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Example Homework 1

Problem 1.

(a) Prove that there is no rational number whose square is 2.

(b) Let n be an integer greater than 1. Prove that the nth root of any prime number p is irrational.

Solution. (a) Suppose that there is a rational number whose square is 2. Then we can find coprime nonzero integers a and b such that $a^2/b^2 = 2$. Then $a^2 = 2b^2$, so a^2 is even. It follows that a is even, so $a = 2k$ for some integer k. Then $2b^2 = 4k^2$, so $b^2 = 2k^2$, which implies that b^2 is even. But then b itself is even, which is a contradiction since a is even and a and b are coprime. Thus, there is no rational number whose square is 2.

(b) The polynomial $f = x^n - p$ is irreducible over $\mathbb{Z}$ by Eisenstein's criterion. By Gauss's Lemma, f is also irreducible over $\mathbb{Q}$. In particular, f has no roots in $\mathbb{Q}$, so the nth root of p is irreducible.

Problem 2 (Ahlfors §4.2.3 #2, p. 123)**.** Prove that a function which is analytic in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some n and all sufficiently large $|z|$ reduces to a polynomial.

Solution. Since f is entire, we may write f as a power series centered at 0 which converges for every complex number z:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Let $R > 0$ be large enough such that $|f(z)| < |z|^n$ whenever $|z| \geqslant R$. Let γ be the positively oriented circle of radius R centered at the origin, parametrized as

$$\gamma(t) = Re^{it}, \qquad t \in [0, 2\pi].$$

For each $k \geqslant 0$ we have

$$\begin{aligned} a_k = \frac{f^{(k)}(0)}{n!} &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{k+1}}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t)^{k+1}}\gamma'(t)\,dt \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(Re^{it})}{R^{k+1}e^{i(t+1)t}} iRe^{it}\,dt = \frac{1}{2\pi R^k}\int_0^{2\pi} \frac{f(Re^{it})}{e^{ikt}}\,dt, \end{aligned}$$

and hence

$$|a_k| = \frac{1}{2\pi R^k}\left|\int_0^{2\pi}\frac{f(Re^{it})}{e^{ikt}}\,dt\right| \leqslant \frac{1}{2\pi R^k}\int_0^{2\pi}\frac{|f(Re^{it})|}{|e^{ikt}|}\,dt$$
$$= \frac{1}{2\pi R^k}\int_0^{2\pi}|f(Re^{it})|\,dt \leqslant \frac{1}{2\pi R^k}\int_0^{2\pi}R^n\,dt = \frac{R^n}{R^k} = \frac{1}{R^{k-n}}.$$

If $k > n$, then letting $R \to \infty$ gives $|a_k| = 0$. Thus, we have $a_k = 0$ for $k > n$, so that

$$f(z) = a_0 + a_1 x + \cdots + a_n x^n.$$

That is, f is a polynomial.

Problem 3. Prove that there are infinitely many prime numbers.

Solution. Let P be the set of prime numbers. To prove that P has infinitely many elements, it suffices to prove that the series

$$\sum_{p\in P}\frac{1}{p} = \lim_{x\to\infty}\sum_{\substack{p\in P\\ p\leqslant x}}\frac{1}{p} \tag{3.1}$$

diverges. Suppose, for the sake of contradiction, that (3.1) converges. Then there is a positive real number M such that

$$\sum_{\substack{p\in P\\ p>M}}\frac{1}{p} < \frac{1}{2}. \tag{3.2}$$

Let $p_1, \ldots, p_n$ be the distinct prime numbers contained in the interval $[1, M]$, and let $Q = p_1 \cdots p_n$. For every positive integer N, define

$$S(N) = \sum_{n=1}^{N}\frac{1}{1+nQ}$$

and let $P(N)$ be the finite set of primes q which divide $1 + nQ$ for at least one $n \in \{1, \ldots, N\}$. For each positive integer n, the integers Q and $1 + nQ$ are coprime, so none of the prime numbers $p_1, \ldots, p_n$ divide $1 + nQ$. It follows that

$$P(N) \subseteq P \cap (M, \infty) \tag{3.3}$$

for every positive integer N. If m is another positive integer, then we define $A(N, m)$ to be the set of all integers n such that $1 \leqslant n \leqslant N$ and such that $1 + nQ$ has exactly m (not necessarily distinct) prime divisors. Consider the sum

$$S(N, m) = \sum_{n\in A(N,m)}\frac{1}{1+nQ}.$$

Observe that $S(N,m) = 0$ for m sufficiently large (since $A(N,m)$ is empty for m sufficiently large) and that

$$S(N) = \sum_{m=1}^{\infty} S(N,m). \tag{3.4}$$

If $n \in A(N,m)$, then $1 + nQ = (q_1 \cdots q_m)^{-1}$ for some prime numbers $q_1, \ldots, q_m \in P(N)$, so

$$S(N,m) \leqslant \sum_{q_1,\ldots,q_m \in P(N)} \frac{1}{q_1 \cdots q_m} = \left(\sum_{p \in P(N)} \frac{1}{p} \right)^m .$$

It then follows from (3.2) and (3.3) that

$$S(N,m) \leqslant \left(\sum_{p \in P(N)} \frac{1}{p} \right)^m \leqslant \left(\sum_{\substack{p \in P \\ p > M}} \frac{1}{p} \right)^m < \frac{1}{2^m}$$

Now (3.4) implies that

$$S(N) = \sum_{m=1}^{\infty} S(N,m) \leqslant \sum_{m=1}^{\infty} \frac{1}{2^m} = 1.$$

Thus, the sequence $(S(N))_{N=1}^{\infty}$ is a monotone increasing bounded sequence, so it converges to the limit

$$\lim_{N \to \infty} S(N) = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{1+nQ} = \sum_{n=1}^{\infty} \frac{1}{1+nQ}.$$

However, the series above diverges (e.g., by the integral test), so we have reached a contradiction. It follows that the series (3.1) diverges.

Problem 4 (Atiyah-Macdonald 1.1)**.** Let x be a nilpotent element of a ring A. Show that $1 + x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

Solution. Let $y = -x$. Then y is also nilpotent, so choose a positive integer n such that $y^n = 0$. We have

$$(1-y)(1+y+y^2+\cdots+y^{n-1}) = 1 - y^n = 1$$

which shows that $1 + x = 1 - y$ is a unit.

Next, let $x \in A$ be nilpotent and $u \in A$ a unit. Then ux is nilpotent, so $1 + ux$ is a unit by the first part. Therefore

$$u + x = u^{-1}(1 + ux)$$

is a product of units, so it is a unit.

Problem 5. Let S be a dense subset of $\mathbb{R}$. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and if $f(x) = 0$ for all $x \in S$, then $f(x) = 0$ for all $x \in \mathbb{R}$.

Solution. Let $x \in \mathbb{R}$ be given, and pick any $\varepsilon > 0$. Since f is continuous at x, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $y \in \mathbb{R}$ such that $|x - y| < \delta$. Since S is dense in $\mathbb{R}$, there exists a number $y \in S \cap (x - \delta, x + \delta)$. Then $f(y) = 0$ and $|x - y| < \delta$. Therefore, $|f(x)| = |f(x) - f(y)| < \varepsilon$. Since $|f(x)| < \varepsilon$ for an arbitrary $\varepsilon > 0$, it follows that $f(x) = 0$. Since x was arbitrary, this shows that $f(x) = 0$ for all $x \in \mathbb{R}$.