Iterated syntactic parametricity translation

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1 Introduction

This note aims at eventually interpreting the cubical type theory described in Moeneclaey's master thesis (called the "source") through a delimited iterated "univalent" internal parametricity translation into Extensional Type Theory (called the "target").

Its key ingredient is a dependently-typed definition of cubical sets in type theory which is used to interpret different levels of complexity of the full (univalent) cubical type theory of Moeneclaey's master thesis. Cubical sets shall carry the structure of the parametricity translation, which is informally defined as mapping a type to the pair of two types and a heterogeneous relation over these types, and mapping a term of some type to the pair of two terms in the corresponding duplication of the type, together with a proof that these terms are related for the heterogeneous relation over the duplicated types. Further iterations of parametricity then canonically build squares, cubes, hypercubes in all dimensions.

In a first stage, Section 3, we give a basic external parametricity model of type theory. We shall fix a dimension n and interpret types as n-cubes of higher-dimensional relations (up to dimension n) and terms as n-cubes of related terms. This corresponds to iterating parametricity n times.

In a second stage, Section 4, we shall move to a basic internal parametricity interpretation modelling a basic form of Bernardy, Coquand and Moulin parametric type theory [3] with abstraction and application of a dimension variable, but no contraction nor swap of dimension variables. By internal parametricity model we thus mean a model providing a reflexivity operator allowing to embed objects of dimension n in the dimension $n+1$. Many new ingredients shall be needed:

- It shall interpret types as cubes whose points are infinite sequences of higher-dimensional relations, $A_0 : \mathsf{U}_l$, $A_1 : A \rightarrow A \rightarrow \mathsf{U}_l$, $A_2 : \Pi a_{LL} : A.\, \Pi a_{LR} : A.\, \Pi a_{RL} : A.\, \Pi a_{RR} : A.\, A_1\, a_{LL}\, a_{LR} \rightarrow A_1\, a_{RL}\, a_{RR} \rightarrow A_1\, a_{LL}\, a_{RL} \rightarrow A_1\, a_{LR}\, a_{RR} \rightarrow \mathsf{U}_l$, ... called "bridges" [], in all dimensions.
- It shall interpret judgements as infinite sequences of judgements in all dimensions such that some additional computational properties are satisfied.

In a last stage, we shall consider additional structure needed to capture the whole of Bernardy, Coquand and Moulin parametric type theory (permutations), a cartesian variant of the latter (thanks to diagonals) as well as a full univalent cubical type theory (by using equivalence instead of bridges).

The final translation can be seen as a delimited variant in unbounded dimensions of previous (non-delimited) syntactic translations by:

- Gandy [5] and Takeuti [9]: setoid translation, capturing function extensionality and propositional extensionality[1]
- Section 5 of Barras, Coquand and Huber [2] modifies the setoid translation so that it supports dependent types and a proof-relevant (non-impredicative) universe up to isomorphism (i.e. equality between types is the existence of an isomorphism). The translation is formulated as a translation to semi-simplicial sets truncated half way between dimension 1 and 2. The translation is to intentional type theory (ITT) and it does not validate the reduction rules for J nor validate substitution under a λ-abstraction.

  In comparison, our translation uses cubes rather than simplices (see Shulman and Riehl [] for an interpretation of simplices on top of cubes using declaration of variables in the interval of the form $j \leq i$ where i is the name of the previous variable declared, and 1 if j is the first variable declared) but mutatis mutandis, both use Kan composition rather than ordinary composition and inverses. The main differences are that: 1) our translation is incremental and iterated, locally extending the dimension when a new interval variable is declared 2) our composition is required to be regular (in the sense of [4, Section 10]) so that the reduction rules for Martin-Löf's eliminator J are validated.

  TODO: why Barras, Coquand and Huber's translation does not validate substitution under a λ-abstraction?
- Sozeau and Tabareau [8]: this also extends Gandy and Takeuti translations but using a definition of 1-groupoid with composition and inverses rather than a semi-simplicial sets with a Kan composition.
- Altenkirch, Boulier, Kaposi and Tabareau [1] gives a machine-checked setoid translation in ITT extended with a definitionally proof-irrelevant universe of proposition. The translation derives from previous work of the first author in the context of Observational Type Theory []. This is similar to Gandy et Takeuti's translation but for a different source language (ITT rather than Church's simple theory of types). In particular, it supports dependent product.

[1]The corresponding translation applies to Church's simple type theory, which, as a type system, can be seen as System F_ω with an impredicative "universe" Prop and a "universe" Type of simple types based on Prop. Equality in Prop is over elements typed in types of Type. The proof level is not dependent, so there is no equality of proofs to consider. The translation thus applies only to the simply-typed λ-calculus built from the arrow type and the impredicative sort Prop. (actually, as a consequence of satisfying propositional extensional types in Prop are provably-irrelevant. Technically, the translation only requires to show that the existence of maps back and forth a proposition extends from propositional atoms to implication and universal quantification over types of the Type level.

Our translation can alternatively be seen as a (delimited, arbitrarily highly n-truncated) syntactic formulation of (undelimited, untruncated) presheaves semantics in set theory (or in any other metalanguage featuring a notion of sets, e.g. type theory with h-sets):

- Sections 7 and 8 of Barras, Coquand and Huber [2] presents an interpretation of ITT with univalence within Kan semi-simplicial set in all dimensions.
- Cohen, Coquand, Huber and Mörtberg [4] presents an interpretation of a variant of ITT with cubical equality within Kan cubical sets in all dimensions. The source language being interpreted differs in the following:
  - Composition in our approach is only for "tubes" while it supports arbitrary systems of faces in [4].
  - Univalence, which connects equivalence of types and equality of types is obtained via a "Glue" operator while it is by definition in our case.

  Regarding the interpretation, besides the fact that [4] gives an interpretation within the metalanguage while our interpretation is in another object language, we see the following differences:
  - We interpret a judgement with n axis variables in the typing context by a cubical set truncated to dimension n while [4] interprets all judgements by a non-truncated cubical set independently of the number of axis variables in the context.
  - The authors of [4] reason at a level of abstraction which circumscribes the combinatoric and equational structure of cubical sets in the definition of the site category of the presheaf. In our case, the equational structure of cubical sets is enforced using type dependencies rather than equations. As a consequence, the combinatorics of cubical sets become intertwined with the definition of cubical sets.

    For instance, as a comparison, a cube in a semi-cubical set[2] presented as a presheaf is an object a together with, for each iteration of faces τ considered modulo the equational theory on faces, another object a_τ which itself comes equipped for each face map τ' of a subobject $(a_\tau)_{\tau'}$ such that $(a_\tau)_{\tau'} = a_{(\tau\tau')}$ holds. In our definition of cubical sets, the later equality holds by construction using type dependencies. Otherwise said, while the presheaf construction represents a cube a by a flat collection of all objects of the form $(...(a_{\tau_1})...)_{\tau_n}$ together with all equations asserting all necessary equations between these objects, our construction adds objects incrementally using instead type dependencies to enforce that the necessary equations hold.

    Also, while [4] treats faces, degeneracies, symmetries, diagonals, reversal and connections at the level of the site of the presheaf, we start from a semi-cubical basis (i.e. with only faces) and introduce reflexivity/degeneracies, symmetries and diagonals as axtra structure (we do not have reversal - TODO: connections?).

    We believe that, in any case, the combinatorics of cubical sets cannot be avoided and the level of details we present is anyway necessary for a full syntactic translation. Alternative syntactic translations closer to the structure of presheaves (i.e. building objects coming together with their collection of cubical maps and their associated equations) are certainly possible and they would eventually require to also characterize the combinatorical and equational structure of cubical sets. Whatever way the information is structured, there are irreductible bits of information to consider.

It relies on an adaptation of the dependent construction of semi-simplicial sets in type theory [6] to a semi-strict (regular) cubical form of ω-groupoids.

[2] meaning here a cubical set with only face maps

2 Target of the interpretation: Extensional Type Theory

We shall construct our interpretation as a syntactic model formulated in Extensional Type Theory (ETT) [7]. Using Extensional Type Theory, in the sense of a theory built on the equality reflection rule

$$\frac{\Gamma \vdash p : t =_A u}{\Gamma \vdash t \equiv u : A}$$

is of course not the important point. What matters is that equality is strict, and this could be also obtained by reasoning in an intensional type theory [7] extended with an axiom ensuring the strictness of the equality, such as the Unicity of Identity Proofs ($\mathsf{UIP} : \Pi A.\Pi x, y : A.\Pi p, q : x =_A y.p = q$). Indeed, it is well known that ETT can be embedded in Intensional Type Theory extended with functional extensionality and UIP [?], and it is probably even the case, even if we do not investigate it here, that functional extensionality and UIP can be equipped with reduction rules which would give a canonicity theorem for a strict type theory.

We shall assume a few basic constructions in the target: a cumulative hierarchy of universes U_l with U_l of type U_{l+1}, Π-types, Σ-types, a unit type with single distinguished inhabitant $\star$, natural number, inductive families over natural numbers (so as to be able to inductively define the order on natural numbers), a coinductive type of (dependent) streams, as well as a homogeneous identity type $t =_A u$. We write $\mathsf{hd}(t)$ for the first component of a Σ-type and $\mathsf{tl}(t)$ for the second component. The unit type is supposed to live in the first universe and, by cumulativity, also in all other universes.

As syntactic sugar expressible in terms of Σ-types we shall write $A \times B$ for the product of two types and write t_L and t_R for the two projections of t. As a convenient syntactic sugar expressible in terms of Σ-types, we shall support record types made of fields of the form $\{\mathsf{fst} : A; \mathsf{snd} : B\}$, with introduction $\{\mathsf{fst} \triangleq t; \mathsf{snd} \triangleq u\}$ and with projections written $t.\mathsf{fst}$ and $t.\mathsf{snd}$ (names and numbers of fields may of course vary).

We may also use notations such as $\Sigma(a_0, a_1) : A \times A.\, B(a_0, a_1)$ to mean $\Sigma a : A \times A.\, B(a.1, a.2)$.

We shall write $\overrightarrow{p}$ for the rewriting operator along an equality proof p of the target language.

We shall make some abuse of notations with equalities in Σ-types which we shall identify with Σ-types of equalities on the respective components.

2.1 Syntax

We use an extrinsic definition of type theory which has the advantage of isolating the computational and logical parts of derivations. The inference rules are given on Figure 1 where we refer to variables by de Bruijn indices. In the rest of the paper, for readability, we will however use named variables, knowing that the translation to indices is standard.

3 Heterogeneous cubes and external parametricity

3.1 The cubical structure of parametricity

One step of parametricity can be seen as interpreting a point by a line. Iterating parametricity shall lead to map terms typed in a context with n declarations of an axis as n-cubes. We shall interpret a typing derivation with n axis declarations as an n-cube of typing derivations in the target. For instance, a typing derivation with exactly one declaration of an axis will be interpreted by 3 typing derivations in the target, two of them typing points and the third one typing a proof, seen as a line, together with the fact that these two points are related (as in ordinary parametricity), while a typing derivation with n declarations of an axis shall be interpreted as a 3^n derivations, split into 2^n derivations of points, $2^{n-1} \times \begin{pmatrix} 1 \\ n \end{pmatrix}$ derivations of lines between those points, $2^{n-2} \times \begin{pmatrix} 2 \\ n \end{pmatrix}$ derivations of squares between those lines, etc. until having a single (i.e. $2^{n-n} \times \begin{pmatrix} n \\ n \end{pmatrix}$) n-cube connecting the whole. Note that the points are not typed in the same

$$
\begin{array}{lcl}
\Gamma & ::= & \cdot \mid \Gamma, A \\
t, u, v, A, B, C & ::= & n \mid \mathsf{U}_l \mid \Pi A.\, B \mid \lambda^A.\, t \mid u\, t \mid \Sigma A.\, B \mid \langle t, u \rangle \mid t.1 \mid t.2
\end{array}
$$

$$
\frac{}{\Gamma, A, \Gamma' \vdash |\Gamma'| : A}
\qquad
\frac{\Gamma \vdash t : A \qquad \Gamma \vdash A \equiv B}{\Gamma \vdash t : B}
$$

$$
\frac{\Gamma \vdash A : \mathsf{U}_l \qquad \Gamma, A \vdash B : \mathsf{U}_l}{\Gamma \vdash \Pi A.\, B : \mathsf{U}_l}
\qquad
\frac{\Gamma \vdash A : \mathsf{U}_l \qquad \Gamma, A \vdash t : B}{\Gamma \vdash \lambda^A.\, t : \Pi A.\, B}
\qquad
\frac{\Gamma \vdash u : \Pi A.\, B \qquad \Gamma \vdash t : A}{\Gamma \vdash u\, t : B[t]}
$$

$$
\frac{\Gamma \vdash A : \mathsf{U}_l \qquad \Gamma, A \vdash B : \mathsf{U}_l}{\Gamma \vdash \Sigma A.\, B : \mathsf{U}_l}
\qquad
\frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B[t]}{\Gamma \vdash \langle t, u \rangle : \Sigma A.\, B}
\qquad
\frac{\Gamma \vdash t : \Sigma A.\, B}{\Gamma \vdash t.1 : A}
\qquad
\frac{\Gamma \vdash t : \Sigma A.\, B}{\Gamma \vdash t.2 : B[t.1]}
$$

$$
\frac{\Gamma, A \vdash u : B \qquad \Gamma \vdash t : A}{\Gamma \vdash (\lambda^A.u)\, t \equiv t[u] : B[u]}
$$

$$
\frac{\Gamma, a : A \vdash B : \mathsf{U}_l \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : B[t]}{\Gamma \vdash \langle t, u \rangle.1 \equiv t : A}
\qquad
\frac{\Gamma, a : A \vdash B : \mathsf{U}_l \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : B[t]}{\Gamma \vdash \langle t, u \rangle.2 \equiv u : B[t/a]}
$$

$+$ universes $+$ inductive types (for natural numbers) $+$ coinductive types (for streams)

Figure 1: Source type theory

context as the lines which connect them: a variable of some type in the source context is interpreted as a variable of the same type in the derivation of a point but as a variable of line over points in this type for the derivation of the line. This difference of typing context in each component of the so-obtained n-cube gives to parametricity a semantic flavor. In the presence of transport, we suspect that a more purely syntactic form of parametricity where all components of the n-cube are typed in the same context is possible. Then, an n-cube of derivations in the same context is isomorphic to the single derivation of the tupling of all components of the n-cubes as an inhabitant of a Σ-type iterated n times.

Each operation of the source language relative to a connective shall be interpreted as an operation applied in parallel on each derivation of the n-cube of derivations. Abstraction over a variable of axis shall be interpreted as turning an n-cube of derivation over some type A into an $(n-1)$-cube of derivations over lines over A. Application of a variable of axis to a proof of an equality shall be interpreted as taking a diagonal of the square defined by this axis and by the line which this equality denotes.

Abstraction over an ordinary variable shall require some care. Interpreting abstraction of an n-cube as an n-cube of abstractions such that a face of dimension p of the n-cube takes a p-cube as arguments shall drop too much information for the diagonals to be definable in dependent products. Thus, abstraction shall instead be interpreted as the abstraction together with a precomputation of the diagonals in all the axes of the n-cube. On the other side, dependent pairing of n-cubes will compositionally be defined as n-cubes of dependent pairs.

Types shall be interpreted as n-cubes of equivalent types. More concretely, a 0-cube shall be a type, a 1-cube shall be two types A and B, both equipped with a proper homogeneous relation, together with a form of Galois connection structure over the two types made of an "oblique" heterogeneous relation over A and B and of coercions back and forth between A and B that agree on the oblique relation. Additionally, this Galois connection is supposed to be regular, meaning here that if the two types are identical, the coercions are the identity. A 2-cube shall be four types forming a square, four proofs that nearby pairs are equivalences and a proof that the four equivalences are consistent, i.e., in fine, that the square commutes. The later itself shall consist of a relation over the four types and four equivalences and 2-dimensional coherent coercions back and forth between the four equivalences. Note in particular that such a relation is defined over the border of a square, i.e. the $3^2 - 1$ objects made from the four types and four equivalences between those types. In the general case, an n-cube of types will be characterized by an n-dimensional heterogeneous relation over n-boxes, i.e. of the border of an n-cube, i.e. an n-cube without a filler inside, plus n-dimensional coercions back and forth.

Each n-cube of derivations (i.e. again, 3^n components) can thus be decomposed into an n-dimensional filler (one component) of its border which is an n-box made from the faces (and iterated faces) of the n-cube.

A typing judgement $t : A$ shall be interpreted as follows: if one sees each p-dimensional face t' of the interpretation of t as an n-cube as the pair of a filler and of a p-box, then, t' shall be an inhabitant of the relation underlying the corresponding face of A applied to the aforementioned p-box.

The interpretation of the universe shall be a delicate part. Since a type A has type U_l for some universe level l in the source, the interpretation of U_l as an n-cube has to characterize what it means for a type to be interpreted as an n-cube.

In the following, we are interpreting cubical type theory in ETT and thus use a "type-theoretic" terminology, such as t is a term of type A. The reader used to set theory can in petto reformulate such a statement as t is a element of the set A.

