Modeling the Lightbulb Problem as a Vector Space

Challenge Report 1

Math 115A

Arteen Abrishami

Problem Statement

There are n lightbulbs, arranged in a circle. All are off, but there is a switch at the base of each. Flipping a particular lightbulb's switch toggles the on/off state of itself and those adjacent to it. In other words, flipping a switch at the base of a lightbulb (l_i) toggles the on/off states of the lightbulbs $\{l_{i-1}, l_i, l_{i+1}\}$. For the purposes of this problem, we assume n is non-trivial, in that $n \in \mathbb{Z}_{>0}$.

It is important to note that addition in $\mathbb{F}_2$ is defined as:

$$+_{\mathbb{F}_2} : \mathbb{F}_2 \times \mathbb{F}_2 \to \mathbb{F}_2$$

$$a +_{\mathbb{F}_2} b := (a + b) \bmod 2$$

for $a, b \in \mathbb{F}_2$.

Part 1

(a)

The set of on/off states of the lightbulbs forms a vector space V over the finite field $\mathbb{F}_2$. We can define the vector space as:

$$V = (\mathbb{F}_2)^n = \{(a_1, ..., a_n) \mid a_i \in \mathbb{F}_2\}$$

We define the *zero* vector as

$$\mathbf{0} = (0, ..., 0)$$

where $0 = 0_{\mathbb{F}_2}$. This can be used to represent all n lightbulbs being off. We define vector addition component-wise, in the usual way:

$$+_V : V \times V \to V$$

$$(a_1, ..., a_n) +_V (b_1, ..., b_n) := (a_1 +_{\mathbb{F}_2} b_1, ..., a_n +_{\mathbb{F}_2} b_n)$$

We define scalar multiplication, also component-wise:

$$\cdot_V : \mathbb{F}_2 \times V \to V$$

$$\lambda \cdot_V (a_1, ..., a_n) := (\lambda \cdot_{\mathbb{F}_2} a_1, ..., \lambda \cdot_{\mathbb{F}_2} a_n)$$

(b)

By definition, there are n components in any $v \in V$, with each having two possible states: $\mathbb{F}_2 = \{0, 1\}$. By the multiplication principle, this means that there are 2^n distinct vectors in V. Therefore, we can say that the size of V is 2^n.

(c)

We define a distinguished vector $\mathbf{t}_i \in V$, which we shall call the *toggle* vector about a subscript $i \in \{1, 2, ..., n\}$ for $n \geq 3$. This is defined:

$$\mathbf{t}_i = (0, ..., 0, t_{i-1}, t_i, t_{i+1}, 0, ..., 0)$$

where $t_{i-1}, t_i, t_{i+1} = 1_{\mathbb{F}_2}$. The semantics of subscript addition are as follows:

$$i + j := (i + j) \bmod n$$

s.t. the subscript $0 := n$ so that $0 \bmod n = 0 = n$ and $n + 1 = (n + 1) \bmod n = 1$ and $1 - 1 = (1 - 1) \bmod n = n$. In other words, subscript addition is defined to wrap around (exhibit modulo behavior). For $0 < n \leq 2$, we have a case where it may be that any of $\{i-1, i, i+1\}$ are equal. Thus, the vector $\mathbf{t}_i$ with subscript behavior is only defined for $n \geq 3$.

For $n = 1$, we define the toggle vector $\mathbf{t}$ as:

$$\mathbf{t} = (1)$$

For $n = 2$, we define the toggle vector $\mathbf{t}$ as

$$\mathbf{t} = (1, 1)$$

In general, we do not contain the cases where $0 < n \leq 2$, since the vector $\mathbf{t} = \mathbf{1}$. The vector $\mathbf{1}$ is defined in **Part 2**.

Part 2

In order to model this system, we must define a distinguished vector $\mathbf{1} \in V$, which we shall call the *one* vector, as

$$\mathbf{1} = (1, ..., 1)$$

where $1 = 1_{\mathbb{F}_2}$. The vector $\mathbf{1}$ corresponds to all the lights being on, just as $\mathbf{0}$ corresponds to all the lights being off.

