Exam Solution

Course: AE4874 Fundamentals of Astrodynamics
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Author:
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0 Introduction

This contains an exam solution. If you wish to contribute to this exam solution:

1. Create a github account, (you can create an "anonymous" one).
2. git clone ...
3. edit your changes in the document.
4. open cmd, and browse to inside the folder you downloaded and edited
5. git pull (updates your local repository=copy of folder, to the latest version in github cloud)
6. git status shows which files you changed.
7. git add "/some folder with a space/someFileYouChanged.tex"
8. git commit -m "Included solution to question 1c."
9. git push

It can be a bit initimidating at first, so feel free to click on "issue" in the github browser of this repository and ask :) (You can also use that to say "Hi, I'm having a bit of help with this particular equation, can someone help me out?")

If you don't know how to edit a latex file on your own pc iso on overleaf, look at the "How to use" section of `https://github.com/a-t-0/AE4872-Satellite-Orbit-Determination`.

0.1 Consistency

To make everything nice and structured, please use very clear citations:

1. If you copy/use an equation of some slide or document, please add the following data:
   (a) Url (e.g. if simple wiki or some site)
   (b) Name of document
   (c) (Author)
   (d) PAGE/SLIDE number so people can easily find it again
   (e) equation number (so people can easily find it again)
2. If you use an equation from the slides/a book that already has an equation number, then hardcode that equation number in this solution manual so people directly see which equation in the lecture material it is, this facilitates remembering the equations.
3. Here is an example is given in eq. (10.32[**?**]) (See file references.bib [**?**]).

$$R_n^E = \sum_{t=1}^{n} l_m(p_t, z_t) - \underset{i}{min} \sum_{t=1}^{n} {z_t}^i \tag{10.32[?]}$$

1 Relative motion in nbody: nbody U gravitational attraction. dI/dt

This entire question is based on Chapter 2: Many-body problem in the reader by K. Wakker.

1.1 a. Equations of motion

First, it's important to state the assumptions made.

- All masses are considered as point masses
- Only gravitational forces caused by the n bodies occur (no external forces, no other bodies outside the n-body system)

For the equations of motion, Newton's second law and Newton's law of gravitation are needed. First consider Newton's second law

$$\vec{F} = m\vec{a} \tag{1}$$

Expressed for body i

$$\vec{F_i} = m_i\ddot{\vec{r_i}} \tag{2}$$

Next, consider Newton's law of gravitation

$$\vec{F_i} = -G\frac{m_i m_j}{r_{ij}^3}\vec{r}_{ij} \tag{3}$$

Substituting eq. 2 into eq. 3 yields the following expression:

$$m_i\ddot{\vec{r_i}} = G\Sigma_{i\neq j}^{n}\frac{m_i m_j}{r_{ij}^3}\vec{r}_{ij} \tag{4}$$

1.2 b. Gravitational field strength

Consider the equation for gravitational field strength

$$U_i = -\Sigma_{j\neq i}G\frac{m_j}{m_{ij}} + U_{i0} \tag{5}$$

Now consider again Newton's law of gravitation (eq. 3) and divide it by mass:

$$\frac{\vec{F}}{m_i} = \vec{g}_i = \Sigma_{j\neq i}G\frac{m_j}{r_{ij}^3}\vec{r}_{ij} \tag{6}$$

Field strength is also expressed by

$$\vec{g}_i = -\vec{\nabla}_i u_i \tag{7}$$

And therefore

$$U_i = -g_i + U_{i0} = -\Sigma_{j\neq i}G\frac{m_j}{r_{ij}^3}\vec{r}_{ij} + U_{i0} \tag{8}$$

It is important to note that the potential U_i is a scalar function.

1.3 c. Field strength conservative or not?

The field strength changes over time since a potential is only described at a certain position in the inertial reference frame. This also means that the total energy of the body described by the potential is subject to change. The sum of kinetic and potential energy of one body is therefore also not constant, however for the entire system it is constant.