3.2 Structure of cubes

The notion of *cube* goes together with the notions of *box* and of *tube*. A n-cube *fills* an n-box and, conversely, the border of an n-cube is an n-box. A $(n+1)$-box whose two opposite faces are missing is called an n-tube. The data of an n-cube together with the n-box that it inhabits shall be called a full n-cube. A 0-cube is called a *point*. A 1-cube connects two points and is called a *line* with the two points forming the bordering 1-box of the line. A 2-cube is a filled *square*. The unfilled square is a 2-box which can be seen as a 1-tube made of two parallel lines together with two extra lines closing the tube. A 3-cube is a filled 3-dimensional cube. The unfilled cube is a 3-box which can be seen as a 2-tube made of an unfilled square stretched in one direction so that it looks like a squared hollow cylinder together with two extra squares closing the tube.

Cubes, boxes and tubes are relative to a type of points and to a *relational structure* in all dimensions telling when a cube inhabits (or fills) a box. There are *homogeneous* and *heterogeneous* cubes and boxes. By a homogeneous cube or box, we mean a (full) cube or box in which all points are in the same type and all relations in all dimensions are homogeneous. In an heterogeneous cube or box, each point might be in a different type and the higher-dimensional relational structure over these types is itself heterogeneous. However, in the latter case, even though the types and higher-dimensional relations is heterogeneous, they can in turn be themselves organized as a full cube or box, which, this time, is homogeneous (i.e. a higher-dimensional relational structure can be seen as a cube over another higher-dimensional relational structure in a upper level, the same way as a type is itself a term in some universe, and this universe itself a term in some upper universe, and so on).

In the syntax of cubical type theory, the cubical structure can be seen in two forms. A judgement in a context with n axis variables denotes an n-cube and this is the "metalevel" use of n-cubes. In particular, in a judgement $\Gamma \vdash t : A$ with n axis variables declared in Γ, the type A denotes an n-cube of relations up to dimension n (with types as points) and t is a full heterogeneous n-cube over the homogeneous n-cube of relations denoted by A.

In the syntax of cubical type theory, there is also an internalizing of the metalevel language in the object language by means of the equality symbol. The equality over some type denotes a relation and, when iterated, it denotes a higher-dimensional relation. Such equality in dimension n expects an n-box as argument, and, altogether, they express the type of fillers of the n-box. For instance, in a judgement of the form $\Gamma \vdash \alpha : p =_{\lambda i.(ri =_A si)} q$ can be seen as a (non full) 2-cube α inhabiting the 2-box built from p, q, r and s. More generally, if the context Γ itself declares m variables of axis, a judgement about an equality nested n times denotes a full heterogeneous m-cube whose points are themselves (non full) heterogeneous n-cubes.

As for an n-tube, it can be seen as the chain of the $2n$ n-cubes forming the tube.

The characterization of n-boxes and full n-cubes can be obtained by a recursive definition reminiscent from parametricity: a full $(n+1)$-cube is obtained by taking two full n-cubes plus a third full n-cube connecting the two full n-cubes. Since it connects full n-cubes, the third full n-cube is actually a full n-cube over lines, so it is shifted by one relatively to the underlying relational structure. Let us write $\mathsf{cube}^{n,p}$ for a full $(n-p)$-cube over points which are full p-cubes. Working informally, we thus have the equation:

$$\mathsf{cube}^{n+1,0} \triangleq (\mathsf{cube}^{n,0} \times \mathsf{cube}^{n,0}) \times \mathsf{cube}^{n+1,1}$$

and, iterating and generalizing the process, we have actually:

$$\mathsf{cube}^{n+1,p} \triangleq (\mathsf{cube}^{n,p} \times \mathsf{cube}^{n,p}) \times \mathsf{cube}^{n+1,p+1}$$

Now, the third cube (of dimension $((n+1)-(p+1))$, i.e. $n-p$) connects the first two $(n-p)$-cubes, so it has to depend on them. So, the equation is rather something of the form:

$$\mathsf{cube}^{n+1,p} \triangleq \Sigma a : (\mathsf{cube}^{n,p} \times \mathsf{cube}^{n,p}).\, \mathsf{cube}^{n+1,p+1}(a)$$

Then, by iterating, we obtain:

$$\mathsf{cube}^{n+1,p} \triangleq \Sigma a : (\mathsf{cube}^{n,p} \times \mathsf{cube}^{n,p}).\, \Sigma b : (\mathsf{cube}^{n,p+1}(a) \times \mathsf{cube}^{n,p+1}(a)).\, \mathsf{cube}^{n+1,p+2}(a,b)$$

and so on. Thus, we need first to make the definition of $\mathsf{cube}^{n,p}$ dependent over the part of the full cube built up to this point and secondly to accumulate this information. At the end of the accumulation of the process, i.e. at the time of building $\mathsf{cube}^{n+1,n}$, a complete n-box will have been built. So, let us call $\mathsf{hetbox}^{n,p}$ the partial n-box built up to the stage p of the construction and rename $\mathsf{cube}^{n,p}$ into $\mathsf{cube}^{n,p}$ (read as "cube over") to indicate a depencency over a box. Thus, we refine our definition into something like:

$$\begin{array}{lcl}
\mathsf{hetbox}^{n+1,0} & \triangleq & \mathsf{unit} \\
\mathsf{hetbox}^{n+1,p+1} & \triangleq & \Sigma d : \mathsf{hetbox}^{n,p}.\, (\mathsf{cube}^{n,p}(d) \times \mathsf{cube}^{n,p}(d)) \\
 & & \\
\mathsf{cube}^{n+1,p}(d : \mathsf{hetbox}^{n+1,p}) & \triangleq & \Sigma a : (\mathsf{cube}^{n,p}(d) \times \mathsf{cube}^{n,p}(d)).\, \mathsf{cube}^{n+1,p+1}(d,a)
\end{array}$$

where unit is a canonical singleton type used to initiate the construction of a list of pairs of n-cubes.

Now, the two occurrences of $\mathsf{cube}^{n,p}(d)$ cannot be any cubes. They have to represent two opposite cubes and we need a way to express this. The answer is simple. The partial box has two main faces and these two cubes have to respectively fill the left face and the right face of the partial box. We thus have to rely on operations $\mathsf{subhetbox}_l^L$ and $\mathsf{subhetbox}_l^R$ which extract the sides of a box, leading to the refined definition:

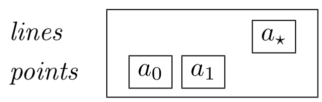
$$\mathsf{cube}^{n+1,p}(d : \mathsf{hetbox}^{n+1,p}) \quad \triangleq \quad \Sigma a : (\mathsf{cube}^{n,p}(\mathsf{subhetbox}_l^L(d)) \times \mathsf{cube}^{n,p}(\mathsf{subhetbox}_l^R(d))).\, \mathsf{cube}^{n+1,p+1}(d, a)$$

The full explicit definition shall be given in the next sections but, to give an idea, let us show the structure of an n-cube for the first three dimensions.

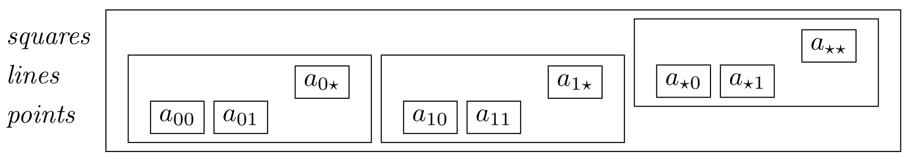
A full 0-cube (of type $\mathsf{cube}^{0,0}(\star)$) is made of just one point:

points $\boxed{a}$

A full 1-cube (of type $\mathsf{cube}^{1,0}(\star)$) is made of two points (of type $\mathsf{cube}^{0,0}(\star)$) connected by a line (of type $\mathsf{cube}^{1,1}(a_0, a_1)$):



A full 2-cube (of type $\mathsf{cube}^{2,0}(\star)$) is made of two full 1-cubes (of type $\mathsf{cube}^{1,0}(\star)$) connected by a lifted 1-cube (of type $\mathsf{cube}^{2,1}(((a_{00}, a_{01}), a_{0\star}), (a_{10}, a_{11}), a_{1\star})))$ made of two lines (of respective types $\mathsf{cube}^{1,1}(a_{00}, a_{10})$ and $\mathsf{cube}^{1,1}(a_{10}, a_{11})$) connecting the former points and a square (of more complicated dependent type $\mathsf{cube}^{2,2}((((a_{00}, a_{01}), a_{0\star}), ((a_{10}, a_{11}), a_{1\star})), (a_{\star 0}, a_{\star 1})))$ connecting the former lines:



In the next section, we define cubes of types and cubes of terms which will form the basis for interpreting judgements of cubical type theory.

3.3 Cubes of types and cubes of terms

We mutually define cubes of types and cubes of terms by means of a large mutual definition dispatched on Figures 2, 3, 4, 5, 26 and 7.

Cubes of types are cubes of relations in all dimensions, with types in dimension 0. For instance, a 2-cube of types is a square with four types at the vertices, four heterogeneous relations at the edges and one heterogeneous relation of dimension 2 filling the square:

$$\begin{tikzcd}
A_{LL} : \mathsf{U}_l \arrow[r, "A_{\star L} : A_{LL} \times A_{RL} \to \mathsf{U}_l"] \arrow[d, "A_{L\star} : A_{LL} \times A_{LR} \to \mathsf{U}_l"'] \arrow[dr, phantom, "\xRightarrow{A_{\star\star}}"] & A_{RL} : \mathsf{U}_l \arrow[d, "A_{R\star} : A_{RL} \times A_{RR} \to \mathsf{U}_l"] \\
A_{LR} : \mathsf{U}_l \arrow[r, "A_{\star R} : A_{LR} \times A_{RR} \to \mathsf{U}_l"] & A_{RR} : \mathsf{U}_l
\end{tikzcd}$$

where the inner relation has type

$$\Sigma a : (\Sigma a_L : (A_{LL} \times A_{LR}).\, A_{L\star}\, a_{LL}\, a_{LR}) \times (\Sigma a_R : (A_{RL} \times A_{RR}).\, A_{R\star}\, a_{RL}\, a_{RR}).\, (A_{\star L}\, a_{LL}\, a_{RL} \times A_{\star R}\, a_{LR}\, a_{RR}) \to \mathsf{U}_l$$

Cubes of terms depend on a cube of types. They are made of terms of the corresponding types in the vertices, of proofs that pairs of adjacent terms are in the corresponding relation between their types at the edges, of proofs that square of adjacent such proofs are in the corresponding relation between the corresponding four types. In the case of a square, this gives:

$$\begin{tikzcd}
a_{LL} : A_{LL} \arrow[r, "a_{\star L} : A_{\star L}\, a_{LL}\, a_{RL}"] \arrow[d, "a_{L\star} : A_{L\star}\, a_{LL}\, a_{LR}"'] \arrow[dr, phantom, "\xRightarrow{a_{\star\star}}"] & a_{RL} : A_{RL} \arrow[d, "a_{R\star} : A_{R\star}\, a_{RL}\, a_{RR}"] \\
a_{LR} : A_{LR} \arrow[r, "a_{\star R} : A_{\star R}\, a_{LR}\, a_{RR}"] & a_{RR} : A_{RR}
\end{tikzcd}$$

where the inner term has type $A_{\star\star}\,((a_{LL}, a_{LR}), a_{L\star}), ((a_{RL}, a_{RR}), a_{R\star}))\,(a_{\star L}, a_{\star R})$.

The construction of cubes of terms and cubes of types has to be mutual because the type of a relation in dimension n depends on the border of a cube of terms (what is called a box). The construction otherwise follows the pattern given in Section 3.2. In particular, cubes are defined mutually with the notion of box, using an auxiliary (non recursive) abbreviation called layer.

The layer level requires a face operation over boxes to be defined. Called Subbox for types and $\mathsf{subhetbox}$ for terms, it is itself defined by compositionally following the recursive structure mutually involving boxes, layers and cubes, with the non-canonical cases being the case of a higher-dimensional relation or of a term.

Nevertheless, the correctness of typing for the layer case require a coherence condition over face operations. Called Cohbox for types and $\mathsf{cohhetbox}$ for terms, it is again defined by compositionally following the recursive structure mutually involving boxes, layers and cubes, with, again, the non purely structural case being the case of a higher-dimensional relation or of a term.

Note that, again, a new coherence condition is needed for typing the layer cause. This is a coherence about equality proofs and this is where we use that ETT implicitly satisfies the Unicity of Identity Proofs principle.

Note finally that, formally, the definition has to be done by well-founded induction (because the types required in the construction at level n depends on level $n-1$). It also requires hd and tl to be themselves defined as part of the induction on n since they are defined only when $n \geq 1$.

3.4 Derived notions: full boxes, tubes, full cubes, fillers

On top of the basic definitions defined in Section 3.3, we can define on Figure 8 a few other concepts at a more standard level of abstraction. In particular, the notions of (full) box of types (Box), (full) box of terms (hetbox), (full) cube of types ($\mathsf{fullCube}$) and (full) cube of terms ($\mathsf{fullhetcube}$) will be the high-level bricks of the interpretation.

3.5 A first parametricity interpretation

We sketch in this section the external parametricity translation.

4 Basic delimited iterated internal parametricity

In this section, we extend the previous interpretation so that it interprets abstraction over a variable of dimension. In particular, this includes interpreting the reflexivity.

The construction is in several steps:

- In a first step, we give a dependently-typed definition of truncated cubical sets, which, in addition to faces, has (only) degeneracies. The definition is split over Section 4.1 for the bare structure of faces, Section 4.2 for a few intermediate definitions and Section 4.3 for the degeneracies structure.

- In a second step, we take the limit of this construction to get non-truncated cubical sets (Section 4.4).

Full n-cubes of types

$$\begin{array}{lll}
\mathsf{fullCube}^n_l & : & \mathsf{U}_{l+1} \\
\mathsf{fullCube}^n_l & \triangleq & \Sigma D : \mathsf{fullBox}^n_l .\, \mathsf{Filler}^n_l
\end{array}$$

Full n-boxes of types

$$\begin{array}{lll}
\mathsf{fullBox}^n_l & : & \mathsf{U}_{l+1} \\
\mathsf{fullBox}^n_l & \triangleq & \mathsf{Box}^{n,n}_l
\end{array}$$

Filler of a full n-box of types

$$\begin{array}{llll}
\mathsf{Filler}^n_l & (D : \mathsf{fullBox}^n_l) & : & \mathsf{U}_{l+1} \\
\mathsf{Filler}^n_l & D & \triangleq & \mathsf{fullhetbox}^n_l(D) \to \mathsf{U}_l
\end{array}$$

Auxiliary definitions

$$\begin{array}{llll}
\mathsf{Box}^{n,p,[p\leq n]}_l & & : & \mathsf{U}_{l+1} \\
\mathsf{Box}^{n,0}_l & & \triangleq & \mathsf{unit} \\
\mathsf{Box}^{n,p'+1}_l & & \triangleq & \Sigma D : \mathsf{Box}^{n,p'}_l .\, \mathsf{Layer}^{n,p'}_l(D) \\
\\
\mathsf{Layer}^{n,p,[p<n]}_l & (D : \mathsf{Box}^{n,p}_l) & : & \mathsf{U}_{l+1} \\
\mathsf{Layer}^{n,p}_l & D & \triangleq & \begin{array}{l} \mathsf{Cube}^{n-1,p}_l(\mathsf{Subbox}^{n,p}_{l,L,p}(D)) \\ \times\ \mathsf{Cube}^{n-1,p}_l(\mathsf{Subbox}^{n,p}_{l,R,p}(D)) \end{array} \\
\\
\mathsf{Cube}^{n,p,[p\leq n]}_l & (D : \mathsf{Box}^{n,p}_l) & : & \mathsf{U}_{l+1} \\
\mathsf{Cube}^{n,p,[p=n]}_l & D & \triangleq & \mathsf{Filler}^n_l(D) \\
\mathsf{Cube}^{n,p,[p<n]}_l & D & \triangleq & \Sigma B : \mathsf{Layer}^{n,p}_l(D) .\, \mathsf{Cube}^{n,p+1}_l(D,B)
\end{array}$$

Figure 2: Full n-box and p-prefix of a partial n-box of higher-order relations

Full n-boxes of terms over a full n-boxes of types

$$
\begin{array}{llcl}
\mathsf{fullhetbox}^{n}_{l} & (D : \mathsf{fullBox}^{n}_{l}) & : & \mathsf{U}_{l} \\
\mathsf{fullhetbox}^{n}_{l} & D & \triangleq & \mathsf{hetbox}^{n,n}_{l}(D)
\end{array}
$$

Filler of a full n-box of terms over some filled full n-box of types

$$
\begin{array}{llcl}
\mathsf{hetfiller}^{n}_{l} & \begin{array}{l}(D : \mathsf{fullBox}^{n}_{l}) \\ (E : \mathsf{Filler}^{n,n}_{l}(D)) \\ (d : \mathsf{fullhetbox}^{n}_{l}(D))\end{array} & : & \mathsf{U}_{l} \\
\mathsf{hetfiller}^{n}_{l} & D\ E\ d & \triangleq & E(d)
\end{array}
$$

