

## Example Homework 1

### Problem 1.

- (a) Prove that there is no rational number whose square is 2.
- (b) Let  $n$  be an integer greater than 1. Prove that the  $n$ th root of any prime number  $p$  is irrational.

**Solution.** (a) Suppose that there is a rational number whose square is 2. Then we can find coprime nonzero integers  $a$  and  $b$  such that  $a^2/b^2 = 2$ . Then  $a^2 = 2b^2$ , so  $a^2$  is even. It follows that  $a$  is even, so  $a = 2k$  for some integer  $k$ . Then  $2b^2 = 4k^2$ , so  $b^2 = 2k^2$ , which implies that  $b^2$  is even. But then  $b$  itself is even, which is a contradiction since  $a$  is even and  $a$  and  $b$  are coprime. Thus, there is no rational number whose square is 2.

(b) The polynomial  $f = x^n - p$  is irreducible over  $\mathbb{Z}$  by Eisenstein's criterion. By Gauss's Lemma,  $f$  is also irreducible over  $\mathbb{Q}$ . In particular,  $f$  has no roots in  $\mathbb{Q}$ , so the  $n$ th root of  $p$  is irreducible.

### Problem 2. Let $S$ be a dense subset of $\mathbb{R}$ .

- (i) Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and  $f(x) = 0$  for all  $x \in S$ .
  - (A) Prove, using the *definition* of denseness, that  $f(x) = 0$  for all  $x \in \mathbb{R}$ .
  - (B) Prove, using the fact that a subset of  $\mathbb{R}$  is dense if and only if it is *sequentially dense*, that  $f(x) = 0$  for all  $x \in \mathbb{R}$ .
- (ii) Suppose  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  are continuous and  $f(x) = g(x)$  for all  $x \in S$ . Prove that  $f(x) = g(x)$  for all  $x \in \mathbb{R}$ .

**Solution.** (i)(A) Let  $x \in \mathbb{R}$  be given, and pick any  $\varepsilon > 0$ . Since  $f$  is continuous at  $x$ , there exists a  $\delta > 0$  such that  $|f(x) - f(y)| < \varepsilon$  for all  $y \in \mathbb{R}$  such that  $|x - y| < \delta$ . Since  $S$  is dense in  $\mathbb{R}$ , there exists a number  $y \in S \cap (x - \delta, x + \delta)$ . Then  $f(y) = 0$  and  $|x - y| < \delta$ . Therefore,  $|f(x)| = |f(x) - f(y)| < \varepsilon$ . Since  $|f(x)| < \varepsilon$  for an arbitrary  $\varepsilon > 0$ , it follows that  $f(x) = 0$ . Since  $x$  was arbitrary, this shows that  $f(x) = 0$  for all  $x \in \mathbb{R}$ .

(i)(B) Since  $S$  is a dense subset of  $\mathbb{R}$ , it is also sequentially dense (i.e., every real number is a limit of some sequence in  $S$ ). Let  $x \in \mathbb{R}$  be given, and choose a sequence  $\{x_n\}$  in  $S$  that converges to  $x$ . For all  $n \in \mathbb{N}$ , we have  $f(x_n) = 0$  since  $x_n \in S$ . Thus,  $\{f(x_n)\} \rightarrow 0$ . However, since  $f$  is continuous and  $\{x_n\} \rightarrow x$ , it must be the case that  $\{f(x_n)\} \rightarrow f(x)$ , so  $f(x) = 0$ . Since  $x$  was arbitrary,  $f(x) = 0$  for all  $x \in \mathbb{R}$ .

(ii) Define  $h : \mathbb{R} \rightarrow \mathbb{R}$  by  $h(x) = f(x) - g(x)$ . Then  $h$  is continuous and  $h(x) = 0$  for all  $x \in S$  (since  $f(x) = g(x)$  for all  $x \in S$ ). By part (i), it follows that  $h(x) = 0$  for all  $x \in \mathbb{R}$ , so therefore  $f(x) = g(x)$  for all  $x \in \mathbb{R}$ .

**Problem 3** (Ahlfors §4.2.3 #2, p. 123). Prove that a function which is analytic in the whole plane and satisfies an inequality  $|f(z)| < |z|^n$  for some  $n$  and all sufficiently large  $|z|$  reduces to a polynomial.

**Solution.** Since  $f$  is entire, we may write  $f$  as a power series centered at 0 which converges for every complex number  $z$ :

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Let  $R > 0$  be large enough such that  $|f(z)| < |z|^n$  whenever  $|z| \geq R$ . Let  $\gamma$  be the positively oriented circle of radius  $R$  centered at the origin, parametrized as

$$\gamma(t) = Re^{it}, \quad t \in [0, 2\pi].$$

For each  $k \geq 0$  we have

$$\begin{aligned} a_k &= \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z^{k+1}} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t)^{k+1}} \gamma'(t) dt \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(Re^{it})}{R^{k+1} e^{i(k+1)t}} iRe^{it} dt = \frac{1}{2\pi R^k} \int_0^{2\pi} \frac{f(Re^{it})}{e^{ikt}} dt, \end{aligned}$$

and hence

$$\begin{aligned} |a_k| &= \frac{1}{2\pi R^k} \left| \int_0^{2\pi} \frac{f(Re^{it})}{e^{ikt}} dt \right| \leq \frac{1}{2\pi R^k} \int_0^{2\pi} \frac{|f(Re^{it})|}{|e^{ikt}|} dt \\ &= \frac{1}{2\pi R^k} \int_0^{2\pi} |f(Re^{it})| dt \leq \frac{1}{2\pi R^k} \int_0^{2\pi} R^n dt = \frac{R^n}{R^k} = \frac{1}{R^{k-n}}. \end{aligned}$$

