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NEWTONIAN MECHANICS APPLICATIONS IN ASTROPHYSICS
牛顿力学在天体物理学中的应用

Classical Mechanics course by Prof. Fan XiLong (范锡龙)

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Physics Department, Wuhan University – Fall 2020

Picture credit: 蒋效铭

Outline

PART 1

1. Overview on the Universe
2. Newtonian Mechanics
   1. Hydrostatic Equilibrium
   2. The Virial Theorem
   3. The Jeans Criterion
   4. Free-Fall Time

PART 2

1. Galaxy rotation curves
2. Beyond Newtonian Cosmology
   1. Relativity
   2. Standard Cosmology
   3. Theories of modified gravity
3. Cosmological Simulations

Who am I?

- Postdoc in the Chemical Evolution of Galaxies at Wuhan University, China
- PhD from University of Trieste, Italy: included a Dust Evolution model in Cosmological Simulations of Galaxy Clusters
- Undergraduate research at IIT and Argonne National Lab in Chicago: Supernova Cosmology
  - selection algorithm efficiency for DES
  - filter transmission efficiency for LSST
  - modified gravity vs quintessence theory

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© Images: NASA/ESA (NGC4214, M27, Cepheus B cloud) Graphics: Eda Gjergo

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Size of the Universe

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Scale of Cosmic Objects

https://www.htwins.net/scale2/

Scale of Cosmic Objects

Name	Symbol	Conversion	Commonly applied to the following scales
Astronomical Unit	AU	$\sim 1.4959787066 \times 10^{11}$ m ~ 8.3 light minutes	Solar system
Light-year	ly	$9.460730472 \times 10^{15}$ m	universal
Parsec	pc	3.0856776×10^{16} m 3.26167 ly	stars
kiloparsec	kpc	10^3 pc	galaxies
megaparsec	Mpc	10^6 pc	galaxy clusters
gigaparsec	Gpc	10^9 pc	large scales

Table 1: Common units in astronomy.

Our Universe

Image Credit: NASA/WMAP Science team. Edit: Subaru Prime Focus Spectrograph

The composition of the Universe changed with time

cosmic scale factor

$a(t)$

$a(t) = 1$

$a(t) = 0$

Physical coordinates

$$\mathbf{r}(t) = a(t)\,\mathbf{x}$$

Comoving coordinates

© DESC

The composition of the Universe changed with time

© DESC

Our Galaxy: the Milky Way

https://astrobob.areavoices.com

Our Galaxy: the Milky Way

The Milky Way observed at different wavelengths

Our Galaxy: the Milky Way

Image Credit: Stefan Payne-Wardenaar. Data: ESA/GAIA

Video of the Milky Way Warp: the dynamics of our galaxy is very much active!

(J. Skowron / OGLE / Astronomical Observatory, University of Warsaw)

First Example of Classical Mechanics In Astrophysics

HOW ARE STARS HELD TOGETHER

What holds stars together?

- An equilibrium between forces of pressure and gravity

Credit: Brian Woodahl (http://woodahl.physics.iupui.edu/Astro105/)

Forces on a small mass component within stars

- We assume spherical symmetry in the following computation.
- The cylinder represents an infinitesimal volume of a star
- The bottom is closer to the center of the star, the top is closer to the surface.
- The bottom surface of the cylinder is at a distance r from the center of the star.
- (it is drawn as a cylinder, but the derivation holds for any shape)

Image Credit: Chris Mihos (http://burro.cwru.edu/Academics/Astr221)

Forces on a small mass component within stars

- This volume of star perceives pressure from top and bottom
- (we ignore the pressure from the sides: due to homogeneity the forces cancel out)
- It is also subject to gravity toward the center of the star.
- We can define the mass m of this element as a product between its density ρ and volume $V = \mathrm{d}r\mathrm{d}A$

Image Credit: Chris Mihos (http://burro.cwru.edu/Academics/Astr221)

