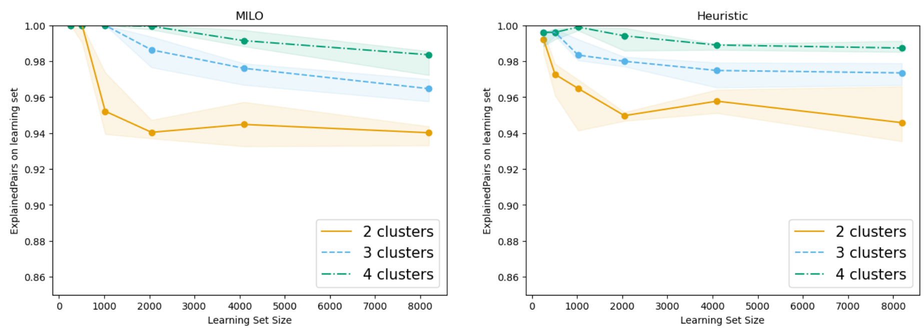
Appendices

A Additional results on synthetic experiments

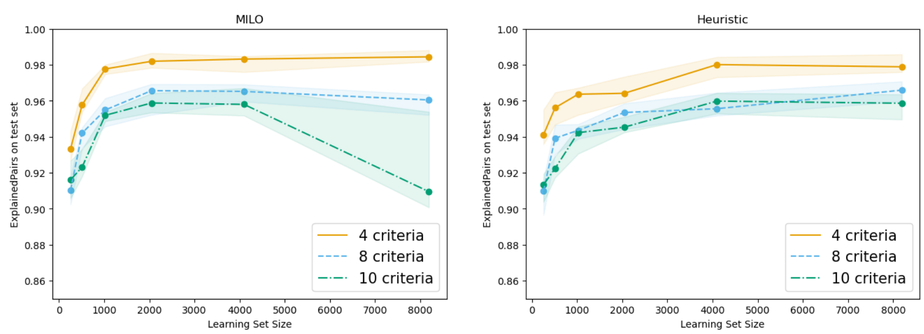
Fig. 8 below provides additional results concerning the numerical experiment on synthetic datasets; it represents, for the mathematical programming formulation and the heuristic,

- the ability to restore the learning set (Fig. 8(a)), with 8 criteria, 5% noise and varying number of cluster, and
- the performance on the test set (Fig. 8(b)), with 3 clusters, 5% noise and varying number of criteria.

The shaded areas represent the first and last quartiles.



(a) Performances on learning set with 8 criteria, 5% noise and different cluster numbers



(b) Performances on test set with 3 clusters, 5% noise and different number of criteria

Fig. 8. Proportion of ExplainedPairs (%) for different scenarios. Shaded areas represent the values of first and 3rd quartiles over experiments repetitions.

We observe, that for both algorithms, increasing the number of clusters makes it more difficult to restore the learning set (Fig. 8(a)), and that an increasing number of criteria negatively impacts the performance on the test set (it is even more the case for the MILO approach than the heuristic with large learning sets, when the timeout is reached before finding an optimal solution).

B Detailed proofs of complexity results

B.1 Detailed proof of Proposition 1

Proposition 1. *for $K \geq 3$, K-DM MRS is NP-complete.*

Proof. reduction from K-COLORABILITY [11, 5]. Given a graph $\mathcal{G} = (V,E)$, we produce an instance $(N,\mathcal{P})$ of K-DM MRS as follows.