1.4 d. Integrals of motion

The first term on the left hand side of (1) in the exam is representative of the kinetic energy E_k, the second term on the left hand side is representative of the potential energy E_p. The sum of these terms is constant for the system:

$$E_k + E_p = C \tag{9}$$

Where

$$E_k = \frac{1}{2}\Sigma_i m_i V_i^2 \tag{10}$$

And

$$E_p = -\frac{1}{2}G\Sigma_i\Sigma_{j\neq i}\frac{m_i m_j}{r_{ij}} \tag{11}$$

1.5 e. Close approach of two or more bodies

When two or more bodies have a close approach, the term r_{ij} Approaches zero and thus the magnitude of the potential energy increases. Due to conservation of energy, the kinetic energy has to increase in turn, which translates to a high velocity of the bodies. This velocity can reach a significant magnitude so that an escape of the system by one or more of the bodies may be possible.

1.6 f. Stability condition

Eq. 9 is now expressed as:

$$E_k = C - E_p \tag{12}$$

Therefore:

$$\frac{d^2I}{dt^2} = 4C - 4E_p + 2E_p = 4C - 2E_p \tag{13}$$

Simultaneously,

$$\frac{d^2I}{dt^2} = 2C + 2E_k \tag{14}$$

Therefore, the term representing potential energy is always negative and the term representing kinetic energy is always positive. Considering the polar moment of inertia I:

$$I = \Sigma_i m_i \vec{r}_i \cdot \vec{r}_i = \Sigma_i m_i r_i^2 \tag{15}$$

Now consider the fact that for stability, the growth of I may not be unbounded. First consider what happens for C>0 by looking at eq. 13 and 14:

$$\frac{d^2I}{dt^2} > 0 \tag{16}$$

This means that the first derivative of I may at some point become positive, even if it is negative, resulting in unbounded growth of I. Now let's consider the case where C<0: Now, considering eq. 13 and 14, the second derivative of I may either be positive or negative. In case where

$$\frac{d^2I}{dt^2} > 0 \tag{17}$$

The system is unstable as discussed earlier. In case where

$$\frac{d^2I}{dt^2} < 0 \tag{18}$$

The system is stable. However, since in this case of C< 0 the sign of the second derivative of I is subject to change, stability is still possible when the first and second derivative of I are varying locally such that unbounded growth is not possible. For example: as long as the first derivative of I is negative while the second derivative of I is positive, the system is still stable.

1.7 g. Long-term stability

Consider again the second derivative of the polar moment of inertia

$$\frac{d^2I}{dt^2} = 4E_k + 2E_p \tag{19}$$

Now consider its average over time from t0 to t1 where t0 = 0.

$$\frac{1}{t_1}\int_{t_0}^{t_1} \frac{d^2I}{dt^2}dt = \frac{4}{t_1}\int_{t_0}^{t_1} E_k dt + \frac{2}{t_1}\int_{t_0}^{t_1} E_p dt \tag{20}$$

Thus

$$\left[\frac{1}{t_1}\frac{dI}{dt}\right]_{t_0}^{t_1} = 4\tilde{E}_k + 2\tilde{E}_p \tag{21}$$

Now substitute the first derivative of the polar moment of inertia

$$\frac{dI}{dt} = 2\Sigma_i m_i \vec{r}_i \cdot \vec{V}_i = 2\Sigma_i m_i \vec{r}_i \cdot \dot{\vec{r}}_i \tag{22}$$

Into eq. 21

$$\frac{1}{t_1}\left[2\Sigma_i m_i \vec{r}_i \cdot \dot{\vec{r}}_i\right]_{t_0}^{t_1} = 4\tilde{E}_k + 2\tilde{E}_p \tag{23}$$

And

$$\frac{1}{t_1}\left[\Sigma_i m_i \vec{r}_i \cdot \dot{\vec{r}}_i\right]_{t_0}^{t_1} = 2\tilde{E}_k + \tilde{E}_p \tag{24}$$

The conditions for stability are that no escapes and collisions occur. Therefore, distance and velocity always have to be finite and thus over large time intervals, the left-hand term will approach zero:

$$0 = 2\tilde{E}_k + \tilde{E}_p \tag{25}$$

Recall from eq. 9 the total energy of the system and consider its average:

$$\tilde{E}_k + \tilde{E}_p = C \tag{26}$$

Now substituting 25 into 26:

$$\tilde{E}_k = -C = -\frac{1}{2}\tilde{E}_p \tag{27}$$

Q.E.D. the Virial theorem.

