Leonhard Euler
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# Math 101 Homework 1

**Problem 1.**
(a) Prove that there is no rational number whose square is 2.
(b) Let $p$ be a prime number and $n$ an integer greater than 1. Prove that there is no rational number whose $n$th power is $p$.

**Problem 2.** Let $S$ be a dense subset of $\mathbb{R}$.
(i) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) = 0$ for all $x \in S$.
  (A) Prove, using the *definition* of denseness, that $f(x) = 0$ for all $x \in \mathbb{R}$.
  (B) Prove, using the fact that a subset of $\mathbb{R}$ is dense if and only if it is *sequentially dense*, that $f(x) = 0$ for all $x \in \mathbb{R}$.
(ii) Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous and $f(x) = g(x)$ for all $x \in S$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

**Problem 3** (Ahlfors §4.2.3 #2, p. 123)**.** Prove that a function which is analytic in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some $n$ and all sufficiently large $|z|$ reduces to a polynomial.

**Problem 4.** Prove that there are infinitely many prime numbers.

**Problem 5.** Let $p_n$ be the $n$th prime number. Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{p_n} \tag{5.1}$$

diverges.

**Problem 6** (Atiyah-Macdonald 1.1)**.** Let $x$ be a nilpotent element of a ring $A$. Show that $1 + x$ is a unit of $A$. Deduce that the sum of a nilpotent element and a unit is a unit.