Auxiliary definitions

$$
\begin{array}{llcl}
\mathsf{hetbox}^{n,p,[p\leq n]}_{l} & (D : \mathsf{Box}^{n,p}_{l}) & : & \mathsf{U}_{l} \\
\mathsf{hetbox}^{n,0}_{l} & D & \triangleq & \mathsf{unit} \\
\mathsf{hetbox}^{n,p'+1,[p'<n]}_{l} & (D, B) & \triangleq & \Sigma d : \mathsf{hetbox}^{n,p'}_{l}(D).\ \mathsf{hetlayer}^{n,p'}_{l}(D)(B)(d) \\
 & & & \\
\mathsf{hetlayer}^{n,p,[p<n]}_{l} & \begin{array}{l}(D : \mathsf{Box}^{n,p}_{l}) \\ (B : \mathsf{Layer}^{n,p}_{l}(D)) \\ (d : \mathsf{hetbox}^{n,p}_{l}(D))\end{array} & : & \mathsf{U}_{l} \\
\mathsf{hetlayer}^{n,p}_{l} & D\ B\ d & \triangleq & \begin{array}{l}\mathsf{hetcube}^{n-1,p}_{l}(\mathsf{Subbox}^{n,p}_{l,L,p}(D))(B_L)(\mathsf{subhetbox}^{n,p}_{l,L,p}(D)(d)) \\ \times\ \mathsf{hetcube}^{n-1,p}_{l}(\mathsf{Subbox}^{n,p}_{l,R,p}(D))(B_R)(\mathsf{subhetbox}^{n,p}_{l,R,p}(D)(d))\end{array} \\
 & & & \\
\mathsf{hetcube}^{n,p,[p\leq n]}_{l} & \begin{array}{l}(D : \mathsf{Box}^{n,p}_{l}) \\ (C : \mathsf{Cube}^{n,p}_{l}(D)) \\ (d : \mathsf{hetbox}^{n,p}_{l}(D))\end{array} & : & \mathsf{U}_{l} \\
\mathsf{hetcube}^{n,p,[p=n]}_{l} & D\ C\ d & \triangleq & \mathsf{hetfiller}^{n}_{l}(D)(C)(d) \\
\mathsf{hetcube}^{n,p,[p<n]}_{l} & D\ (B, C)\ d & \triangleq & \Sigma b : \mathsf{hetlayer}^{n,p}_{l}(D)(d).\ \mathsf{hetcube}^{n,p+1}_{l}(D, B)(C)(d, b)
\end{array}
$$

Figure 3: Full n-box and p-prefix of a partial n-box of terms over an n-box of higher-order relations

$$
\begin{array}{llcl}
\mathsf{Subbox}^{n,p,[p\leq q<n]}_{l,\epsilon,q} & (D : \mathsf{Box}^{n,p}_{l}) & : & \mathsf{Box}^{n-1,p}_{l} \\
\mathsf{Subbox}^{n,0}_{l,\epsilon,q} & \star & \triangleq & \star \\
\mathsf{Subbox}^{n,p'+1}_{l,\epsilon,q} & (D, B) & \triangleq & (\mathsf{Subbox}^{n,p'}_{l,\epsilon,q}(D), \mathsf{Sublayer}^{n,p'}_{l,\epsilon,q}(D)(B)) \\
 & & & \\
\mathsf{Sublayer}^{n,p,[p<q<n]}_{l,\epsilon,q} & \begin{array}{l}(D : \mathsf{Box}^{n,p}_{l}) \\ (B : \mathsf{Layer}^{n,p}_{l}(D))\end{array} & : & \mathsf{Layer}^{n-1,p}_{l}(\mathsf{Subbox}^{n,p}_{l,\epsilon,q}(D)) \\
\mathsf{Sublayer}^{n,p}_{l,\epsilon,q} & D\ (C_L, C_R) & \triangleq & \begin{array}{l}(\overrightarrow{\mathsf{Cohbox}^{n,p}_{l,\epsilon,L,q,p}(D)}(\mathsf{Subcube}^{n-1,p}_{l,\epsilon,q-1}(\mathsf{Subbox}^{n,p}_{l,L,p}(D))(C_L)), \\ \overrightarrow{\mathsf{Cohbox}^{n,p}_{l,\epsilon,R,q,p}(D)}(\mathsf{Subcube}^{n-1,p}_{l,\epsilon,q-1}(\mathsf{Subbox}^{n,p}_{l,R,p}(D))(C_R)))\end{array} \\
 & & & \\
\mathsf{Subcube}^{n,p,[p\leq q<n]}_{l,\epsilon,q} & \begin{array}{l}(D : \mathsf{Box}^{n,p}_{l}) \\ (C : \mathsf{Cube}^{n,p}_{l}(D))\end{array} & : & \mathsf{Cube}^{n-1,p}_{l}(\mathsf{Subbox}^{n,p}_{l,\epsilon,q}(D)) \\
\mathsf{Subcube}^{n,p,[p=q]}_{l,\epsilon,q} & D\ (B, \_) & \triangleq & B_{\epsilon} \\
\mathsf{Subcube}^{n,p,[p<q]}_{l,\epsilon,q} & D\ (B, C) & \triangleq & (\mathsf{Sublayer}^{n,p}_{l,\epsilon,q}(D)(B), \mathsf{Subcube}^{n,p+1}_{l,\epsilon,q}(D, B)(C))
\end{array}
$$

Figure 4: q-Projection of the p-prefix of an n-box of higher-order relations

$$
\begin{array}{llcl}
\mathsf{subhetbox}^{n,p,[p \leq q < n]}_{l,\epsilon,q} & \begin{array}{l}(D : \mathsf{Box}^{n,p}_{l}) \\ (d : \mathsf{hetbox}^{n,p}_{l}(D))\end{array} & : & \mathsf{hetbox}^{n-1,p}_{l}(\mathsf{Subbox}^{n,p}_{l,\epsilon,q}(D)) \\
\mathsf{subhetbox}^{n,0}_{l,\epsilon,q} & D \star & \triangleq & \star \\
\mathsf{subhetbox}^{n,p'+1}_{l,\epsilon,q} & (D,B)\ (d,b) & \triangleq & (\mathsf{subhetbox}^{n,p'}_{l,\epsilon,q}(D)(d), \mathsf{subhetlayer}^{n,p'}_{l,\epsilon,q}(D)(B)(d)(b)) \\
\\
\mathsf{subhetlayer}^{n,p,[p < q < n]}_{l,\epsilon,q} & \begin{array}{l}(D : \mathsf{Box}^{n,p}_{l}) \\ (B : \mathsf{Layer}^{n,p}_{l}) \\ (d : \mathsf{hetbox}^{n,p}_{l}(D)) \\ (b : \mathsf{hetlayer}^{n,p}_{l}(D)(B)(d))\end{array} & : & \mathsf{hetlayer}^{n-1,p}_{l}(\mathsf{Subbox}^{n,p}_{l,\epsilon,q}(D))(B_\epsilon)(\mathsf{subhetbox}^{n,p}_{l,\epsilon,q}(D)(d)) \\
\mathsf{subhetlayer}^{n,p}_{l,\epsilon,q} & D\ B\ d\ b & \triangleq & \begin{array}{l}
(\overrightarrow{\substack{(\mathsf{Cohbox}^{n,p}_{l,\epsilon,L,q,p}(D)(d), \\ \mathsf{cohhetbox}^{n,p}_{l,\epsilon,L,q,p}(D)(d))}}\ (\mathsf{subhetcube}^{n-1,p}_{l,\epsilon,q-1} \begin{pmatrix}\mathsf{Subbox}^{n,q}_{l,L,p}(D) \\ B_L \\ \mathsf{subhetbox}^{n,p}_{l,L,p}(D)(d) \\ b_L\end{pmatrix}), \\
\overrightarrow{\substack{(\mathsf{Cohbox}^{n,p}_{l,\epsilon,R,q,p}(D)(d), \\ \mathsf{cohhetbox}^{n,p}_{l,\epsilon,R,q,p}(D)(d))}}\ (\mathsf{subhetcube}^{n-1,p}_{l,\epsilon,q-1} \begin{pmatrix}\mathsf{Subbox}^{n,q}_{l,R,p}(D) \\ B_R \\ \mathsf{subhetbox}^{n,p}_{l,R,p}(D)(d) \\ b_R\end{pmatrix}))
\end{array} \\
\\
\mathsf{subhetcube}^{n,p,[p \leq q < n]}_{l,\epsilon,q} & \begin{array}{l}(D : \mathsf{Box}^{n,p}_{l}) \\ (C : \mathsf{Cube}^{n,p}_{l}(D)) \\ (d : \mathsf{hetbox}^{n,p}_{l}(D)) \\ (c : \mathsf{hetcube}^{n,p}_{l}(D)(C)(d))\end{array} & : & \mathsf{hetcube}^{n-1,p}_{l}(\mathsf{Subbox}^{n,p}_{l,\epsilon,q}(D))(\mathsf{Subcube}^{n,p}_{l,\epsilon,q}(C))(\mathsf{subhetbox}^{n,p}_{l,\epsilon,q}(D)(d)) \\
\mathsf{subhetcube}^{n,p,[p=q]}_{l,\epsilon,q} & D\ E\ d\ (b, \_) & \triangleq & b_\epsilon \\
\mathsf{subhetcube}^{n,p,[p<q]}_{l,\epsilon,q} & D\ (B,C)\ d\ (b,c) & \triangleq & (\mathsf{subhetlayer}^{n,p}_{l,\epsilon,q}(D)(B)(d)(b), \mathsf{subhetcube}^{n,p+1}_{l,\epsilon,q}(D,B)(C)(d,b)(c))
\end{array}
$$

Figure 5: Definition of a cube of terms (faces)

$$
\begin{array}{llcl}
\mathsf{Cohbox}^{n,p\ [p \leq r < q < n]}_{l,\epsilon,\epsilon',q,r} & (D : \mathsf{Box}^{n,p}_{l}) & : & \begin{array}{l}\mathsf{Subbox}^{n-1,p}_{l,\epsilon,q-1}(\mathsf{Subbox}^{n,p}_{l,\epsilon',r}(D)) \\ \equiv_{ETT} \mathsf{Subbox}^{n-1,p}_{l,\epsilon',r-1}(\mathsf{Subbox}^{n,p}_{l,\epsilon,q}(D))\end{array} \\
\mathsf{Cohbox}^{n,0}_{l,\epsilon,\epsilon',q,r} & D \star & \triangleq & \mathsf{refl}_{ETT}(\star) \\
\mathsf{Cohbox}^{n,p'+1}_{l,\epsilon,\epsilon',q,r} & D\ (d,b) & \triangleq & (\mathsf{Cohbox}^{n,p'}_{l,\epsilon,\epsilon',q,r}(D)(d), \mathsf{Cohlayer}^{n,p'}_{l,\epsilon,\epsilon',q,r}(D)(d)(b)) \\
\\
\mathsf{Cohlayer}^{n,p\ [p < r < q < n]}_{l,\epsilon,\epsilon',q,r} & \begin{array}{l}(D : \mathsf{Box}^{n,p}_{l}) \\ (B : \mathsf{Layer}^{n,p}_{l}(D))\end{array} & : & \begin{array}{l}\mathsf{Sublayer}^{n-1,p}_{l,\epsilon,q-1}(\mathsf{Subbox}^{n,p}_{l,\epsilon',r}(D))(\mathsf{Sublayer}^{n,p}_{l,\epsilon',r}(D)(B)) \\ \equiv_{ETT} \mathsf{Sublayer}^{n-1,p}_{l,\epsilon',r-1}(\mathsf{Subbox}^{n,p}_{l,\epsilon,q}(D))(\mathsf{Sublayer}^{n,p}_{l,\epsilon,q}(D)(B))\end{array} \\
\mathsf{Cohlayer}^{n,p}_{l,\epsilon,\epsilon',q,r} & D\ C & \triangleq & \begin{array}{l}(\mathsf{Cohcube}^{n-1,p}_{l,\epsilon,\epsilon',q-1,r-1}(\mathsf{Subbox}^{n,p}_{l,L,p}(D))(C_L), \\ \ \mathsf{Cohcube}^{n-1,p}_{l,\epsilon,\epsilon',q-1,r-1}(\mathsf{Subbox}^{n,p}_{l,R,p}(D))(C_R))\end{array} \\
\\
\mathsf{Cohcube}^{n,p\ [p \leq r < q < n]}_{l,\epsilon,\epsilon',q,r} & \begin{array}{l}(D : \mathsf{Box}^{n,p}_{l}) \\ (C : \mathsf{Cube}^{n,p}_{l}(D))\end{array} & : & \begin{array}{l}\mathsf{Subcube}^{n-1,p}_{l,\epsilon,q-1}(\mathsf{Subbox}^{n,p}_{l,\epsilon',q}(D))(\mathsf{Subcube}^{n,p}_{l,\epsilon',r}(D)(C)) \\ \equiv_{ETT} \mathsf{Subcube}^{n-1,p}_{l,\epsilon',r-1}(\mathsf{Subbox}^{n,p}_{l,\epsilon,q}(D))(\mathsf{Subcube}^{n,p}_{l,\epsilon,q}(D)(C))\end{array} \\
\mathsf{Cohcube}^{n,p,[p=r]}_{l,\epsilon,\epsilon',q,r} & D\ (B, \_) & \triangleq & \mathsf{refl}_{ETT}(\mathsf{Subcube}^{n-1,p}_{l,\epsilon,q-1}(\mathsf{Subbox}^{n,p}_{l,\epsilon',q}(D))(B_\epsilon)) \\
\mathsf{Cohcube}^{n,p,[p<r]}_{l,\epsilon,\epsilon',q,r} & D\ (B,C) & \triangleq & (\mathsf{Cohlayer}^{n,p}_{l,\epsilon,\epsilon',q,r}(D)(B),\ \mathsf{Cohcube}^{n,p+1}_{l,\epsilon,\epsilon',q,r}(D,B)(C))
\end{array}
$$

Figure 6: Definition of a cube of higher-order relations (compatibility of faces)

$$
\begin{array}{llcl}
\mathsf{cohhetbox}^{\substack{n,\,p \\ [p\,\le\, r\,<\,q\,<\,n]}}_{l,\epsilon,\epsilon',q,r} & \begin{array}{l}(D : \mathsf{Box}^{n,p}_l) \\ (d : \mathsf{hetbox}^{n,p}_l(D))\end{array} & : & \begin{array}{l}\mathsf{subhetbox}^{n-1,p}_{l,\epsilon,q-1}(\mathsf{Subbox}^{n,p}_{l,\epsilon',r}(D))(\mathsf{subhetbox}^{n,p}_{l,\epsilon',r}(D)(d)) \\ \equiv_{ETT} \mathsf{subhetbox}^{n-1,p}_{l,\epsilon',r-1}(\mathsf{Subbox}^{n,p}_{l,\epsilon,q}(D))(\mathsf{subhetbox}^{n,p}_{l,\epsilon,q}(D)(d))\end{array} \\
\mathsf{cohhetbox}^{n,0}_{l,\epsilon,\epsilon',q,r} & D\ \star & \triangleq & \mathsf{refl}_{ETT}(\star) \\
\mathsf{cohhetbox}^{n,p'+1}_{l,\epsilon,\epsilon',q,r} & (D,B)\ (d,b) & \triangleq & (\mathsf{cohhetbox}^{n,p'}_{l,\epsilon,\epsilon',q,r}(D)(d), \mathsf{cohhetlayer}^{n,p'}_{l,\epsilon,\epsilon',q,r}(D)(B)(d)(b)) \\
\\
\mathsf{cohhetlayer}^{\substack{n,\,p \\ [p\,<\, r\,<\,q\,<\,n]}}_{l,\epsilon,\epsilon',q,r} & \begin{array}{l}(D : \mathsf{Box}^{n,p}_l) \\ (B : \mathsf{Layer}^{n,p}_l) \\ (d : \mathsf{hetbox}^{n,p}_l(D)) \\ (b : \mathsf{hetlayer}^{n,p}_l(D)(B)(d))\end{array} & : & \begin{array}{l}\mathsf{subhetlayer}^{n-1,p}_{l,\epsilon,q-1}(\mathsf{Subbox}^{n,p}_{l,\epsilon',r}(D))(\mathsf{Sublayer}^{n,p}_{l,\epsilon',r}(D)(B)) \\ \quad (\mathsf{subhetbox}^{n,p}_{l,\epsilon',r}(D)(d))(\mathsf{subhetlayer}^{n,p}_{l,\epsilon',r}(D)(B)(d)(b)) \\ \equiv_{ETT} \mathsf{subhetlayer}^{n-1,p}_{l,\epsilon',r-1}(\mathsf{Subbox}^{n,p}_{l,\epsilon,q}(D))(\mathsf{Sublayer}^{n,p}_{l,\epsilon,q}(D)(B)) \\ \quad (\mathsf{subhetbox}^{n,p}_{l,\epsilon,q}(D)(d))(\mathsf{subhetlayer}^{n,p}_{l,\epsilon,q}(D)(B)(d)(b))\end{array} \\
\mathsf{cohhetlayer}^{n,p}_{l,\epsilon,\epsilon',q,r} & D\ B\ d\ c & \triangleq & \begin{array}{l}(\mathsf{cohhetcube}^{n,p}_{l,\epsilon,\epsilon',q,r}(D)(B_L)(\mathsf{subhetbox}^{n,p}_{l,L,p}(D)(d))(c_L), \\ \ \mathsf{cohhetcube}^{n,p}_{l,\epsilon,\epsilon',q,r}(D)(B_R)(\mathsf{subhetbox}^{n,p}_{l,R,p}(D)(d))(c_R))\end{array} \\
\\
\mathsf{cohhetcube}^{\substack{n,\,p \\ [p\,\le\, r\,<\,q\,<\,n]}}_{l,\epsilon,\epsilon',q,r} & \begin{array}{l}(D : \mathsf{Box}^{n,p}_l) \\ (C : \mathsf{Cube}^{n,p}_l) \\ (d : \mathsf{hetbox}^{n,p}_l(D)) \\ (c : \mathsf{hetcube}^{n,p}_l(D)(C)(d))\end{array} & : & \begin{array}{l}\mathsf{subhetcube}^{n-1,p}_{l,\epsilon,q-1}(\mathsf{Subbox}^{n,p}_{l,\epsilon',r}(D))(\mathsf{Subcube}^{n,p}_{l,\epsilon',r}(D)(C)) \\ \quad (\mathsf{subhetbox}^{n,p}_{l,\epsilon',r}(D)(d))(\mathsf{subhetcube}^{n,p}_{l,\epsilon',r}(D)(C)(d)(c)) \\ \equiv_{ETT} \mathsf{subhetcube}^{n-1,p}_{l,\epsilon',r-1}(\mathsf{Subbox}^{n,p}_{l,\epsilon,q}(D))(\mathsf{Subcube}^{n,p}_{l,\epsilon',q}(D)(C)) \\ \quad (\mathsf{subhetbox}^{n,p}_{l,\epsilon,q}(D)(d))(\mathsf{subhetcube}^{n,p}_{l,\epsilon,q}(D)(C)(d)(c))\end{array} \\
\mathsf{cohhetcube}^{n,p,[p=r]}_{l,\epsilon,\epsilon',q,r} & D\ (B,\_)\ d\ (b,\_) & \triangleq & \mathsf{refl}_{ETT}(\mathsf{subhetcube}^{n-1,p}_{l,\epsilon,q-1}(\mathsf{Subbox}^{n,p}_{l,\epsilon',p}(D))(B)(\mathsf{subhetbox}^{n,p}_{l,\epsilon',p}(D)(d))(b_\epsilon)) \\
\mathsf{cohhetcube}^{n,p,[p<r)}_{l,\epsilon,\epsilon',q,r} & D\ (B,C)\ d\ (b,c) & \triangleq & (\mathsf{cohhetlayer}^{n,p}_{l,\epsilon,\epsilon',q,r}(D)(B)(d)(b),\ \mathsf{cohhetcube}^{n,p+1}_{l,\epsilon,\epsilon',q,r}(D,B)(C)(d,b)(c))
\end{array}
$$