Forces on a small mass component within stars

- The net forces on this mass elements are zero, therefore:

$$P(r)dA - P(r+dr)dA - F_{grav} = 0$$

$$-[P(r+dr) - P(r)]dA - ma_g = 0$$

$$-dPdA - \left[\rho(g)\left(drdA\right)\right] g(r) = 0$$

Image Credit: Chris Mihos (http://burro.cwru.edu/Academics/Astr221)

Hydrostatic Equilibrium

$$\boxed{dP = -g(r)\rho(r)dr}$$

- Both g – the gravitational acceleration –, and ρ – the star's density – are positive
- So the pressure decreases with increasing radius (the closer to the center, the higher the pressure)
- But g can also be expressed as:

$$F_{grav} = mg(r) = \frac{GmM(r)}{r^2} \rightarrow g(r) = \frac{GM(r)}{r^2}$$

The Virial Theorem Derivation

- Multiply both sides by the volume of a sphere of radius r.
- $V = \frac{4}{3}\pi r^3$
- How would you express the rate of change of the mass inside a spherical shell between r and $r + \mathrm{d}r$?

$$\mathrm{d}M = \rho(r)\left(4\pi r^2 \mathrm{d}r\right)$$

The RHS: substitute volume, acceleration, and differential

$$V\mathrm{d}P = -\left(\frac{4}{3}\pi r^3\right)\rho(r)\left(\frac{GM(r)}{r^2}\right)\left(\frac{\mathrm{d}M}{4\pi r^2\rho(r)}\right)$$

- We substitute the radius-dependent expression for volume, the gravitational acceleration $g(r)$, and we switch the differential from radius $\mathrm{d}r$ to mass $\mathrm{d}M$.

$$V\mathrm{d}P = -\frac{GM(r)}{3r}\mathrm{d}M$$

- Now change variables the right hand side (RHS) for the differential from radius $\mathrm{d}r$ to mass $\mathrm{d}M$.
- But the integral of the RHS is proportional to the gravitational potential energy U_{grav}

$$U_{grav} = -\int_0^{M_{star}} \frac{GM(r)}{r}\mathrm{d}M$$

Gas, thermal energy, and pressure: simplifying the LHS

$$V\mathrm{d}P = -\frac{1}{3}\frac{GM(r)}{r}\mathrm{d}M$$

- How can we tackle the pressure differential in the left hand side (LHS)?
- Integrate $V\mathrm{d}P$ by parts.

$$\int_{P_0}^{0} V\mathrm{d}P\,. = PV|_{center}^{surface} - \int_{0}^{V_{surface}} P\mathrm{d}V$$

- We can now integrate w.r.t. volume instead of pressure.

A bit of statistical physics

- What relation holds in an ideal gas between pressure P, volume V, temperature T, and number of particles N?

- $P = \frac{kNT}{V}$
- Where $k = 1.380658 \times 10^{-9} J/K$ is the Boltzmann constant.

- The equipartition theorem relates the average kinetic energy density ϵ of particles in a gas at temperature T with:

$$\epsilon = \frac{3}{2} kT \frac{N}{V}$$

- So it follows that the LHS:

$$\int_0^{V_{tot}} P \mathrm{d}V = \int_0^{V_{tot}} \frac{kNT}{V} \mathrm{d}V = \frac{2}{3} \int_0^{V_{tot}} \epsilon \mathrm{d}V$$

Virial Theorem: Putting it all together

- The integral is none other than the total thermal energy K of the star:

$$K = \int_0^{V_{tot}} \epsilon \mathrm{d}V$$

- So the LHS:

$$V\mathrm{d}P = -\int_0^{V_{tot}} P\mathrm{d}V = -\frac{2}{3}K$$

- While the RHS:

$$-\frac{1}{3}\int_0^{M_{star}} \frac{GM(r)}{r}\mathrm{d}M = \frac{1}{3}U_{grav}$$

- Combining the two:

$$2K + U_{grav} = 0$$

The Virial Theorem

$$2K_{therm} + U_{grav} = 0$$

- Comes up over and over in astrophysical systems.
  - Governs gravitational collapse of gas
  - Determines the minimum mass of a star
  - Helps compute the total mass of galaxies (analytic solutions for spirals and ellipticals)
  - Governs timescales of structure formation
- It also has limitations
  - I applies only to systems of point particles.
  - It does not apply to particles that are not gravitationally bound.