- *Criteria:* let $N := V \cup \{\tau_0,\tau_1,\tau_2\} \cup \bigcup_{k\in[K]}\{\omega_k\}$. We note $\Omega_{-k} := \bigcup_{k'\in[K]:k'\neq k}\{\omega_k\}$ and $N_{-k} := N \setminus \{\omega_k\}$.
- *Preference statements:* let $\mathcal{P} := \bigcup_{v\in V}\pi_v \cup \bigcup_{k\in[K]}\mathcal{P}_k^E \cup \bigcup_{k\in[K]}\mathcal{P}_k^\tau \cup \bigcup_{k\in K}\pi_k$ where:
  - for all $v \in V$, π_v denotes the statement $\{v\} \succ \{\tau_1\}$;
  - for all $k \in [K]$, $\mathcal{P}_k^E$ and $\mathcal{P}_k^\tau$ denote the following collections of statements $\mathcal{P}_k^E := \bigcup_{\{u,v\}\in E} \{\{\tau_2\} \succ \{u,v\} \cup \Omega_{-k}\}\}$, $\mathcal{P}_k^\tau := \{\{\tau_1\} \succ \{\tau_0\} \cup \Omega_{-k}, \{\tau_0,\tau_1\} \succ \{\tau_2\} \cup \Omega_{-k}\}$
  - finally, for all $k \in [K]$, π_k^ω denotes the statement $\{\omega_k\} \succ N_{-k}$.
- *Claim:* $\mathcal{G}$ is a positive instance of K-COLORABILITY $\iff$ $N,\mathcal{P}$ is a positive instance of K-DM MRS.
- *Hint:* observe the statements $\bigcup_{k\in[K]}\pi_k^\omega$ are pairwise inconsistent, so each segment contains exactly one of those, defining the color k associated to the vertices v whose statement π_v appear in the same bundle as π_k^ω. Statements in $\mathcal{P}_k^E$ or $\mathcal{P}_k^\tau$ are pairwise inconsistent with $\pi_{k'}^\omega$ with $k \neq k'$ thus appear in the same segment as π_k^ω and ensure the coloring is correct.

Proof of $\Rightarrow$. Let $V_1,\ldots V_K$ be a partition of V into K colors such that no two vertices forming an edge have the same color. Let $\mathcal{P}_k := \bigcup_{v\in V_k}\pi_v \cup \mathcal{P}_k^E \cup \mathcal{P}_k^\tau \cup \pi_k$. The collections of statements $\mathcal{P}_k$ partition $\mathcal{P}$ by construction. Moreover, let $W_k(v) = \begin{cases} 1, \text{if } v \in V_k \\ 0, \text{else} \end{cases}$, $W_k(\omega_{k'}) = \begin{cases} 1, \text{if } k = k' \\ 0, \text{else} \end{cases}$, $W_k(\tau_0) = 0.8$, $W_k(\tau_1) = 0.9$, $W_k(\tau_2) = 1.6$. These weights obviously represent the statements $\bigcup_{v\in V_k}\pi_v \cup \mathcal{P}_k^\tau \cup \pi_k$ in the majority rule. Suppose they do not represent a statement of $\mathcal{P}_k^E$, namely $\{\tau_2\} \succ \{u,v\} \cup \Omega_{-k}$ with $\{u,v\} \in E$. Hence $1.6 \leq W_k(u) + W_k(v) + 0$. As $W_k(u), W_k(v) \in \{0,1\}$, this is only possible if $W_k(u) = 1 = W_k(v)$, i.e. if u and v have the same color k: a contradiction.

Proof of $\Leftarrow$. Suppose $N,\mathcal{P}$ is a positive instance of K-DM MRS . Observe the statements π_k^ω and $\pi_{k'}^\omega$ with $k \neq k'$ are pairwise incompatible with the majority rule. Thus, there is exactly one of those statement in each subset of the K-partition of $\mathcal{P}$. W.l.o.g. we denote $\mathcal{P}_k$ the subset containing the statement π_k^ω. Each vertex $v \in V$ can be mapped to the color $k \in [K]$ corresponding to the subset $\mathcal{P}_k$ containing the statement π_v. This mapping is a correct K-coloring of $\mathcal{G}$. Indeed, suppose there are two vertices u,v with the same color k and forming an edge $\{u,v\} \in E$. Each statement in $\mathcal{P}_k^E \cup \mathcal{P}_k^\tau$ is incompatible to $\pi_{k'}^\omega$ when $k' \neq k$. Hence, $\mathcal{P}_k^E \cup \mathcal{P}_k^\tau \subset \mathcal{P}_k$. Thus, $\mathcal{P}_k$ contains the statements $\{u\} \succ \{\tau_1\}$, $\{v\} \succ \{\tau_1\}$, $\{\tau_2\} \succ \{u,v\} \cup \Omega_{-k}, \{\tau_1\} \succ \{\tau_0\} \cup \Omega_{-k}, \{\tau_0,\tau_1\} \succ \{\tau_2\} \cup \Omega_{-k}$. Their representation with a function W_k in the majority rule model entails: $W_k(u) > W_k(\tau_1)$, $W_k(v) > W_k(\tau_1)$, $W_k(\tau_2) > W_k(u) + W_k(v)$, $W_k(\tau_1) > W_k(\tau_0)$ and $W_k(\tau_0) + W_k(\tau_1) > W_k(\tau_2)$. Summation of those comparisons yields $W_k(u) + W_k(v) + W_k(\tau_0) + 2W_k(\tau_1) + W_k(\tau_2) > W_k(u) + W_k(v) + W_k(\tau_0) + 2W_k(\tau_1) + W_k(\tau_2)$: a contradiction.

