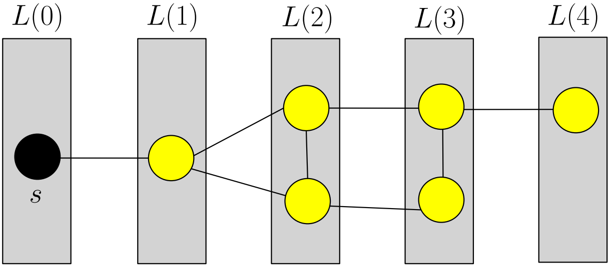
# Massive Data Algorithmics

## Lecture 11: BFS and DFS

# Breadth-First Search(BFS)

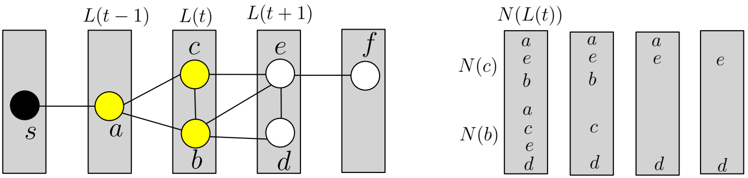
- One of the most basic graph-traversal methods
  - input: $G(V,E)$, undirected
  - one starting point: $s$
  - compute: BFS-levels $L(i)$, where $L(i)$ node with dist. $i$ from $s$




- Standard implementation for internal memory: $O(|V|+|E|)$ time

# Breadth-First Search(BFS)

- $N(L(t))$: all neighbors of nodes in $L(t)$
- Idea: all reached nodes in $N(L(t))$ belong to $L(t)$ or $L(t-1)$
- **Procedure** BFS
  1: Compute $N(L(t))$ : $O(|L(t)| + |N(L(t))|/B)$
  2: Eliminate duplicates in $N(L(t))$ by sorting: $O(\text{sort}(|N(L(t))|))$ I/Os
  3: Eliminate nodes already in $L(t)$ by sorting: $O(\text{sort}(|L(t)|))$ I/Os
  4: Eliminate nodes already in $L(t-1)$ by sorting: $O(\text{sort}(|L(t-1)|))$ I/Os



# Breadth-First Search(BFS)

- Analysis
  - $\sum_t |N(L(t))| \leq 2|E|$
  - $\sum_t |L(t)| \leq |V|$

  $\Rightarrow O(|V| + \text{sort}(|V| + |E|))$ I/Os

# Breadth-First Search(BFS):Improvment

- Main problem: In line $1$ of BFS procedure, we pay at least one I/O per vertex
- Idea: Cluster vertices, for each cluster read adjacent vertices to the cluster together

# Breadth-First Search(BFS):Improvment

- Main problem: In line $1$ of BFS procedure, we pay at least one I/O per vertex
- Idea: Cluster vertices, for each cluster read adjacent vertices to the cluster together

# Clustering

- Idea: diameter of each cluster does not exceed a specific number
- Choose $0 < \mu < 1$
- $V'$ is the set of cluster centers (masters). Starting vertex $s$ is inserted to $V'$.
- Select a vertex as a master with probability $\mu$ and put into $V'$: $E(|V'|) = 1 + \mu|V|$
- Put $V'$ into list $L(0)$ and compute levels $L(i)$ using the BFS procedure with following modifications
  - Instead of accessing the adjacency list of each vertex at $L(i)$, scan $E$ and $L(i)$ and retrieve adjacent vertices to $L(i)$: $O(\text{scan}(|E|))$ I/Os
  - Sort to remove duplicates: $O(\text{sort}(|E_i|))$ I/Os

Expected $1/\mu$ iterations $\Rightarrow O(\text{sort}(|E|) + \text{scan}(|E|)/\mu)$ I/Os

# Clustering

- The expected diameter of any cluster is $2/\mu$
  - There is a path from $s$ to vertex $v$: $P: s, x_k, x_{k-1}, \cdots, x_1, v$
  - Then each vertex belongs to a cluster
  - $j$ smallest index so $x_j$ is a master
  - $E(j) = 1/\mu$ since each vertex is master with probability $\mu$
  - Then expected diameter is $2/\mu$

# BFS: Improvement

- Maintain each cluster $C_i$ in a file $F_i$
  - $F_i$ maintain all adjacent vertices (not necessary in $C_i$) to vertices in $C_i$
  - With each edge maintain the starting location $F_i$

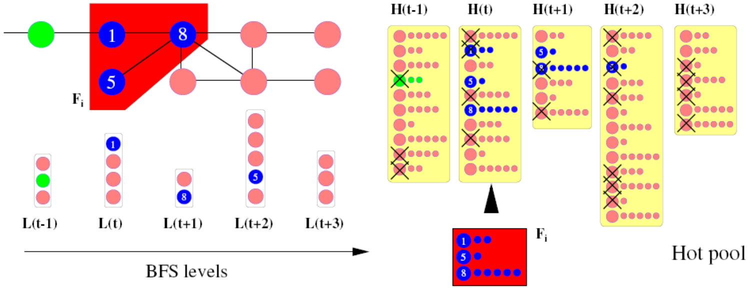
$\Rightarrow O(\mu|V| + \text{sort}(E))$ I/Os

- Hot Pool $H$: maintain edges in sorted order
  - If a cluster has a vertex adjacent to a vertex in $L(t)$ the whole cluster is maintained in $H$.
- List $L(t)$ is maintained sorted

