \mathsf{iscontr}(\Sigma a : A.\, a =_* b) \end{array} \right\}$$

  where we rely on Altenkirch-Kaposi-advocated symmetric definition of equivalence that *generalises bridges with an extra structure which directly provides Kan completion of (full) open boxes*.

- or both (e.g. Cavallo-Harper 2019 and 2020)

- optionally with cartesian (contraction rule for dimensions) and symmetric (exchange rule for dimensions) structure

First main ingredient of the translation: the definition in ETT of *cubical sets* as a dependent stream of higher-dimensional cubical relations (with additional structure)

Cubical sets

$$
\begin{array}{lll}
\mathsf{cubset}_l & & : \ \mathsf{U}_{l+1} \\
\mathsf{cubset}_l & & \triangleq \ \mathsf{cubset}_l^{\geq 0}(\star)
\end{array}
$$

$$
\begin{array}{lll}
\mathsf{cubset}_l^{\geq n} & (D : \mathsf{cubset}_l^{<n}) & : \ \mathsf{U}_{l+1} \\
\mathsf{cubset}_l^{\geq n} & D & \triangleq \ \Sigma R : \mathsf{cubset}_l^{=n}(D).\, \mathsf{cubset}_l^{\geq n+1}(D, R)
\end{array}
$$

Truncated cubical sets

$$
\begin{array}{lll}
\mathsf{cubset}_l^{<n} & & : \ \mathsf{U}_{l+1} \\
\mathsf{cubset}_l^{<0} & & \triangleq \ \mathsf{unit} \\
\mathsf{cubset}_l^{<n'+1} & & \triangleq \ \Sigma D : \mathsf{cubset}_l^{<n'}.\, \mathsf{cubset}_l^{=n}(D))
\end{array}
$$

$$
\begin{array}{lll}
\mathsf{cubset}_l^{=n} & (D : \mathsf{cubset}_l^{<n}) & : \ \mathsf{U}_{l+1} \\
\mathsf{cubset}_l^{=n} & D & \triangleq \ \{=_*: \mathsf{fullbox}_l^n(D) \to \mathsf{U}_l; \ldots \text{ and other structure } \ldots\}
\end{array}
$$

where $\mathsf{fullbox}_l^n$ is defined by a rather technical mutual recursive construction (see later)

Second main ingredient: *cubical sets dependent over a cubical set*

Dependent cubical sets

$$
\begin{array}{lll}
\mathsf{depcubset}_l & (X : \mathsf{cubset}_l) & : \mathsf{U}_{l+1} \\
\mathsf{depcubset}_l & X & \triangleq \mathsf{depcubset}_l^{\geq 0}(\star)(X)(\star)
\end{array}
$$

$$
\begin{array}{lll}
 & (D : \mathsf{cubset}_l^{<n}) & \\
\mathsf{depcubset}_l^{\geq n} & (X : \mathsf{cubset}_l^{\geq n}(D)) & : \mathsf{U}_{l+1} \\
 & (P : \mathsf{depcubset}_l^{<n}(D)) & \\
\mathsf{depcubset}_l^{\geq n} & (D, E)\ (R, X)\ P & \triangleq \Sigma R' : \mathsf{depcubset}_l^{=n}(D)(E)(P).\, \mathsf{depcubset}_l^{\geq n+1}(D, R)(X)(P, R')
\end{array}
$$

Truncated dependent cubical sets

$$
\begin{array}{lll}
\mathsf{depcubset}_l^{<n} & (D : \mathsf{cubset}_l^{<n}) & : U_{l+1} \\
\mathsf{depcubset}_l^{<0} & D & \triangleq \mathsf{unit} \\
\mathsf{depcubset}_l^{<n'+1} & (D, E) & \triangleq \Sigma P : \mathsf{depcubset}_l^{<n'}(D).\, \mathsf{depcubset}_l^{=n}(D)(E)(P)
\end{array}
$$

$$
\begin{array}{lll}
 & (D : \mathsf{cubset}_l^{<n}) & \\
\mathsf{depcubset}_l^{=n} & (E : \mathsf{cubset}_l^{=n}(D)) & : \mathsf{U}_l \\
 & (P : \mathsf{depcubset}_l^{<n}(D)) & \\
\mathsf{depcubset}_l^{=n} & D\ E\ P & \triangleq \{=_*: \Pi d : \mathsf{fullbox}_l^n(D).\Pi c : E.\ =_* (d).\mathsf{fullhetbox}_l^n(\mathsf{appbox}_l^n(D)(P)(d)) \to \mathsf{U}_l; \ldots\}
\end{array}
$$

where $\mathsf{fullhetbox}_l^n$ and appbox_l^n are each defined by some rather technical mutual recursive construction involving cubes of higher-dimensional relations and heterogenous cubes over those relations

