# hw4

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# 1 Assignment 4 - Control Theory (Artem Bakhanov)

I have variant A and I used "Artem" as my name and "a.bahanov@innopolis.university" as my email for obtaining the variant. I am very lazy and I used Python for all the symbolic calculations (I love SymPy library).

## 1.1 Initial setup

Here I create symbolic variables I will work with.

[1]:
```
import sympy as sp
import numpy as np

# x and its first and second derivatives
x = sp.Symbol("x")
x_ = sp.Symbol("x'")
x__ = sp.Symbol("x''")

# theta and its first and second derivatives
theta = sp.Symbol("theta")
theta_ = sp.Symbol("\\theta^\prime")
theta__ = sp.Symbol("\\theta^{\prime\prime}")

# other parameters
m = sp.Symbol("m")
M = sp.Symbol("M")
l = sp.Symbol("l")
F = sp.Symbol("F")
g = sp.Symbol("g")
```

## 1.2 Task A

In this task I found matrices $M(q)$, $n(q, \dot{q})$ and $B$.

$$M = \begin{pmatrix} M + m & -lm\cos(\theta) \\ -\cos(\theta) & l \end{pmatrix} = \begin{pmatrix} 8.5 & -3.68\cos(\theta) \\ -\cos(\theta) & 1.15 \end{pmatrix}$$

$$n = \begin{pmatrix} lm \cdot (\theta')^2 \sin(\theta) \\ -g\sin(\theta) \end{pmatrix} = \begin{pmatrix} 3.68 \cdot (\theta')^2 \sin(\theta) \\ -g\sin(\theta) \end{pmatrix}$$

$$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

[2]:
```
M_ = sp.Matrix([[M + m, -m * l * sp.cos(theta)], [-sp.cos(theta), l]])
n_ = sp.Matrix([[m * l * sp.sin(theta) * theta_ ** 2], [-g * sp.sin(theta)]])
B_ = sp.Matrix([[1], [0]])
```

### 1.3 Task B

Since I am very lazy to calculate everything by hand I found the inverse of M. Since f and g are just vectors (actually vector functions but anyway), the first two elements of f are simply $x$ and $\theta$ and ones of g are 0s. The problem is with finding the last two elements. Let us use the result of task A.

$$M(q)\ddot{q} + n(q, \dot{q}) = Bu$$

$$M(q)\ddot{q} = -n(q, \dot{q}) + Bu$$

$$\ddot{q} = M(q)^{-1}(-n(q, \dot{q}) + Bu)$$

where $q = \begin{bmatrix} x & \theta \end{bmatrix}^T$. Since $\ddot{q}$ is what we are trying to find, $-M(q)^{-1}n(q, \dot{q})$ and $M(q)^{-1}B$ are the last elements of $f$ and $g$ respectively. So the final answer:

$$f(z) = \begin{bmatrix} x' \\ \theta' \\ \frac{m\left(g\cos(\theta) - l(\theta')^2\right)\sin(\theta)}{M + m\sin^2(\theta)} \\ \frac{\left(g(M+m) - lm(\theta')^2\cos(\theta)\right)\sin(\theta)}{l\left(M + m\sin^2(\theta)\right)} \end{bmatrix} = \begin{bmatrix} x' \\ \theta' \\ \frac{3.2\left(g\cos(\theta) - 1.15(\theta')^2\right)\sin(\theta)}{3.2\sin^2(\theta) + 5.3} \\ \frac{0.87\left(8.5g - 3.68(\theta')^2\cos(\theta)\right)\sin(\theta)}{3.2\sin^2(\theta) + 5.3} \end{bmatrix}$$

$$g(z) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M + m\sin^2(\theta)} \\ \frac{\cos(\theta)}{l\left(M + m\sin^2(\theta)\right)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3.2\sin^2(\theta) + 5.3} \\ \frac{0.87\cos(\theta)}{3.2\sin^2(\theta) + 5.3} \end{bmatrix}$$

[3]:
```
from IPython.core.interactiveshell import InteractiveShell
InteractiveShell.ast_node_interactivity = 'all'
subs = dict(zip([m, M, l], [3.2, 5.3, 1.15]))

# find M inverse
M_inv = sp.simplify(M_.inv())
M_inv
```

[3]:

$$\begin{bmatrix} \frac{1}{M + m\sin^2(\theta)} & \frac{m\cos(\theta)}{M + m\sin^2(\theta)} \\ \frac{\cos(\theta)}{l\left(M + m\sin^2(\theta)\right)} & \frac{M + m}{l\left(M + m\sin^2(\theta)\right)} \end{bmatrix}$$

