1 First problem: Indefinite integral

a. Rational function.
b. Fractional rational function with transcendental part.

1.1 Rational function

Methods: indefinite coefficients, application under differential.
Warnings: dont forget when you have found the coefficients, substitute them into the equation to make sure that they are correct.
Example: 2022–2023, first task (answer).

$$\int \frac{x^2+2}{(x+1)(x^2-2x+3)}dx = ? \tag{1}$$

$$\int \frac{x^2+2}{(x+1)(x^2-2x+3)}dx = \int \frac{A}{x+1}dx + \int \frac{Bx+C}{x^2-2x+3}dx$$

$$x^2+2 = A(x^2-2x+3) + (Bx+C)(x+1) \tag{2}$$

$$1.\ 2 = 3A + C$$

$$x.\ 0 = -2A + B + C$$

$$x^2.\ 1 = A + B$$

$$A = \frac{1}{2},\ B = \frac{1}{2},\ C = \frac{1}{2} \tag{3}$$

$$\int \frac{x^2+2}{(x+1)(x^2-2x+3)}dx = \int \frac{\frac{1}{2}}{x+1}dx + \int \frac{\frac{x}{2}+\frac{1}{2}}{x^2-2x+3}dx = \frac{1}{2}ln|x+1| + \frac{1}{2}\int \frac{x+1}{x^2-2x+3}dx$$

$$\frac{1}{2}\int \frac{x+1}{x^2-2x+3}dx = \frac{1}{2}\int \frac{x+1}{2x-2}d(ln|x^2-2x+3|) = \frac{1}{4}\int \frac{2x-2+2+2}{2x-2}d(ln|x^2-2x+3|)$$

$$\frac{1}{4}\int d(ln|x^2-2x+3|) + \int \frac{1}{x^2-2x+3}dx = \frac{1}{4}ln|x^2-2x+3| + \int \frac{1}{(x-1)^2+2}dx$$

$$\frac{1}{4}ln|x^2-2x+3| + \frac{1}{\sqrt{2}}\arctan(\frac{x-1}{\sqrt{2}}) + C$$

$$\int \frac{x^2+2}{(x+1)(x^2-2x+3)}dx = \frac{1}{2}ln|x+1| + \frac{1}{4}ln|x^2-2x+3| + \frac{1}{\sqrt{2}}\arctan(\frac{x-1}{\sqrt{2}}) + C \tag{4}$$

1.2 Fractional rational function with transcendental part.

Methods: variable replacement, differentiation, integration by parts, application under differential.
Example: 2022–2023, first task (answer).

$$\int \frac{xexp(x)}{\sqrt{exp(x)-1}}dx = ? \tag{5}$$

$$t = \sqrt{exp(x)-1} \tag{6}$$

$$t^2 + 1 = exp(x) \tag{7}$$

$$ln(t^2+1) = x \tag{8}$$

$$dt = \frac{1}{2}\frac{exp(x)}{\sqrt{exp(x)-1}}dx = \frac{1}{2}\frac{t^2+1}{t}dx$$

$$dx = \frac{2tdt}{1+t^2} \tag{9}$$

$$\int \frac{xexp(x)}{\sqrt{exp(x)-1}}dx = \int \frac{(t^2+1)ln(t^2+1)}{t}\frac{2tdt}{1+t^2} = \int 2ln(1+t^2)dt$$

$$\int 2ln(1+t^2)dt = 2tln(1+t^2) - \int 2td(ln(1+t^2)) = 2tln(1+t^2) - \int 2t\frac{2t}{1+t^2}dt$$

$$2tln(1+t^2) - \int \frac{4t^2}{1+t^2}dt = 2tln(1+t^2) - \int \frac{4t^2+4-4}{1+t^2}dt$$

$$-\int 4dt + \int \frac{4dt}{1+t^2} + 2tln(1+t^2) = 2tln(1+t^2) - 4t + 4\arctan(t) + C$$

$$\int \frac{xexp(x)}{\sqrt{exp(x)-1}}dx = 2tln(1+t^2) - 4t + 4\arctan(t) + C \tag{10}$$

2 Second problem: Differentials and Taylor's formula for functions of several variables

Methods: partial derivatives, Taylor's multivariable formula.
Warnings: dont forget about function value at a point (you must add it to Taylor's formula as constant) and about additional multipliers (if you write first derivative at Taylor's formula you divide by 1, second derivative by 2, third derivative by 6 etc.).
Example: 2022–2023, second task (answer).