B.2 Detailed proof of Proposition 2

Proposition 2. *2-DM MRS is NP-complete.*

Proof. reduction from VERTEX COVER [11, 5]. Given a graph $\mathcal{G}=(V,E)$ and a positive integer $K<|V|$, we produce an instance $(N,\mathcal{P})$ of 2-DM MRS as follows.

- *Criteria:* Let $N := V\cup\{\alpha,\beta,\omega_1,\omega_2\}\cup\{\gamma_1,...,\gamma_K\}$, such that $|N|=|V|+K+4<2|V|+4$.
- *Preference statements:* let $\mathcal{P} := \bigcup_{v\in V}\{\pi_v^+,\pi_v^-\}\cup\bigcup_{\{u,v\}\in E}\pi^E_{\{u,v\}}\cup\{\pi_2^\alpha,\pi_1^\omega,\pi_2^\omega\}\cup\mathcal{P}_1^\alpha$ where:
  - for all vertices $v\in V$, $\pi_v^+ : \{v\}\succ\{\alpha\}$ and $\pi_v^- : \{\beta\}\succ\{v\}$;
  - for all edges $\{u,v\}\in E$, $\pi^E_{u,v} : \{u,v\}\succ\{\alpha,\beta,\omega_2\}$;
  - $\pi_1^\omega : \{\omega_1\}\succ V\cup\{\omega_2,\alpha,\beta\}$, $\pi_2^\omega : \{\omega_2\}\succ V\cup\{\omega_1,\alpha,\beta\}$ and $\pi_2^\alpha : \{\alpha\}\succ\{\beta,\omega_1\}$;
  - $\mathcal{P}_1^\alpha : \big\{\{\alpha\}\succ\{\beta,\omega_2\},\{\alpha\}\cup\bigcup_{k\in[K]}\{\gamma_k\}\succ V\cup\{\omega_2\}\big\}\cup\bigcup_{k=1}^K\{\{\alpha\}\succ\{\gamma_k,\omega_2\}\}$.
- *Claim:* $\mathcal{G}$ has a VERTEX COVER (i.e. a subset $V'\subset V$ such that for each edge $\{u,v\}\in E$ at least one of u and v belongs to V') of size at most $K \iff N,\mathcal{P}$ is a positive instance of 2-DM MRS.
- *Hint:* Observe the statements π_1^ω and π_2^ω cannot appear in the same segment. Thus, comparative statements with ω_1 (resp. ω_2) at the RHS are bundled together. Meanwhile, for each vertex $v\in V$, statements π_v^+ and π_v^- are pairwise inconsistent and appear in distinct bundles. The set $\{v\in V : \pi_v^+$ appears in the same segment as $\pi_1^\omega\}$ is a vertex cover of size at most K.

Proof of $\Rightarrow$. Let $\mathcal{P}_1 := \bigcup_{v\in V'}\pi_v^+\cup\bigcup_{v\notin V'}\pi_v^-\cup\mathcal{P}^E\cup\mathcal{P}_1^\alpha\cup\pi_1^\omega$ and $\mathcal{P}_2 := \bigcup_{v\notin V'}\pi_v^+\cup\bigcup_{v\in V'}\pi_v^-\cup\pi_2^\alpha\cup\pi_2^\omega$. $\mathcal{P}_1,\mathcal{P}_2$ is a bipartition of $\mathcal{P}$. Moreover, let $W_1(v)=\begin{cases}1,\text{if } v\in V'\\0,\text{else}\end{cases}$, $W_1(\alpha)=1-\frac{1}{3K}$, $W_1(\beta)=\frac{1}{6K}$, $W_1(\gamma_k)=1-\frac{1}{2K}$, $W_1(\omega_1)=|V|+2$, $W_1(\omega_2)=0$ and $W_2(v)=\begin{cases}0,\text{if } v\in V'\\1,\text{else}\end{cases}$, $W_2(\alpha)=0.9$, $W_2(\beta)=0.1$, $W_2(\gamma_k)=81$, $W_2(\omega_1)=0$, $W_2(\omega_2)=|V|+2$. The correct representation of the collection of statements $\mathcal{P}_2$ by W_2 in the majority rule model is obvious, as well as the representation of all statements in $\mathcal{P}_1$ by W_1 but those of $\mathcal{P}^E$ and $\{\alpha\}\cup\bigcup_{k\in[K]}\{\gamma_k\}\succ V\cup\{\omega_2\}$. Suppose $\pi^E_{u,v}$ is not correctly represented, i.e. $W_1(u)+W_1(v)\le W_1(\alpha)+W_1(\beta)+W_1(\omega_2)$ or $W_1(u)+W_1(v)\le(1-\frac{1}{3K})+\frac{1}{6K}+0=1-\frac{1}{6K}<1$. As $W_1(u),W_1(v)\in\{0,1\}$, this is only possible if $W_1(u)=W_1(v)=0$, meaning neither u nor v belongs to V', leaving the edge $\{u,v\}\in E$ uncovered: a contradiction. Moreover $|V'|\le K$, hence $\sum_{v\in V}W_i(v)+W_1(\omega_2)=\sum_{v\in V'}1+\sum_{v\notin V'}0+0=|V'|\le K$. On the other hand, $W_1(\alpha)+\sum_{k\in[K]}W_1(\gamma_k)=1-\frac{1}{3K}+K\times(1-\frac{1}{2K})=K+1-\frac{1}{2}-\frac{1}{3K}>K$. Thus, the statement $\{\alpha\}\cup\bigcup_{k\in[K]}\{\gamma_k\}\succ V\cup\{\omega_2\}$ is correctly represented in the majority rule model by W_1.

