Cayley-Hamilton and Diagonalizability

Challenge Report 3

Math 115A

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Problem Statement

Linear maps $T : V \to V$ of finite dimensional vector spaces correspond to $n \times n$ matrices, where $dim(V) = n$. This is equivalent to stating that there is an isomorphism $\mathcal{L}(V, V) \to M_{n\times n}(\mathbb{F})$. Therefore, composition of linear maps corresponds to matrix multiplication.

We will consider what happens when we compose T with itself many times, which corresponds to multiplying the standard matrix $[A]_\beta^\beta$, for a basis β, with itself many times.

Definition Let k be a non-negative integer, and let $[A] \in M_{n\times n}(\mathbb{F})$. $[A]^k$ denotes the product of k copies of $[A]$. For example, $[A]^2 = [A][A]$.

Furthermore, we define $[A]^0$ to be the $n \times n$ identity matrix.

One can deduce several relationships about powers of matrices - one of the most important theorems is the following:

Theorem (Cayley-Hamilton) Let $[A] \in M_{n\times n}(\mathbb{F})$, and let $\chi_A(t)$ be the characteristic polynomial of $[A]$. Then

$$\chi_A([A]) = [0]$$

That is, if $\chi_A(t) = \sum_{i=0}^{n} c_i t^i$, then

$$\sum_{i=0}^{n} c_i [A]^i = [0]$$

In other words, a matrix always satisfies its own characteristic polynomial.

Another construction we can do with matrices is as follows:

Definition Let $[A] \in M_{n\times n}(\mathbb{C})$. The **matrix exponential of** $[A]$ is denoted $e^{[A]} \in M_{n\times n}(\mathbb{C})$, and is defined by

$$e^{[A]} := \sum_{k=0}^{\infty} \frac{1}{k!}[A]^k$$

The matrix exponential has several nice properties. Notably, it is used to solve systems of linear ordinary differential equations.

Remark Recall that the function $e^x : \mathbb{C} \to \mathbb{C}$ has the Taylor expansion

$$e^x := \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$

Hence, the matrix exponential is a generalization of the exponential function.

We will prove the Cayley-Hamilton theorem for **diagonalizable** matrices, alongside properties of their matrix exponentials, with many examples.

Part 1 Cayley-Hamilton on a Matrix $[A]$

Consider the matrix

$$[A] = \begin{bmatrix} 3 & 7 & 2 \\ -2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \in M_{3\times 3}(\mathbb{C})$$

(a) Characteristic Polynomial

By definition, for a 3×3 matrix, the characteristic polynomial is $\chi_A(t) := det(A - tI_3)$.

We can see that

$$\begin{aligned} \chi_A(t) &= det(A - tI_3) \\ &= det(\begin{bmatrix} 3 & 7 & 2 \\ -2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} - t \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}) \end{aligned}$$

$$= det(\begin{bmatrix} 3-t & 7 & 2 \\ -2 & 3-t & 1 \\ 1 & 1 & 2-t \end{bmatrix})$$
$$= \begin{vmatrix} 3-t & 7 & 2 \\ -2 & 3-t & 1 \\ 1 & 1 & 2-t \end{vmatrix}$$
$$= -t^3 + 8t^2 - 32t + 40$$

(b) Cayley-Hamilton

We would like to show that $\chi_A([A]) = [0]$ in order to show that the Cayley-Hamilton theorem holds for $[A]$.

Note that

$$[A]^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in M_{3\times3}(\mathbb{C})$$
$$[A] = \begin{bmatrix} 3 & 7 & 2 \\ -2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \in M_{3\times3}(\mathbb{C})$$
$$[A]^2 = \begin{bmatrix} -3 & 44 & 17 \\ -11 & -4 & 1 \\ 3 & 12 & 7 \end{bmatrix} \in M_{3\times3}(\mathbb{C})$$
$$[A]^3 = \begin{bmatrix} -80 & 128 & 72 \\ -24 & -88 & -24 \\ -8 & 64 & 32 \end{bmatrix} \in M_{3\times3}(\mathbb{C})$$

