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# Math 101 Homework 1

**Problem 1** (Euclid)**.** Prove that there are infinitely many prime numbers.

**Solution.** Let $n$ be a natural number, and suppose we can find $n$ prime numbers $p_1, \ldots, p_n$. Then $N = p_1 \cdots p_n + 1$ is coprime to each $p_i$, so it has a prime factor $q$ different from any of the $p_i$'s. Then $p_1, \ldots, p_n, q$ is a list of $n+1$ prime numbers. It follows that, given any finite set of prime numbers, we can always find a prime number not in that set. Therefore, there are infinitely many prime numbers. This contradicts the assumption that there are only finitely many primes.

**Problem 2.**

(a) Prove that $\sqrt{2}$ is irrational.

(b) Let $n$ be an integer greater than 1. Prove that the $n$th root of any prime number $p$ is irrational.

**Solution.** (a) If $\sqrt{2} = a/b$ for coprime integers $a$ and $b$, then $2b^2 = a^2$, so $a^2$ is even. Then $a$ is also even, so $b^2 = 2c^2$ for some integer $c$. But then $b^2$ is even, hence $b$ is even. This contradicts the assumption that $a$ and $b$ are coprime.

(b) The polynomial $f = x^n - p$ is irreducible over $\mathbb{Z}$ by Eisenstein's criterion. By Gauss's Lemma, $f$ is also irreducible over $\mathbb{Q}$. In particular, $f$ has no roots in $\mathbb{Q}$, so the $n$th root of $p$ is irreducible.