$$f(x,y)=\tan(x+sin(xy^2)),\ M=(\frac{\pi}{4},0),\ df=?\ df^2=?\ f=?+o((x-\frac{\pi}{4})^2+y^2),\ x\to\frac{\pi}{4},\ y\to 0 \tag{11}$$

$$\frac{\partial}{\partial x}f\Big|_M=\frac{1+cos(xy^2)y^2}{cos^2(x+sin(xy^2))}\Big|_M=2$$

$$\frac{\partial}{\partial y}f\Big|_M=\frac{cos(xy^2)x2y}{cos^2(x+sin(xy^2))}\Big|_M=0$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}f\Big|_M=y\cdot ...\Big|_M=0$$

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}f\Big|_M=\frac{cos^2(x+sin(xy^2))(cos(xy^2)2y-y^2sin(xy^2)x2y)-}{cos^4(x+sin(xy^2))}$$

$$\frac{-(1+cos(xy^2)y^2)2cos(x+sin(xy^2))(-sin(x+sin(xy^2)))(cos(xy^2)x2y)}{cos^4(x+sin(xy^2))}\Big|_M=y\cdot ...\Big|_M=0$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x}f\Big|_M=\frac{(cos^2(x+sin(xy^2)))(-sin(xy^2)y^2y^2)-}{cos^4(x+sin(xy^2))}$$

$$\frac{-(1+cos(xy^2)y^2)(2cos(x+sin(xy^2))(-sin(x+sin(xy^2)))(1+...y)}{cos^4(x+sin(xy^2))}\Big|_M=\frac{-1\cdot 2cos(\pi/4)(-sin(\pi/4))}{cos^4(\pi/4)}=$$

$$=\frac{4\cdot 2}{2}=4$$

$$\frac{\partial}{\partial y}\frac{\partial}{\partial y}f\Big|_M=\pi$$

$$df=2dx,\ d^2f=4dx^2+\pi dy^2 \tag{12}$$

$$f(M)=1$$

$$f=1+2(x-\frac{\pi}{4})+2(x-\frac{\pi}{4})^2+\frac{\pi}{2}y^2+o((x-\frac{\pi}{4})^2+y^2),\ x\to\frac{\pi}{4},\ y\to 0 \tag{13}$$

3 Third problem: Length, area and volume calculation using definite integral

3.1 Length

Methods: .
Warnings: .
Example: 2022–2023, third task (answer).

$$x = sin^3(\frac{t}{2}),\ x = cos^3(\frac{t}{2}), z = \frac{3}{4}(t + sin(t)),\ 0 \le t \le \pi,\ |L| = ? \tag{14}$$

$$|L| = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} dt \tag{15}$$

$$\frac{dx}{dt} = 3sin^2(t/2)cos(t/2)\frac{1}{2}$$

$$\frac{dy}{dt} = 3cos^2(t/2)(-sin(t/2))\frac{1}{2}$$

$$\frac{dz}{dt} = \frac{3}{4}(1 + cos(t))$$

$$|L| = \int_0^\pi \sqrt{\frac{9}{4}sin^2(t/2)cos^2(t/2)(sin^2(t/2) + cos^2(t/2)) + \frac{9}{16} + \frac{9}{16}cos^2(t) + \frac{9}{8}cos(t)dt}$$

$$|L| = \int_0^\pi \sqrt{\frac{9}{4}sin^2(t/2)cos^2(t/2) + \frac{9}{16} + \frac{9}{16}cos^2(t) + \frac{9}{8}cos(t)dt}$$

$$|L| = \int_0^\pi \sqrt{\frac{9}{16}sin^2(t) + \frac{9}{16} + \frac{9}{16}cos^2(t) + \frac{9}{8}cos(t)dt}$$

$$|L| = \int_0^\pi \sqrt{\frac{9}{8} + \frac{9}{8}cos(t)dt}$$

$$|L| = \int_0^\pi \frac{3}{2\sqrt{2}}\sqrt{1 + cos(t)}dt$$

$$cos^2(t/2) = \frac{1 + cos(t)}{2}$$

$$|L| = \int_0^\pi \frac{3}{2}|cos(t/2)|dt$$

$$|L| = \int_0^\pi 3|cos(t/2)|dt/2$$

$$|L| = \int_0^{\pi/2} 3|cos(\phi)|d\phi$$

$$|L| = 3 \tag{16}$$

4 Sixth problem: Convergence of a constant sign number series

Method: to find a suitable convergence test.
Need to know: d'Alembert's convergence test, Cauchy's convergence test.
Warnings: most convergence tests works only with constant sign number series.
Example: 2022–2023, sixth task (answer).