Part 3

Suppose $n \equiv 0 \pmod 3$, with the minimum possible value of n being 3. Given that we begin with all n lightbulbs off, we would like to turn all n lightbulbs on while toggling the minimum number of switches. We can model this using our definition of addition and the *toggle* vector $\mathbf{t}_i$. Through a series of additions, arbitrarily beginning with $\mathbf{0}$ (all lightbulbs off), we can derive a summation that is generally equal to the vector $\mathbf{1}$ (all lightbulbs on):

$$\mathbf{0} + \mathbf{t}_2 + \mathbf{t}_5 + \mathbf{t}_8 + ... + \mathbf{t}_{n-1} = \mathbf{t}_2 + \mathbf{t}_5 + \mathbf{t}_8 + ... + \mathbf{t}_{n-1}$$

By definition, $n = 3k$ for $k \in \mathbb{Z}_{>0}$:

$$\mathbf{t}_2 + \mathbf{t}_5 + \mathbf{t}_8 + ... + \mathbf{t}_{n-1} = \sum_{i=0}^{k-1} \mathbf{t}_{2+3i}$$

$$\sum_{i=0}^{k-1} \mathbf{t}_{2+3i} = \mathbf{1}$$

We can verify that this series is optimal by noting that k is equal to the number of $\mathbf{t}_i$ required, with each corresponding to a *toggle* action, as described in the problem statement. Since you can at most toggle 3 lightbulbs at a time, then $k = n/3$ must be optimal. In other words, suppose that an algorithm exists with $j < k$ toggle actions. Since $j < n/3$, then all n lightbulbs were not turned on.

Part 4

(a)

Suppose $n \equiv 1 \pmod 3$, with the minimum possible value of n being 1. As an intermediary step in accomplishing the same task as **Part 3** (where we go from all n lightbulbs off to all n lightbulbs on, with the minimum number of toggles), we must first reach a state where there is exactly one lightbulb on. This intermediary state can be modeled by $\mathbf{a} \in V$ s.t $\mathbf{a} = (1, 0, 0, ..., 0)$, a vector where the first component is $1_{\mathbb{F}_2}$, with the rest being $0_{\mathbb{F}_2}$. As in **Part 3**, we can derive a summation that is generally equal to the vector $\mathbf{a}$:

$$\mathbf{0} + \mathbf{t}_1 + \mathbf{t}_2 + \mathbf{t}_4 + \mathbf{t}_5 + \mathbf{t}_7 + \mathbf{t}_8 + ... + \mathbf{t}_n = \mathbf{t}_1 + \mathbf{t}_2 + \mathbf{t}_4 + \mathbf{t}_5 + \mathbf{t}_7 + \mathbf{t}_8 + ... + \mathbf{t}_n$$

By definition, $n = 3k + 1$ for $k \in \mathbb{Z}_{>0}$ s.t $k = \frac{n-1}{3}$. We note that there is no subscript that is an integral multiple of 3. Doing so, we can put it into the following notation:

$$\mathbf{t}_1 + \mathbf{t}_2 + \mathbf{t}_4 + \mathbf{t}_5 + \mathbf{t}_7 + \mathbf{t}_8 + ... + \mathbf{t}_n = (\mathbf{t}_1 + \mathbf{t}_2 + \mathbf{t}_3 + ...\mathbf{t}_n) \; - \; (\mathbf{t}_3 + \mathbf{t}_6 ... + \mathbf{t}_{n-1})$$

$$(\mathbf{t}_1 + \mathbf{t}_2 + \mathbf{t}_3 + ...\mathbf{t}_n) \; - \; (\mathbf{t}_3 + \mathbf{t}_6 ... + \mathbf{t}_{n-1}) = \sum_{i=1}^{n} \mathbf{t}_i \; - \; \sum_{i=1}^{k} \mathbf{t}_{3i}$$

$$\sum_{i=1}^{n} \mathbf{t}_i \; - \; \sum_{i=1}^{k} \mathbf{t}_{3i} = \mathbf{a} = (1, 0, 0, ..., 0)$$

In words, we can consider the algorithm displayed above to be as such: begin with $\mathbf{t}_1$. For the vector $\mathbf{t}_i$ at arbitrary position i, if $i \equiv 1 \bmod 2$, we $+1$ to the subscript i to get the next subscript $i+1$, and if $i \equiv 0 \bmod 2$, we $+2$ to the subscript i to get the next subscript as $i+2$. In other words, we alternate adding 1 or 2 to each subscript to get the next, beginning at 1 and ending in n. This will always require $2k+1$ vectors $\mathbf{t}_i$.

