2.1

$$\dot{x} = Ax + Bu$$

$$\sigma = Gx \qquad u_{eq} - ?$$

$$\dot{\sigma} = \nabla_x \sigma \cdot \dot{x} = G(Ax + Bu) = 0$$

a) $A = \begin{bmatrix} 2 & 19 \\ 3 & 29 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $G = \begin{bmatrix} 9 & 12 \end{bmatrix}$

$$\begin{bmatrix} 9 & 12 \end{bmatrix} \begin{bmatrix} 2 & 19 \\ 3 & 29 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 9 & 12 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} u = 0$$

$$54x_1 + 519x_2 + 54u = 0 \Rightarrow u = -x_1 - \frac{519}{54}x_2$$

b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 9 \\ 1 & -2 \\ -1 & 0 \end{bmatrix}$ $G = \begin{bmatrix} 1 & 29 & 0 \\ 1 & 12 & 0 \end{bmatrix}$

$$u = -(GB)^{-1} GAx = \begin{bmatrix} -0.133x_1 - 0.733x_2 - 1.93x_3 \\ -0.667x_1 + 0.133x_2 + 0.533x_3 \end{bmatrix}$$

2.2

$$\dot{x}_1 = x_1 + x_2 + x_3 + u_1 + 10u_2$$

$$\dot{x}_2 = x_2 + 3x_3 + u_1 - 2u_2 .$$

$$\dot{x}_3 = x_1 + x_3 - u_1$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 10 \\ 1 & -2 \\ -1 & 0 \end{bmatrix}$$

$$\dot{x} = Ax + Bu$$

$$\sigma_1 = x_1 + 10x_2$$

$$\sigma_2 = x_1 + 5x_2$$

$$\sigma = \begin{bmatrix} 1 & 10 & 0 \\ 1 & 5 & 0 \end{bmatrix} x \quad , \quad G = \begin{bmatrix} 1 & 10 & 0 \\ 1 & 5 & 0 \end{bmatrix}$$

$$\dot{\sigma} = \nabla_x \sigma \cdot \dot{x} = G \cdot \dot{x} = G(Ax + Bu) = 0$$

$$u = -(GB)^{-1} GAx = \begin{bmatrix} -0.167x_1 - x_2 - 2.67x_3 \\ -0.0833x_1 + 0.167x_3 \end{bmatrix}$$

2.3

$$L\dot{i} = -v + v_{in} \cdot u \qquad \dot{x}_1 = \frac{1}{L}\left(-x_2 + V_{in} u\right)$$

$$C\dot{v} = i - \frac{v}{R} \qquad \dot{x}_2 = \frac{1}{C}\left(x_1 - \frac{x_2}{R}\right) \qquad \sigma = x_1 - i_{des}$$

$$\dot{\sigma} = \nabla_x \sigma \cdot \dot{x} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \dot{x}_1 = \frac{1}{L}\left(-x_2 + V_{in} u\right)$$

$$\frac{1}{L}\left(-x_2 + V_{in} u_{eq}\right) = 0 \Rightarrow u_{eq} = \frac{1}{V_{in}} x_2$$

