Artem Mavrin
Fall 2016

Math 101 Homework 1

Problem 1 (Euclid)**.** Prove that there are infinitely many prime numbers.

Solution. Let n be a natural number, and suppose we can find n prime numbers $p_1, \ldots, p_n$. Then $N = p_1 \cdots p_n + 1$ is coprime to each p_i, so it has a prime factor q different from any of the p_i's. Then $p_1, \ldots, p_n, q$ is a list of $n+1$ prime numbers. It follows that, given any finite set of prime numbers, we can always find a prime number not in that set. Therefore, there are infinitely many prime numbers. This contradicts the assumption that there are only finitely many primes.

Problem 2.

(a) Prove that $\sqrt{2}$ is irrational.

(b) Let n be an integer greater than 1. Prove that the nth root of any prime number p is irrational.

Solution. (a) If $\sqrt{2} = a/b$ for coprime integers a and b, then $2b^2 = a^2$, so a^2 is even. Then a is also even, so $b^2 = 2c^2$ for some integer c. But then b^2 is even, hence b is even. This contradicts the assumption that a and b are coprime.

(b) The polynomial $f = x^n - p$ is irreducible over $\mathbb{Z}$ by Eisenstein's criterion. By Gauss's Lemma, f is also irreducible over $\mathbb{Q}$. In particular, f has no roots in $\mathbb{Q}$, so the nth root of p is irreducible.