Therefore

$$\chi_A([A]) = -[A]^3 + 8[A]^2 - 32[A] + 40[A]^0$$
$$= -\begin{bmatrix} -80 & 128 & 72 \\ -24 & -88 & -24 \\ -8 & 64 & 32 \end{bmatrix} + 8\begin{bmatrix} -3 & 44 & 17 \\ -11 & -4 & 1 \\ 3 & 12 & 7 \end{bmatrix} - 32\begin{bmatrix} 3 & 7 & 2 \\ -2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} + 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 80 & -128 & -72 \\ 24 & 88 & 24 \\ 8 & -64 & -32 \end{bmatrix} + \begin{bmatrix} -24 & 132 & 136 \\ -88 & -32 & 8 \\ 24 & 96 & 56 \end{bmatrix} + \begin{bmatrix} -96 & -224 & -64 \\ 64 & -96 & -32 \\ -32 & -32 & -64 \end{bmatrix} + \begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= [0] \in M_{3\times 3}(\mathbb{C})$$

as desired.

Part 2 Cayley-Hamilton on a Matrix $[B]$

Consider the matrix

$$[B] = \begin{bmatrix} 2 & 5 & 1 \\ 2 & 1 & 5 \\ -2 & -3 & -3 \end{bmatrix} \in M_{3\times 3}(\mathbb{C})$$

(a) Characteristic Polynomial

By definition, for a 3×3 matrix, the characteristic polynomial is $\chi_B(t) := det(B - tI_3)$.

We can see that

$$\begin{aligned} \chi_B(t) &= det(B - tI_3) \\ &= det(\begin{bmatrix} 2 & 5 & 1 \\ 2 & 1 & 5 \\ -2 & -3 & -3 \end{bmatrix} - t \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}) \\ &= det(\begin{bmatrix} 2-t & 5 & 1 \\ 2 & 1-t & 5 \\ -2 & -3 & -3-t \end{bmatrix}) \\ &= \begin{vmatrix} 2-t & 5 & 1 \\ 2 & 1-t & 5 \\ -2 & -3 & -3-t \end{vmatrix} \\ &= -t^3 \end{aligned}$$

(b) Matrix Exponential

First observe that the matrix $[B]$ is **nilpotent**. That is, $\exists k \in \mathbb{Z}_{\geq 0}, [B]^k = [0]$.

$$[B]^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in M_{3\times 3}(\mathbb{C})$$

$$[B]^1 = \begin{bmatrix} 2 & 5 & 1 \\ 2 & 1 & 5 \\ -2 & -3 & -3 \end{bmatrix} \in M_{3\times 3}(\mathbb{C})$$

$$[B]^2 = \begin{bmatrix} 12 & 12 & 24 \\ -4 & -4 & -8 \\ -4 & -4 & -8 \end{bmatrix} \in M_{3\times 3}(\mathbb{C})$$

$$[B]^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M_{3\times 3}(\mathbb{C})$$

$$[B]^k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M_{3\times 3}(\mathbb{C}), k \in \mathbb{Z}_{\geq 3}$$

where $e^{[B]} := \sum_{k=0}^{\infty} \frac{1}{k!}[B]^k$ and $0! := 1$, therefore

$$\begin{aligned}
e^{[B]} &= \sum_{k=0}^{\infty} \frac{1}{k!}[B]^k \\
&= \frac{1}{0!}[B]^0 + \frac{1}{1!}[B]^1 + \frac{1}{2!}[B]^2 + \frac{1}{3!}[B]^3 + \dots \\
&= \frac{1}{0!}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{1!}\begin{bmatrix} 2 & 5 & 1 \\ 2 & 1 & 5 \\ -2 & -3 & -3 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 12 & 12 & 24 \\ -4 & -4 & -8 \\ -4 & -4 & -8 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \dots \\
&= \begin{bmatrix} 9 & 11 & 13 \\ 0 & 0 & 1 \\ -4 & -5 & -6 \end{bmatrix}
\end{aligned}$$

Part 3 Matrix Exponential of $[D]$

Let $[D] = [d_{ij}] \in M_{n\times n}(\mathbb{C})$ be an arbitrary diagonal matrix where $d_{ij} = 0_\mathbb{C}$ if $i \neq j$. We will calculate the matrix exponential $e^{[D]}$.

