Intro to G.T.

Ch1. Intro to Decision Theory

Def: Decision problem $:(A, \succsim)$, A is set of alternative, $\succsim$ is weak preference.

pro. 1.2.1: A be countable, exist utility function u representing $\succsim$.

p.f. $h_{ij} = \begin{cases} 1 & \text{if } a_i \succsim a_j \\ 0 & \text{otherwise} \end{cases}$, $u(a_i) = \sum_{j=1}^{\infty} \frac{1}{2^j} h_{ij}$.

Def: order dense set: $B \subset A$, $\forall a_1, a_2 \in A$, $a_2 \succ a_1$, $\exists b \in B$, $a_2 \succsim b \succsim a_1$

Def: (a_1, a_2) is gap: $b \succsim a_2$ or $a_1 \succsim b$. a_1, a_2 are gap extreme $(a_2 \succ a_1)$
A^*: set of gap extreme.

Lem 1.2.2: (i) If B is countable, so well A^*
(ii) If u exist, A^* is countable.

p.f. (i): A_1^*, A_2^* be the lower and upper extreme of A^*.
If (a_1, a_2) is a gap. $\exists b$ either $b = a_1$ or $b = a_2$.
$\Rightarrow A_{1,2}^* \setminus B$ is smaller then B, $\Rightarrow A^*$ is countable.
(ii) $u(a_2) > u(a_1) \Rightarrow \exists\, q \in Q$ s.t. $u(a_2) > q \geq u(a_1)$, thus countable.

Thm 1.2.3: $\succsim$ can be represented by $u \Leftrightarrow \exists B$, countable ordered dense set in A.

Pf. $(\Leftarrow)$ first (last) element: $\forall a'$ s.t. $a' \succsim a$ $(a \succsim a')$.
B is the countable ..., $\bar{B} = B \cup (\text{first, last})$ A^* is countable.
$\Rightarrow B^* = \bar{B} \cup A^*$ is countable. we have $\bar{u}$ rep $\succsim$ in B^*:
Def $u(a) := \sup\{\bar{u}(b) \mid b \in B^*, a \succsim b\}$.
1° If $a \notin B^*$, $a_2 \succ a \succ a_1$, $a_2 \succ b_2 \succ a \succ b_1 \succsim a_1$.
S $\{\bar{u}(b) \mid b \in B^*, a \succsim b\}$ is non-empty, upper-bounded.
2° If $a_2 \succ a_1$, $a_1 \notin B^*$ $\Rightarrow a_2 \succsim b_2 \succ a_1$, since $a_1 \notin A^*$.
$\Rightarrow \exists \bar{a}$ s.t. $b_2 \succ \bar{a} \succ a \Rightarrow \bar{a} \succsim b_1 \succ a \Rightarrow$
$u(a_2) \geq u(b_2) > u(b_1) > u(a_1) \Rightarrow u(a_2) > u(a_1)$
$(\Rightarrow)$ $\bar{Q}^2 = \{(q_1, q_2)$ s.t. $q_2 > u(a) > q_1\}$. $g: \bar{Q}^2 \to A$
(map (q_1, q_2) to such a), We have $\bar{B} = g(\bar{Q}^2)$,
$B = A^* \cup \bar{B}$.
Take (a_1, a_2) not gap. $\exists q_1, q_2$ and $\bar{a}$ s.t. $a_2 \succ \bar{a} \succ a_1$
$u(a_2) > q_2 > u(\bar{a}) > q_1 > u(a_1)$. $\Rightarrow (q_1, q_2) \in \bar{Q}^2$.
$g(q_1, q_2) = b$ and $a_2 \succ b \succsim a_1 \Rightarrow B$ is order dense.

Def: linear utility function $\bar{u}$: $\bar{u}(tx + (1-t)y) = t\bar{u}(x) + (1-t)\bar{u}(y)$.

Def: $(\succsim)$ Independent: $x \succ y$, $tx + (1-t)z \succ ty + (1-t)z$.

Def: $(\succsim)$ continuous: $x \succ y \succ z$, $y \sim tx + (1-t)z$ for some t.

Prop 1.3.1: $\succsim$ indep. $s > t \in [0,1]$, $y \succ x \Rightarrow sy + (1-s)x \succ ty + (1-t)x$

p.f. $sy + (1-s)x = ty + (1-t)\left(\frac{s-t}{1-t}y + \frac{1-s}{1-t}x\right) \underset{\text{(indep.)}}{\succ} ty + (1-t)x$.

