

Math 101 Homework 1

Problem 1 (Euclid). Prove that there are infinitely many prime numbers.

Solution. Let n be a natural number, and suppose we can find n prime numbers p_1, \dots, p_n . Then $N = p_1 \cdots p_n + 1$ is coprime to each p_i , so it has a prime factor q different from any of the p_i 's. Then p_1, \dots, p_n, q is a list of $n + 1$ prime numbers. It follows that, given any finite set of prime numbers, we can always find a prime number not in that set. Therefore, there are infinitely many prime numbers. This contradicts the assumption that there are only finitely many primes.

Problem 2.

- (a) Prove that $\sqrt{2}$ is irrational.
- (b) Let n be an integer greater than 1. Prove that the n th root of any prime number p is irrational.

Solution. (a) Let $n = p = 2$ in part (b).

(b) The polynomial $f = x^n - p$ is irreducible over \mathbb{Z} by Eisenstein's criterion. By Gauss's Lemma, f is also irreducible over \mathbb{Q} . In particular, f has no roots in \mathbb{Q} , so the n th root of p is irreducible.