First note

$$[D]^0 = \begin{bmatrix} 1_\mathbb{C} & \cdots & 0_\mathbb{C} \\ \vdots & \ddots & \vdots \\ 0_\mathbb{C} & \cdots & 1_\mathbb{C} \end{bmatrix} \in M_{n\times n}(\mathbb{C})$$

$$[D]^1 = \begin{bmatrix} d_{11} & \cdots & 0_\mathbb{C} \\ \vdots & \ddots & \vdots \\ 0_\mathbb{C} & \cdots & d_{nn} \end{bmatrix} \in M_{n\times n}(\mathbb{C})$$

$$[D]^2 = \begin{bmatrix} {d_{11}}^2 & \cdots & 0_\mathbb{C} \\ \vdots & \ddots & \vdots \\ 0_\mathbb{C} & \cdots & {d_{nn}}^2 \end{bmatrix} \in M_{n\times n}(\mathbb{C})$$

$$[D]^3 = \begin{bmatrix} {d_{11}}^3 & \cdots & 0_\mathbb{C} \\ \vdots & \ddots & \vdots \\ 0_\mathbb{C} & \cdots & {d_{nn}}^3 \end{bmatrix} \in M_{n\times n}(\mathbb{C})$$

$$[D]^k = \begin{bmatrix} {d_{11}}^k & \cdots & 0_\mathbb{C} \\ \vdots & \ddots & \vdots \\ 0_\mathbb{C} & \cdots & {d_{nn}}^k \end{bmatrix} \in M_{n\times n}(\mathbb{C}), k \in \mathbb{Z}_{\geq 0}$$

Recall $e^{[D]} := \sum_{k=0}^{\infty} \frac{1}{k!}[D]^k$ and $e^{d_{ij}} := \sum_{k=0}^{\infty} \frac{1}{k!}{d_{ij}}^k$. We have

$$\begin{aligned} e^{[D]} &= \sum_{k=0}^{\infty} \frac{1}{k!}[D]^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} {d_{11}}^k & \cdots & 0_\mathbb{C} \\ \vdots & \ddots & \vdots \\ 0_\mathbb{C} & \cdots & {d_{nn}}^k \end{bmatrix} \\ &= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!}{d_{11}}^k & \cdots & 0_\mathbb{C} \\ \vdots & \ddots & \vdots \\ 0_\mathbb{C} & \cdots & \sum_{k=0}^{\infty} \frac{1}{k!}{d_{nn}}^k \end{bmatrix} && \text{by component-wise addition} \end{aligned}$$

$$= \begin{bmatrix} e^{d_{11}} & \cdots & 0_{\mathbb{C}} \\ \vdots & \ddots & \vdots \\ 0_{\mathbb{C}} & \cdots & e^{d_{nn}} \end{bmatrix}$$

Part 4 Matrix Exponential of $[A]$

Let $[A] \in M_{n\times n}(\mathbb{C})$ be diagonalizable, with diagonal matrix $[D]$. By definition, $\exists$ invertible matrix $[Q]$ such that $[A] = [Q]^{-1}[D][Q]$. Let $[I] \in M_{n\times n}(\mathbb{C})$ denote the identity matrix.

