

# W203, Test 1

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Q 2.1 Solve for the constant

Given that

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x, y) dx dy = 1 \quad (1)$$

we can write for this case:

$$\int_{x=0}^{x=2} \int_{y=0}^{y=2} c * x^2 y dx dy = 1 \quad (2)$$

consequently:

$$c * \left| \frac{x^3}{3} \right|_{x=0}^{x=2} \left| \frac{y^2}{2} \right|_{y=0}^{y=2} = \frac{c}{6} * 2^3 * 2^2 = 1 \quad (3)$$

Therefore

$$c = \frac{3}{16}$$

Q 2.2 Probability a la mode

We can derive PDF for Y from joint PDF of X and Y by finding  $f_Y(y)$ , marginal PDF for Y:

$$f_Y(y) = \int_0^2 f_{X,Y}(x, y) dx \quad (4)$$

$$\begin{aligned} f_Y(y) &= \int_0^2 \frac{3}{16} x^2 y dx = \frac{3}{16} * \left| \frac{x^3}{3} * y \right|_{x=0}^{x=2} = \\ &= \begin{cases} \frac{y}{2} & 0 \leq y \leq 2 \\ 0, & elsewhere \end{cases} \end{aligned}$$

Since this is a continuous differentiable function on the interval  $[0; 2]$ , it should have maximum on this interval (Extreme value theorem). Since this is an increasing linear function, it will have its maximum at the maximum of its argument, i.e

$$f_{Y_{max}}(y) = \frac{2}{2} = 1$$

Q 2.3 Expectation of Y,  $E[Y]$

By definition, expectation of a continuous random variable:

$$E[Y] = \int_{-\infty}^{+\infty} y * f_Y(y) dy \quad (5)$$

applying this definition to our case, using Eq. 4:

$$E[Y] = \int_0^2 y * \frac{y}{2} dy = \left| \frac{y^3}{2 * 3} \right|_0^2 = \frac{4}{3} \quad (6)$$

Q 2.4 Variance of Y,  $V[Y]$

Using a simple modification of the definition for variance of a continuous random variable, and applying LOTUS and result 6:

$$V[Y] = E[y^2] - E^2[y] = \int_0^2 y^2 \frac{y}{2} dy - \left( \frac{4}{3} \right)^2 = \left| \frac{y^4}{4 * 2} \right|_0^2 - \frac{16}{9} = \frac{8}{31} \quad (7)$$