Homogeneous cubes of terms over cubical sets vs heterogenous cubes of terms over cubes of types

n-truncated (semi-)cubical sets

$$\begin{array}{lcl} A & : & \mathsf{U}_l \\ A_\star & : & A \times A \to \mathsf{U}_l \\ A_{\star\star} & & \\ \ldots & & \end{array}$$

$\hookrightarrow$

n-cubes of types

$$\begin{tikzcd} A_{LL} : \mathsf{U}_l \arrow[r, "A_{\star L} : A_{LL} \times A_{RL} \to \mathsf{U}_l"] \arrow[d, swap, "A_{L\star} : A_{LL} \times A_{LR} \to \mathsf{U}_l"] \arrow[dr, phantom, "\xRightarrow{A_{\star\star}}"] & A_{RL} : \mathsf{U}_l \arrow[d, "A_{R\star} : A_{RL} \times A_{RR} \to \mathsf{U}_l"] \\ A_{LR} : \mathsf{U}_l \arrow[r, "A_{\star R} : A_{LR} \times A_{RR} \to \mathsf{U}_l"] & A_{RR} : \mathsf{U}_l \end{tikzcd}$$

homogeneous n-cubes of terms
over some n-cubes cubical set

$$\begin{tikzcd} a_{LL} : A \arrow[r, "a_{\star L} : A_\star\, a_{LL}\, a_{RL}"] \arrow[d, swap, "a_{L\star} : A_\star\, a_{LL}\, a_{LR}"] \arrow[dr, phantom, "\xRightarrow{a_{\star\star}}"] & a_{RL} : A \arrow[d, "a_{R\star} : A_\star\, a_{RL}\, a_{RR}"] \\ a_{LR} : A \arrow[r, "a_{\star R} : A_\star\, a_{LR}\, a_{RR}"] & a_{RR} : A \end{tikzcd}$$

$\subset$

heterogenous n-cubes of terms
over some n-cube of types

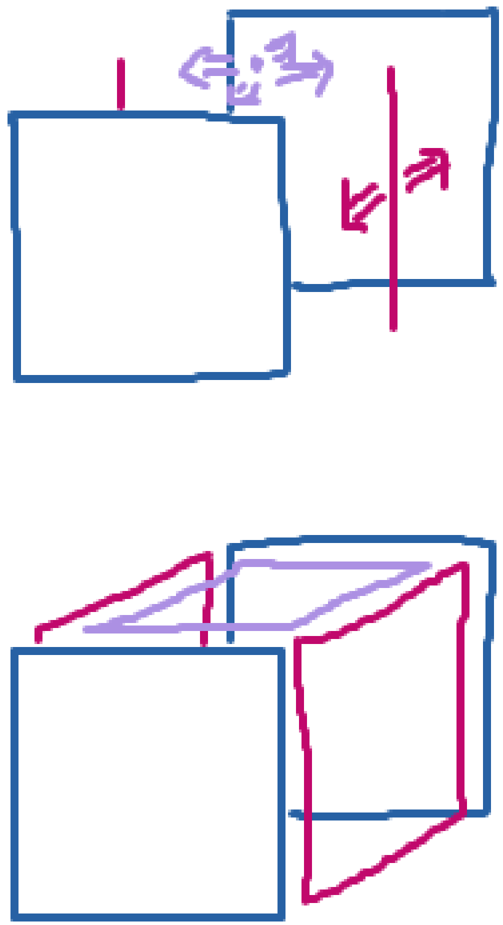
$$\begin{tikzcd} a_{LL} : A_{LL} \arrow[r, "a_{\star L} : A_{\star L}\, a_{LL}\, a_{RL}"] \arrow[d, swap, "a_{L\star} : A_{L\star}\, a_{LL}\, a_{LR}"] \arrow[dr, phantom, "\xRightarrow{a_{\star\star}}"] & a_{RL} : A_{RL} \arrow[d, "a_{R\star} : A_{R\star}\, a_{RL}\, a_{RR}"] \\ a_{LR} : A_{LR} \arrow[r, "a_{\star R} : A_{\star R}\, a_{LR}\, a_{RR}"] & a_{RR} : A_{RR} \end{tikzcd}$$

For incrementality of the construction, we define all three kinds of cubes separatedly but we plan to prove correspondences between them in a second step.

The basis of the translation

The basis of the translation is to interpret contexts with p dimension variables as p-cubes of cubical sets. This requires taking the Σ-type of a cubical set and of a dependent cubical set over this cubical set, and more generally to see boxes of cubical sets as cubical sets of boxes. This is still in progress.