As an example, consider when $n = 7$:

$$\mathbf{t}_1 + \mathbf{t}_2 + \mathbf{t}_4 + \mathbf{t}_5 + \mathbf{t}_7 = \mathbf{a} = (1, 0, 0, 0, 0, 0, 0)$$

As a corollary, you may also obtain the vector $\mathbf{b} \in V$, $\mathbf{b} = (0, 0, ..., 0, 1)$, simply by first adding 2 to the subscript of $\mathbf{t}_1$ rather than 1, and alternating in the same way.

(b)

Now we consider how to turn all n lights from a state where the first light is on, represented by the vector $\mathbf{a}$ we derived in part (a). Consider:

$$\mathbf{a} + \mathbf{t}_3 + \mathbf{t}_6 + \mathbf{t}_9 + ... + \mathbf{t}_{n-1} = \mathbf{a} + \sum_{i=0}^{k-1} \mathbf{t}_{3+3i} = \mathbf{1}$$

This summation should look familiar from **Part 3**. We have effectively performed the same method for toggling, but have begun at $\mathbf{t}_i$ about subscript 3 as opposed to subscript 2. We know that, the series:

$$\mathbf{t}_3 + \mathbf{t}_6 + \mathbf{t}_9 + ... + \mathbf{t}_{n-1} = \sum_{i=0}^{k-1} \mathbf{t}_{3+3i} = (0, 1, 1, ..., 1)$$

consists of k vectors $\mathbf{t}_i$. Since the previous step to obtain $\mathbf{a}$ took $2k+1$ vectors $\mathbf{t}_i$, we see that $(2k+1) + k = 3k+1 \leq n$, meaning that there were at most n vectors $\mathbf{t}_i$ involved in obtaining the vector $\mathbf{1}$. In other words, it took at most n toggles in order to turn all n lightbulbs on, where $n \equiv 1 \pmod 3$.

Part 5

(a)

To wrap up our discussion, suppose $n \equiv 2 \pmod 3$, with the minimum possible value of n being 2. We must perform a similar intermediary step as our task in **Part 4** in order to complete the equivalent task of **Part 3**; however, in this case, our intermediary vector will represent that which has 2 lightbulbs on. Let $\mathbf{c} \in V$, $\mathbf{c} = (1, 1, 0, 0, ..., 0)$.

Note that $n = 3k + 2$, $k \in \mathbb{Z}_{>0}$ s.t $k = \frac{n-2}{3}$. Recall the vector $\mathbf{b} = (0, 0, ..., 0, 1)$ from the corollary in **Part 4**, which can be obtained in the following way:

$$\mathbf{t}_2 + \mathbf{t}_3 + \mathbf{t}_5 + \mathbf{t}_6 + \mathbf{t}_8 ... + \mathbf{t}_n = \sum_{i=1}^{n} \mathbf{t}_i \; - \; \sum_{i=0}^{k} \mathbf{t}_{1+3i} = \mathbf{b}$$

Note that $2k + 1$ *toggle* vectors $\mathbf{t}_i$ were required in the following series to obtain the vector $\mathbf{b}$, equal to the number that were required to obtain the vector $\mathbf{a}$ in **Part 4**. From here, we obtain the vector $\mathbf{c}$ as follows:

$$\mathbf{b} + \mathbf{t}_1 = \mathbf{c} = (1, 1, 0, 0, ..., 0)$$

requiring in total $2k + 2$ vectors $\mathbf{t}_i$.

(b)

Finally, we perform the trivial task of turning the remaining lights on, resulting in the vector $\mathbf{1}$. This can be done as follows:

$$\mathbf{c} + \mathbf{t}_4 + \mathbf{t}_7 + ... + \mathbf{t}_{n-1} = \mathbf{c} + \sum_{i=0}^{k-1} \mathbf{t}_{4+3i} = \mathbf{1}$$

where k additional vectors $\mathbf{t}_i$ were required. We can see that a total of $(2k+2) + k = 3k+2$ vectors were required, where $3k + 2 \leq n$, meaning that at most n vectors $\mathbf{t}_i$ were required to obtain the vector $\mathbf{1}$ in the case that $n \equiv 2 \pmod 3$. This is equivalent to saying that the number of switches to go from all lights off to all lights on took at most n flips.

With that, we conclude our discussion of the lightbulb problem. $\square$