Example to work out in class

Calculate the potential gravitational energy of a star of mass M_* and radius R_* assuming it possesses a <u>constant density</u> ρ

Solution

Calculate the potential gravitational energy of a star of mass M_* and radius R_* assuming it possesses a constant density ρ

$$U_{grav} = -\int_0^{M_{star}} \frac{GM(r)}{r} \mathrm{d}M$$

HINTS

- How do you express the gravitational energy?
- What is the average density within the radius r?
- From the previous equation, how can you express r?
- Write the gravitational energy in terms of $M(r)$ and solve the integral.
- Given that we assume the density is constant, how would you express it in terms of total mass and radius of the star?
- Put it all together to answer the problem

$$\bar{\rho}(r) = \frac{M(r)}{\frac{4}{3}\pi r^3} \rightarrow r = \left(\frac{M(r)}{\frac{4}{3}\pi\rho}\right)^{1/3}$$

$$U_{grav} = -\frac{3}{5}\frac{GM_*^2}{R_*}$$

Second Example of Classical Mechanics In Astrophysics

UNDER WHICH PHYSICAL CONDITIONS DOES A FLUID (LIKE AN INTERSTELLAR CLOUD) COLLAPSE?

To answer the question: Apply the Virial Theorem!

- When will gravitational energy dominate over thermal energy?

$$-U_{grav} > 2K$$

- Under this condition, how will the cloud behave?

It will collapse

- How do we express the gravitational potential energy, assuming homogeneous density?

$$U_{grav} = -\frac{3}{5}\frac{GM_*^2}{R_*}$$

What about the thermal energy?

- Again, from the equipartition theorem:

$$K = \frac{3}{2} kNT$$

- We can express the number of particles N as a function of the mean molecular weight μ. m_H is the weight of a Hydrogen atom

$$N = \frac{M}{\mu m_H}$$

- From the previous exercise, what is R for this isothermal gas at uniform density?

$$R = \left(\frac{M(r)}{\frac{4}{3}\pi\rho} \right)^{1/3}$$

The Jeans Criterion

- Let's write the Virial Theorem, substituting these quantities:

$$-U_{grav} > 2K$$

$$\frac{3}{5}\frac{GM^2}{R} > \frac{3}{2}kNT$$

$$\frac{3}{5}GM^2\left(\frac{M(r)}{\frac{4}{3}\pi\rho}\right)^{-1/3} > \frac{3MkT}{\mu m_H}$$

- Lastly, by isolating M, we obtain the Jeans criterion:

$$\boxed{M > M_J = \left(\frac{5kT}{\mu m_H G}\right)^{3/2}\left(\frac{3}{4\pi\rho}\right)^{1/2}}$$

- Or equivalently for density:

$$\rho > \rho_J = \left(\frac{5kT}{\mu m_H G}\right)^{3}\left(\frac{3}{4\pi M^2}\right)$$

In-class exercise: derive the Jeans radius

Or at home if we don't have time.

It's simple algebra, but this radius comes up often in both observation and simulation papers.

$$R_J = \left(\frac{15kT}{4\pi\rho\mu m_H G} \right)^{1/2}$$

Jeans' Mass example:

Calculate Jeans' mass for an average molecular cloud. Typically, molecular clouds have masses on the order of 1000 $M_\odot$ or more, temperatures on the order of 10K and number densities of approximately 1000 H_2 molecules per cm^3.