Figure 7: Definition of a cube of terms (compatibility of faces)

Type of n-tubes of types

$$
\begin{array}{llcl}
\mathsf{Tube}^n_l & & : & \mathsf{U}_{l+1} \\
\mathsf{Tube}^n_l & & \triangleq & \mathsf{Box}^{n+1,n}_l
\end{array}
$$

Type of n-tubes of terms over a $(n+1)$-box of types

$$
\begin{array}{llcl}
\mathsf{hettube}^n_l & (D : \mathsf{fullBox}^{n+1}_l) & : & \mathsf{U}_l \\
\mathsf{hettube}^n_l & D & \triangleq & \mathsf{hetbox}^{n+1,n}_l(D)
\end{array}
$$

Type of full n-cubes of terms over a full n-cube of types

$$
\begin{array}{llcl}
\mathsf{fullhetcube}^n_l & (C : \mathsf{fullCube}^n_l) & : & \mathsf{U}_l \\
\mathsf{fullhetcube}^n_l & (D,E) & \triangleq & \Sigma d : \mathsf{fullhetbox}^n_l(D).\, E(d)
\end{array}
$$

Type of grounded n-cubes of types

$$
\begin{array}{llcl}
\mathsf{groundedCube}^n_l & & : & \mathsf{U}_l \\
\mathsf{groundedCube}^n_l & & \triangleq & \mathsf{Cube}^{n,0}_l(\star)
\end{array}
$$

Type of grounded n-cubes of terms over a full n-cube of types

$$
\begin{array}{llcl}
\mathsf{groundedhetcube}^n_l & (C : \mathsf{fullCube}^n_l) & : & \mathsf{U}_l \\
\mathsf{groundedhetcube}^n_l & (D,E) & \triangleq & \mathsf{hetcube}^{n,0}_l(D)(E)(\star)
\end{array}
$$

Figure 8: Standard derived notions

Type of truncated cubical sets with degeneracies

$$
\begin{array}{llll}
\mathsf{cubset}_l^{<n} & & : & \mathsf{U}_{l+1} \\
\mathsf{cubset}_l^{<0} & & \triangleq & \mathsf{unit} \\
\mathsf{cubset}_l^{<n'+1} & & \triangleq & \Sigma D : \mathsf{cubset}_l^{<n'} . \, \mathsf{cubset}_l^{=n'}(D)
\end{array}
$$

Structure carried at each dimension

$$
\begin{array}{llll}
\mathsf{cubset}_l^{=n} & (D : \mathsf{cubset}_l^{<n}) & : & \mathsf{U}_l \\
\mathsf{cubset}_l^{=n} & D & \triangleq & \left\{ \begin{array}{l} \mathsf{rel} : \mathsf{fullbox}_l^n(D) \to \mathsf{U}_l); \\ \mathsf{refl} : n \geq 1 \to \mathsf{reflexive}_l^n(D)(R) \end{array} \right\}
\end{array}
$$

Figure 9: Definition of a truncated cubical set (main part of the higher-dimensional relation structure)

- In a third step, we give the definition of cubical sets dependent on a cubical set (Section **??**).
- From the latter, we can give in a fifth step a definition of Σ types over cubical sets (Sections A.1).
- In a final step, we can give for all n an interpretation of each universe as a n-cube (Sections 4.5).

4.1 Truncated core cubical sets

In this section, we give the core of the definition of truncated cubical sets with reflexivities whose part relative to reflexivities will be defined in Section 4.3.

The definition is dispatched over Figures 9, 10, 11 and 12. It describes the structure of the underlying higher-dimensional relations on which cubes and boxes are built, together with the definition of homogeneous n-boxes, homogeneous n-cubes, together with face operations on boxes and cubes, together with commutation properties of the face operations. All are mutually defined as types of the target language ETT. Note that such relational structure, cubes and boxes are relative to a universe.

Note the presence of a coherence condition cohbox_l to ensuring that both sides of the equality in $\mathsf{sublayer}_l$ and $\mathsf{subcube}_l$ are in the same type. The proof of cohbox_l itself requires an higher-dimensional coherence condition which we obtain by working here in ETT where all proofs of an equality are identified (principle of Unicity of Identity Proofs). Note that if the proofs of the same equality were not equated, there would be a need for arbitrary many higher-dimensional coherences (see e.g. [6] for a discussion on the de facto need for recursive higher-dimensional coherence conditions in formulating higher-dimensional structures in type theory). Note also that for a given n, the coherence conditions evaluate to a reflexivity proof, so that the construction evaluates to an effective sequence of types of iterated relations not mentioning subbox_l nor cohbox_l anymore.

When reflexivities are excepted, we call the structure thus defined *bare truncated cubical sets*: *bare* because it can be seen as defining a cubical equivalent to semi-simplicial sets with only faces as part of the structure (otherwise said, another terminology could have been "semi-cubical" sets); *truncated* because we consider only such cubical sets up to some fixed dimension.

4.2 Derived notions

On top of the basic definitions on Figures 10, 11 and 12, we can define on Figures 13 and 14 a few other concepts at a more standard level of abstraction. In particular, a homogeneous n-box in the above sense is defined to be an object of type $\mathsf{fullbox}_l^n(D_n)$ for $D_n \triangleq (A, =_A, ..., =_A^n)$ the initial segment of n-th first iterated equalities over A (by convention, a homogeneous 0-box shall be the canonical singleton type). A n-tube is of type $\mathsf{box}_l^{n+1,n}(D_{n+1})$. An homogeneous (proper) n-cube in some n-box d is of type $\mathsf{cube}_l^{n+1,n}(\mathsf{hd}(D_{n+1}))(\mathsf{tl}(D_{n+1}))(d)$ and a full n-cube is of type $\mathsf{cube}_l^{n+1,0}(\mathsf{hd}(D_{n+1}))(\mathsf{tl}(D_{n+1})))(\star)$.

Full homogeneous n-boxes

$$
\begin{array}{llcl}
\mathsf{fullbox}_l^{n} & (D : \mathsf{cubset}_l^{<n}) & : & \mathsf{U}_l \\
\mathsf{fullbox}_l^{n} & D & \triangleq & \mathsf{box}_l^{n,n}(D)
\end{array}
$$

Homogeneous partial n-boxes and homogeneous partial n-cubes

$$
\begin{array}{llcl}
\mathsf{box}_l^{n,p,[p\leq n]} & (D : \mathsf{cubset}_l^{<n}) & : & \mathsf{U}_l \\
\mathsf{box}_l^{n,0} & D & \triangleq & \mathsf{unit} \\
\mathsf{box}_l^{n,p'+1} & D & \triangleq & \Sigma d : \mathsf{box}_l^{n,p'}(D).\, \mathsf{layer}_l^{n,p'}(D)(d) \\
 & & & \\
\mathsf{layer}_l^{n,p,[p<n]} & \begin{array}{l}(D : \mathsf{cubset}_l^{<n}) \\ (d : \mathsf{box}_l^{n,p}(D))\end{array} & : & \mathsf{U}_l \\
\mathsf{layer}_l^{n,p} & D\ d & \triangleq & \begin{array}{l}\mathsf{cube}_l^{n-1,p}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}_{l,L,p}^{n,p}(D)(d)) \\ \times\ \mathsf{cube}_l^{n-1,p}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}_{l,R,p}^{n,p}(D)(d))\end{array} \\
 & & & \\
\mathsf{cube}_l^{n,p,[p\leq n]} & \begin{array}{l}(D : \mathsf{cubset}_l^{<n}) \\ (E : \mathsf{cubset}_l^{=n}(D)) \\ (d : \mathsf{box}_l^{n,p}(D))\end{array} & : & \mathsf{U}_l \\
\mathsf{cube}_l^{n,p,[p=n]} & D\ E\ d & \triangleq & E.\mathsf{rel}(d) \\
\mathsf{cube}_l^{n,p,[p<n]} & D\ E\ d & \triangleq & \Sigma b : \mathsf{layer}_l^{n,p}(D)(d).\, \mathsf{cube}_l^{n,p+1}(D)(E)(d,b)
\end{array}
$$

Figure 10: Definition of a truncated cubical set (boxes)

$$
\begin{array}{llcl}
\mathsf{subbox}_{l,\epsilon,q}^{n,p,[p\leq q<n]} & \begin{array}{l}(D : \mathsf{cubset}_l^{<n}) \\ (d : \mathsf{box}_l^{n,p}(D))\end{array} & : & \mathsf{box}_l^{n-1,p}(\mathsf{hd}(D)) \\
\mathsf{subbox}_{l,\epsilon,q}^{n,0} & D\ \star & \triangleq & \star \\
\mathsf{subbox}_{l,\epsilon,q}^{n,p'+1} & D\ (d,b) & \triangleq & (\mathsf{subbox}_{l,\epsilon,q}^{n,p'}(D)(d), \mathsf{sublayer}_{l,\epsilon,q}^{n,p'}(D)(d)(b)) \\
 & & & \\
\mathsf{sublayer}_{l,\epsilon,q}^{n,p,[p<q<n]} & \begin{array}{l}(D : \mathsf{cubset}_l^{<n}) \\ (d : \mathsf{box}_l^{n,p}(D)) \\ (b : \mathsf{layer}_l^{n,p}(D)(d))\end{array} & : & \mathsf{layer}_l^{n-1,p}(\mathsf{hd}(D))(\mathsf{subbox}_{l,\epsilon,q}^{n,p}(D)(d)) \\
\mathsf{sublayer}_{l,\epsilon,q}^{n,p} & D\ d\ c & \triangleq & \begin{array}{l}(\overrightarrow{\mathsf{cohbox}_{l,\epsilon,L,q,p}^{n,p}(D)(d)}(\mathsf{subcube}_{l,\epsilon,q-1}^{n-1,p}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}_{l,L,p}^{n,p}(D)(d))(c_L)), \\ \ \overrightarrow{\mathsf{cohbox}_{l,\epsilon,R,q,p}^{n,p}(D)(d)}(\mathsf{subcube}_{l,\epsilon,q-1}^{n-1,p}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}_{l,R,p}^{n,p}(D)(d))(c_R)))\end{array} \\
 & & & \\
\mathsf{subcube}_{l,\epsilon,q}^{n,p,[p\leq q<n]} & \begin{array}{l}(D : \mathsf{cubset}_l^{<n}) \\ (E : \mathsf{cubset}_l^{=n}(D)) \\ (d : \mathsf{box}_l^{n,p}(D)) \\ (b : \mathsf{cube}_l^{n,p}(D)(E)(d))\end{array} & : & \mathsf{cube}_l^{n-1,p}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}_{l,\epsilon,q}^{n,p}(D)(d)) \\
\mathsf{subcube}_{l,\epsilon,q}^{n,p,[p=q]} & D\ E\ d\ (b, \_) & \triangleq & b_\epsilon \\
\mathsf{subcube}_{l,\epsilon,q}^{n,p,[p<q]} & D\ E\ d\ (b, c) & \triangleq & (\mathsf{sublayer}_{l,\epsilon,q}^{n,p}(D)(d)(b), \mathsf{subcube}_{l,\epsilon,q}^{n,p+1}(D)(E)(d,b)(c))
\end{array}
$$

Figure 11: Definition of a homogeneous bare cubical set (q-th projection)

$\mathsf{cohbox}^{n,\,p}_{[p\,\le\,r\,<\,q\,<\,n]\atop l,\epsilon,\epsilon',q,r}$	$(D : \mathsf{cubset}^{<n}_l)$ $(d : \mathsf{box}^{n,p}_l(D))$	$:$	$\mathsf{subbox}^{n-1,p}_{l,\epsilon,q-1}(\mathsf{hd}(D))(\mathsf{subbox}^{n,p}_{l,\epsilon',r}(D)(d))$ $\equiv_{ETT}\ \mathsf{subbox}^{n-1,p}_{l,\epsilon',r-1}(\mathsf{hd}(D))(\mathsf{subbox}^{n,p}_{l,\epsilon,q}(D)(d))$
$\mathsf{cohbox}^{n,0}_{l,\epsilon,\epsilon',q,r}$	$D\ \star$	$\triangleq$	$\mathsf{refl}_{ETT}(\star)$
$\mathsf{cohbox}^{n,p'+1}_{l,\epsilon,\epsilon',q,r}$	$D\ (d,b)$	$\triangleq$	$(\mathsf{cohbox}^{n,p'}_{l,\epsilon,\epsilon',q,r}(D)(d), \mathsf{cohlayer}^{n,p'}_{l,\epsilon,\epsilon',q,r}(D)(d)(b))$
$\mathsf{cohlayer}^{n,\,p}_{[p\,<\,r\,<\,q\,<\,n]\atop l,\epsilon,\epsilon',q,r}$	$(D : \mathsf{cubset}^{<n}_l)$ $(d : \mathsf{box}^{n,p}_l(D))$ $(b : \mathsf{layer}^{n,p}_l(D)(d))$	$:$	$\mathsf{sublayer}^{n-1,p}_{l,\epsilon,q}(\mathsf{hd}(D))(\mathsf{subbox}^{n,p}_{l,\epsilon',r}(D)(d))(\mathsf{sublayer}^{n,p}_{l,\epsilon',r}(D)(d)(b))$ $\equiv_{ETT}\ \mathsf{sublayer}^{n-1,p}_{l,\epsilon',r}(\mathsf{hd}(D))(\mathsf{subbox}^{n,p}_{l,\epsilon,q}(D)(d))(\mathsf{sublayer}^{n,p}_{l,\epsilon,q}(D)(d)(b))$
$\mathsf{cohlayer}^{n,p}_{l,\epsilon,\epsilon',q,r}$	$D\ d\ c$	$\triangleq$	$(\mathsf{cohcube}_{l,\epsilon,\epsilon',q-1,r-1,n-1,p}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,p}_{l,L,p}(D)(d))(c_L),$ $\mathsf{cohcube}_{l,\epsilon,\epsilon',q-1,r-1,n-1,p}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,p}_{l,R,p}(D)(d))(c_R))$
$\mathsf{cohcube}_{l,\epsilon,\epsilon',q,r,\,{n,\,p \atop [p\,\le\,r\,<\,q\,<\,n]}}$	$(D : \mathsf{cubset}^{<n}_l)$ $(E : \mathsf{cubset}^{=n}_l(D))$ $(d : \mathsf{box}^{n,p}_l(D))$ $(b : \mathsf{cube}^{n,p}_l(D)(E)(d))$	$:$	$\mathsf{subcube}^{n-1,p}_{l,\epsilon,q-1}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,p}_{l,\epsilon',r}(D)(d))(\mathsf{subcube}^{n,p}_{l,\epsilon',r}(D)(E)(d)(b))$ $\equiv_{ETT}\ \mathsf{subcube}^{n-1,p}_{l,\epsilon',r-1}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,p}_{l,\epsilon,q}(D)(d))(\mathsf{subcube}^{n,p}_{l,\epsilon,q}(D)(E)(d)(b))$
$\mathsf{cohcube}_{l,\epsilon,\epsilon',q,r,n,p,[p=r]}$	$D\ E\ d\ (b, \_)$	$\triangleq$	$\mathsf{refl}_{ETT}(\mathsf{subcube}^{n-1,p}_{l,\epsilon,q-1}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,p}_{l,\epsilon',p}(D)(d))(b_\epsilon))$
$\mathsf{cohcube}_{l,\epsilon,\epsilon',q,r,n,p,[p<r]}$	$D\ E\ d\ (b,c)$	$\triangleq$	$(\mathsf{cohlayer}^{n,p}_{l,\epsilon,\epsilon',q,r}(D)(d)(b),\ \mathsf{cohcube}_{l,\epsilon,\epsilon',q,r,n,p+1}(D)(E)(d,b)(c))$

Figure 12: Definition of a homogeneous bare cubical set (commutation of q-th projection and r-th projection)

Note that the components of an n-box or n-tube are tuples associated to the left. For cubes, the components can be associated to the left, and we call this a full n-cube or to the right, which we call grounded n-cube, and which corresponds to "complete" partial cubes, i.e. to partial cubes over the empty box. Figure 13 show how to compute the n-box surrounding a full n-cube over a prefix D_{n+1} of iterated equalities (see border).