$$\sum_{n=1}^{\infty} \frac{C_{3n}^{n}}{7^n} \tag{17}$$

$$\lim_{n\to\infty} \frac{C_{3n+3}^{n+1}}{7^{n+1}} \frac{7^n}{C_{3n}^{n}} = \lim_{n\Rightarrow\infty} \frac{1}{7} \frac{(3n+3)!}{(n+1)!((3n+3)-(n+1))!} \frac{n!(3n-n)!}{3n!}$$

$$\lim_{n\to\infty} \frac{1}{7} \frac{(3n+1)(3n+2)(3n+3)}{(n+1)(2n+1)(2n+2)} = \frac{3\cdot 3\cdot 3}{7\cdot 2\cdot 2} = \frac{27}{28} < 1$$

$\Rightarrow$ number series converges using d'Alembert's ratio test.

5 Eighth problem: Taylor series

Method: calculate derivative of a complex part, expand the result in Taylor series, integrate result, multiply result to not complex part.
Need to know: derivative table (trigonometric), Taylor series table, Cauchy–Hadamard theorem.
Warnings: don't forget to correctly calculate derivative of complex function (that you must multiply derivative of argument), don't forget that you must add constant after integration (function value in the null), if you have x to the power 2n in your series, you must take square root of R that you calculated with Cauchy–Hadamard theorem
Example: 2022–2023, eight task (answer).

$$f = x^2 \arccos \frac{\sqrt{2+x}}{2} = \sum ..? \tag{18}$$

$$\frac{d}{dx} \arccos \frac{\sqrt{2+x}}{2} = \frac{-1}{\sqrt{1-\frac{2+x}{4}}} \frac{1}{4\sqrt{2+x}} = \frac{-1}{2\sqrt{4-2-x}\sqrt{2+x}} = \frac{-1}{2\sqrt{4-x^2}}$$

$$\frac{-1}{2\sqrt{4-x^2}} = -\frac{1}{4} \sum_{k=0}^{\infty} C_{-1/2}^{k} (\frac{-x^2}{4})^n = \sum_{k=0}^{\infty} C_{-1/2}^{k} (-1)^{n+1} \frac{x^{2n}}{4^{n+1}}$$

$$\arccos \frac{\sqrt{2+x}}{2} = \arccos(\frac{\sqrt{2}}{2}) + \sum_{k=0}^{\infty} C_{-1/2}^{k} (-1)^{n+1} \frac{x^{2n+1}}{4^{n+1}(2n+1)}$$

$$f = \frac{\pi}{4} x^2 + \sum_{k=0}^{\infty} C_{-1/2}^{k} (-1)^{n+1} \frac{x^{2n+3}}{4^{n+1}(2n+1)} \tag{19}$$

$$\lim_{n\to\infty} |\frac{C_{-1/2}^{n}}{4^{n+1}(2n+1)} \frac{4^{n+2}(2n+3)}{C_{-1/2}^{n+1}}| = \lim_{n\to\infty} |4 \frac{n+1}{-1/2-n}| = 4$$

$$4 \sim x^{2n} \Rightarrow 2 \sim x^n$$

$$R = 2 \tag{20}$$

6 Sixth problem: Theory

6.1 d'Alembert's convergence test

$$1.\ let\ a_n > 0\ n \in \mathbb{N}$$

$$2.\ let\ \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda\ exist$$

$$then\ if$$

$$q < 1\ \sum_{n=0}^{\infty} a_n\ converges$$

$$q > 1\ \sum_{n=0}^{\infty} a_n\ diverges$$

$$q = 1\ \sum_{n=0}^{\infty} a_n\ can\ converge\ or\ diverge$$

7 Eighth problem: Theory

7.1 Trigonometric derivative table

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}arcctgx = \frac{-1}{1+x^2}$$

7.2 Taylor series table

$$(1+x)^\alpha = \sum_{n=0}^{\infty} C_\alpha^n x^n$$

$$C_\alpha^n = \frac{\alpha(\alpha-1)...(\alpha-(n-1))}{n!}$$

7.3 Cauchy–Hadamard theorem

$$R = \lim_{n\to\infty} |\frac{c_n}{c_{n+1}}|$$