2 Two body elliptical orbits: Two body H. velocity and acceleration

This entire question is sourced from chapters 2.3, 5.1, 6.2 and 7.1.

2.1 a. Circular velocity and escape velocity as function of r

Starting with

$$e^2 = 1 - \frac{rV^2}{\mu}\left(2 - \frac{rV^2}{\mu}\right)\cos^2\gamma \tag{28}$$

The following trigonometric identity is used:

$$\sin^2\gamma + \cos^2\gamma = 1 \tag{29}$$

$$e^2 = \cos^2\gamma + \sin^2\gamma - \frac{rV^2}{\mu}\left(2 - \frac{rV^2}{\mu}\right)\cos^2\gamma \tag{30}$$

$$e^2 = \sin^2\gamma + \cos^2\gamma\left[1 - \frac{2rV^2}{\mu} + \left(\frac{rV^2}{\mu}\right)^2\right] \tag{31}$$

Rewriting to

$$e^2 = \sin^2\gamma + \cos^2\gamma\left[\left(1 - \frac{rV^2}{\mu}\right)\left(1 - \frac{rV^2}{\mu}\right)\right]^2 \tag{32}$$

$$e^2 = \sin^2\gamma + \cos^2\gamma\left[1 - \frac{rV^2}{\mu}\right]^2 \tag{33}$$

For circular orbit, plugging in e=0 yields:

$$\sin^2\gamma = -\cos^2\gamma\left[1 - \frac{rV^2}{\mu}\right]^2 \tag{34}$$

$$\frac{\sin^2\gamma}{\cos^2\gamma} = -\left[1 - \frac{rV^2}{\mu}\right]^2 \tag{35}$$

$$\tan^2\gamma = -\left[1 - \frac{rV^2}{\mu}\right]^2 \tag{36}$$

$$\gamma = \arctan\left(\sqrt{-\left[1 - \frac{rV^2}{\mu}\right]^2}\right) \tag{37}$$

For real solutions, the following condition applies:

$$\frac{rV^2}{\mu} = 1 \tag{38}$$

Where V is the circular velocity V_c. Rewriting yields:

$$V_c = \sqrt{\frac{\mu}{r}} \tag{39}$$

Next, realizing that for escape velocity, e = 1, this value is plugged into the expression from eq.65.

$$e^2 = 1 = 1 - \frac{rV^2}{\mu}\left(2 - \frac{rV^2}{\mu}\right)\cos^2\gamma \tag{40}$$

$$\frac{rV^2}{\mu}\left(2 - \frac{rV^2}{\mu}\right)\cos^2\gamma = 0 \tag{41}$$

The two solutions to this equation are:

$$\frac{rV^2}{\mu}\cos^2\gamma = 0 \vee \left(2 - \frac{rV^2}{\mu}\right) = 0 \tag{42}$$

The first solution is considered:

$$\frac{rV^2}{\mu}\cos^2\gamma = 0 \tag{43}$$

Where, assuming neither r nor V are equal to zero, the path angle is required to be equal to 90 degrees. The physical meaning would be that the body would rectilinearly move away from the centre of attraction, which bears no physical meaning and therefore this solution is a trivial one. Now, the second solution is considered:

$$\left(2 - \frac{rV^2}{\mu}\right) = 0 \tag{44}$$

Rewriting yields:

$$V = \sqrt{\frac{2\mu}{r}} \tag{45}$$

And since

$$V_c = \sqrt{\frac{\mu}{r}} \tag{46}$$

And therefore

$$V_{esc} = \sqrt{2}V_c \tag{47}$$

2.2 b. Derivation of the expression for the semi-major axis for an elliptical orbit