(a) Formula for $[A]^k$

We give a formula for $[A]^k$ in terms of $[D]$. We do this by expanding the diagonalization and factorizing out the matrices $[Q]$ and $[Q]^{-1}$.

$$\begin{aligned}
[A]^k &= ([Q]^{-1}[D][Q])^k \\
&= ([Q]^{-1}[D][Q])([Q]^{-1}[D][Q])\cdots([Q]^{-1}[D][Q])([Q]^{-1}[D][Q]) \\
&= [Q]^{-1}[D]([Q][Q]^{-1})[D]([Q]\cdots[Q]^{-1})[D]([Q][Q]^{-1})[D][Q] && \text{by associativity} \\
&= [Q]^{-1}[D]([I])[D]([I])\cdots([I])[D]([I])[D][Q] && \text{by property of the inverse} \\
&= [Q]^{-1}[D][D]\cdots[D][D][Q] && \text{by identity} \\
&= [Q]^{-1}[D]^k[Q]
\end{aligned}$$

(b) Formula for $e^{[A]}$

We give a formula for $e^{[A]}$ in terms of $e^{[D]}$.

$$\begin{aligned}
e^{[A]} &= \sum_{k=0}^{\infty} \frac{1}{k!}[A]^k \\
&= \frac{1}{0!}[A]^0 + \frac{1}{1!}[A]^1 + \frac{1}{2!}[A]^2 + \dots \\
&= [I] + [A] + \frac{1}{2!}[A]^2 + \dots \\
&= [Q]^{-1}[I][Q] + [Q]^{-1}[D][Q] + \frac{1}{2!}([Q]^{-1}[D][Q])([Q]^{-1}[D][Q]) + \dots && \text{by property of the inverse} \\
&= [Q]^{-1}([I] + [D] + \frac{1}{2!}[D]^2 + \dots)[Q] && \text{by distributivity}
\end{aligned}$$

$$= [Q]^{-1}(\frac{1}{0!}[D]^0 + \frac{1}{1!}[D] + \frac{1}{2!}[D]^2 + \dots)[Q]$$
$$= [Q]^{-1}(\sum_{k=0}^{\infty} \frac{1}{k!}[D]^k)[Q]$$
$$= [Q]^{-1}(e^{[D]})[Q]$$

(c) Calculation for $e^{[A]}$

Let

$$[A] = \begin{bmatrix} 3 & 7 & 2 \\ -2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \in M_{3\times 3}(\mathbb{C})$$

Note that $[A]$ is diagonalizable. This is because the roots of $\chi_A(t) = -t^3 + 8t^2 - 32t + 40$ are exactly $2, 3+\sqrt{11}i, 3-\sqrt{11}i$, where the number of distinct eigenvalues is equal to the dimension of $[A]$. Recall that $i^2 = -1$. We can write $[A]$ as follows

$$\begin{aligned} [A] =& [Q]^{-1}[D][Q] \\ =& \begin{bmatrix} 1 & 7+2\sqrt{11}i & 7-2\sqrt{11}i \\ -1 & -4+\sqrt{11}i & -4-\sqrt{11}i \\ 3 & 3 & 3 \end{bmatrix} \\ & \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3+\sqrt{11}i & 0 \\ 0 & 0 & 3-\sqrt{11}i \end{bmatrix} \\ & \begin{bmatrix} \frac{-1}{12} & \frac{1}{6} & \frac{5}{12} \\ \frac{1}{24} - i\frac{\sqrt{11}}{88} & -\frac{1}{12} - i\frac{\sqrt{11}}{44} & -\frac{1}{24} - i\frac{\sqrt{11}}{264} \\ \frac{1}{24} + i\frac{\sqrt{11}}{88} & -\frac{1}{12} + i\frac{\sqrt{11}}{44} & -\frac{1}{24} + i\frac{\sqrt{11}}{264} \end{bmatrix} \end{aligned}$$

Consider that $e^{[A]} = [Q]^{-1}(e^{[D]})[Q]$.