[4]:
```
f_2 = sp.simplify(-M_inv * n_)  # the last two elements of f
g_2 = sp.simplify(M_inv * B_)  # the last two elements of g
```

```python
# f and g
f_ = sp.simplify(sp.Matrix([[x_], [theta_], [f_2[0]], [f_2[1]]]))
g_ = sp.simplify(sp.Matrix([[0], [0], [g_2[0]], [g_2[1]]]))

# printing f and g out
f_
g_
```

[4]:

$$\begin{bmatrix} x' \\ \theta' \\ \frac{m\left(-(\theta')^2 l + g\cos(\theta)\right)\sin(\theta)}{M + m\sin^2(\theta)} \\ -\frac{\left((\theta')^2 lm\cos(\theta) - g(M+m)\right)\sin(\theta)}{l\left(M + m\sin^2(\theta)\right)} \end{bmatrix}$$

[4]:

$$\begin{bmatrix} 0 \\ 0 \\ \frac{1}{M + m\sin^2(\theta)} \\ \frac{\cos(\theta)}{l\left(M + m\sin^2(\theta)\right)} \end{bmatrix}$$

### 1.4 Task C

Linearazation was made automatically again.

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{gm}{M} & 0 & 0 \\ 0 & \frac{g(M+m)}{Ml} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 5.92 & 0 & 0 \\ 0 & 13.68 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ \frac{1}{Ml} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.19 \\ 0.16 \end{bmatrix}$$

[5]:

```python
# states
states = [x, theta, x_, theta_]

# input
inputs = [F]

# matrices Ad and Bd
# here I create a Jacobian matrix.
# Each line is a partial derivative of f_i (f)
Ad = sp.Matrix([[sp.diff(f, x) for x in states] for f in f_])

# Bd is simply g since it is already multiplied by u. There is no need to
    calculate Jacobian.
Bd = g_
```

```python
# dict of subtitutions for desired states
desired_states = dict(zip([x, theta, x_, theta_], [0, 0, 0, 0]))

# all substitutions (check variant A)
subs = dict(zip([m, M, l, g], [3.2, 5.3, 1.15, 9.81]))

# matrix A and B (without parameters substituting)
Ad.subs(desired_states)
Bd.subs(desired_states)
```

[5]:
$$\begin{bmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\0 & \frac{gm}{M} & 0 & 0\\0 & \frac{g(M+m)}{Ml} & 0 & 0\end{bmatrix}$$

[5]:
$$\begin{bmatrix}0\\0\\\frac{1}{M}\\\frac{1}{Ml}\end{bmatrix}$$

[6]:
```python
# final matrices with substituted numbers

A_ = Ad.subs(subs).subs(desired_states)
B_ = Bd.subs(subs).subs(desired_states)
A = np.array(A_.tolist()).astype(np.float64)
B = np.array(B_.tolist()).astype(np.float64)
A_
B_
```

[6]:
$$\begin{bmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\0 & 5.92301886792453 & 0 & 0\\0 & 13.6808859721083 & 0 & 0\end{bmatrix}$$

[6]:
$$\begin{bmatrix}0\\0\\0.188679245283019\\0.164068908941756\end{bmatrix}$$

### 1.5 Task D

Here I use test presented on the lab. The system is controllable if $ rank([B AB … A^{n-1}B)]) = n $. In our case n = 4. Let us verify it.

[7]:
```python
assert(np.linalg.matrix_rank(np.block([[B, A.dot(B), A.dot(A).dot(B), A.dot(A).
↪dot(A).dot(B)]])) == 4)
```

No exception! It means that the system is controllable.

## 1.6 Task E

Calculating the eigen values of the linearized system.

[8]:
```
np.linalg.eig(A)[0]
```

[8]:
```
array([ 0.        ,  0.        ,  3.69876817, -3.69876817])
```

The system is not stable since there is one eigen value with positive real part.