The recursive process used to build boxes and cubes

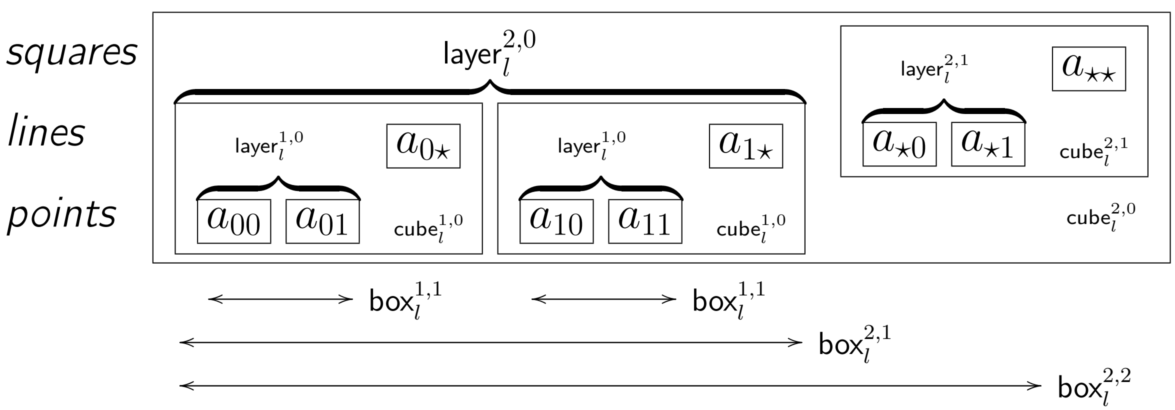


A n-box is made of n layers, each made of two opposite cubes of decreasing size and stretched to get the size of the box

The recursive construction, formally

$$
\begin{array}{lll}
\mathsf{fullbox}_l^n & (D : \mathsf{cubset}_l^{<n}) & : \mathsf{U}_l \\
\mathsf{fullbox}_l^n & D & \triangleq \mathsf{box}_l^{n,n}(D) \\
\\
\mathsf{box}_l^{n,p,[p\leq n]} & (D : \mathsf{cubset}_l^{<n}) & : \mathsf{U}_l \\
\mathsf{box}_l^{n,0} & D & \triangleq \mathsf{unit} \\
\mathsf{box}_l^{n,p'+1} & D & \triangleq \Sigma d : \mathsf{box}_l^{n,p'}(D).\, \mathsf{layer}_l^{n,p'}(D)(d) \\
\\
\mathsf{layer}_l^{n,p,[p<n]} & (D : \mathsf{cubset}_l^{<n})\ (d : \mathsf{box}_l^{n,p}(D)) & : \mathsf{U}_l \\
\mathsf{layer}_l^{n,p} & D\ d & \triangleq \mathsf{cube}_l^{n-1,p}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}_{l,L}^{n,p}(D)(d)) \\
 & & \quad \times\ \mathsf{cube}_l^{n-1,p}(\mathsf{hd}(D))(\mathsf{hd}(D))(\mathsf{subbox}_{l,R}^{n,p}(D)(d)) \\
\\
\mathsf{cube}_l^{n,p,[p\leq n]} & (D : \mathsf{cubset}_l^{<n})\ (E : \mathsf{cubset}_l^{=n}(D))\ (d : \mathsf{box}_l^{n,p}(D)) & : \mathsf{U}_l \\
\mathsf{cube}_l^{n,p,[p=n]} & D\ E\ d & \triangleq E.\, =_\star (d) \\
\mathsf{cube}_l^{n,p,[p<n]} & D\ E\ d & \triangleq \Sigma b : \mathsf{layer}_l^{n,p}(D)(d).\, \mathsf{cube}_l^{n,p+1}(D)(E)(d,b)
\end{array}
$$

which corresponds to the following organisation of the 3^n components of a n-cube (shown for $n = 2$), with box_l associating layers on the left and cube_l associating them on the right:



additionally, each atomic component at dimension n is a $\mathsf{cube}_l^{n,n}$

The recursive construction: restrictions ("faces")