$$\begin{aligned} e^{[A]} =& [Q]^{-1}(e^{[D]})[Q] \\ =& \begin{bmatrix} 1 & 7+2\sqrt{11}i & 7-2\sqrt{11}i \\ -1 & -4+\sqrt{11}i & -4-\sqrt{11}i \\ 3 & 3 & 3 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^{3+\sqrt{11}i} & 0 \\ 0 & 0 & e^{3-\sqrt{11}i} \end{bmatrix}$$

$$\begin{bmatrix} \frac{-1}{12} & \frac{1}{6} & \frac{5}{12} \\ \frac{1}{24} - i\frac{\sqrt{11}}{88} & -\frac{1}{12} - i\frac{\sqrt{11}}{44} & -\frac{1}{24} - i\frac{\sqrt{11}}{264} \\ \frac{1}{24} + i\frac{\sqrt{11}}{88} & -\frac{1}{12} + i\frac{\sqrt{11}}{44} & -\frac{1}{24} + i\frac{\sqrt{11}}{264} \end{bmatrix}$$

We rewrite $e^{[D]}$ using Euler's formula, where $e^{ix} = \cos x + i \sin x$ for $x \in \mathbb{R}$. Also note the negative angle identities, where $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$.

$$\begin{aligned} e^{[D]} &= \begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^{3+\sqrt{11}i} & 0 \\ 0 & 0 & e^{3-\sqrt{11}i} \end{bmatrix} \\ &= \begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^3 e^{\sqrt{11}i} & 0 \\ 0 & 0 & e^3 e^{-\sqrt{11}i} \end{bmatrix} \\ &= \begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^3(\cos\sqrt{11} + i\sin\sqrt{11}) & 0 \\ 0 & 0 & e^3(\cos\sqrt{11} - i\sin\sqrt{11}) \end{bmatrix} \end{aligned}$$

Finally, we can use this to find $e^{[A]}$:

$$\begin{aligned} e^{[A]} =& [Q]^{-1}(e^{[D]})[Q] \\ =& \begin{bmatrix} 1 & 7+2\sqrt{11}i & 7-2\sqrt{11}i \\ -1 & -4+\sqrt{11}i & -4-\sqrt{11}i \\ 3 & 3 & 3 \end{bmatrix} \\ & \begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^3(\cos\sqrt{11} + i\sin\sqrt{11}) & 0 \\ 0 & 0 & e^3(\cos\sqrt{11} - i\sin\sqrt{11}) \end{bmatrix} \\ & \begin{bmatrix} \frac{-1}{12} & \frac{1}{6} & \frac{5}{12} \\ \frac{1}{24} - i\frac{\sqrt{11}}{88} & -\frac{1}{12} - i\frac{\sqrt{11}}{44} & -\frac{1}{24} - i\frac{\sqrt{11}}{264} \\ \frac{1}{24} + i\frac{\sqrt{11}}{88} & -\frac{1}{12} + i\frac{\sqrt{11}}{44} & -\frac{1}{24} + i\frac{\sqrt{11}}{264} \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \frac{(143\cos\sqrt{11}\cdot e - \sin\sqrt{11}\cdot e\cdot\sqrt{11} - 11)\cdot e^2}{132} & \frac{-(11\cos\sqrt{11}\cdot e - 43\sin\sqrt{11}\cdot e\cdot\sqrt{11} - 11)\cdot e^2}{66} & \frac{-(55\cos\sqrt{11}\cdot e - 29\sin\sqrt{11}\cdot e\cdot\sqrt{11} - 5)\cdot e^2}{132} \\ \frac{-(11\cos\sqrt{11}\cdot e + 23\sin\sqrt{11}\cdot e\cdot\sqrt{11} - 11)\cdot e^2}{132} & \frac{(77\cos\sqrt{11}\cdot e - \sin\sqrt{11}\cdot e\cdot\sqrt{11} - 11)\cdot e^2}{66} & \frac{(55\cos\sqrt{11}\cdot e + 7\sin\sqrt{11}\cdot e\cdot\sqrt{11} - 55)\cdot e^2}{132} \\ \frac{(11\cos\sqrt{11}\cdot e + 3\sin\sqrt{11}\cdot e\cdot\sqrt{11} - 11)\cdot e^2}{44} & \frac{-(11\cos\sqrt{11}\cdot e - 3\sin\sqrt{11}\cdot e\cdot\sqrt{11} - 11)\cdot e^2}{22} & \frac{-(11\cos\sqrt{11}\cdot e - \sin\sqrt{11}\cdot e\cdot\sqrt{11} - 55)\cdot e^2}{44} \end{bmatrix}$$

Part 5 Proof of Cayley Hamilton for $[A]$

Let $[A] \in M_{n\times n}(\mathbb{C})$ be an arbitrary diagonalizable matrix, where $[A] = [Q]^{-1}[D][Q]$. As before, $[D] = [d_{ij}]$, where $d_{ij} = 0$ if $i \neq j$. Let $[I] \in M_{n\times n}(\mathbb{C})$ denote the identity matrix.