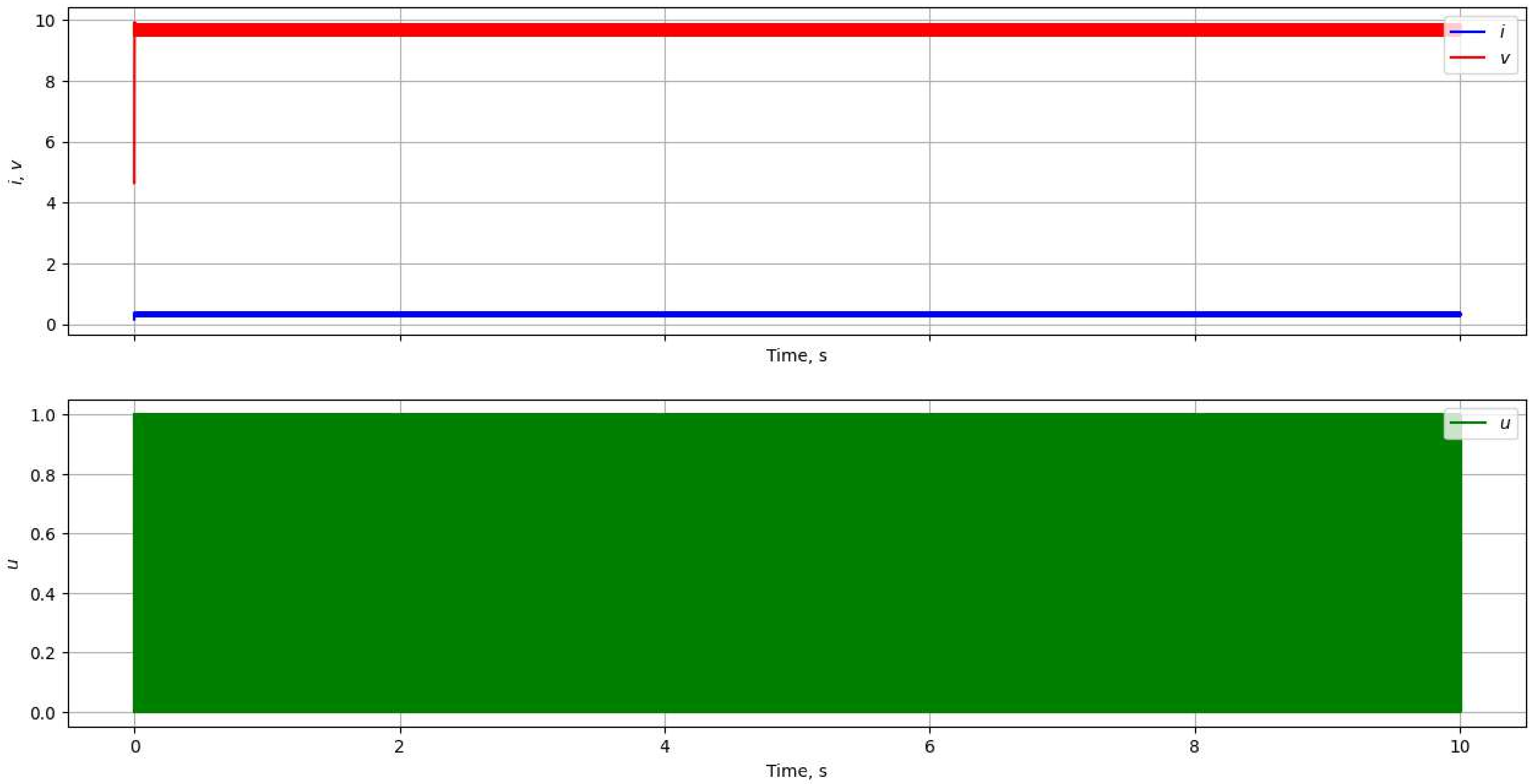
$$\dot{x}_1 = 0$$

$$\dot{x}_2 = \frac{1}{C}\left(x_1 - \frac{x_2}{R}\right) = \frac{1}{C}\left(\right.$$

$$x_{2\,eq} = x_{1\,eq} \cdot R = i_{des} \cdot R$$

$$\dot{x}_2 = \frac{1}{C}\left(i_{des} + \delta x_1 - \frac{i_{des} R + \delta x_2}{R}\right) = \frac{1}{C}\left(\cancelto{0}{\delta x_1} - \frac{1}{R}\delta x_2\right) = -\frac{1}{RC}\delta x_2$$

System is stable



10
8
6
4
2
0
i, v
i
v
Time, s
1.0
0.8
0.6
0.4
0.2
0.0
u
u
0
2
4
6
8
10
Time, s

2.5

a) $\dot{x}_1 = u$

$\dot{x}_2 = (2u^2 - 1)x_2$

$\sigma = x_1$

$u = -\mathrm{sign}(\sigma)$

$\nabla_x \sigma = (1 \quad 0)$

$\dot{\sigma} = \nabla_x \sigma \cdot \dot{x} = (1 \quad 0)\begin{pmatrix} u \\ (2u^2-1)x_2 \end{pmatrix} = u$

$\dot{\sigma} = 0 \Rightarrow u_{eq} = 0 \Rightarrow \dot{x}_1 = 0$ Stable system

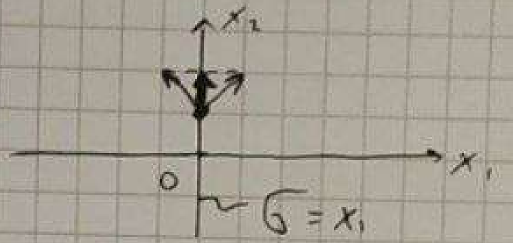
$\dot{x}_2 = -x_2$

Filippov method:

$\dot{x}_1 = -\mathrm{sign}\,\sigma$

$\dot{x}_2 = x_2$

$f_- = \begin{pmatrix} +1 \\ x_2 \end{pmatrix} \quad f_+ = \begin{pmatrix} -1 \\ x_2 \end{pmatrix}$



system is not stable

b) $\dot{x}_1 = u$

$\dot{x}_2 = (u - 2u^3)x_2$

$\nabla_x \sigma = (1 \quad 0)$

$\dot{\sigma} = \nabla_x \sigma \cdot \dot{x} = (1 \quad 0)\begin{pmatrix} u \\ (u-2u^3)x_2 \end{pmatrix} = u$