# BFS: Improvement

- Scan $L(t)$ and $H$ to identify vertices in $L(t)$ whose ALs are not in $H$
  - If $v \in C_j$ is such a vertex, add $F_j$ into list $Q$
  - Sort $Q$ to remove duplicates
- The files in $Q$ is appended to $H'$
- Make $H'$ sorted and merge with $H$
- Scan $L(t)$ and $H$ to extract ALs and to $L(i+1)$
- Sort $L(t+1)$ to remove duplicate.
- Eliminate vertices appear in $L(t)$ and $L(t-1)$

# BFS: Improvement



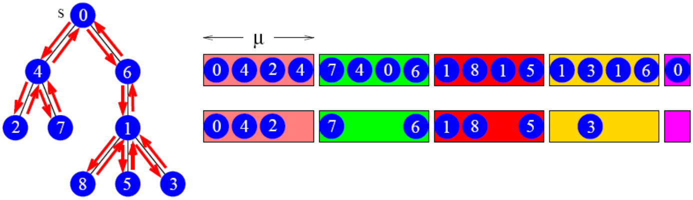
# BFS: Improvement

- Analysis
  - $H$ is scanned in each iteration
  - Each edge is maintained $O(1/\mu)$ iterations in $H$
  - Total cost of scanning $H$ is $O(\text{scan}(E)/\mu)$
  - $O(\mu|V| + \text{sort}(E))$ I/Os to retrieve files
  - the rest in $\text{sort}(E))$ I/Os as before

  $\Rightarrow O(\mu|V| + \text{sort}(E) + \text{scan}(E)/\mu)$ I/Os
- Set $\mu = \sqrt{|E|/B|V|} \Rightarrow O(\sqrt{|V||E|/B} + \text{sort}(|V| + |E|))$ I/Os
- For spars graph: $O(|V|/\sqrt{B} + \text{sort}(|V|)$ I/Os

# Deterministic Clustering

- Compute a spanning tree
- Make a Euler tour
- Chop Euler-tour into $2n/\mu$ pieces
- Eliminate duplicate
- BFS: $O(\sqrt{|V||E|/B} + \mathrm{sort}(|V| + |E|)\log_2\log_2|V|)$ I/Os



# Buffered Repository Tree (BRT)

- Store key-value pairs $(k, v)$
- Support the following operations
  - Insert$((k, v))$: insert given $(k, v)$ into BRT in $O(\frac{1}{B}\log_2(N/B))$ I/Os
  - Extract$(k)$: remove all key-value pairs with key $k$ from BRT and return them in $O(\log_2(N/B) + K/B)$ I/Os

# Buffered Repository Tree (BRT)

- BRT is a (2,4)-tree $T$
- For each node a buffer of size $B$ is maintained
- Its maintenance is like that of buffer trees with few changes
- Since buffer size is small in contrast with the size of buffers in buffer trees, the tree can support search quickly
- Since each node has at most 4 children, a full buffer can be emptied with 4 I/Os

# Directed DFS

```
Push s into Stack Q
While Q is not empty do
    v = Top(Q)
    if there is an unexplored edge (v,w) and w is unvisited then
        push(Q,w) and set w is visited
    else
        Pop(Q,w)
```

# Directed DFS

- A BRT $T$ storing edges of $G$. Each edge has its source vertex as its key. Tree $T$ is initially empty.
- A buffered priority queue $P(v)$ per vertex $v \in G$, which stores the out-edges of $v$ that have not been explored yet and whose other endpoints have not been visited before the last visit to $v$.
- invariant: the edges that are stored in $P(v)$ and are not stored in $T$ are the edges from $v$ to unvisited vertices.

# Directed DFS

- A BRT $T$ storing edges of $G$. Each edge has its source vertex as its key. Tree $T$ is initially empty.
- A buffered priority queue $P(v)$ per vertex $v \in G$, which stores the out-edges of $v$ that have not been explored yet and whose other endpoints have not been visited before the last visit to $v$.
- invariant: the edges that are stored in $P(v)$ and are not stored in $T$ are the edges from $v$ to unvisited vertices.

# Directed DFS

Push $s$ into Stack $Q$<br>
**While** $Q$ is not empty **do**<br>
    $v =$ Top($Q$),<br>
    Extract($v$) from $T$ and call Delete($P(v)$) for each extracted vertex<br>
    $w =$ Deletemin($P(v)$)<br>
    **if** w exists **then**<br>
        push($Q, w$) and insert in-edges of $w$ into T<br>
    else<br>
        Pop($Q, w$)

# Directed DFS

- $|E|$ insertion into $T$
- $|E|$ deletion from $P(v)$s
- Numbers of visits is $O(|V|)$, since DFS-algorithm performs an inorder traversal of DFS-tree
- $O(|V|)$ Extract from $T$
- $O(|V|)$ Deletemin from $P(v)$s
- We have to maintain a buffer of size $B$ for each $P(v) \rightarrow |V|B < M$
- Since it is not necessarily $|V|B < M$, we just maintain the buffer of active node in the memory
- Since the active nodes changes at most $O(|V|)$ time, we pay $O(|V|)$ extra I/Os

$\Rightarrow O((|V| + |E|/B)\log_2 |V|)$ I/Os

# Summary: BFS and DFS

- Undirected BFS
  - $O(|V| + \text{sort}(|V| + |E|))$ I/Os
  - $O(\sqrt{|V||E|/B} + \text{sort}(|V| + |E|))$ I/Os
  - For spars graph: $O(|V|/\sqrt{B} + \text{sort}(|V|)$ I/Os
- Directed BFS and DFS
  - $O((|V| + |E|/B)\log_2 |V|)$ I/Os

# References

- **I/O efficient graph algorithms**
  Lecture notes by Norbert Zeh.
  - Section 6