Figure 13 also shows how to compute the faces of a box or of a cube.

4.3 Truncated cubical sets with reflexivities

The definition of $\mathsf{reflexive}^n_l$ mentioned on Figure 9 is given on Figures 15, 16 and 17.

Types are actually also equipped with a notion of reflexivity which shall precisely associate to a type a relation over this type (consistently with the view that equality in the universe is giving a relation).

4.4 Full cubical sets with reflexivities

We shall interpret 0-cube of types as cubical sets, themselves obtained as the coinductive limit of truncated cubical sets, see Figure 18.

4.5 The n-cube interpretation of universes

Let U_l be a universe. For any n, we define in this section the interpretation of U_l as an n-cube of types, that is, as an inhabitant of $\mathsf{fullCube}^n_{l+1}$.

The points of the n-cube associated to U_l are all the same. Such point represents the type of cubical types in this universe, or more precisely cubical sets since the target of the interpretation is ETT where all types have homotopy level 0, i.e. are sets. The type of such cubical set is the type of infinite sequences of a type X, of a relation $X_\star$ over X, of a square $X_{\star\star}$ over X and $X_\star$, etc. In such sequence, all elements are relations over some box except X. In order to have a uniform definition, we shall also represent X as a dependent product over a box, namely over the 0-box, meaning that we shall take X of type $\Pi x : \mathsf{unit}.\, \mathsf{U}_l$. Additionally, still to fit the general recursive scheme, this infinite sequence will be abstracted over unit so that it itself has the form of such X of type $\Pi x : \mathsf{unit}.\, \mathsf{U}_{l+1}$ (at the next level).

The segments of the n-cube associated to U_l are also all the same. Such segment is an (informative) relation over points, that is a relation over two cubical sets, say $\mathcal{A}(\star) \triangleq (A, A_\star, ...)$ and $\mathcal{B}(\star) \triangleq (B, B_\star, ...)$

Type of full n-cubes

$$
\begin{array}{lllll}
\mathsf{fullcube}^n_l & (D : \mathsf{cubeset}^{<n+1}_l) & & : & \mathsf{U}_l \\
\mathsf{fullcube}^n_l & (D, E) & & \triangleq & \Sigma d : \mathsf{fullbox}^n_l(D).\, E.\mathsf{rel}(d)
\end{array}
$$

Type of n-tubes

$$
\begin{array}{lllll}
\mathsf{tube}^n_l & (D : \mathsf{cubeset}^{<n+1}_l) & & : & \mathsf{U}_l \\
\mathsf{tube}^n_l & D & & \triangleq & \mathsf{box}^{n+1,n}_l(D)
\end{array}
$$

Type of grounded n-cubes

$$
\begin{array}{lllll}
\mathsf{groundedcube}^n_l & (D : \mathsf{cubeset}^{<n+1}_l) & & : & \mathsf{U}_l \\
\mathsf{groundedcube}^n_l & (D, E) & & \triangleq & \mathsf{cube}^{n,0}_l(D)(E)(\star)
\end{array}
$$

Type of fillers of some n-box

$$
\begin{array}{lllll}
\mathsf{properfiller}^n_l & (D : \mathsf{cubeset}^{<n+1}_l)\ (d : \mathsf{fullbox}^n_l(\mathsf{hd}(D))) & & : & \mathsf{U}_l \\
\mathsf{properfiller}^n_l & (D, E)\ d & & \triangleq & E.\mathsf{rel}(d)
\end{array}
$$

Border of a grounded n-cube

$$
\begin{array}{lllll}
\mathsf{border}^n_l & (D : \mathsf{cubeset}^{<n+1}_l)\ (c : \mathsf{groundedcube}^n_l(D)) & & : & \mathsf{fullbox}^n_l(\mathsf{hd}(D)) \\
\mathsf{border}^n_l & D\ c & & \triangleq & \mathsf{border}^{n,0}_l(D)(\star)(c)
\end{array}
$$

where the border of partial cubes is defined by

$$
\begin{array}{llll}
\mathsf{border}^{n,p}_l & (D : \mathsf{cubeset}^{<n+1}_l)\ (d : \mathsf{box}^{n,p}_l(\mathsf{hd}(D)))\ (c : \mathsf{cube}^{n,p}_l(\mathsf{hd}(D))(\mathsf{tl}(D))(d)) & : & \mathsf{fullbox}^n_l(\mathsf{hd}(D)) \\
\mathsf{border}^{n,p,[p=n]}_l & D\ d\ c & \triangleq & d \\
\mathsf{border}^{n,p,[p<n]}_l & D\ d\ (b, c) & \triangleq & \mathsf{border}^{n,p+1}_l(D)(d, b)(c)
\end{array}
$$

Figure 13: Standard derived notions

q-th projection of an n-box

$$
\begin{array}{lllll}
\mathsf{subfullbox}^{n,[q<n]}_{l,\epsilon,q} & (D : \mathsf{cubset}^{<n}_l)\ (d : \mathsf{fullbox}^n_l(D)) & : & \mathsf{fullbox}^{n-1}_l(\mathsf{hd}(D)) \\
\mathsf{subfullbox}^{n}_{l,\epsilon,q} & D\ (d, \_) & \triangleq & \mathsf{subbox}^{n,n-1}_{l,\epsilon,q}(D)(d)
\end{array}
$$

q-th face of an n-cube

$$
\begin{array}{lllll}
\mathsf{subfullcube}^{n,[q<n]}_{l,\epsilon,q} & (D : \mathsf{cubset}^{<n+1}_l)\ (c : \mathsf{fullcube}^n_l(D)) & : & \mathsf{fullcube}^{n-1}_l(\mathsf{hd}(D)) \\
\mathsf{subfullcube}^{n}_{l,\epsilon,q} & (D, E)\ c & \triangleq & \mathsf{subcube}^{n,0}_{l,\epsilon,q}(D)(E)(\star)(c)
\end{array}
$$

where the extension of subbox_l *to* $p > q$ *is*

$$
\begin{array}{lllll}
\mathsf{subbox}^{n,p,[q<p<n]}_{l,\epsilon,q} & (D : \mathsf{cubset}^{<n}_l)\ (d : \mathsf{box}^{n,p}_l(D)) & : & \mathsf{fullbox}^{n-1}_l(\mathsf{hd}(D)) \\
\mathsf{subbox}^{n,q+1}_{l,\epsilon,q} & D\ (d, b) & \triangleq & \mathsf{border}^{n-1,q}_l(\mathsf{hd}(D))(\mathsf{subbox}^{n,q}_{l,\epsilon,q}(D)(d))(b_\epsilon) \\
\mathsf{subbox}^{n,p,[q<p-1]}_{l,\epsilon,q} & D\ (d, b) & \triangleq & \mathsf{subbox}^{n,p-1}_{l,\epsilon,q}(D)(d)
\end{array}
$$

Figure 14: Standard derived notions (continued)

$$
\begin{array}{llll}
\mathsf{reflexive}^{n,[n\geq 1]}_l & \begin{array}{l}(D : \mathsf{cubset}^{<n}_l) \\ (E : \mathsf{cubset}^{=n}_l(D) \to \mathsf{U}_l)\end{array} & : & \mathsf{U}_l \\
\mathsf{reflexive}^{n}_l & D\ E & \triangleq & \Pi b : \mathsf{fullcube}^{n+1}_l(D).\ E.\mathsf{rel}\,(\mathsf{reflfullcube}^n_l(D)(b)) \\[1ex]
\mathsf{reflfullcube}^{n,[n\geq 1]}_l & \begin{array}{l}(D : \mathsf{cubset}^{<n}_l) \\ (b : \mathsf{fullcube}^{n+1}_l(D))\end{array} & : & \mathsf{fullbox}^n_l(D) \\
\mathsf{reflfullcube}^{n}_l & D\ (d, c) & \triangleq & (\mathsf{reflbox}^{n,n-1}_l(D)(d), \begin{array}{l}(\overrightarrow{\mathsf{cohrefleqbox}^{n,n-1}_L(D)(d)}(c), \\ \overrightarrow{\mathsf{cohrefleqbox}^{n,n-1}_R(D)(d)}(c))\end{array}) \\[1ex]
\mathsf{reflbox}^{n,p,[p<n]}_l & \begin{array}{l}(D : \mathsf{cubset}^{<n}_l) \\ (d : \mathsf{box}^{n-1,p}_l(\mathsf{hd}(D)))\end{array} & : & \mathsf{box}^{n,p}_l(D) \\
\mathsf{reflbox}^{n,0}_l & D\ \star & \triangleq & \star \\
\mathsf{reflbox}^{n,p'+1}_l & D\ (d, b) & \triangleq & (\mathsf{reflbox}^{n,p'}_l(D)(d), \mathsf{refllayer}^{n,p'}_l(D)(d)(b)) \\[1ex]
\mathsf{refllayer}^{n,p,[p+1<n]}_l & \begin{array}{l}(D : \mathsf{cubset}^{<n}_l) \\ (d : \mathsf{box}^{n-1,p}_l(\mathsf{hd}(D))) \\ (b : \mathsf{layer}^{n-1,p}_l(\mathsf{hd}(D))(d)\end{array} & : & \mathsf{layer}^{n,p}_l(D)(\mathsf{reflbox}^{n,p}_l(D)(d)) \\
\mathsf{refllayer}^{n,p}_l & D\ d\ b & \triangleq & \begin{array}{l}(\overrightarrow{\mathsf{cohreflbox}^{n,p,p}_L(D)(d)}(\mathsf{reflcube}^{n-1,p}_l(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n-1,p}_{l,L,p}(\mathsf{hd}(D))(d))(c_L)), \\ \overrightarrow{\mathsf{cohreflbox}^{n,p,p}_R(D)(d)}(\mathsf{reflcube}^{n-1,p}_l(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n-1,p}_{l,R,p}(\mathsf{hd}(D))(d))(b_R)))\end{array} \\[1ex]
\mathsf{reflcube}^{n,p,[p<n]}_l & \begin{array}{l}(D : \mathsf{cubset}^{<n}_l) \\ (E : \mathsf{cubset}^{=n,n}_l(D)) \\ (d : \mathsf{box}^{n-1,p}_l(\mathsf{hd}(D))) \\ (c : \mathsf{cube}^{n-1,p}_l(\mathsf{hd}(D))(\mathsf{tl}(D))(d))\end{array} & : & \mathsf{cube}^{n,p}_l(D)(E)(\mathsf{reflbox}^{n,p}_l(D)(d)) \\
\mathsf{reflcube}^{n,p,[p+1=n]}_l & (D, E)\ d\ c & \triangleq & ((\overrightarrow{\mathsf{cohrefleqbox}^{n,n-1}_L(D)(d)}(c), \overrightarrow{\mathsf{cohrefleqbox}^{n,n-1}_R(D)(d)}(c)), E.\mathsf{refl}(d, c)) \\
\mathsf{reflcube}^{n,p,[p+1<n]}_l & D\ d\ (b, c) & \triangleq & (\mathsf{refllayer}^{n,p}_l(D)(d)(b), \mathsf{reflcube}^{n,p+1}_l(D)(E)(d, b)(c))
\end{array}
$$

Figure 15: Specification of reflexivity in a cubical set (reflexivity properly speaking)

$\mathsf{cohrefleqbox}_{\epsilon}^{n,\,p}{}_{[p\,<\,n]}$	$(D : \mathsf{cubset}_l^{<n})$ $(d : \mathsf{box}_l^{n-1,p}(\mathsf{hd}(D)))$	$:$	$\mathsf{subbox}_{l,\epsilon,p}^{n,n-1}(D)(\mathsf{reflbox}_l^{n,p}(D)(d))$ $\equiv_{ETT}\ d$
$\mathsf{cohrefleqbox}_{\epsilon}^{n,0}$	$D\ \star$	$\triangleq$	$\mathsf{refl}_{ETT}(\star)$
$\mathsf{cohrefleqbox}_{\epsilon}^{n,p'+1}$	$D\ (d,b)$	$\triangleq$	$(\mathsf{cohrefleqbox}_{\epsilon}^{n,p'}(D)(d), \mathsf{cohrefleqlayer}_{\epsilon}^{n,p'}(D)(d)(b))$
$\mathsf{cohrefleqlayer}_{\epsilon}^{n,\,p}{}_{[p\,+\,1\,<\,n]}$	$(D : \mathsf{cubset}_l^{<n})$ $(d : \mathsf{box}_l^{n-1,p}(\mathsf{hd}(D)))$ $(b : \mathsf{layer}_l^{n-1,p}(\mathsf{hd}(D))(d))$	$:$	$\mathsf{sublayer}_{l,\epsilon,p}^{n,n-1}(D)(\mathsf{reflbox}_l^{n,p}(D)(d))(\mathsf{refllayer}_l^{n,p}(D)(d)(b))$ $\equiv_{ETT}\ b$
$\mathsf{cohrefleqlayer}_{\epsilon}^{n,p}$	$D\ d\ b$	$\triangleq$	$(\mathsf{cohrefleqcube}_{\epsilon}^{n-1,p}(D)(\mathsf{subbox}_{l,L,p}^{n,p}(D)(d))(b_L),$ $\mathsf{cohrefleqcube}_{\epsilon}^{n-1,p}(D)(\mathsf{subbox}_{l,R,p}^{n,p}(D)(d))(b_R))$
$\mathsf{cohrefleqcube}_{\epsilon}^{n,\,p}{}_{[p\,<\,n]}$	$(D : \mathsf{cubset}_l^{<n})$ $(E : \mathsf{cubset}_l^{=n}(D))$ $(d : \mathsf{box}_l^{n-1,p}(\mathsf{hd}(D)))$ $(c : \mathsf{cube}_l^{n-1,p}(\mathsf{hd}(D))(\mathsf{tl}(D))(d))$	$:$	$\mathsf{subcube}_{l,\epsilon,p}^{n,n-1}(D)(\mathsf{reflbox}_l^{n,p}(D)(d))(\mathsf{reflcube}_l^{n,p}(D)(E)(d)(c))$ $\equiv_{ETT}\ c$
$\mathsf{cohrefleqcube}_{\epsilon}^{n,p,[p+1=p]}$	$D\ E\ d\ c$	$\triangleq$	$\mathsf{refl}_{ETT}(c)$
$\mathsf{cohrefleqcube}_{\epsilon}^{n,p,[p+1<n]}$	$D\ E\ d\ (b,c)$	$\triangleq$	$(\mathsf{cohrefleqlayer}_{\epsilon}^{n,p}(D)(d)(b),\ \mathsf{cohrefleqcube}_{\epsilon}^{n,p+1}(D)(E)(d,b)(c))$

Figure 16: Specification of reflexivity in a cubical set (cancelling coherences)