$$
\begin{array}{lllll}
\mathsf{subbox}^{n,q,p,[p\leq q<n]}_{l,\epsilon} & \begin{array}{l}(D:\mathsf{cubset}^{<n}_{l})\\(d:\mathsf{box}^{n,p}_{l}(D))\end{array} & : & \mathsf{box}^{n-1,p}_{l}(\mathsf{hd}(D)) \\
\mathsf{subbox}^{n,q,0}_{l,\epsilon} & D\ \star & \triangleq & \star \\
\mathsf{subbox}^{n,q,p'+1}_{l,\epsilon} & D\ (d,b) & \triangleq & (\mathsf{subbox}^{n,q,p'}_{l,\epsilon}(D)(d), \mathsf{sublayer}^{n,q,p'}_{l,\epsilon}(D)(d)(b)) \\
\\
\mathsf{sublayer}^{n,q,p,[p<q<n]}_{l,\epsilon} & \begin{array}{l}(D:\mathsf{cubset}^{<n}_{l})\\(d:\mathsf{box}^{n,p}_{l}(D))\\(b:\mathsf{layer}^{n,p}_{l}(D)(d))\end{array} & : & \mathsf{layer}^{n-1,p}_{l}(\mathsf{hd}(D))(\mathsf{subbox}^{n,q,p}_{l,\epsilon}(D)(d)) \\
\mathsf{sublayer}^{n,q,p}_{l,\epsilon} & D\ d\ c & \triangleq & \begin{array}{l}(\overrightarrow{\mathsf{cohbox}^{n,p,q,p}_{l,\epsilon,L}(D)(d)}(\mathsf{subcube}^{n-1,q-1,p}_{l,\epsilon}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,p,p}_{l,L}(D)(d))(c_L)),\\ \overrightarrow{\mathsf{cohbox}^{n,p,q,p}_{l,\epsilon,R}(D)(d)}(\mathsf{subcube}^{n-1,q-1,p}_{l,\epsilon}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,p,p}_{l,bR}(D)(d))(c_R)))\end{array} \\
\\
\mathsf{subcube}^{n,q,p,[p\leq q<n]}_{l,\epsilon} & \begin{array}{l}(D:\mathsf{cubset}^{<n}_{l})\\(E:\mathsf{cubset}^{=n}_{l}(D))\\(d:\mathsf{box}^{n,p}_{l}(D))\\(b:\mathsf{cube}^{n,p}_{l}(D)(E)(d))\end{array} & : & \mathsf{cube}^{n-1,p}_{l}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,q,p}_{l,\epsilon}(D)(d)) \\
\mathsf{subcube}^{n,q,p,[p=q]}_{l,\epsilon} & D\ E\ d\ (b,\_) & \triangleq & b_\epsilon \\
\mathsf{subcube}^{n,q,p,[p<q]}_{l,\epsilon} & D\ E\ d\ (b,c) & \triangleq & (\mathsf{sublayer}^{n,q,p}_{l,\epsilon}(D)(d)(b), \mathsf{subcube}^{n,q,p+1}_{l,\epsilon}(D)(E)(d,b)(c))
\end{array}
$$

where $\mathsf{cohbox}_{l,\epsilon,\epsilon'}$ is a coherence proof and we write $\overrightarrow{\mathsf{cohbox}_{l,\epsilon,\epsilon'}}$ for the rewriting of this proof from left to right