The following expression is to be derived:

$$a = \frac{\mu/2}{\mu/r - V^2/2} \tag{48}$$

Starting with the given equations

$$p = \frac{H^2}{\mu} \tag{49}$$

and

$$H = rV\cos\gamma \tag{50}$$

Substitution yields

$$p = \frac{r^2V^2\cos^2\gamma}{\mu} \tag{51}$$

Now considering the orbital equation

$$r = \frac{p}{1 + e\cos\theta} \tag{52}$$

Considering eq. 52 at respectively 0 and 180 degrees for perigee and apogee:

$$2a = r_a + r_p = \frac{p}{1 + e\cos 0} + \frac{p}{1 + e\cos 180} = \frac{p}{1-e} + \frac{p}{1+e} = \frac{2p}{1-e^2} \tag{53}$$

Therefore

$$a = \frac{p}{1-e^2} \tag{54}$$

Now for the big finale, substitution of eq. 65, 51 and 54 yields:

$$a = \frac{r^2V^2\cos^2\gamma}{\mu\left[1 - \left(1 - \frac{rV^2}{\mu}\left[2 - \frac{rV^2}{\mu}\right]\cos^2\gamma\right)\right]} \tag{55}$$

$$a = \frac{r^2V^2\cos^2\gamma}{rV^2\cos^2\gamma\left[2 - \frac{rV^2}{\mu}\right]} \tag{56}$$

$$a = \frac{r}{2 - \frac{rV^2}{\mu}} = \frac{r}{r\left[\frac{2}{r} - \frac{V^2}{\mu}\right]} = \frac{\mu}{\mu}\frac{1}{\frac{2}{r} - \frac{V^2}{\mu}} \tag{57}$$

$$a = \frac{\mu}{\frac{2\mu}{r} - V^2} = \frac{1/2}{1/2}\frac{\mu}{\frac{2\mu}{r} - V^2} = \frac{\mu/2}{\mu/r - V^2/2} \tag{58}$$

Q.E.D.

2.3 c. Proof of constant energy sum

Considering the conservation of energy

$$E_k + E_p = C \tag{59}$$

Where

$$E_k = \frac{1}{2}mV^2 \tag{60}$$

and

$$E_p = -m\frac{\mu}{r} \tag{61}$$

And because m is constant for a body

$$\frac{1}{2}V^2 - \frac{\mu}{r} = C \tag{62}$$

Considering the equation for the semi-major axis from subquestion (b), it is observed that a and mu always have a positive and nonzero value. Thus it can be stated that

$$\frac{\mu}{r} - \frac{1}{2}V^2 > 0 \tag{63}$$

or

$$\frac{1}{2}V^2 - \frac{\mu}{r} < 0 \tag{64}$$

Comparing equations 62 and 64, the definition is proven, the proof holds for orbits with an eccentricity between 0 and 1 due to the definition of the semi-major axis.

2.4 d. Velocity at pericenter and apocenter

Considering

$$V^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{65}$$

For the pericenter it is known that

$$r_p = \frac{p}{1+e} \tag{66}$$

Substituting

$$p = a(1-e^2) \tag{67}$$

yields

$$r_p = \frac{a(1-e^2)}{1+e} = a(1-e) \tag{68}$$

Substituting this in eq. 65 results in

$$V_p = \sqrt{\mu\left(\frac{2}{a(1-e)} - \frac{1}{a}\right)} = \sqrt{\frac{2\mu}{a(1-e)} - \frac{\mu(1-e)}{a(1-e)}} \tag{69}$$

So

$$V_p = \sqrt{\frac{2\mu - \mu + e\mu}{a(1-e)}} = \sqrt{\frac{\mu(1+e)}{a(1-e)}} = \sqrt{\frac{\mu}{a(1-e)}(1+e)} \tag{70}$$