$u_{eq} = 0 \Rightarrow \dot{x}_1 = 0$ stable, but

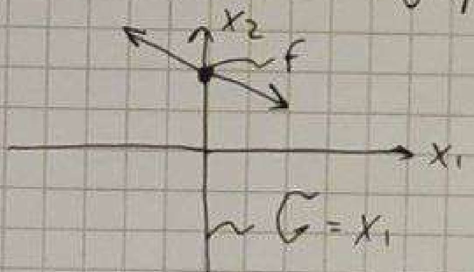
$\dot{x}_2 = 0$ not asymptotically

Filippov method:

$\dot{x}_1 = -\mathrm{sign}\,\sigma$

$\dot{x}_2 = u(1 - 2u^2)x_2 = \mathrm{sign}\,\sigma\, x_2$

$f_- = \begin{pmatrix} 1 \\ -x_2 \end{pmatrix} \quad f_+ = \begin{pmatrix} -1 \\ x_2 \end{pmatrix}$



system is stable

c) $\dot{x}_1 = u$

$\dot{x}_2 = (u - 2u^2)x_2$

$\dot{\sigma} = \nabla_x \sigma \cdot \dot{x} = (1 \quad 0)\begin{pmatrix} u \\ (u-2u^2)x_2 \end{pmatrix} = u$

$\dot{\sigma} = 0 \Rightarrow u_{eq} = 0 \Rightarrow \dot{x}_1 = 0$ $\Big|$ → stable, but

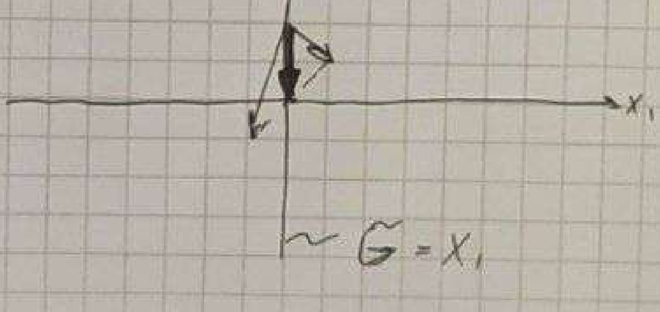
$\dot{x}_2 = 0$ $\Big|$ not asymptotically

Filippov method:

$\dot{x}_1 = -\mathrm{sign}\,\sigma$

$\dot{x}_2 = (-\mathrm{sign}\,\sigma - 2)x_2$

$f_- = \begin{pmatrix} 1 \\ -x_2 \end{pmatrix} \quad f_+ = \begin{pmatrix} -1 \\ -3x_2 \end{pmatrix}$



system is stable

2.9

$$I_1 \dot{\omega}_1 = (I_2 - I_3)\,\omega_2 \omega_3 + u_1 \qquad u = u_0 + u_1$$
$$I_2 \dot{\omega}_2 = (I_3 - I_1)\,\omega_3 \omega_1 + u_2$$
$$I_3 \dot{\omega}_3 = (I_1 - I_2)\,\omega_1 \omega_2 + u_3$$

$$A = 0, \quad B = \begin{bmatrix} \frac{1}{I_1} & 0 & 0 \\ 0 & \frac{1}{I_2} & 0 \\ 0 & 0 & \frac{1}{I_3} \end{bmatrix} \quad \varphi = \begin{bmatrix} (I_2 - I_3)\,\omega_2 \omega_3 \\ (I_3 - I_1)\,\omega_3 \omega_1 \\ (I_1 - I_2)\,\omega_1 \omega_2 \end{bmatrix}$$

$$\dot{x} = \cancel{Ax}^{\,0} + B(u + \varphi)$$

$$u_1 = -\alpha\, \mathrm{sign}(\sigma)$$
$$\sigma = G(x(t) - x(0)) + G \int_0^t B u_0 \, d\tau, \qquad G = B^{-1}.$$

$$\cancel{A^T P}^{\,0} + \cancel{PA}^{\,0} - PBR^{-1}B^T P = -Q \sim \text{LQR controller}$$
$$K = R^{-1} B^T P$$

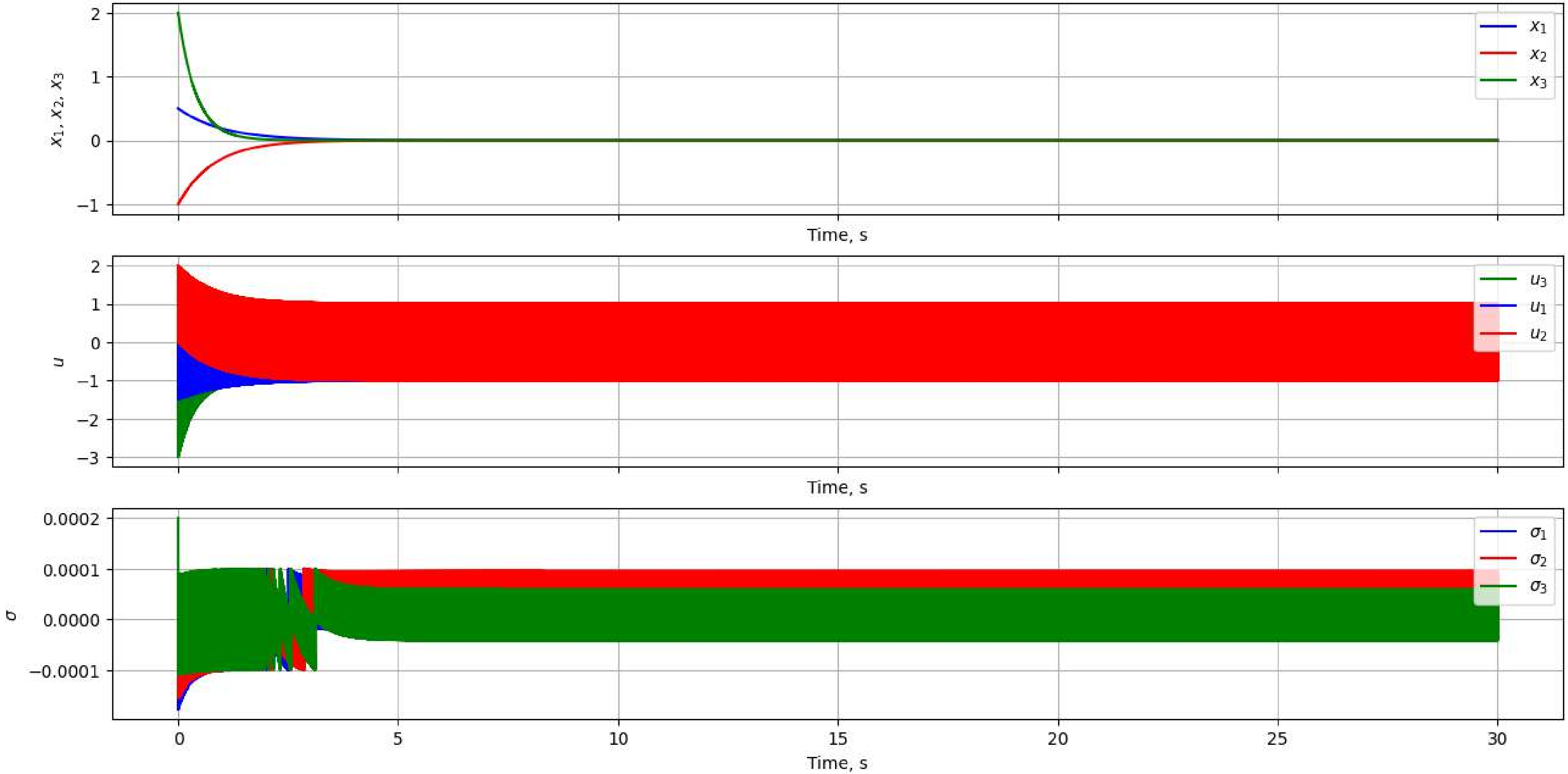
$$\left.\begin{aligned} Q &= E \\ R &= E \end{aligned}\right| \Rightarrow PBB^T P = E$$

$$P = B^{-1} \Rightarrow B^{-1} B \cdot B^T B^{-1} = E$$

$$K = B^T \cdot B^{-1} = E$$

$$u_0 = -Kx = -Ex$$

$$x(0) = \begin{bmatrix} 0.5 \\ -1.0 \\ 2 \end{bmatrix} \quad I_1 = 1, \; I_2 = 0.8, \; I_3 = 0.4$$



2
1
0
−1
x_1, x_2, x_3
x_1
x_2
x_3
Time, s
2
1
0
−1
−2
−3
u
u_3
u_1
u_2
Time, s
0.0002
0.0001
0.0000
−0.0001
σ
σ_1
σ_2
σ_3
0
5
10
15
20
25
30
Time, s