First, we consider a proof of Cayley-Hamilton for the diagonal matrix $[D]$. Let $\chi_D(t)$ be the characteristic polynomial for $[D]$. We wish to show that $\chi_D([D]) = [0] \in M_{n\times n}(\mathbb{C})$.

Consider that $\chi_D(t) := det(D - tI)$. We know that the matrix $[D - tI] \in M_{n\times n}(\mathbb{C})$ is diagonal, with diagonal entries $[d_{ii} - t]$ so

$$\begin{aligned} \chi_D(t) &= det(D - tI) \\ &= \prod_{i=1}^{n} d_{ii} - t \\ &= (d_{11} - t)(d_{22} - t)\cdots(d_{nn} - t) \\ &= (d_{11}t^0 - t)(d_{22}t^0 - t)\cdots(d_{nn}t^0 - t) \end{aligned}$$

Now consider

$$\begin{aligned} \chi_D([D]) =& (d_{11}[D]^0 - [D])(d_{22}[D]^0 - [D])\cdots(d_{nn}[D]^0 - [D]) \\ =& (d_{11}[I] - [D])(d_{22}[I] - [D])\cdots(d_{nn}[I] - [D]) \\ =& (\begin{bmatrix} d_{11} & \cdots & 0_{\mathbb{C}} \\ \vdots & \ddots & \vdots \\ 0_{\mathbb{C}} & \cdots & d_{11} \end{bmatrix} - \begin{bmatrix} d_{11} & \cdots & 0_{\mathbb{C}} \\ \vdots & \ddots & \vdots \\ 0_{\mathbb{C}} & \cdots & d_{nn} \end{bmatrix}) \\ & (\begin{bmatrix} d_{22} & \cdots & 0_{\mathbb{C}} \\ \vdots & \ddots & \vdots \\ 0_{\mathbb{C}} & \cdots & d_{22} \end{bmatrix} - \begin{bmatrix} d_{11} & \cdots & 0_{\mathbb{C}} \\ \vdots & \ddots & \vdots \\ 0_{\mathbb{C}} & \cdots & d_{nn} \end{bmatrix}) \\ & \qquad\qquad \vdots \end{aligned}$$

$$(\begin{bmatrix} d_{nn} & \cdots & 0_{\mathbb{C}} \\ \vdots & \ddots & \vdots \\ 0_{\mathbb{C}} & \cdots & d_{nn} \end{bmatrix} - \begin{bmatrix} d_{11} & \cdots & 0_{\mathbb{C}} \\ \vdots & \ddots & \vdots \\ 0_{\mathbb{C}} & \cdots & d_{nn} \end{bmatrix})$$

$$=(\begin{bmatrix} 0_{\mathbb{C}} & 0_{\mathbb{C}} & \cdots & 0_{\mathbb{C}} \\ \vdots & d_{11}-d_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0_{\mathbb{C}} & 0_{\mathbb{C}} & \cdots & d_{11}-d_{nn} \end{bmatrix}) \qquad \text{where } d_{11}-d_{11}=0_{\mathbb{C}}$$

$$(\begin{bmatrix} d_{22}-d_{11} & 0_{\mathbb{C}} & \cdots & 0_{\mathbb{C}} \\ \vdots & 0_{\mathbb{C}} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0_{\mathbb{C}} & 0_{\mathbb{C}} & \cdots & d_{22}-d_{nn} \end{bmatrix}) \qquad \text{where } d_{22}-d_{22}=0_{\mathbb{C}}$$