If  $k > n$ , then letting  $R \rightarrow \infty$  gives  $|a_k| = 0$ . Thus, we have  $a_k = 0$  for  $k > n$ , so that

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n.$$

That is,  $f$  is a polynomial.

**Problem 4.** Prove that there are infinitely many prime numbers.

**Solution.** Let  $P$  be the set of prime numbers. To prove that  $P$  has infinitely many elements, it suffices to prove that the series

$$\sum_{p \in P} \frac{1}{p} = \lim_{x \rightarrow \infty} \sum_{\substack{p \in P \\ p \leq x}} \frac{1}{p} \tag{4.1}$$

diverges. Suppose, for the sake of contradiction, that (4.1) converges. Then there is a positive real number  $M$  such that

$$\sum_{\substack{p \in P \\ p > M}} \frac{1}{p} < \frac{1}{2}. \quad (4.2)$$

Let  $p_1, \dots, p_n$  be the distinct prime numbers contained in the interval  $[1, M]$ , and let  $Q = p_1 \cdots p_n$ . For every positive integer  $N$ , define

$$S(N) = \sum_{n=1}^N \frac{1}{1+nQ}$$

and let  $P(N)$  be the finite set of primes  $q$  which divide  $1+nQ$  for at least one  $n \in \{1, \dots, N\}$ . For each positive integer  $n$ , the integers  $Q$  and  $1+nQ$  are coprime, so none of the prime numbers  $p_1, \dots, p_n$  divide  $1+nQ$ . It follows that

$$P(N) \subseteq P \cap (M, \infty) \quad (4.3)$$

for every positive integer  $N$ . If  $m$  is another positive integer, then we define  $A(N, m)$  to be the set of all integers  $n$  such that  $1 \leq n \leq N$  and such that  $1+nQ$  has exactly  $m$  (not necessarily distinct) prime divisors. Consider the sum

$$S(N, m) = \sum_{n \in A(N, m)} \frac{1}{1+nQ}.$$

Observe that  $S(N, m) = 0$  for  $m$  sufficiently large (since  $A(N, m)$  is empty for  $m$  sufficiently large) and that

$$S(N) = \sum_{m=1}^{\infty} S(N, m). \quad (4.4)$$

If  $n \in A(N, m)$ , then  $1+nQ = (q_1 \cdots q_m)^{-1}$  for some prime numbers  $q_1, \dots, q_m \in P(N)$ , so

$$S(N, m) \leq \sum_{q_1, \dots, q_m \in P(N)} \frac{1}{q_1 \cdots q_m} = \left( \sum_{p \in P(N)} \frac{1}{p} \right)^m.$$

It then follows from (4.2) and (4.3) that

$$S(N, m) \leq \left( \sum_{p \in P(N)} \frac{1}{p} \right)^m \leq \left( \sum_{\substack{p \in P \\ p > M}} \frac{1}{p} \right)^m < \frac{1}{2^m}$$

Now (4.4) implies that

$$S(N) = \sum_{m=1}^{\infty} S(N, m) \leq \sum_{m=1}^{\infty} \frac{1}{2^m} = 1.$$

Thus, the sequence  $(S(N))_{N=1}^{\infty}$  is a monotone increasing bounded sequence, so it converges to the limit

$$\lim_{N \rightarrow \infty} S(N) = \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{1}{1+nQ} = \sum_{n=1}^{\infty} \frac{1}{1+nQ}.$$

However, the series above diverges (e.g., by the integral test), so we have reached a contradiction. It follows that the series (4.1) diverges.

**Problem 5** (Atiyah-Macdonald 1.1). Let  $x$  be a nilpotent element of a ring  $A$ . Show that  $1 + x$  is a unit of  $A$ . Deduce that the sum of a nilpotent element and a unit is a unit.

**Solution.** Let  $y = -x$ . Then  $y$  is also nilpotent, so choose a positive integer  $n$  such that  $y^n = 0$ . We have

$$(1 - y)(1 + y + y^2 + \cdots + y^{n-1}) = 1 - y^n = 1$$

which shows that  $1 + x = 1 - y$  is a unit.

Next, let  $x \in A$  be nilpotent and  $u \in A$  a unit. Then  $u^{-1}x$  is nilpotent, so  $1 + u^{-1}x$  is a unit by the first part. Therefore

$$u + x = u(1 + u^{-1}x).$$

Since it is a product of units,  $u + x$  is a unit.