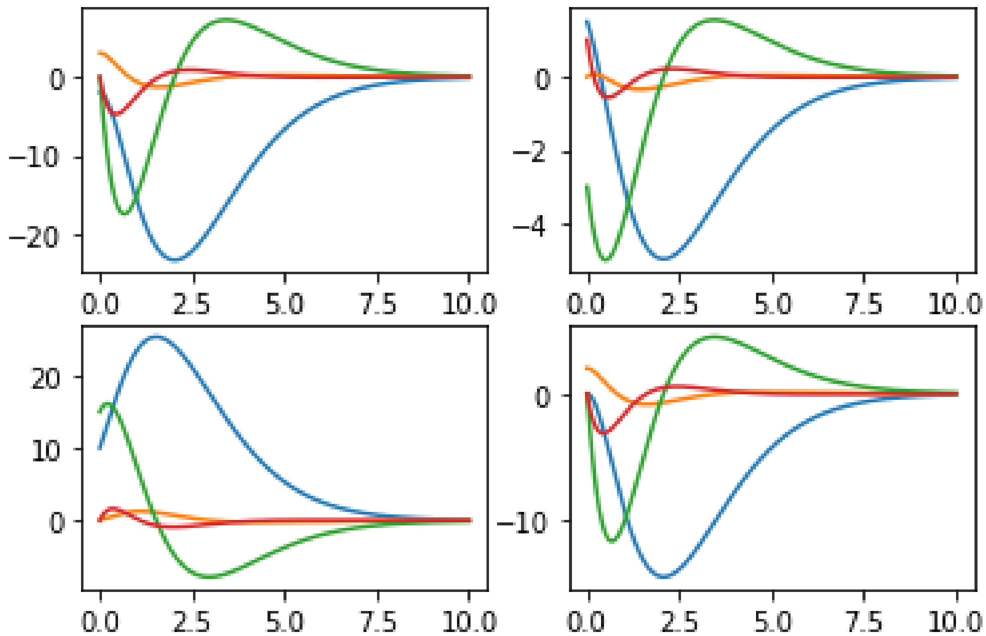
## 1.7 Task F

For this task I used SciPy library with module signal.

[9]:
```
from scipy.signal import place_poles
import matplotlib.pyplot as plt
from scipy.integrate import *
from IPython.core.interactiveshell import InteractiveShell
InteractiveShell.ast_node_interactivity = 'last'

def stepinfo(t,yout):
    print(f"Overshoot: {(yout.max() - yout[-1])}")
    print(f"Rise Time: {t[([0] + [i for i in range(0, len(yout)-1) if␣
↪yout[i]>yout[-1] * .90])[-1]]-t[0]}")
    print(f"Settling Time: {t[next(len(yout)-i for i in range(2,len(yout)-1) if␣
↪abs(yout[-i]/yout[-1])>1.02)]-t[0]}")

def subplot(pos, t, res):
    plt.subplot(pos)
    res = res.T
    for i in range(len(res)):
        plt.plot(t, res[i], label="test")
```

[10]:
```
K = place_poles(A, B, [-1, -1.2, -1.3, -1.4]).gain_matrix
A_cl = A - B.dot(K)
np.linalg.eig(A_cl)[0]

t = np.linspace(0, 10, 1000)

def df(x, t):
    x = np.array(x).reshape(4, 1)
    return list(np.matmul(A_cl, x).T[0])
res1 = odeint(df, [-2, 3, 0, 0], t)
res2 = odeint(df, [1.5, 0, -3, 1], t)
res3 = odeint(df, [10, 0.2, 15, 0], t)
res4 = odeint(df, [0, 2, 0, 0], t)

subplot(221, t, res1)
```

```
subplot(222, t, res2)
subplot(223, t, res3)
subplot(224, t, res4)
plt.show()
```




As you can see, the system goes well to the desired position $(0,0,0,0)^T$

[11]:
```
stepinfo(t, res1.T[0])
stepinfo(t, res1.T[1])
print("Res 2:")
stepinfo(t, res2.T[0])
stepinfo(t, res2.T[1])
```

```
Overshoot: 0.0
Rise Time: 0.0
Settling Time: 9.96996996996997
Overshoot: 2.989736475204101
Rise Time: 9.98998998998999
Settling Time: 9.96996996996997
Res 2:
Overshoot: 1.5261776450246798
Rise Time: 0.36036036036036034
Settling Time: 9.96996996996997
Overshoot: 0.07791883665628939
Rise Time: 9.98998998998999
Settling Time: 9.96996996996997
```