Consider the circular velocity at perigee

$$V_{cp} = \sqrt{\frac{\mu}{r_p}} = \sqrt{\frac{\mu}{a(1-e)}} \tag{71}$$

$$V_{cp}^2 = \frac{\mu}{a(1-e)} \tag{72}$$

Substitution of eq. 72 into eq. 70 yields:

$$V_p = \sqrt{V_{cp}^2(1+e)} = V_{cp}\sqrt{1+e} \tag{73}$$

This is now written as

$$V_p^2 = V_{cp}^2(1+e) \tag{74}$$

And now for apocenter

$$r_a = \frac{p}{1-e} = \frac{a(1-e^2)}{1-e} = \frac{a(1+e)(1-e)}{(1-e)} = a(1+e) \tag{75}$$

Substituting this into eq. (2) as given in the exam, yields

$$V_a^2 = \mu\Big(\frac{2}{r_a} - \frac{1}{a}\Big) = \mu\Big(\frac{2}{a(1+e)} - \frac{1(1+e)}{a(1+e)}\Big) \tag{76}$$

$$= \mu\Big(\frac{1-e}{a(1+e)}\Big) = \frac{\mu}{a(1+e)}(1-e) \tag{77}$$

Circular velocity at apogee is

$$V_{ca} = \sqrt{\frac{\mu}{r_a}} = \sqrt{\frac{\mu}{a(1+e)}} \tag{78}$$

$$V_{ca}^2 = \frac{\mu}{a(1+e)} \tag{79}$$

Substituting in the same way as demonstrated for the perigee velocity:

$$V_a^2 = V_{ca}^2(1+e) \tag{80}$$

2.5 e. Ratio between maximum/minimum velocity in an elliptical orbit

$$\frac{V_{max}}{V_{min}} = \frac{1+e}{1-e} \tag{81}$$

Knowing that the highest velocity is at the pericenter and the lowest velocity at the apocenter, it is stated that:

$$V_{max} = V_p = V_{cp}\sqrt{1+e} = \sqrt{\frac{\mu(1+e)}{a(1-e)}} \tag{82}$$

$$V_{min} = V_a = V_{ca}\sqrt{1-e} = \sqrt{\frac{\mu(1-e)}{a(1+e)}} \tag{83}$$

Therefore

$$\frac{V_{max}}{V_{min}} = \frac{\sqrt{\frac{\mu(1+e)}{a(1-e)}}}{\sqrt{\frac{\mu(1-e)}{a(1+e)}}} = \sqrt{\frac{\frac{\mu}{a}}{\frac{\mu}{a}}}\sqrt{\frac{\frac{1+e}{1-e}}{\frac{1-e}{1+e}}} = \sqrt{\frac{1+e}{1-e}\frac{\frac{1}{1-e}}{\frac{1}{1+e}}} = \sqrt{\frac{(1+e)^2}{(1-e)^2}} = \frac{1+e}{1-e} \tag{84}$$

2.6 f. Semi-major axis for hyperbolic trajectory

First, a sketch of a hyperbolic trajectory:

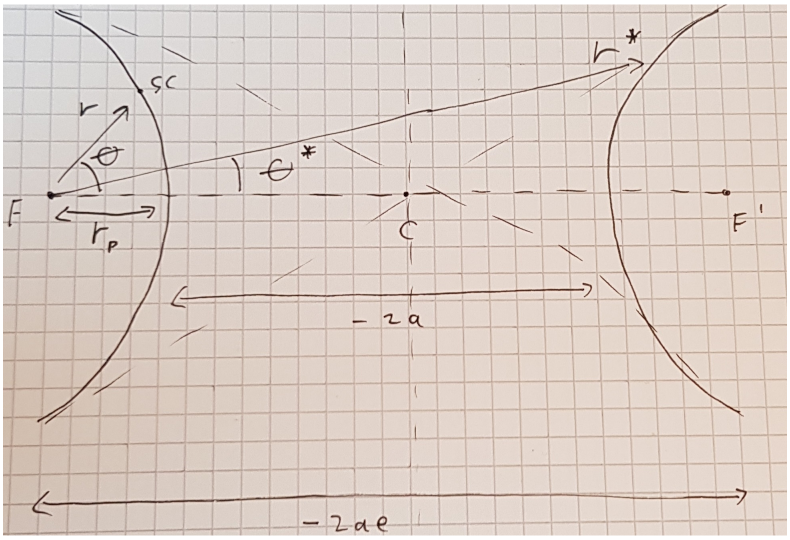

Figure 1: Sketch, Artemis2020, 2019

For a hyperbolic trajectory,

$$e > 1 \tag{85}$$

holds. At

$$\theta_p = 0 \tag{86}$$

So

$$r_p = \frac{p}{1 + e\cos 0} = \frac{p}{1+e} = a(1-e) \tag{87}$$

Distance CF from center to focal point for elliptical orbits:

$$FC = r_p - a = a(1-e) - a = -ae \tag{88}$$

For a hyperbolic orbit it holds that (see sketch)

$$-2ae = 2r_p - 2a \tag{89}$$

$$2a - 2ae = 2a(1-e) = 2r_p \tag{90}$$

Thus

$$a = \frac{r_p}{1-e} \tag{91}$$

For this type of trajectory, a negative semi-major axis is assumed. This is the shortest distance between both curves shown, expressed as:

$$-2a = r^*(\theta^* = 0) - r(\theta = 0) \tag{92}$$

Calculating r*

$$r^* = \frac{-p}{1 - e\cos 0} = \frac{-p}{1-e} \tag{93}$$

Calculating r

$$r = \frac{p}{1 + e\cos 0} = \frac{p}{1+e} \tag{94}$$

Combining these three equations yields

$$-2a = \frac{-p}{1-e} - \frac{p}{1+e} = \frac{-p(1+e) - p(1-e)}{(1-e)(1+e)} = \frac{-p - pe - p + pe}{1-e^2} = \frac{-2p}{1-e^2} \tag{95}$$

Therefore

$$2a = \frac{2p}{1-e^2} \tag{96}$$

And thus

$$a = \frac{p}{1-e^2} \tag{97}$$

Q.E.D.

2.7 g. Local velocity in hyperbolic orbit

It is asked to prove the following equation

$$V^2 = V_{esc}^2 + V_\infty^2 \tag{98}$$

First, consider the vis-vica equation

$$V^2 = \mu\Big[\frac{2}{r} - \frac{1}{a}\Big] \tag{99}$$

At infinity, the first term inside the square brackets approaches zero. Therefore:

$$V_\infty^2 = -\frac{\mu}{a} \tag{100}$$

Consider now the definition of the escape velocity

$$V_{esc} = \sqrt{2}V_c = \sqrt{\frac{2\mu}{r}} \tag{101}$$

Hence

$$V_{esc}^2 = \frac{2\mu}{r} \tag{102}$$

Plugging these equations into the vis-viva equation:

$$V^2 = \frac{2\mu}{r} - \frac{\mu}{r} = V_\infty^2 - V_{esc}^2 \tag{103}$$

2.8 h. Ratio between maximum and minimum velocity in hyperbolic orbit

Prove that:

$$\frac{V_{max}}{V_{min}} = \sqrt{\frac{e+1}{e-1}} \tag{104}$$

Realizing once again that maximum velocity occurs at the pericenter,

$$V_{max}^2 = V_p^2 = V_{cp}^2(1+e) = \sqrt{\frac{\mu(1+e)}{a(1-e)}} \tag{105}$$

Realizing that minimum velocity occurs at the apocenter, which for a hyperbolic orbit is at infinity,

$$V_{min} = V_\infty = \sqrt{-\frac{\mu}{a}} \tag{106}$$

Combining yields:

$$\frac{V_{max}}{V_{min}} = \frac{\sqrt{\frac{\mu(1+e)}{a(1-e)}}}{\sqrt{-\frac{\mu}{a}}} = \frac{\sqrt{\frac{\mu}{a}}}{\sqrt{\frac{\mu}{a}}}\frac{-\sqrt{1+e}}{1-e} = \sqrt{\frac{e+1}{e-1}} \tag{107}$$

Q.E.D.

