Seminars on Theory of Probability

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1 Discrete Probability

1.1 Sets & Combinatorial Problems

[1]: p. 29 Exercise 3.1.1; p. 30 Exercise 3.1.6, 3.1.8; p. 35 Exercise 3.2.5, 3.2.6, 3.2.7; p. 40 Exercise 3.3.1, 3.3.2, 3.3.4, 3.3.5; p. 41 Exercise 3.3.6, 3.3.7, 3.3.8, 3.3.9, 3.3.11; p. 50 Exercise 3.5.1, 3.5.2, 3.5.3; p.51 Exercise 3.5.7

1.2 Probability Assignments by Combinatorial Methods

[1]: p. 72 Exercise 4.2.1., 4.2.2., 4.2.4., 4.2.5.; p. 73 Exercise Exercise 4.2.8., 4.2.9., 4.2.10., 4.2.11.; p. 74 Exercise 4.2.12., 4.2.15., 4.2.16.; p. 75 Exercise 4.2.17., 4.2.18., 4.2.19., 4.2.22.
[3]: p. 29 Exercise 1.7.3, 1.7.5, 1.7.7, 1.7.8, 1.7.10; p. 30 Exercise 1.7.12, 1.7.14, 1.7.18, 1.7.24(!),
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1.3 Algebra of Events

[1]: p.60 Exercise 4.1.8,
[2]: p. 21 Exercise 2.1; p. 22 Exercise 2.7, 2.9; p. 23 Exercise 2.11, 2.12
[3]: p. 28 Exercise 1.7.2; p. 31 Exercise 1.7.25
[4]: p. 35 Exercise 2; p. 36 Exercise 8; p. 37 Exercise 10,

1.4 Independence & Conditional Probabilities

[1]: p .81 Exercise 4.3.1, 4.3.4; p. 82 Exercise 4.3.8, 4.3.9; p.88 Exercise 4.4.1; p.89 Exercise 4.4.3, 4.4.4, 4.4.7; p.90 Exercise 4.4.8, 4.4.9, 4.4.11, 4.4.12, 4.4.13
[2]:p. 37 Exercise 3.2, 3.5, p. 38 Exercise 3.9, 3.10
[3]: p. 32 Exercise 1.7.26, 1.7.28, 1.7.29; p. 32 Exercise 1.7.33.
[4]: p. 37 Exercise 12, 14

1.5 The Theorem of Total Probability and the Theorem of Bayes

[1]: p. 100 Exercise 4.5.1, 4.5.2, 4.5.3; p.101 Exercise 4.5.6, 4.5.7, 4.5.8, 4.5.9; p.102 Exercise 4.5.10, 4.5.11, 4.5.12; p. 114 Exercise 5.1.12,
[3]: p. 30 Exercise 1.7.17; p. 32 Exercise 1.7.31; p. 33 Exercise 1.7.35
[4]: p. 40 Exercise 24,

2 Discrete Random Variables

[1]: p.112, Exercise 5.1.1, 5.1.2, 5.1.3, 5.1.4; p. 113 Exercise 5.1.5, 5.1.6, 5.1.7, 5.1.9, 5.1.10, 5.1.11
[2]: p. 51 Exercise 4.2, 4.3
[3]: p. 64 Exercise 2.7.1, 2.7.2; p. 65 Exercise 2.7.6, 2.7.7.(a), 2.7.9 (a), 2.7.10; p. 66 Exercise 2.7.12; p. 67 Exercise 2.7.21; p. 68 Exercise 2.7.23, 2.7.24, 2.7.25, 2.7.28; p. 70 Exercise 2.7.41

3 Continuous Random Variables

[1]: p. 121 Exercise 5.2.1, 5.2.2, 5.2.3, 5.2.4,
[2]: p. 68 Exercise 5.1, 5.2, 5.3, 5.5, 5.7
[3]: p.131 Exercise 5.12.2, 5.12.3, 5.12.4, 5.12.5, 5.12.6, 5.12.7,
#**1** Consider a density ($\lambda > 0$)

$$f(x) = \begin{cases} c\exp(-\lambda x), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Find

1. the constant c
2. the distribution function $F(x)$
3. $\mathsf{E}X$ and $\mathsf{Var}(X)$
4. quantile Q_p, median, 1st & 3rd quartiles

#**2** Consider a density ($\lambda > 0$)

$$f(x) = \begin{cases} cx^{\lambda}, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find

1. the constant c
2. the distribution function $F(x)$
3. the following probabilities

$$\mathsf{P}(X < \frac{1}{2}) \qquad \mathsf{P}(X \ge \frac{1}{3}) \qquad \mathsf{P}(X \le \frac{1}{4})$$

4. $\mathsf{E}X$ and $\mathsf{Var}(X)$
5. quantile Q_p, median, 1st & 3rd quartiles

#**3** Consider a density ($\mu, \lambda > 0$)

$$f(x) = \begin{cases} cx^{\mu}(1 - x^{\lambda}), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find

1. the constant c
2. the distribution function $F(x)$
3. the following probabilities

$$\mathsf{P}(X < \frac{1}{2}) \qquad \mathsf{P}(X \ge \frac{1}{3}) \qquad \mathsf{P}(X \le \frac{1}{4})$$

4. $\mathsf{E}X$ and $\mathsf{Var}(X)$

4 Standard Distributions

#1 Let $X \sim U[-1, 1]$. Calculate the probabilities

$$\mathsf{P}(X < -\frac{1}{2}) \qquad \mathsf{P}(X \geq \frac{1}{3}) \qquad \mathsf{P}(-\frac{1}{3} \leq X \leq \frac{3}{4})$$

#2 Let $X \sim U[0, 2]$. Calculate the probabilities

$$\mathsf{P}(X < 1.5) \qquad \mathsf{P}(X \geq \frac{1}{3}) \qquad \mathsf{P}(\frac{1}{5} \leq X \leq 1.5)$$

#3 Let $Z \sim N(0, 1)$.

1. Calculate the probabilities

$$\mathsf{P}(Z < 1.5) \qquad \mathsf{P}(Z \geq \frac{1}{3}) \qquad \mathsf{P}(-1 \leq Z \leq 1)$$

2. Found z_1 s.t. $\mathsf{P}(Z < z_1) = 0.975$ (5%-critical value)
3. Found z_2 s.t. $\mathsf{P}(Z < z_2) = 0.95$ (10%-critical value)
4. Found z_3 s.t. $\mathsf{P}(Z < z_3) = 0.995$ (1%-critical value)

#3 Let $X^2 \sim \chi^2_2$.

1. Calculate the probabilities

$$\mathsf{P}(X^2 < 1.5) \qquad \mathsf{P}(X^2 \geq 1) \qquad \mathsf{P}(X^2 \leq 2)$$

2. Found x_1 s.t. $\mathsf{P}(X^2 < x_1) = 0.9$ (10%-critical value)
3. Found x_2 s.t. $\mathsf{P}(X^2 < x_2) = 0.95$ (5%-critical value)
4. Found x_3 s.t. $\mathsf{P}(X^2 < x_3) = 0.99$ (1%-critical value)

#4 Let $X^2 \sim \chi^2_8$.

1. Calculate the probabilities

$$\mathsf{P}(X^2 < 1.5) \qquad \mathsf{P}(X^2 \geq 1) \qquad \mathsf{P}(X^2 \leq 2)$$

2. Found x_1 s.t. $\mathsf{P}(X^2 < x_1) = 0.9$ (10%-critical value)

3. Found x_2 s.t. $\mathsf{P}(X^2 < x_2) = 0.95$ (5%-critical value
4. Found x_3 s.t. $\mathsf{P}(X^2 < x_3) = 0.99$ (1%-critical value)

#**5** Let $T \sim t_{10}$.

1. Calculate the probabilities

$$\mathsf{P}(T < 1.5) \qquad \mathsf{P}(T \geq \frac{1}{3}) \qquad \mathsf{P}(-1 \leq T \leq 1)$$

2. Found t_1 s.t. $\mathsf{P}(T < t_1) = 0.975$ (5%-critical value)
3. Found t_2 s.t. $\mathsf{P}(T < t_2) = 0.95$ (10%-critical value)
4. Found t_3 s.t. $\mathsf{P}(T < t_3) = 0.995$ (1%-critical value)

#**6** Let $f \sim F_{3,20}$.

1. Calculate the probabilities

$$\mathsf{P}(f < 1.5) \qquad \mathsf{P}(f \geq 1) \qquad \mathsf{P}(f \leq 2)$$

2. Found F_1 s.t. $\mathsf{P}(f < F_1) = 0.9$ (10%-critical value)
3. Found F_2 s.t. $\mathsf{P}(f < F_2) = 0.95$ (5%-critical value
4. Found F_3 s.t. $\mathsf{P}(f < F_3) = 0.99$ (1%-critical value)

5 Mathematical Statistics

5.1 Hypothesis testing

[1]: p. 314, Exercise 8.4.3, 8.4.4

References

[1] Géza Schay, «Introduction to Probability with Statistical Applications», 2 ed Birkhäuser, 2016

[2] F.M. Dekking, C. Kraaikamp, H.P. Lopuhaä, L.E. Meester, «A Modern Introduction to Probability and Statistics», Springer-Verlag, 2005

[3] Ronald Meester, «A Natural Introduction to Probability Theory», 2 ed, Birkhäuser, 2008

[4] Ron C. Mittelhammer, «Mathematical Statistics for Economics and Business», 2 ed, Springer-Verlag, 2013