$\mathsf{cohreflbox}_{\epsilon}^{n,\,q,\,p}{}_{[p\,\le\,q\,\le\,n]}$	$(D : \mathsf{cubset}_l^{<n})$ $(d : \mathsf{box}_l^{n-1,p}(\mathsf{hd}(D)))$	$:$	$\mathsf{subbox}_{l,\epsilon,q}^{n,p}(D)(\mathsf{reflbox}_l^{n,p}(D)(d))$ $\equiv_{ETT}\ \mathsf{reflbox}_l^{n-1,p}(\mathsf{hd}(D))(\mathsf{subbox}_{l,\epsilon,q-1}^{n-1,p}(\mathsf{hd}(D))(d))$
$\mathsf{cohreflbox}_{\epsilon}^{n,q,0}$	$D\ \star$	$\triangleq$	$\mathsf{refl}_{ETT}(\star)$
$\mathsf{cohreflbox}_{\epsilon}^{n,q,p'+1}$	$D\ (d,b)$	$\triangleq$	$(\mathsf{cohreflbox}_{\epsilon}^{n,q,p'}(D)(d), \mathsf{cohrefllayer}_{\epsilon}^{n,q,p'}(D)(d)(b))$
$\mathsf{cohrefllayer}_{\epsilon}^{n,\,q,\,p}{}_{[p\,<\,q\,\le\,n]}$	$(D : \mathsf{cubset}_l^{<n})$ $(d : \mathsf{box}_l^{n-1,p}(\mathsf{hd}(D)))$ $(b : \mathsf{layer}_l^{n-1,p}(\mathsf{hd}(D))(d))$	$:$	$\mathsf{sublayer}_{l,\epsilon,q}^{n,p}(D)(\mathsf{reflbox}_l^{n,p}(D)(d))(\mathsf{refllayer}_l^{n,p}(D)(d)(b))$ $\equiv_{ETT}\ \mathsf{refllayer}_l^{n-1,p}(\mathsf{hd}(D))(\mathsf{subbox}_{l,\epsilon,q-1}^{n-1,p}(\mathsf{hd}(D))(d))(\mathsf{sublayer}_{l,\epsilon,q}^{n,p}(\mathsf{hd}(D))(d)(b))$
$\mathsf{cohrefllayer}_{\epsilon}^{n,q,p}$	$D\ d\ c$	$\triangleq$	$(\mathsf{cohreflcube}_{\epsilon}^{n-1,q-1,p}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}_{l,L,p}^{n,p}(D)(d))(c_R),$ $\mathsf{cohreflcube}_{\epsilon}^{n-1,q-1,p}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}_{l,R,p}^{n,p}(D)(d))(c_R))$
$\mathsf{cohreflcube}_{\epsilon}^{n,\,q,\,p}{}_{[p\,\le\,q\,\le\,n]}$	$(D : \mathsf{cubset}_l^{<n})$ $(E : \mathsf{cubset}_l^{=n}(D)$ $(d : \mathsf{box}_l^{n-1,p}(\mathsf{hd}(D)))$ $(b : \mathsf{cube}_l^{n-1,p}(\mathsf{hd}(D))(d))$	$:$	$\mathsf{subcube}_{l,\epsilon,q}^{n,p}(D)(\mathsf{reflbox}_l^{n,p}(D)(d))(\mathsf{reflcube}_l^{n,p}(D)(d)(b))$ $\equiv_{ETT}\ \mathsf{reflcube}_l^{n-1,p}(\mathsf{hd}(D))(\mathsf{subbox}_{l,\epsilon,q-1}^{n-1,p}(\mathsf{hd}(D))(d))(\mathsf{subcube}_{l,\epsilon,q-1}^{n-1,p}(\mathsf{hd}(D))(d)(b))$
$\mathsf{cohreflcube}_{\epsilon}^{n,q,p,[p=q]}$	$D\ d\ (b,\_)$	$\triangleq$	$\mathsf{refl}_{ETT}(\mathsf{reflcube}_l^{n-1,p}(\mathsf{hd}(D))(\mathsf{subbox}_{l,\epsilon',q-1}^{n-1,p}(\mathsf{hd}(D))(d))(b_\epsilon))$
$\mathsf{cohreflcube}_{\epsilon}^{n,q,p,[p<q)}$	$D\ d\ (b,c)$	$\triangleq$	$(\mathsf{cohrefllayer}_{\epsilon}^{n,q,p}(D)(d)(b),\ \mathsf{cohreflcube}_{\epsilon}^{n,q,p+1}(D)(E)(d,b)(c))$

Figure 17: Specification of reflexivity in a cubical set (commutation with faces) (TODO: check whether it is normal that we don't need reflexivity in directions other than the top one; point out where UIP is used)

Full (non-truncated) cubical sets with degeneracies

cubset_l		$:$	U_{l+1}
cubset_l		$\triangleq$	$\mathsf{cubset}_l^{\ge 0}(\star)$
$\mathsf{cubset}_l^{\ge n}$	$(D : \mathsf{cubset}_l^{<n})$	$:$	U_{l+1}
$\mathsf{cubset}_l^{\ge n}$	D	$\triangleq$	$\Sigma R : \mathsf{cubset}_l^{=n}(D).\, \mathsf{cubset}_l^{\ge n+1}(D,R)$

Figure 18: Definition of a bare cubical set (coinductive structure)

(we obtain the points by applying the inhabitant $\star$ of unit since this is the pattern we followed). The relation represents the type of cubical sets dependent over the two cubical sets. Such dependent cubical set is the infinite sequence of a relation P of type $T \triangleq A \times B \to \mathsf{U}_l$ (a relation over a heterogeneous 1-box), then of a relation $P_\star$ over P of type $T_\star(P) \triangleq (\Sigma((a_0, a_1), a_\star) : (\Sigma a : (A \times A).A_\star(a))) \times (\Sigma((b_0, b_1), b_\star) : (\Sigma b : (B \times B).B_\star(b))) \times (P(a_0, b_0) \times P(a_1, b_1)) \to \mathsf{U}_l$ (a relation over an heterogeneous 2-box), etc. All in all, the segment in the universe is thus the relation $\lambda(\mathcal{A}, \mathcal{B}).(\Sigma P : T_0.\Sigma P_\star : T_\star(P)....)$.

We see in particular that the types involved in the definition of the boxes come from 2 kinds of sources: the P, $P_\star$, ... come from the type of dependent cubical set being defined while the A, $A_\star$ and the B, $B_\star$ come from each individual cubical set given as arguments. The recursive process used to define the interpretation of U_l as a n-cube is given on Figures 19, 20, 21, 22, 23, 24 and 25. In particular, the construction of the box involved in the construction of the type of the p relation of the cubical set dependent over a n-box of cubical sets is in three steps:

- the higher p layers of the resulting $(n+p)$-box exclusively involve the p first relations P, $P_\star$, ...
- the lower n layers follow the structure of the n layers of the n-box of cubical sets,
- each q-cube (for $q \leq n$) involved in these layers is a cube built on the p first dimensions of the corresponding, q-dimensional component of the n-box.

Note the following properties:

Proposition 1 $\mathsf{depcubset}^0_l(\star)$ *coincides with* cubset_l.

Proposition 2 $\mathsf{UnivfullBox}^n_l : \mathsf{fullhetbox}^n_l(\mathsf{UnivfullBox}^n_{l+1})$

Proposition 3 *The cubical set obtained by taking the sequence of* $\mathsf{UnivFiller}^n_l$ *is thus of type* $\mathsf{UnivFiller}^0_{l+1}(\star)$*, i.e. of type* $\mathsf{fullhetbox}^0_{l+1}(\mathsf{UnivfullCube}^0_l)$*, thus implementing the typing rule* $\mathsf{U}_l : \mathsf{U}_{l+1}$.

Conversely, $\mathsf{UnivfullCube}^n_l$ is the expansion of the former cubical set, so we have: n-expansion(l-universe-cubical-set) : $\mathsf{fullhetbox}^n_{l+1}(n - expansion(l + 1 - universe - cubical - set)$

4.6 Dependent product of cubical sets

5 Univalent iterated internal parametricity

A Dependent cubical sets over homogeneous cubical sets

A.1 Dependent sums of cubical sets

See Figure 31, 32 and 33.

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n-cube associated to the universe U_l

$$
\begin{array}{lcl}
\mathsf{UnivfullCube}_l^n & : & \mathsf{fullCube}_{l+1}^n \\
\mathsf{UnivfullCube}_l^n & \triangleq & (\mathsf{UnivfullBox}_l^n, \mathsf{UnivFiller}_l^n) \\
\\
\mathsf{UnivfullBox}_l^n & : & \mathsf{fullBox}_{l+1}^n \\
\mathsf{UnivfullBox}_l^n & \triangleq & \mathsf{UnivBox}_l^{n,n} \\
\\
\mathsf{UnivFiller}_l^n & : & \mathsf{Filler}_{l+1}^n \\
\mathsf{UnivFiller}_l^n & \triangleq & \lambda D : \mathsf{fullhetbox}_l^n(\mathsf{UnivfullBox}_l^n).\, \mathsf{depcubset}_l^n(D)
\end{array}
$$

---

incremental construction of the n-box associated to universe U_l

$$
\begin{array}{lcl}
\mathsf{UnivBox}_l^{n,p} & : & \mathsf{Box}_{l+1}^{n,p} \\
\mathsf{UnivBox}_l^{n,0} & \triangleq & \star \\
\mathsf{UnivBox}_l^{n,p'+1} & \triangleq & (\mathsf{UnivBox}_l^{n,p'}, \mathsf{UnivLayer}_l^{n,p'}) \\
\\
\mathsf{UnivLayer}_l^{n,p,[p<n]} & : & \mathsf{Layer}_{l+1}^{n,p}(\mathsf{UnivBox}_l^{n,p}) \\
\mathsf{UnivLayer}_l^{n,p} & \triangleq & (\mathsf{UnivCube}_l^{n-1,p}, \mathsf{UnivCube}_l^{n-1,p}) \\
\\
\mathsf{UnivCube}_l^{n,p,[p\leq n]} & : & \mathsf{Cube}_{l+1}^{n,p}(\mathsf{UnivBox}_l^{n,p}) \\
\mathsf{UnivCube}_l^{n,p,[p=n]} & \triangleq & \mathsf{UnivFiller}_l^n \\
\mathsf{UnivCube}_l^{n,p,[p<n]} & \triangleq & (\mathsf{UnivLayer}_l^{n,p}, \mathsf{UnivCube}_l^{n,p+1})
\end{array}
$$

Figure 19: The n-cube associated to the universe l

Full (non-truncated) cubical sets over an n-box of cubical sets

$$\begin{array}{lll} \mathsf{depcubset}^n_l & (D : \mathsf{fullhetbox}^n_l(\mathsf{UnivfullBox}^n_l)) & : \mathsf{U}_{l+1} \\ \mathsf{depcubset}^n_l & D & \triangleq \mathsf{depcubset}^{n,\geq 0}_l(D)(\star) \end{array}$$

Cubical set p-suffix over an n-box of cubical sets

$$\begin{array}{lll} \mathsf{depcubset}^{n,\geq p}_l & \begin{array}{l}(D : \mathsf{fullhetbox}^n_l(\mathsf{UnivfullBox}^n_l)) \\ (P : \mathsf{depcubset}^{n,<p}_l(D))\end{array} & : \mathsf{U}_{l+1} \\ \mathsf{depcubset}^{n,\geq p}_l & D\ P & \triangleq \Sigma R : \mathsf{depcubset}^{n,=p}_l(D)(P).\, \mathsf{depcubset}^{n,\geq p+1}_l(D)(P,R) \end{array}$$

p-truncated cubical sets over an n-box of cubical sets

$$\begin{array}{lll} \mathsf{depcubset}^{n,<p}_l & (D : \mathsf{fullhetbox}^n_l(\mathsf{UnivfullBox}^n_l)) & : \mathsf{U}_{l+1} \\ \mathsf{depcubset}^{n,<0}_l & D & \triangleq \mathsf{unit} \\ \mathsf{depcubset}^{n,<p'+1}_l & D & \triangleq \Sigma P : \mathsf{depcubset}^{n,<p'}_l(D).\, \mathsf{depcubset}^{n,=p'}_l(D)(P) \end{array}$$

Dependent structure carried at each dimension

$$\begin{array}{lll} \mathsf{depcubset}^{n,=p}_l & \begin{array}{l}(D : \mathsf{fullhetbox}^n_l(\mathsf{UnivfullBox}^n_l)) \\ (P : \mathsf{depcubset}^{n,<p}_l(D))\end{array} & : \mathsf{U}_{l+1} \\ \mathsf{depcubset}^{n,=p}_l & D\ P & \triangleq \mathsf{fulldepBox}^{n,p}_l(D)(P) \to \mathsf{U}_l \end{array}$$

Figure 20: Dependent cube set over a n-box of cubical sets

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Full n-boxes dependent over a n-type and a p-cubical set

$$\begin{array}{llcl}
\mathsf{fulldepBox}_l^{n,p} & \begin{array}{l}(D:\mathsf{fullhetbox}_l^n(\mathsf{UnivfullBox}_l^n)) \\ (P:\mathsf{depcubset}_l^{n,<p}(D))\end{array} & : & \mathsf{U}_l \\
\mathsf{fulldepBox}_l^{n,p} & D\ P & \triangleq & \mathsf{depBox}_l^{n,p,p}(D)(P)
\end{array}$$

q-partial n-boxes dependent over a n-type and a p-cubical set

$$\begin{array}{llcl}
\mathsf{depBox}_l^{n,p,q,[q\leq p]} & \begin{array}{l}(D:\mathsf{fullhetbox}_l^n(\mathsf{UnivfullBox}_l^n)) \\ (P:\mathsf{depcubset}_l^{n,<p}(D))\end{array} & : & \mathsf{U}_l \\
\mathsf{depBox}_l^{n,p,0} & D\ P & \triangleq & \mathsf{liftfullBox}_l^{n,p}(D) \\
\mathsf{depBox}_l^{n,p,q'+1} & D\ P & \triangleq & \Sigma d : \mathsf{depBox}_l^{n,p,q'}(D)(P).\, \mathsf{depLayer}_l^{n,p,q'}(D)(P)(d) \\
 & & & \\
\mathsf{depLayer}_l^{n,p,q,[q<p]} & \begin{array}{l}(D:\mathsf{fullhetbox}_l^n(\mathsf{UnivfullBox}_l^n)) \\ (P:\mathsf{depcubset}_l^{n,<p}(D)) \\ (d:\mathsf{depBox}_l^{n,p,q}(D)(P))\end{array} & : & \mathsf{U}_l \\
\mathsf{depLayer}_l^{n,p,q} & D\ P\ d & \triangleq & \begin{array}{l}\mathsf{depCube}_l^{n,p-1,q}(D)(\mathsf{hd}(P))(\mathsf{tl}(P))(\mathsf{depsubbox}_{l,L,q}^{n,p,q}(D)(P)(d)) \\ \times\ \mathsf{depCube}_l^{n,p-1,q}(D)(\mathsf{hd}(P))(\mathsf{tl}(P))(\mathsf{depsubbox}_{l,R,q}^{n,p,q}(D)(P)(d))\end{array} \\
 & & & \\
\mathsf{depCube}_l^{n,p,q,[q\leq p]} & \begin{array}{l}(D:\mathsf{fullhetbox}_l^n(\mathsf{UnivfullBox}_l^n)) \\ (P:\mathsf{depcubset}_l^{n,<p}(D)) \\ (C:\mathsf{depcubset}_l^{n,=p}(D)) \\ (d:\mathsf{depBox}_l^{n,p,q}(D)(P))\end{array} & : & \mathsf{U}_l \\
\mathsf{depCube}_l^{n,p,q,[q=p]} & D\ P\ C\ d & \triangleq & C(d) \\
\mathsf{depCube}_l^{n,p,q,[q<p]} & D\ P\ C\ d & \triangleq & \Sigma b : \mathsf{depLayer}_l^{n,p,q}(D)(P)(d).\, \mathsf{depCube}_l^{n,p,q+1}(D)(P)(C)(d,b)
\end{array}$$

Figure 21: $(n+p)$-box involved in dimension p of a cubical set dependent over an n-box of cubical sets

p-lifting of a n-box dependent over a n-type

$\mathsf{liftfullBox}_l^{n,p}$	$(D : \mathsf{fullhetbox}_l^n(\mathsf{UnivfullBox}_l^n))$	$:$	U_l
$\mathsf{liftfullBox}_l^{n,p}$	D	$\triangleq$	$\mathsf{liftBox}_l^{n,p,n}(D)$
$\mathsf{liftBox}_l^{n,p,r,[r\leq n]}$	$(D : \mathsf{hetbox}_l^{n,r}(\mathsf{UnivBox}_l^{n,r}))$	$:$	U_l
$\mathsf{liftBox}_l^{n,p,0}$	$\star$	$\triangleq$	unit
$\mathsf{liftBox}_l^{n,p,r'+1}$	(D, B)	$\triangleq$	$\Sigma d : \mathsf{liftBox}_l^{n,p,r'}(D).\, \mathsf{liftLayer}_l^{n,p,r'}(D)(B)(d)$
$\mathsf{liftLayer}_l^{n,p,r,[r<n]}$	$(D : \mathsf{hetbox}_l^{n,r}(\mathsf{UnivBox}_l^{n,r}))$ $(B : \mathsf{hetlayer}_l^{n,r}\begin{pmatrix}\mathsf{UnivBox}_l^{n,r}\\ \mathsf{UnivLayer}_l^{n,r}\end{pmatrix}(D))$ $(d : \mathsf{liftBox}_l^{n,p,r}(D))$	$:$	U_l
$\mathsf{liftLayer}_l^{n,p,r}$	$D\ B\ d$	$\triangleq$	$\mathsf{liftCube}_l^{n-1,p,r}(\mathsf{subhetbox}_{l,L,r}^{n,r}(D))(B_L)(\mathsf{depsubliftbox}_{l,L,r}^{n,p,r}(D)(d))$ $\times\ \mathsf{liftCube}_l^{n-1,p,r}(\mathsf{subhetbox}_{l,R,r}^{n,r}(D))(B_R)(\mathsf{depsubliftbox}_{l,R,r}^{n,p,r}(D)(d))$
$\mathsf{liftCube}_l^{n,p,r,[r\leq n]}$	$(D : \mathsf{hetbox}_l^{n,r}(\mathsf{UnivBox}_l^{n,r}))$ $(C : \mathsf{hetcube}_l^{n,r}\begin{pmatrix}\mathsf{UnivBox}_l^{n,r}\\ \mathsf{UnivCube}_l^{n,r}\end{pmatrix}(D))$ $(d : \mathsf{liftBox}_l^{n,p,r}(D))$	$:$	U_l
$\mathsf{liftCube}_l^{n,p,r,[r=n]}$	$D\ C\ d$	$\triangleq$	$\mathsf{expandCube}_l^{n,p,0}(D)(\star)(C(D))(d)$
$\mathsf{liftCube}_l^{n,p,r,[r<n]}$	$D\ (B, C)\ d$	$\triangleq$	$\Sigma b : \mathsf{liftLayer}_l^{n,p,r}(D)(d).\, \mathsf{liftCube}_l^{n,p,r+1}(D, B)(C)(d, b)$