### 1.8 Task G

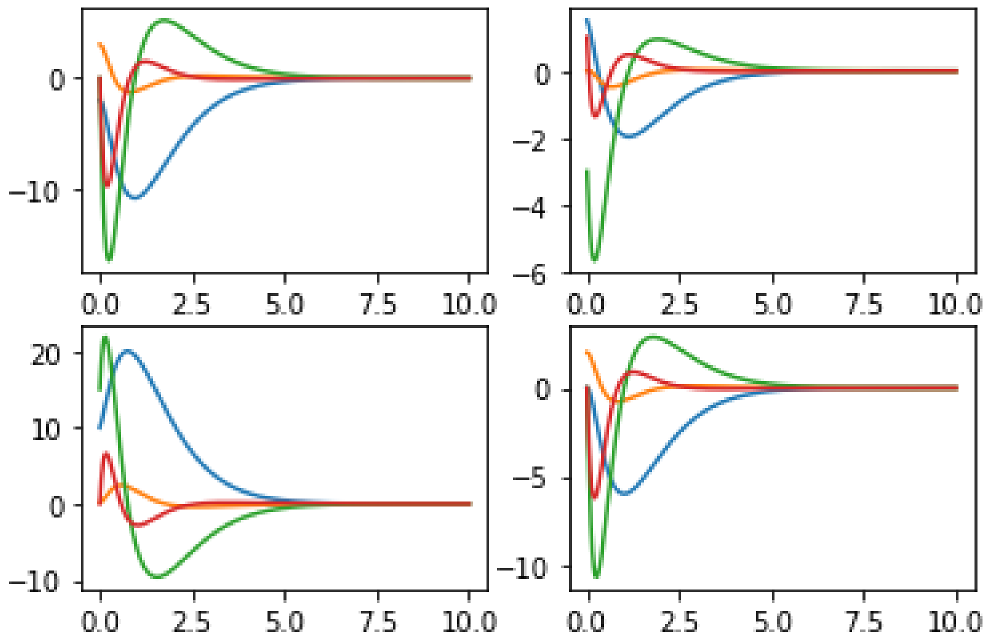
For this task I designed an LQR controller. The matrices Q and R were adjusted by hand. In this task I can tell that it the system stabilizes much faster and better (look below).

[12]:
```
from control import *
from scipy.signal import place_poles
import matplotlib.pyplot as plt
from scipy.integrate import *
from IPython.core.interactiveshell import InteractiveShell
InteractiveShell.ast_node_interactivity = 'last'
from math import *

def subplot(pos, t, res):
    plt.subplot(pos)
    res = res.T
    for i in range(len(res)):
        plt.plot(t, res[i], label="test")
```

[13]:
```
R = np.array([
    [25, 0, 0, 0],
    [0, 0.0003, 0, 0],
    [0, 0, 25, 0],
    [0, 0, 0, 0.0003]
])
Q = np.array([[0.1]])
K, S, E = lqr(A, B, R, Q)
K = np.array(K)
print(K)

A_cl = A - B.dot(K)
print(f"Eig: {np.linalg.eig(A_cl)[0]}")

t = np.linspace(0, 10, 1000)

def df(x, t):
    x = np.array(x).reshape(4, 1)
    return list(np.matmul(A_cl, x).T[0])
res1 = odeint(df, [-2, 3, 0, 0], t)
res2 = odeint(df, [1.5, 0, -3, 1], t)
res3 = odeint(df, [10, 0.2, 15, 0], t)
res4 = odeint(df, [0, 2, 0, 0], t)

subplot(221, t, res1)
subplot(222, t, res2)
subplot(223, t, res3)
subplot(224, t, res4)
```

```
plt.show()
```

```
[[-15.8113883  328.85784213 -32.26899057 101.27906058]]
Eig: [-4.70013752+0.j         -3.36443707+0.j         -1.23184079+0.30313026j
 -1.23184079-0.30313026j]
```




[14]:
```
print("Res 1:")
stepinfo(t, res1.T[0])
stepinfo(t, res1.T[1])
print("Res 2:")
stepinfo(t, res2.T[0])
stepinfo(t, res2.T[1])
```

```
Res 1:
Overshoot: 0.0
Rise Time: 0.0
Settling Time: 9.98998998998999
Overshoot: 2.9999180579347073
Rise Time: 9.98998998998999
Settling Time: 9.97997997997998
Res 2:
Overshoot: 1.5000286375733218
Rise Time: 0.2902902902902903
Settling Time: 9.98998998998999
Overshoot: 0.03647336282589482
Rise Time: 9.98998998998999
```

Settling Time: 9.97997997997998

Here I compared these two methods on first two initial conditions: as you can see the second method (LQR controller) is much better since it has smaller overshoot and settling time.

[ ]:

[16]:
```python
sp.diff(g_, theta)
```

[16]:

$$\left[\begin{matrix}0\\0\\- \frac{2 m \sin{\left(\theta \right)} \cos{\left(\theta \right)}}{\left(M + m \sin^{2}{\left(\theta \right)}\right)^{2}}\\- \frac{2 m \sin{\left(\theta \right)} \cos^{2}{\left(\theta \right)}}{l \left(M + m \sin^{2}{\left(\theta \right)}\right)^{2}} - \frac{\sin{\left(\theta \right)}}{l \left(M + m \sin^{2}{\left(\theta \right)}\right)}\end{matrix}\right]$$

[ ]: