W203, Test 1

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Q 2.1 Solve for the constant
Given that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = 1 \tag{1}$$

we can write for this case:

$$\int_{x=0}^{x=2}\int_{y=0}^{y=2} c * x^2\,y\,dx\,dy = 1 \tag{2}$$

consequently:

$$c * \Big|_{x=0}^{x=2}\frac{x^3}{3}\Big|_{y=0}^{y=2}\frac{y^2}{2} = \frac{c}{6} * 2^3 * 2^2 = 1 \tag{3}$$

Therefore

$$c = \frac{3}{16}$$

Q 2.2 Probability a la mode
We can derive PDF for Y from joint PDF of X and Y by finding $f_Y(y)$, marginal PDF for Y:

$$f_Y(y) = \int_0^2 f_{X,Y}(x,y)\,dx \tag{4}$$

$$f_Y(y) = \int_0^2 \frac{3}{16}\,x^2\,y\,dx = \frac{3}{16} * \Big|_{x=0}^{x=2}\frac{x^3 * y}{3} =$$
$$= \begin{cases} \frac{y}{2}\ 0 \leq y \leq 2 \\ 0,\ elsewhere \end{cases}$$

Since this is a continuous differnetiable function on the interval $[0;2]$, it should have maxiumum on this interval (Extreme value theorem). Since this is an increasing linear function, it will have its maximum at the maximum of its argument, i.e

$$f_{Y_{max}}(y) = \frac{2}{2} = 1$$

Q 2.3 Expectation of Y, E[Y]
By definition, expectation of a continuous random variable:

$$E[Y] = \int_{-\infty}^{+\infty} y * f_Y(y)\,dy \tag{5}$$

applying this definition to our case, using Eq. 4:

$$E[Y] = \int_0^2 y * \frac{y}{2}\,dy = \Big|_0^2 \frac{y^3}{2*3} = \frac{4}{3} \tag{6}$$

Q 2.4 Variance of Y, E[Y]
Using a simple modification of the definition for variance of a continuous random variable, and applying LOTUS and result 6:

$$V[Y] = E[y^2] - E^2[y] = \int_0^2 y^2\,\frac{y}{2}\,dy - \left(\frac{4}{3}\right)^2 = \Big|_0^2\left[\frac{y^4}{4*2}\right] - \frac{16}{9} = \frac{8}{31} \tag{7}$$