3 Two body elliptical orbits:two body du grav potential

Most information for this question is found in chapter 5.10 - Relativistic effects of the reader by K. Wakker.

3.1 a. The meaning of various parameters.

The "distance-potential":

$$u = \frac{1}{r} \tag{108}$$

Distance between both bodies:

$$r \tag{109}$$

Angle of vector r w.r.t. reference direction

$$\phi \tag{110}$$

Gravitational parameter *(G*M)*

$$\mu \tag{111}$$

Angular momentum:

$$H \tag{112}$$

Speed of light in vacuum:

$$c \tag{113}$$

Arbitrary constant to enable easy notation of first-order relativistic effects:

$$\alpha \tag{114}$$

Relativistic effect:

$$3\frac{\mu}{c^2}u^2 \tag{115}$$

Solving the following leads to the orbital equation:

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu}{H^2} \tag{116}$$

3.2 b. Proof of second right-hand term always being smaller than first right-hand term

Consider the normal velocity

$$V_\phi = V\cos\Phi \tag{117}$$

Consider the right-hand side of equation (1) in the question, expressed as:

$$\frac{\mu}{H^2}\left[1 + 3\frac{H^2u^2}{c^2}\right] \tag{118}$$

Now for the angular momentum

$$H = rV_\Phi \tag{119}$$

Rewriting (1)

$$\frac{d^2u}{d\Phi^2} + u = \frac{\mu}{H^2}\left[1 + 3\frac{r^2V_\Phi^2}{c^2}\frac{1}{r^2}\right] \tag{120}$$

$$\frac{d^2u}{d\Phi^2} + u = \frac{\mu}{H^2}\left[1 + 3\frac{V_\Phi^2}{c^2}\right] \tag{121}$$

Because

$$V_\Phi^2 << c^2 \tag{122}$$

It is clear that

$$3\frac{V_\Phi^2}{c^2} << 1 \tag{123}$$

Therefore it is stated that the second term in square brackets of eq. 121, representing the term in question, is always smaller than the first term.

3.3 c. Finding a first-order solution for the Keplerian orbit equation

For this, the order of successive approximation is used.

- First, a zeroth-order approximation is found by neglecting the term including alpha.
- This solution for u is substituted into the right-hand side of (1) from the question.
- Solve the homogeneous form
- Solve the particular form
- Apply boundary conditions
- Add homogeneous+particular form for the solution.

3.4 d. Term contributions to first-order solution

The relativistic effects are described by the second square bracket terms.

$$\alpha\frac{\mu^2}{H^4}\Big[1+\frac{1}{2}e^2\Big] \tag{124}$$

The above term represents a small constant term of little influence.

$$\alpha\frac{\mu^2}{H^4}\Big[e\Phi\sin(\Phi-\omega)\Big] \tag{125}$$

The above term represents a fluctuation of which the amplitude grows larger as Phi increases.

$$\alpha\frac{\mu^2}{H^4}\Big[-\frac{1}{b}e^2\cos^2(\Phi-\omega)\Big] \tag{126}$$

The above term represents a pure oscillation with constant amplitude of

$$\frac{\alpha\mu^2e^2}{6H^4} \tag{127}$$

Therefore, the second term which is an ever-increasing fluctuation will in the long term dominate the relativistic effect.