Figure 22: Extraction of the p first dimensions of a n-box of cubical sets

p-expansion of a q-dimensional relation over cubical sets

$\mathsf{expandCube}_l^{n,p,q,[q\leq p]}$	$(D : \mathsf{fullhetbox}_l^n(\mathsf{UnivfullBox}_l^n))$ $(P : \mathsf{depcubset}_l^{n,<q}(D))$ $(S : \mathsf{depcubset}_l^{n,\geq q}(D)(P))$ $(d : \mathsf{depBox}_l^{n,p,q}(D)(P))$	$:$	U_l
$\mathsf{expandCube}_l^{n,p,q,[q=p]}$	$D\ P\ (R, \_)$	$\triangleq$	$R(d)$
$\mathsf{expandCube}_l^{n,p,q,[q<p]}$	$D\ P\ (R, S)$	$\triangleq$	$\Sigma b : \mathsf{expandLayer}_l^{n,p,q}(D)(d).\, \mathsf{expandCube}_l^{n,p,q+1}(D)(P, R)(S)(d, b)$
$\mathsf{expandLayer}_l^{n,p,q,[q<p]}$	$(D : \mathsf{fullhetbox}_l^n(\mathsf{UnivfullBox}_l^n))$ $(P : \mathsf{depcubset}_l^{n,<q}(D))$ $(S : \mathsf{depcubset}_l^{n,\geq q}(D)(P))$ $(d : \mathsf{depBox}_l^{n,p,q}(D)(P))$	$:$	U_l
$\mathsf{expandLayer}_l^{n,p,q}$	$D\ P\ S\ d$	$\triangleq$	$\mathsf{expandCube}_l^{n,p-1,q}(D)(P)(S)(\mathsf{depsubbox}_{l,L,q}^{n,p,q}(D)(P)(d))$ $\times\ \mathsf{expandCube}_l^{n,p-1,q}(D)(P)(S)(\mathsf{depsubbox}_{l,R,q}^{n,p,q}(D)(P)(d))$

Figure 23: Build a p-cube out of the p first dimensions of a n-box of cubical sets

q-Projection of the s-prefix of the inner part of dimension p of a box dependent on a n-box of cubical sets in its inner part

$$
\begin{array}{lllll}
\mathsf{epsubbox}_{l,\epsilon,q}^{n,p,s,[s\leq q<p]} & \begin{array}{l}(D : \mathsf{fullhetbox}_l^n(\mathsf{UnivfullBox}_l^n)) \\ (P : \mathsf{depcubset}_l^{n,<p}(D)) \\ (d : \mathsf{depBox}_l^{n,p,s}(D)(P))\end{array} & : & \mathsf{depBox}_l^{n,p-1,s}(D)(P) \\
\mathsf{epsubbox}_{l,\epsilon,q}^{n,p,0} & D\ P\ d & \triangleq & \mathsf{depsubliftbox'}_{l,\epsilon,q}^{n,p,n]}(D)(d) \\
\mathsf{epsubbox}_{l,\epsilon,q}^{n,p,s'+1} & D\ P\ (d,b) & \triangleq & (\mathsf{depsubbox}_{l,\epsilon,q}^{n,p,s'}(D)(P)(d), \mathsf{depsublayer}_{l,\epsilon,q}^{n,p,s'}(D)(P)(d)(b)) \\
\mathsf{epsublayer}_{l,\epsilon,q}^{n}p,s,[s<r<n] & \begin{array}{l}(D : \mathsf{fullhetbox}_l^n(\mathsf{UnivfullBox}_l^n)) \\ (P : \mathsf{depcubset}_l^{n,<p}(D)) \\ (d : \mathsf{depBox}_l^{n,p,s}(D)(P)) \\ (b : \mathsf{depLayer}_l^{n,p,s}(D)(P)(d))\end{array} & : & \mathsf{depLayer}_l^{n,p-1,s}(D)(P)(\mathsf{depsubbox}_{l,\epsilon,q}^{n,p,s}(d)) \\
\mathsf{epsublayer}_{l,\epsilon,q}^{n,p,s} & D\ P\ d\ b & \triangleq & \begin{array}{l}\ldots\ (\overrightarrow{\mathsf{depcohbox}_{l,\epsilon,L,q,p}^{n,p}(D)}(\mathsf{depsubcube}_{l,\epsilon,q}^{n,p-1,s}(\mathsf{depsubbox}_{l,L,p}^{n,p}(D))(C_L)), \\ \quad \overrightarrow{\mathsf{depcohbox}_{l,\epsilon,R,q,p}^{n,p}(D)}(\mathsf{depsubcube}_{l,\epsilon,q-1}^{n-1,p}(\mathsf{depsubbox}_{l,R,p}^{n,p}(D))(C_R)))\end{array} \\
\mathsf{epsubcube}_{l,\epsilon,q}^{n}p,[p\leq q<n] & \begin{array}{l}(D : \mathsf{hetbox}_l^{n,p}) \\ (C : \mathsf{hetcube}_l^{n,p}(D))\end{array} & : & \mathsf{hetcube}_l^{n-1,p}(\mathsf{depsubbox}_{l,\epsilon,q}^{n}p(D)) \\
\mathsf{epsubcube}_{l,\epsilon,q}^{n}p,[p=q] & D\ (B,\_) & \triangleq & B_\epsilon \\
\mathsf{epsubcube}_{l,\epsilon,q}^{n}p,[p<q] & D\ (B,C) & \triangleq & (\mathsf{depsublayer}_{l,\epsilon,q}^{n,p}(D)(B), \mathsf{depsubcube}_{l,\epsilon,q}^{n,p+1}(D,B)(C))
\end{array}
$$

q-Projection of the s-prefix of the outer part of dimension n of a box dependent on a n-box of cubical sets in its inner part of dimension p

$$
\begin{array}{llll}
\mathsf{epsubliftbox'}_{l,\epsilon,q}^{n,p,s,[s\leq n]} & \begin{array}{l}(D : \mathsf{hetbox}_l^{n,s}(\mathsf{UnivBox}_l^{n,s})) \\ (d : \mathsf{liftBox}_l^{n,p,s}(D))\end{array} & : & \mathsf{liftBox}_l^{n,p-1,s}(D) \\
\mathsf{epsubliftbox'}_{l,\epsilon,q}^{n,p,0} & D\ \star & \triangleq & \star \\
\mathsf{epsubliftbox'}_{l,\epsilon,q}^{n,p,s'+1} & (D,B)\ (d,b) & \triangleq & (\mathsf{depsubliftbox'}_{l,\epsilon,q}^{n,p,s'}(D)(d), \mathsf{depsubliftlayer'}_{l,\epsilon,q}^{n,p,s'}(D)(B)(d)(b)) \\
\mathsf{epsubliftlayer'}_{l,\epsilon,r}^{n,p,s,[s<r<n]} & \begin{array}{l}(D : \mathsf{hetbox}_l^{n,s}(\mathsf{UnivBox}_l^{n,s})) \\ (B : \mathsf{hetlayer}_l^{n,s}\begin{array}{l}(\mathsf{UnivBox}_l^{n,s}) \\ (\mathsf{UnivLayer}_l^{n,s})\end{array}(D)) \\ (d : \mathsf{liftBox}_l^{n,p,s}(D)) \\ (b : \mathsf{liftLayer}_l^{n,p,s}(D)(B)(d))\end{array} & : & \mathsf{liftLayer}_l^{n,p-1,s}(D)(B)(\mathsf{depsubliftbox'}_{l,\epsilon,q}^{n,p,s}(D)(d)) \\
\mathsf{epsubliftlayer'}_{l,\epsilon,r}^{n,p,s} & D\ B\ d\ b & \triangleq & \left(\begin{array}{l}\overrightarrow{\mathsf{depcohliftbox'}_{l,\epsilon,L,r,s}^{n,p,t}(D)}\left(\begin{array}{l}\mathsf{depsubliftcube'}_{l,\epsilon,r-1}^{n-1,p,s} \\ \quad(\mathsf{subhetbox}_{l,L,r}^{n,p,s}(\mathsf{UnivBox}_l^{n,s})(D)) \\ B_L \\ \mathsf{depsubliftbox'}_{l,L,r}^{n,p,s}(D)(d) \\ b_L\end{array}\right), \\ \overrightarrow{\mathsf{depcohliftbox'}_{l,\epsilon,R,r,s}^{n,p,t}(D)}\left(\begin{array}{l}\mathsf{depsubliftcube'}_{l,\epsilon,r-1}^{n-1,p,s} \\ \quad(\mathsf{subhetbox}_{l,R,r}^{n,p,s}(\mathsf{UnivBox}_l^{n,s})(D)) \\ B_R \\ \mathsf{depsubliftbox'}_{l,R,r}^{n,p,s}(D)(d) \\ b_R\end{array}\right)\end{array}\right)
\end{array}
$$

Figure 24: q-Projection of the s-prefix of the inner part of dimension p of a box dependent on a n-box of cubical sets in its inner part

$$\cdots$$

$$\mathsf{epsubliftbox}^{n,p,s,[s\leq r<n]}_{l,\epsilon,r} \quad \begin{array}{l}(D : \mathsf{hetbox}^{n,s}_{l}(\mathsf{UnivBox}^{n,s}_{l}))\\ (d : \mathsf{liftBox}^{n,p,s}_{l}(D))\end{array} \quad : \quad \mathsf{liftBox}^{n-1,p,s}_{l}(\mathsf{subhetbox}^{n,s}_{l,\epsilon,r}(\mathsf{UnivBox}^{n,s}_{l})(D))$$

$$\mathsf{epsubliftbox}^{n,p,0}_{l,\epsilon,r} \quad D\ \star \quad \triangleq \quad \star$$

$$\mathsf{epsubliftbox}^{n,p,s'+1}_{l,\epsilon,r} \quad (D,B)\ (d,b) \quad \triangleq \quad (\mathsf{depsubliftbox}^{n,p,s'}_{l,\epsilon,r}(D)(d), \mathsf{depsubliftlayer}^{n,p,s'}_{l,\epsilon,r}(D)(B)(d)(b))$$

$$\mathsf{epsubliftlayer}^{n,p,s,[s<r<n]}_{l,\epsilon,r} \quad \begin{array}{l}(D : \mathsf{hetbox}^{n,s}_{l}(\mathsf{UnivBox}^{n,s}_{l}))\\ (B : \mathsf{hetlayer}^{n,s}_{l}\begin{array}{l}(\mathsf{UnivBox}^{n,s}_{l})\\(\mathsf{UnivLayer}^{n,s}_{l})\end{array}(D))\\ (d : \mathsf{liftBox}^{n,p,s}_{l}(D))\\ (b : \mathsf{liftLayer}^{n,p,s}_{l}(D)(B)(d))\end{array} \quad : \quad \mathsf{liftLayer}^{n-1,p,s}_{l}(\mathsf{subhetbox}^{n,p,s}_{l,\epsilon,r}(\mathsf{UnivBox}^{n,s}_{l})(D))(B_\epsilon)(\mathsf{depsubliftbox}^{n,p,s}_{l,\epsilon,r}(D)(d$$

$$\mathsf{epsubliftlayer}^{n,p,s}_{l,\epsilon,r} \quad D\ B\ d\ b \quad \triangleq \quad \left(\begin{array}{l} \overrightarrow{\mathsf{depcohliftbox}^{n,p,t}_{l,\epsilon,L,r,s}(D)}\begin{pmatrix}\mathsf{depsubliftcube}^{n-1,p,s}_{l,\epsilon,r-1}\\ \quad(\mathsf{subhetbox}^{n,p,s}_{l,L,r}(\mathsf{UnivBox}^{n,s}_{l})(D))\\ B_L\\ \mathsf{depsubliftbox}^{n,p,s}_{l,L,r}(D)(d)\\ b_L\end{pmatrix},\\ \overrightarrow{\mathsf{depcohliftbox}^{n,p,t}_{l,\epsilon,R,r,s}(D)}\begin{pmatrix}\mathsf{depsubliftcube}^{n-1,p,s}_{l,\epsilon,r-1}\\ \quad(\mathsf{subhetbox}^{n,p,s}_{l,R,r}(\mathsf{UnivBox}^{n,s}_{l})(D))\\ B_R\\ \mathsf{depsubliftbox}^{n,p,s}_{l,R,r}(D)(d)\\ b_R\end{pmatrix}\end{array}\right)$$

$$\mathsf{epsubliftcube}^{n,p,s,[s\leq r<n]}_{l,\epsilon,r} \quad \begin{array}{l}(D : \mathsf{hetbox}^{n,s}_{l}(\mathsf{UnivBox}^{n,s}_{l}))\\ (C : \mathsf{hetcube}^{n,s}_{l}\begin{array}{l}(\mathsf{UnivBox}^{n,s}_{l})\\(\mathsf{UnivCube}^{n,s}_{l})\end{array}(D))\\ (d : \mathsf{liftBox}^{n,p,s}_{l}(D))\\ (c : \mathsf{liftCube}^{n,p,s}_{l}(D)(B)(d))\end{array} \quad : \quad \mathsf{liftCube}^{n-1,p,s}_{l}(\mathsf{subhetbox}^{n,p,s}_{l,\epsilon,r}(D))(B_\epsilon)(\mathsf{depsubliftbox}^{n,p,s}_{l,\epsilon,r}(D)(d))$$

$$\mathsf{epsubliftcube}^{n,p,s,[s=r]}_{l,\epsilon,r} \quad D\ (B,\_) \quad \triangleq \quad B_\epsilon$$

$$\mathsf{epsubliftcube}^{n,p,s,[s<r]}_{l,\epsilon,r} \quad D\ (B,C) \quad \triangleq \quad (\mathsf{depsubliftlayer}^{n,p}_{l,\epsilon,q}(D)(B), \mathsf{depsubliftcube}^{n,q,p+1}_{l,\epsilon,q}(D,B)(C))$$

Figure 25: r-Projection of the s-prefix of the outer p-lifted part of a box dependent on a n-box of cubical sets along in its outer part

$$\boxed{\mathsf{depcohliftbox}^{n,p,t}_{l,\epsilon,\epsilon',q,s} \quad : \quad \ldots}$$

Figure 26: Definition of a cube of higher-order relations (compatibility of faces)

$$
\begin{array}{llcl}
\mathsf{appfullcube}_l^n & \begin{array}{l}(D : \mathsf{fullCube}_l^n)\\ (P : \mathsf{depcubset}_l^{n,<}(D))\\ (d : \mathsf{fullcube}_l^n(D))\end{array} & : & \mathsf{fullCube}_l^n \\
\mathsf{appfullcube}_l^n & (D, E)\ (P, R)\ (d, b) & \triangleq & (\mathsf{appfullbox}_l^{n,q}(D)(P)(d), R\,d\,b) \\
\\
\mathsf{appfullbox}_l^{n,q} & \begin{array}{l}(D : \mathsf{fullCube}_l^n)\\ (P : \mathsf{depcubset}_l^{n,<}(D))\\ (d : \mathsf{fullbox}_l^n(D))\end{array} & : & \mathsf{fullBox}_l^n \\
\mathsf{appfullbox}_l^{n,q} & D\ P\ d & \triangleq & \mathsf{appbox}_l^{n,n}(D)(P)(d) \\
\\
\mathsf{appbox}_l^{n,p,[p\leq n]} & \begin{array}{l}(D : \mathsf{fullCube}_l^n)\\ (P : \mathsf{depcubset}_l^{n,<}(D))\\ (d : \mathsf{box}_l^{n,p}(D))\end{array} & : & \mathsf{Box}_l^{n,p} \\
\mathsf{appbox}_l^{n,0} & D\ \star\ \star & \triangleq & \star \\
\mathsf{appbox}_l^{n,p'+1} & D\ P\ (d, b) & \triangleq & (\mathsf{appbox}_l^{n,p'}(D)(P)(d), \mathsf{applayer}_l^{n,p'}(D)(P)(d)(b)) \\
\\
\mathsf{applayer}_l^{n,p,[p\leq n]} & \begin{array}{l}(D : \mathsf{fullCube}_l^n)\\ (P : \mathsf{depcubset}_l^{n,<}(D))\\ (d : \mathsf{box}_l^{n,p}(D))\\ (b : \mathsf{layer}_l^{n,p}(D)(d))\end{array} & : & \mathsf{Layer}_l^{n,p}(\mathsf{appbox}_l^{n,p'}(D)(P)(d)) \\
\mathsf{applayer}_l^{n,p} & D\ P\ d\ c & \triangleq & \begin{array}{l}(\mathsf{appcube}_l^{n-1,p}\begin{array}{l}(\mathsf{hd}(D))\\ (\mathsf{hd}(P))\\ (\mathsf{tl}(P))\\ (\mathsf{subbox}_{l,L,n-1}^{n,n-1}(d))\end{array}(c_L),\\ \ \mathsf{appcube}_l^{n-1,p}\begin{array}{l}(\mathsf{hd}(D))\\ (\mathsf{hd}(P))\\ (\mathsf{tl}(P))\\ (\mathsf{subbox}_{l,R,n-1}^{n,n-1}(d))\end{array}(c_R))\end{array}
\end{array}
$$

$$+ \text{ proofs of commutations } \mathsf{subbox}_{l,\epsilon,q}/\mathsf{appbox}$$

$$
\begin{array}{llcl}
\mathsf{appcube}_l^{n,p,[p\leq n]} & \begin{array}{l}(D : \mathsf{fullCube}_l^n)\\ (P : \mathsf{depcubset}_l^{n,<}(D))\\ (R : \mathsf{depcubset}_l^{n,=}(D)(P))\\ (d : \mathsf{box}_l^{n,p}(D))\\ (c : \mathsf{cube}_l^{n,p}(D)(d))\end{array} & : & \mathsf{Cube}_l^{n,p}(\mathsf{appbox}_l^{n,p'}(D)(P)(d)) \\
\mathsf{appcube}_l^{n,p,[p=n]} & D\ P\ R\ d\ c & \triangleq & (R\,d\,c) \\
\mathsf{appcube}_l^{n,p,[p<n]} & D\ P\ R\ d\ (b, c) & \triangleq & (\mathsf{applayer}_l^{n,p}(D)(P)(d)(b), \mathsf{appcube}_l^{n,p+1}(D)(P)(R)(d)(c))
\end{array}
$$

Figure 27: Definition of a dependent truncated cubical set (application)

Type of dependent truncated cubical sets with degeneracies

$$
\begin{array}{lll}
\mathsf{phomocubset}_l^{<n} \quad (D : \mathsf{cubset}_l^{<n}) & : & \mathsf{U}_{l+1} \\
\mathsf{phomocubset}_l^{<0} \quad D & \triangleq & \mathsf{unit} \\
\mathsf{phomocubset}_l^{<n'+1} \quad (D, E) & \triangleq & \Sigma P : \mathsf{dephomocubset}_l^{<n'}(D).\, \mathsf{dephomocubset}_l^{=n}(D)(E)(P)
\end{array}
$$

Dependent structure carried at each dimension

$$
\begin{array}{llll}
 & (D : \mathsf{cubset}_l^{<n}) & & \\
\mathsf{phomocubset}_l^{=n} & (E : \mathsf{cubset}_l^{=n}(D)) & : & \mathsf{U}_l \\
 & (P : \mathsf{dephomocubset}_l^{<n}(D)) & & \\
\mathsf{phomocubset}_l^{=n} & D\ E\ P & \triangleq & \Pi d : \mathsf{fullbox}_l^n(D).\Pi c : \mathsf{properfiller}_l^n(D, E)(d).\mathsf{fullhetbox}_l^n(\mathsf{homoappfullbox}_l^n(D)(P)(d)) \rightarrow
\end{array}
$$

Figure 28: Definition of a dependent cube of higher-order relations

$$
\begin{array}{llcl}
\mathsf{homoappfullcube}_l^n & \begin{array}{l}(D : \mathsf{cubset}_l^{<n}) \\ (P : \mathsf{dephomocubset}_l^{<n}(D)) \\ (d : \mathsf{fullcube}_l^n(D))\end{array} & : & \mathsf{fullCube}_l^n \\
\mathsf{homoappfullcube}_l^n & (D, E)\ (P, R)\ (d, b) & \triangleq & (\mathsf{homoappfullbox}_l^n(D)(P)(d), R\,d\,b) \\
\\
\mathsf{homoappfullbox}_l^n & \begin{array}{l}(D : \mathsf{cubset}_l^{<n}) \\ (P : \mathsf{dephomocubset}_l^{<n}(D)) \\ (d : \mathsf{fullbox}_l^n(D))\end{array} & : & \mathsf{fullBox}_l^n \\
\mathsf{homoappfullbox}_l^n & D\ P\ d & \triangleq & \mathsf{homoappbox}_l^{n,n}(D)(P)(d) \\
\\
\mathsf{homoappbox}_l^{n,p,[p \le n]} & \begin{array}{l}(D : \mathsf{cubset}_l^{<n}) \\ (P : \mathsf{dephomocubset}_l^{<n}(D)) \\ (d : \mathsf{box}_l^{n,p}(D))\end{array} & : & \mathsf{Box}_l^{n,p} \\
\mathsf{homoappbox}_l^{n,0} & D\ \star\ \star & \triangleq & \star \\
\mathsf{homoappbox}_l^{n,p'+1} & D\ P\ (d, b) & \triangleq & (\mathsf{homoappbox}_l^{n,p'}(D)(P)(d), \mathsf{homoapplayer}_l^{n,p'}(D)(P)(d)(b)) \\
\\
\mathsf{homoapplayer}_l^{n,p,[p \le n]} & \begin{array}{l}(D : \mathsf{cubset}_l^{<n}) \\ (P : \mathsf{dephomocubset}_l^{<n}(D)) \\ (d : \mathsf{box}_l^{n,p}(D)) \\ (b : \mathsf{layer}_l^{n,p}(D)(d))\end{array} & : & \mathsf{Layer}_l^{n,p}(\mathsf{homoappbox}_l^{n,p'}(D)(P)(d)) \\
\mathsf{homoapplayer}_l^{n,p} & D\ P\ d\ c & \triangleq & \begin{array}{l}(\mathsf{homoappcube}_l^{n-1,p} \begin{array}{l}(\mathsf{hd}(D)) \\ (\mathsf{hd}(P)) \\ (\mathsf{tl}(P)) \\ (\mathsf{subbox}_{l,L,n-1}^{n,n-1}(d))\end{array} (c_L), \\ \ \mathsf{homoappcube}_l^{n-1,p} \begin{array}{l}(\mathsf{hd}(D)) \\ (\mathsf{hd}(P)) \\ (\mathsf{tl}(P)) \\ (\mathsf{subbox}_{l,R,n-1}^{n,n-1}(d))\end{array} (c_R))\end{array}
\end{array}
$$

$$+ \text{ proofs of commutations } \mathsf{subbox}_{l,\epsilon,q}/\mathsf{homoappbox}$$

$$
\begin{array}{llcl}
\mathsf{homoappcube}_l^{n,p,[p \le n]} & \begin{array}{l}(D : \mathsf{cubset}_l^{<n}) \\ (P : \mathsf{dephomocubset}_l^{<n}(D)) \\ (R : \mathsf{dephomocubset}_l^{=n}(D)(P)) \\ (d : \mathsf{box}_l^{n,p}(D)) \\ (c : \mathsf{cube}_l^{n,p}(D)(d))\end{array} & : & \mathsf{Cube}_l^{n,p}(\mathsf{homoappbox}_l^{n,p'}(D)(P)(d)) \\
\mathsf{homoappcube}_l^{n,p,[p=n]} & D\ P\ R\ d\ c & \triangleq & (R\,d\,c) \\
\mathsf{homoappcube}_l^{n,p,[p<n]} & D\ P\ R\ d\ (b, c) & \triangleq & (\mathsf{homoapplayer}_l^{n,p}(D)(P)(d)(b), \mathsf{homoappcube}_l^{n,p+1}(D)(P)(R)(d)(c))
\end{array}
$$

Figure 29: Definition of a dependent truncated cubical set (application)

$$
\begin{array}{llcl}
\multicolumn{4}{c}{\text{Full (non-truncated) cubical sets with degeneracies}} \\
\\
\mathsf{dephomocubset}_l & (X : \mathsf{cubset}_l) & : & \mathsf{U}_{l+1} \\
\mathsf{dephomocubset}_l & X & \triangleq & \mathsf{dephomocubset}_l^{\geq 0}(\star)(X)(\star) \\
\\
\mathsf{dephomocubset}_l^{\geq n} & \begin{array}{l}(D : \mathsf{cubset}_l^{<n}) \\ (X : \mathsf{cubset}_l^{\geq n}(D)) \\ (P : \mathsf{dephomocubset}_l^{<n}(D))\end{array} & : & \mathsf{U}_{l+1} \\
\mathsf{dephomocubset}_l^{\geq n} & D\ (R, X)\ P & \triangleq & \Sigma R' : \mathsf{dephomocubset}_l^{=n}(D)(P).\, \mathsf{dephomocubset}_l^{\geq n+1}(D,R)(X)(P,R')
\end{array}
$$

Figure 30: Definition of a dependent cubical set (coinductive structure)

$$
\begin{array}{llcl}
\multicolumn{4}{c}{\textit{Dependent sum of truncated cubical sets}} \\
\\
\mathsf{cubset}_l^{<n} & \begin{array}{l}(D : \mathsf{cubset}_l^{<n}) \\ (P : \mathsf{dephomocubset}_l^{<n}(D))\end{array} & : & \mathsf{cubset}_l^{<n} \\
\mathsf{cubset}_l^{<0} & \star\ \star & \triangleq & \star \\
\mathsf{cubset}_l^{<n'+1} & (D, R)\ (P, S) & \triangleq & (\mathsf{sigcubset}_l^{<n'}(D)(P).\, \mathsf{sigcubset}_l^{=n'}(D)(P)(R)(S)) \\
\\
\multicolumn{4}{c}{\textit{Structure carried at each dimension}} \\
\\
\mathsf{cubset}_l^{=n} & \begin{array}{l}(D : \mathsf{cubset}_l^{<n}) \\ (P : \mathsf{dephomocubset}_l^{<n}(D)) \\ (S : \mathsf{cubset}_l^{=n}(D)) \\ (S : \mathsf{dephomocubset}_l^{=n}(D)(P))\end{array} & : & \mathsf{cubset}_l^{=n}(\mathsf{sigcubset}_l^{<n}(D)(P)) \\
\mathsf{cubset}_l^{=n} & D\ P\ R\ S & \triangleq & \left\{\begin{array}{l}
\mathsf{rel} \triangleq \lambda d : \mathsf{fullbox}_l^{n}(\mathsf{sigcubset}_l^{<n}(D)(P)).\begin{array}{l}\Sigma b : R.\mathsf{rel}(\mathsf{fstbox}_l^{n}(D)(P)(d)). \\ (S\ (\mathsf{fstbox}_l^{n}(D)(P)(d))\ b).\mathsf{rel}(\mathsf{sndbox}_l^{n}(D)(P)(d))\end{array} \\
\mathsf{refl}^{[n\geq 1]} \triangleq \lambda b : \mathsf{fullcube}_l^{n+1}(\mathsf{sigcubset}_l^{<n}(D)(P)).\begin{array}{l}(R.\mathsf{refl}(\mathsf{fstfullcube}_l^{n}(b)), \\ (S\ (\mathsf{fstfullcube}_l^{n}(b))).\mathsf{refl}(\mathsf{sndfullcube}_l^{n}(b)))\end{array}
\end{array}\right. \\
\\
\multicolumn{4}{c}{\text{up to curryfication of } \mathsf{fstfullcube}_l^{n}(b) + \text{typing to check}}
\end{array}
$$

Figure 31: Definition of a dependent sum of truncated cubical sets

$$
\begin{array}{llcl}
\mathsf{fstfullcube}_l^{n,p,[p\leq n]} & \begin{array}{l}(D : \mathsf{cubset}_l^{<n})\\ (P : \mathsf{dephomocubset}_l^{<n}(D))\\ (d : \mathsf{fullcube}_l^{n,p}(\mathsf{sigcubset}_l^{<n}(D)(P)))\end{array} & : & \mathsf{fullcube}_l^{n}(D)\\
\mathsf{fstfullcube}_l^{n,p} & D\ P\ (d,b) & \triangleq & (\mathsf{fstbox}_l^{n,p}(D)(d), b.\mathsf{fst})
\end{array}
$$

$$
\begin{array}{llcl}
\mathsf{sndfullcube}_l^{n,p,[p\leq n]} & \begin{array}{l}(D : \mathsf{cubset}_l^{<n})\\ (P : \mathsf{dephomocubset}_l^{<n}(D))\\ (d : \mathsf{fullcube}_l^{n,p}(\mathsf{sigcubset}_l^{<n}(D)(P)))\end{array} & : & \mathsf{fullcube}_l^{n}(\mathsf{homoappfullcube}_l n(D)(P)(\mathsf{fstfullcube}_l^{n}(D)(P)(d)))\\
\mathsf{sndfullcube}_l^{n,p} & D\ P\ (d,b) & \triangleq & (\mathsf{sndbox}_l^{n,p}(D)(d), b.\mathsf{snd})
\end{array}
$$

$$
\begin{array}{llcl}
\mathsf{fstbox}_l^{n,p,[p\leq n]} & \begin{array}{l}(D : \mathsf{cubset}_l^{<n})\\ (P : \mathsf{dephomocubset}_l^{<n}(D))\\ (d : \mathsf{box}_l^{n,p}(D))\end{array} & : & \mathsf{box}_l^{n}\\
\mathsf{fstbox}_l^{n,0} & D\ \star\ \star & \triangleq & \star\\
\mathsf{fstbox}_l^{n,p'+1} & D\ P\ (d,b) & \triangleq & (\mathsf{fstbox}_l^{n,p'}(D)(P)(d), \mathsf{fstlayer}_l^{n,p'}(D)(P)(d)(b))
\end{array}
$$

$$
\begin{array}{llcl}
\mathsf{fstlayer}_l^{n,p,[p\leq n]} & \begin{array}{l}(D : \mathsf{cubset}_l^{<n})\\ (P : \mathsf{dephomocubset}_l^{<n}(D))\\ (d : \mathsf{box}_l^{n,p}(D))\\ (b : \mathsf{layer}_l^{n,p}(D)(d))\end{array} & : & \mathsf{layer}_l^{n,p}(\mathsf{fstbox}_l^{n,p'}(D)(P)(d))\\
\mathsf{fstlayer}_l^{n,p} & D\ P\ d\ c & \triangleq & \begin{array}{l}(\mathsf{fstcube}_l^{n-1,p}\begin{array}{l}(\mathsf{hd}(D))\\ (\mathsf{hd}(P))\\ (\mathsf{tl}(P))\\ (\mathsf{subbox}_{l,L,n-1}^{n,n-1}(d))\end{array}(c_L),\\ \ \mathsf{fstcube}_l^{n-1,p}\begin{array}{l}(\mathsf{hd}(D))\\ (\mathsf{hd}(P))\\ (\mathsf{tl}(P))\\ (\mathsf{subbox}_{l,R,n-1}^{n,n-1}(d))\end{array}(c_R))\end{array}
\end{array}
$$

+ proofs of commutations $\mathsf{subbox}_{l,\epsilon,q}/\mathsf{fstbox}$

$$
\begin{array}{llcl}
\mathsf{fstcube}_l^{n,p,[p\leq n]} & \begin{array}{l}(D : \mathsf{cubset}_l^{<n})\\ (P : \mathsf{dephomocubset}_l^{<n}(D))\\ (E' : \mathsf{sigcubset}_l^{=n,p}(D)(P))\\ (d : \mathsf{box}_l^{n,p}(D))\\ (c : \mathsf{cube}_l^{n,p}(D)(d))\end{array} & : & \mathsf{cube}_l^{n,p}(\mathsf{fstbox}_l^{n,p'}(D)(P)(d))\\
\mathsf{fstcube}_l^{n,p,[p=n]} & D\ P\ (E', \_)\ d\ c & \triangleq & E'\\
\mathsf{fstcube}_l^{n,p,[p<n]} & D\ P\ E'\ d\ (b,c) & \triangleq & (\mathsf{fstlayer}_l^{n,p}(D)(P)(d)(b), \mathsf{fstcube}_l^{n,p+1}(D)(P)(d)(E')(c))
\end{array}
$$

+ sndbox, etc.

Figure 32: Definition of a homogeneous dependent sum of truncated cubical sets (projections)

Full (non-truncated) dependent sum of cubical sets with degeneracies

$$\begin{array}{llcl}
\mathsf{sigcubset}_l & (X : \mathsf{cubset}_l)\ (P : \mathsf{dephomocubset}_l(X)) & : & \mathsf{U}_{l+1} \\
\mathsf{sigcubset}_l & X\ P & \triangleq & \mathsf{sigcubset}_l^{\geq 0}(\star)(X)(\star)
\end{array}$$

...

Figure 33: Definition of a dependent sum of cubical sets (coinductive structure)