Consider that $m_H = 1.674 \times 10^{-24}$ g

And that the solar mass $M_\odot = 1.9891 \times 10^{33}$g

The Boltzmann constant $k = 1.380658 \times 10^{-9} J/K$

$G = 6.674 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$

What will be the density of the cloud? In units of [g cm^{-3}]

$$\rho = 2m_H N/V \approx 3 \times 10^{-21}\,\mathrm{g\,cm^{-3}}$$

The Jeans' mass will be:

$$M_J \approx 20 M_\odot$$

So this cloud <u>will</u> collapse

(it's much more massive than M_J – in fact, several stars will form from this single cloud)

Jeans' Density example:

Calculate Jeans' density for a diffuse hydrogen (or HI) cloud. Typically, diffuse hydrogen clouds have masses of less than 100 $M_\odot$, temperatures on the order of 100K and number densities of less than 1000 H atoms per cm^3.

Consider that $m_H = 1.674 \times 10^{-24}$ g

And that the solar mass $M_\odot = 1.9891 \times 10^{33}$g

The Boltzmann constant $k = 1.380658 \times 10^{-9} J/K$

$G = 6.674 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$

What will be the density of the cloud? In units of [g cm^{-3}]

$$\rho = m_H N/V \approx 2 \times 10^{-21}\,\mathrm{g\,cm^{-3}}$$

Its Jeans' density will be:

$$\rho_J \approx 10^{-18}\,\mathrm{g\,cm^{-3}}$$

As this neutral hydrogen cloud is less dense than the Jeans' density, it will be stable and it will <u>not</u> collapse.

Third Example of Classical Mechanics In Astrophysics

HOW LONG DOES IT TAKE FOR A FLUID (e.g INTERSTELLAR CLOUD) COLLAPSE?

Free-Fall Time

The Free-fall time is defined as the time it takes a cloud to collapse from an original shape to a single point

Pressure, magnetic fields, and momenta will all affect real cloud collapse times !!!

(Also, hydrostatic equilibrium will kick in much earlier than the collapse to a single point – in which case a stellar black hole would form and we'd need to take general relativity into account)

Free-Fall Time derivation

Let's consider what happens to a small mass m, initially at rest on the surface of a spherical cloud, freely contracting under gravity

The Kinetic energy of m is equivalent to the difference between the collapsed and the initial gravitational potential energy

$$K = \frac{1}{2}m\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = U_f - U_0 = \frac{GMm}{r} - \frac{GMm}{R}$$

Solving w.r.t. time

- We can isolate the velocity of the particle:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\left(\frac{GMm}{r} - \frac{GMm}{R}\right)^{1/2}$$

- Hence obtaining a differential equation for the free fall time

$$t_{ff} = -\int_R^0 \left(\frac{GMm}{r} - \frac{GMm}{R}\right)^{-1/2} dr$$

- Let's perform a change of variable $x = r/R$:

$$t_{ff} = \left(\frac{R^3}{2GM}\right)^{1/2} \int_0^1 \left(\frac{x}{1-x}\right)^{1/2} dx$$

- You may recall that's a definite integral equal to $\pi/2$
- Assume the mass for a homogeneous cloud:

$$M = \frac{4}{3}\pi R^3 \rho$$

Free-Fall Time

$$t_{ff} = \frac{\pi}{2}\left(\frac{R^3}{2GM}\right)^{1/2} = \left(\frac{3\pi}{32G\rho}\right)^{1/2}$$

Despite the assumptions and crude model, this expression to first order identifies reliable cloud collapse timescales.

Free -Fall time example

How long will it take for a molecular cloud as heavy as the sun to collapse?

Calculate the Jeans' density:

$$\rho_J = \left(\frac{5kT}{\mu m_H G}\right)^3 \left(\frac{3}{4\pi M^2}\right)$$

$$\rho \approx 2 \times 10^{-18} \text{ g cm}^{-3}$$

$$t_{ff} = \left(\frac{3\pi}{32G\rho}\right)^{1/2} \approx 50\ 000 \text{ yr}$$