3.5 e. Long-term effect approximation

Since the first and third term in the second square brackets can be neglected, the relativistic effect is described by:

$$u=\frac{\mu}{H^2}\Big[1+e\cos(\Phi-\omega)\Big]+\alpha\frac{\mu^2}{H^4}e\Phi\sin(\Phi-\omega) \tag{128}$$

$$u=\frac{\mu}{H^2}\Big[1+e\cos(\Phi-\omega)+\alpha\frac{\mu}{H^2}e\Phi\sin(\Phi-\omega)\Big] \tag{129}$$

Including the following substitution

$$\beta=\alpha\frac{\mu}{H^2} \tag{130}$$

Leads to:

$$\frac{\mu}{H^2}\Big[1+e\cos(\Phi-\omega)+\beta e\Phi\sin(\Phi-\omega)\Big] \tag{131}$$

Now, assuming beta is very small it is stated that

$$\sin\beta\Phi\approx\beta\Phi \tag{132}$$

And

$$\cos\beta\Phi\approx 1 \tag{133}$$

Using both approximations it is stated that

$$u=\frac{\mu}{H^2}\Big[1+e\cos(\Phi-\omega)\cos(\beta\Phi)+e\sin(\Phi-\omega)\sin(\beta\Phi)\Big] \tag{134}$$

The following trig-identities are used to achieve the solution:

$$\cos(\alpha)\cos(\beta)=\frac{1}{2}\big[\cos(\alpha-\beta)+\cos(\alpha+\beta)\big] \tag{135}$$

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\big[\cos(\alpha-\beta) - \cos(\alpha+\beta)\big] \tag{136}$$

Therefore

$$u = \frac{\mu}{H^2}\Big[1 + e\frac{1}{2}\big[\cos(\Phi-\omega-\beta\Phi) + \cos(\Phi-\omega+\beta\Phi)\big] + e\frac{1}{2}\big[\cos(\Phi-\omega-\beta\Phi) - \cos(\Phi-\omega+\beta\Phi)\big]\Big] \tag{137}$$

$$u = \frac{\mu}{H^2}\Big[1 + e\cos(\Phi-\omega-\beta\Phi)\Big] \tag{138}$$

Q.E.D.

3.6 f. Motion of body i

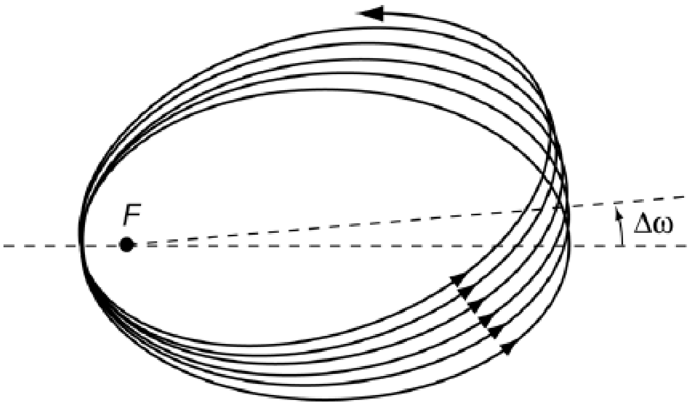

Figure 2: Relativistic precession of the orbit's major axis, figure 5.10 from Wakker

Given that

$$p = \frac{H^2}{\mu} \tag{139}$$

And

$$u = \frac{1}{r} \tag{140}$$

Using the solution from subquestion (e) the following can be stated:

$$r = \frac{p}{1 + e\cos(\Phi-\omega-\beta\Phi)} \tag{141}$$

Which can be interpreted as body i moving as a conic section around k, where the instantaneous argument of the pericenter is given by

$$\omega_{inst} = \omega + \beta\Phi \tag{142}$$

3.7 g. Relativistic change of the argument of pericenter

After one full revolution of body i around body k,

$$\Phi_1 = \Phi_0 + 2\pi \tag{143}$$

And therefore

$$\Delta\Phi = 2\pi \tag{144}$$

Knowing from the answer of subquestion (f),

$$\omega_{inst} = \omega + \beta\Phi \tag{145}$$

And therefore

$$\Delta\omega = \beta\Delta\Phi \tag{146}$$

Substituting eq. 144 into eq. 146 results in

$$\Delta\omega = 2\pi\beta \tag{147}$$

Substituting beta in results in

$$\Delta\omega = 6\pi\frac{\mu^2}{H^2c^2} \tag{148}$$

Q.E.D.

Conclusion