$$\vdots \qquad \text{where } d_{ii}-d_{ii}=0_{\mathbb{C}}$$

$$(\begin{bmatrix} d_{nn}-d_{11} & 0_{\mathbb{C}} & \cdots & 0_{\mathbb{C}} \\ \vdots & d_{nn}-d_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0_{\mathbb{C}} & 0_{\mathbb{C}} & \cdots & 0_{\mathbb{C}} \end{bmatrix}) \qquad \text{where } d_{nn}-d_{nn}=0_{\mathbb{C}}$$

$$=(\begin{bmatrix} 0_{\mathbb{C}} & 0_{\mathbb{C}} & \cdots & 0_{\mathbb{C}} \\ \vdots & 0_{\mathbb{C}} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0_{\mathbb{C}} & 0_{\mathbb{C}} & \cdots & d_{11}-d_{nn} \end{bmatrix}) \qquad \text{by multiplying the first two matrices}$$

$$\vdots$$

$$(\begin{bmatrix} d_{nn}-d_{11} & 0_{\mathbb{C}} & \cdots & 0_{\mathbb{C}} \\ \vdots & d_{nn}-d_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0_{\mathbb{C}} & 0_{\mathbb{C}} & \cdots & 0_{\mathbb{C}} \end{bmatrix})$$

$$
\begin{aligned}
&= \begin{bmatrix} 0_{\mathbb{C}} & \cdots & 0_{\mathbb{C}} \\ \vdots & \ddots & \vdots \\ 0_{\mathbb{C}} & \cdots & 0_{\mathbb{C}} \end{bmatrix} && \text{by multiplying all } n \text{ matrices} \\
&= [0] \in M_{n\times n}(\mathbb{C})
\end{aligned}
$$

Having proven that $\chi_D([D]) = [0] \in M_{n\times n}(\mathbb{C})$, we now prove that $\chi_A([A]) = [0] \in M_{n\times n}(\mathbb{C})$, where $[A] = [Q]^{-1}[D][Q]$.

Let $\chi_A(t)$ be the characteristic polynomial of the matrix $[A]$. We know that the characteristic polynomial is independent of choice of *basis*, where the matrices $[A]$ and $[D]$ represent the same transformation T under different *bases*, by the definition of diagonalizibility. Therefore

$$
\begin{aligned}
\chi_A(t) &= \chi_D(t) \\
&= (d_{11}t^0 - t)(d_{22}t^0 - t)\cdots(d_{nn}t^0 - t)
\end{aligned}
$$

We prove that $\chi_A([A]) = [0] \in M_{n\times n}(\mathbb{C})$ through a process of substitution, factorization of $[Q]$ and $[Q]^{-1}$, and the usage of the fact that $\chi_D([D]) = (d_{11}[I]-[D])(d_{22}[I]-[D])\cdots(d_{nn}[I]-[D]) = [0] \in M_{n\times n}(\mathbb{C})$.

$$
\begin{aligned}
\chi_A([A]) =&(d_{11}[A]^0 - [A])(d_{22}[A]^0 - [A])\cdots(d_{nn}[A]^0 - [A]) \\
=&(d_{11}[Q]^{-1}[I][Q] - [Q]^{-1}[D][Q]) \\
&(d_{22}[Q]^{-1}[I][Q] - [Q]^{-1}[D][Q]) \\
&\qquad\qquad\vdots \\
&(d_{nn}[Q]^{-1}[I][Q] - [Q]^{-1}[D][Q]) && \text{by diagonalizibility} \\
=&[Q]^{-1}(d_{11}[I] - [D])(d_{22}[I] - [D])\cdots(d_{nn}[I] - [D])[Q] && \text{by distributivity} \\
=&[Q]^{-1}(\chi_D([D]))[Q] \\
=&[Q]^{-1}([0])[Q] && \chi_D([D]) = [0] \\
=&[0] \in M_{n\times n}(\mathbb{C})
\end{aligned}
$$

□