The recursive construction: coherences

$$
\begin{array}{lll}
\mathsf{cohbox}^{\substack{n,q,r,p\\ [p\leq r<q<n]}}_{l,\epsilon,\epsilon'} & \begin{array}{l}(D:\mathsf{cubset}^{<n}_{l})\\ (d:\mathsf{box}^{n,p}_{l}(D))\end{array} & : \begin{array}{l}\mathsf{subbox}^{n-1,q-1,p}_{l,\epsilon}(\mathsf{hd}(D))(\mathsf{subbox}^{n,r,p}_{l,\epsilon'}(D)(d))\\ =\mathsf{subbox}^{n-1,r-1,p}_{l,\epsilon'}(\mathsf{hd}(D))(\mathsf{subbox}^{n,q,p}_{l,\epsilon}(D)(d))\end{array}\\
\mathsf{cohbox}^{n,q,r,0}_{l,\epsilon,\epsilon'} & D\ \star & \triangleq\ \mathsf{refl}\,\star\\
\mathsf{cohbox}^{n,q,r,p'+1}_{l,\epsilon,\epsilon'} & D\ (d,b) & \triangleq\ (\mathsf{cohbox}^{n,q,r,p'}_{l,\epsilon,\epsilon'}(D)(d),\mathsf{cohlayer}^{n,q,r,p'}_{l,\epsilon,\epsilon'}(D)(d)(b))\\
\\
\mathsf{cohlayer}^{\substack{n,q,r,p\\ [p<r<q<n]}}_{l,\epsilon,\epsilon'} & \begin{array}{l}(D:\mathsf{cubset}^{<n}_{l})\\ (d:\mathsf{box}^{n,p}_{l}(D))\\ (b:\mathsf{layer}^{n,p}_{l}(D)(d))\end{array} & : \begin{array}{l}\mathsf{sublayer}^{n-1,q,p}_{l,\epsilon}(\mathsf{hd}(D))(\mathsf{subbox}^{n,r,p}_{l,\epsilon'}(D)(d))(\mathsf{sublayer}^{n,r,p}_{l,\epsilon'}(D)(d)(b))\\ =\mathsf{sublayer}^{n-1,r,p}_{l,\epsilon'}(\mathsf{hd}(D))(\mathsf{subbox}^{n,q,p}_{l,\epsilon}(D)(d))(\mathsf{sublayer}^{n,q,p}_{l,\epsilon}(D)(d)(b))\end{array}\\
\mathsf{cohlayer}^{n,q,r,p}_{l,\epsilon,\epsilon'} & D\ d\ c & \triangleq\ \begin{array}{l}(\mathsf{cohcube}^{n-1,q-1,r-1,p}_{l,\epsilon,\epsilon'}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,p,p}_{l,L}(D)(d))(c_L),\\ \ \mathsf{cohcube}^{n-1,q-1,r-1,p}_{l,\epsilon,\epsilon'}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,p,p}_{l,R}(D)(d))(c_R))\end{array}\\
\\
\mathsf{cohcube}^{\substack{n,q,r,p\\ [p\leq r<q<n]}}_{l,\epsilon,\epsilon'} & \begin{array}{l}(D:\mathsf{cubset}^{<n}_{l})\\ (E:\mathsf{cubset}^{=n}_{l}(D))\\ (d:\mathsf{box}^{n,p}_{l}(D))\\ (b:\mathsf{cube}^{n,p}_{l}(D)(E)(d))\end{array} & : \begin{array}{l}\mathsf{subcube}^{n-1,q-1,p}_{l,\epsilon}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,r,p}_{l,\epsilon'}(D)(d))(\mathsf{subcube}^{n,r,p}_{l,\epsilon'}(D)(E)(d)(b))\\ =\mathsf{subcube}^{n-1,r-1,p}_{l,\epsilon'}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,q,p}_{l,\epsilon}(D)(d))(\mathsf{subcube}^{n,q,p}_{l,\epsilon}(D)(E)(d)(b))\end{array}\\
\mathsf{cohcube}^{n,q,r,p,[p=r]}_{l,\epsilon,\epsilon'} & D\ E\ d\ (b,\_) & \triangleq\ \mathsf{refl}\,\mathsf{subcube}^{n-1,q-1,p}_{l,\epsilon}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,p,p}_{l,\epsilon'}(D)(d))(b_\epsilon)\\
\mathsf{cohcube}^{n,q,r,p,[p<r]}_{l,\epsilon,\epsilon'} & D\ E\ d\ (b,c) & \triangleq\ (\mathsf{cohlayer}^{n,q,r,p}_{l,\epsilon,\epsilon'}(D)(d)(b),\ \mathsf{cohcube}^{n,q,r,p+1}_{l,\epsilon,\epsilon'}(D)(E)(d,b)(c))
\end{array}
$$

where we rely on the strictness of equality in ETT to enforce irrelevance of proof of coherence (otherwise, coherences in higher dimensions are recursively needed)

Other definitions

There are similar definitions organised over the structure of boxes, layers, cubes, and coming with coherences for about every bit of the global structure (for cubes of types, heterogenous cubes, reflexivities/degeneracies, dependent cubical sets, ...), which eventually give a rather huge construction. We are wondering at whether it could be made more compact.

Conclusions and open questions

- When completed, we should get a "perfect" match between cubical sets in "indexed" (dependently-typed) form and iterated parametricity translations, as imagined by Altenkirch-Kaposi 2014 (see also Polonsky 2014, Adams 2016).

- Definitional bridge/equivalence :

  We expect being able to justify definitional equivalence in ETT, the same way (according to Cavallo-Harper) as Bernardy-Coquand-Moulin semantically justify definitional bridges (even if their syntax provides only a definitional isomorphism), without having to mediate through glue, gel or weld.

- Definitional functional extensionality:

  We suspect being able to justify definitional functional extensionality in ETT in the non-cartesian case.

- Regularity:

  We do not see an obstacle to having regularity in the universe and to have cubical equality compatible with a J-style reduction rule.

- Scalability:

  We focussed on bridges and equivalences, but we may expect the framework to support various kinds of algebraic structures (e.g. truncated equality, or, maybe, eventually, a non-symmetric, directed form of equivalence).

- Work-in-progress paper at `pauillac.inria.fr/~herbelin/articles/param-draft.pdf`.