Proof of $\Leftarrow$. Let $\mathcal{P}_1,\mathcal{P}_2$ the partition of the preference statements $\mathcal{P}$ into two subsets, each represented in the majority rule model with the function W. As π_1^ω and π_2^ω are pairwise inconsistent with the majority rule, let $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) the collection of statements containing π_1^ω (resp. π_2^ω). Observe that each statement in $\mathcal{P}_1^\alpha$ is inconsistent with π_2^ω in the majority rule model, thus $\mathcal{P}_1^\alpha\subset\mathcal{P}_1$. Observe that π_2^α is pairwise inconsistent with π_1^ω, thus $\pi_2^\alpha\in\mathcal{P}_2$. Define $V' := \{v\in V : \pi_v^+\in\mathcal{P}_1\}$.

Suppose V' is not a cover of $\mathcal{G}$, i.e. there is an edge $\{u,v\}\in E$ such that neither u nor v belongs to V'. As the comparative statement $\pi^E_{u,v}$ is incompatible to π_2^ω, it belongs to $\mathcal{P}_1$, as well as the statements $\mathcal{P}^\alpha$. The set of statements $\{\pi^E_{u,v},\pi_u^-,\pi_v^-\}\cup\mathcal{P}^\alpha$

is inconsistent with the majority rule. Hence, either π_u^- or π_v^- belongs to $\mathcal{P}_2$. W.l.o.g. suppose $\pi_u^- \in \mathcal{P}_2$. Then $\pi_u^+ \notin \mathcal{P}_2$, because π_u^-, π_u^+ and π_2^α cannot be represented together in the majority rule model. Thus $\pi_u^+ \in \mathcal{P}_1$: a contradiction.

Suppose $|V'| > K$, i.e. $|V'| \geq K+1$. From each statement $\{\alpha\} \succ \{\gamma_k, \omega_2\} \in \mathcal{P}_1$ we obtain the comparison $W_1(\alpha) > W_1(\gamma_k)$ for all $k \in [K]$. Hence, summation over $k \in [K]$ yields $K \times W_1(\alpha) > \sum_{k\in[K]} W_1(\gamma_k)$. As the statement $\{\alpha\} \cup \bigcup_{k\in[K]}\{\gamma_k\} \succ V \cup \{\omega_2\}$ belongs to $\mathcal{P}_1$, we have that $W_1(\alpha) + \sum_{k\in[K]} W_1(\gamma_k) > \sum_{v\in V} W_1(v)$. By transitivity, $\sum_{v\in V} W_1(v) < (K+1) \times W_1(\alpha)$. On the other hand, from $\pi_v^+ \in \mathcal{P}_1$ for $v \in V'$ we obtain $v \in V' \Rightarrow W_1(v) > W_1(\alpha)$. Summing up this comparisons over $v \in V$, we obtain $\sum_{v\in V} W_1(v) \geq \sum_{v\in V'} W_1(v) > \sum_{v\in V'} W_1(\alpha) = |V'| \times W_1(\alpha) \geq (K+1) \times W_1(\alpha)$: a contradiction.