HOMEWORK 5, DUE MARCH 26, 2019

ANALYSIS II

(1) (a) The hodge star operator on a differential k-form in $\mathbf{R}^n$ is defined to be a map from $\Omega^k(R^n)$ to $\Omega^{n-k}(R^n)$ for $0 \leq k \leq n$ defined by the following: Let $y, \cdots, y_n$ be the coordinate on $\mathbf{R}^n$. Let dy_I be a basic k-form with $I = (i_1, \cdots, i_k)$. $\star dy_I = dy_J$ where $J = (j_1, \cdots, j_{n-k})$ such that $dy_I \wedge dy_J = dy_1 \wedge \cdots \wedge dy_n$. Suppose $\omega = \sum_I f_I dy_I$. Then $\star w = \sum_I f_I(\star dy_I)$. Show that if $\omega = \sum_I f_I dy_I$ and $\eta = \sum_I g_I dy_I$ where I runs thru ascending k tuples in $\{1, \cdots, n\}$ then $\omega \wedge \star\eta = (\sum_I f_I g_I) dy_1 \wedge \cdots \wedge dy_n$.
(b) In Maxwell's theory of electricity and magnetism, developed in the late nineteenth century, the electric field $E = \langle E_1, E_2, E_3 \rangle$ and the magnetic field $B = \langle B_1, B_2, B_3 \rangle$ in a vacuum $\mathbf{R}^3$ with no charge or current satisfy the following equations: Note that $E = E(x, y, z, t), B = B(x, y, z, t)$.
$\nabla \times E = -\partial_t B$, $\nabla \times B = \partial_t E,$, $div E = 0, div B = 0$.
Let $\mathbf{R}^4$ be space-time with coordinates (x, y, z, t). But this space-time is endowed with a Minkowski metric. The hodge star operator needed to be modified. Let $y_1 = x, y_2 = y, y_3 = z, y_4 = t$. We define $\star dy_I = dy_J$ with $dy_I \wedge dy_J = dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4$ if none of the subindex in I is 4.
We define $\star dy_I = dy_J$ if $dy_I \wedge dy_J = -dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4$ if one of the subindex in I is 4. The basic two forms are spanned by $dx \wedge dy, dx \wedge dz, dx \wedge dt, dy \wedge dz, dy \wedge dt, dz \wedge dt$. Thus we have $\star(dx \wedge dy) = dz \wedge dt, \star(dx \wedge dz) = dt \wedge dy, dx \wedge dt = -dy \wedge dz, dy \wedge dz = dx \wedge dt, dy \wedge dt = dx \wedge dz, dz \wedge dt = -dx \wedge dy$. Then both E and B can be viewed as differential 1-form and 2-form on $\mathbf{R}^4$ defined by $E = E_1 dx + E_2 dy + E_3 dz$ and $B = B_1 dy \wedge dz + B_2 \wedge dz \wedge dx + B_3 dx \wedge dy$. Define F to be the 2-form $F = E \wedge dt + B$. Show that Maxwell's equations are equivalent to the equation $dF = 0$ and $d \star F = 0$ ($\star$ is the star-operator defined in part(b).)

(2) (a) Let $\alpha = \sum_I f_I dy_I$ be a closed k-form whose coefficients f_I are smooth functions defined on $\mathbf{R}^n \setminus \{0\}$ that are all homogeneous of the same degree $p \neq -k$ (i.e. $f_I(tx) = t^p f_I(x)$.) Let $\beta = \frac{1}{p+k} \sum_I \sum_{l=1}^k (-1)^{l+1} y_{i_l} f_I dy_{i_1} \wedge \cdots \hat{dy_{i_l}} \cdots \wedge dy_{i_k}$. Show that $d\beta = \alpha$.
(b) Let $\alpha = (2xyz - x^2 y) dy \wedge dz + (xz^2 - y^2 z) dz \wedge dx + (2xyz - xy^2) dx \wedge dy$. Check that α is closed and find a 1-form β such that $d\beta = \alpha$.

(3) Do problem 63 and 67 (spherical shell is defined on problem 59 on p379) on p380 from the book

(4) Let U be an open set in $\mathbf{R}^n$. Let $[\alpha]$ be a class in $H^k(U)$ and let $[\beta]$ be a class in $H^l(U)$. Define the product of $[\alpha]$ and $[\beta]$ to be the class $[\alpha]\cdot[\beta] = [\alpha \wedge \beta] \in H^{k+l}(U)$ Show that $[\alpha]\cdot[\beta]$ is well-defined, i.e. independent of the choice of the representatives of the classes $[\alpha]$ and$[\beta]$. (One need to show that (1) if α and β are closed then $\alpha \wedge \beta$ is closed. and (2) if $\omega_1 \in \Omega^{k-1}(U)$ and $\omega_2 \in \Omega^{l-1}(U)$ then there exists $\omega_3 \in \Omega^{k+l-1}(U)$ such that $(\alpha + d\omega_1) \wedge (\beta + d\omega_2) = \alpha \wedge \beta + d\omega_3$.