Thm 1.3.3. Assume $\succsim$ is indep. and conti. $\exists\, \bar{u}$ represent $\succsim$, and $\bar{u}$ is unique up to positive affine transform.

p.f. pick any $x_1 \prec x_2$, easy to prove $u(x_1) = 0$, $u(x_2) = 1$ is well-define, linear utility. Then extend to X

Def: $\Delta A := \{x \in [0,1]^A \mid \{a \in A \mid x(a) > 0\}, \sum_{a \in A} x(a) = 1\}$
$\Rightarrow$ this can be decision problem: $x \succsim y \Leftrightarrow \sum u(a) x(a) \geq \sum u(a) y(a)$

Prop 1.3.4: $\exists\, u$ represent $\succsim$ $\Leftrightarrow$ exist linear utility u represent $\succsim$!

Introduction to GT

Chapter 2 Strategy Game.

Def: Strategy Game: $G=(A,u)$ (N players):

① A := set of strategy profiles
② u := payoff: $u(a)\to R^N$
③ $f: A\to R$, strategy to outcome.
④ $\{\succsim\}_{i\in N}$ preference, complete, transitive, rep.able
⑤ $\{U_i\}$ utility of player. $\Rightarrow u(a)=U(f(a))$

Example (Prisoner's Dilemma)

Example (Cournot) Each producer produce a_i, $TC:=C_i(a_i)$
Price is $\pi(\sum a_i)$. $G=(A,u)$. $A_i=[0,\infty)$.
$u_i(a)=\pi(\sum a_j)\cdot a_i - C_i(a_i)$

Example: (1st price), $u_i(a)=\begin{cases} v_i-a_i & i=\min\{j\in N,\ a_j=\max_{k\in N} a_k\} \\ 0 \end{cases}$ to break tie.
$(v_1>v_2\ldots>v_n>0)$

Example: (2nd price) $u_i(a)=0\wedge v_i-\max\{a_j, j\neq i\}$.

Def: NE: $\forall i\in N$, $\bar a_i\in A_i$, $u_i(a^*)\geq u_i(a^*_{-i},\bar a_i)$

EXM: $(0,v_1+1,0,0\ldots0)$ is a NE for 2nd price.

Def: $f: X\to 2^Y$ is correspondence, $\{x_k\}\to\bar x$
f is upper hemi conti: $\forall Y^*\supset f(\bar x)$, $f(x_k)\subset Y^*$
" lower " $:\forall Y^*\cap f(\bar x)\neq\emptyset$, $f(x_k)\cap Y^*\neq\emptyset$
$\Rightarrow$ if both, f is conti.

Thm: Kakutani fix-point theorem:
$X\subseteq R^n$ is non-empty, compact convex, $F: X\rightrightarrows X$ UHC, nonempty value, closed-, convex-value correspondence, F has fixed-point.
(proof omitted here)

Def: Best Reply Corres: $Br_i(a_{-i})=\arg\max_{a_i\in A_i} u_i(a_i,a_{-i})$, $BR=\prod BR_i$

Def: Quasi-concave: $\forall r$, $\{f(x)\geq r\}$ is convex or $f(\alpha a+(1-\alpha)\bar a)\geq \min\{f(a), f(\bar a)\}$ (Equivalent)

Prop 2.2.2: ① A_i is non-empty, compact set. ② u_i conti.
③ $u_i(a_i,\cdot)$ is quasi-concave.
$\Rightarrow BR_i$ is ^{a}upper-hemi conti, ^{b}non-empty-, ^{c}closed-, ^{d}convex-valued

P.f. (b), (c) is trivial because A_i compact, u_i conti.
(d) Suppose $a_i, \bar a_i\in BR(a_i)$. $u(ta_i+(1-t)\bar a_i)\geq\min(u(a_i), u(\bar a_i))$
$\Rightarrow$ it belongs to $BR(a_i)$, too.
(a) suppose not, $\exists\{a_k\}\to\bar a$, $\forall B\supset BR_i(\bar a)$, $\exists K>k_0\ \forall k_0$ s.t. $BR_i(a_K)\not\subset B$. $\Rightarrow\exists\{a'\}\in A_i$, $a'_n\in BR_i(a_{K(n)})/B$.
By B-W theorem, assume $\{a'\}\to\bar a'$. $\because B^*$ open, $\bar a'\in A_i/B'$
$\therefore \bar a'\notin BR_i(\bar a)$.
$\because u_i(a_{K(n)}, a'_n)\geq u_i(a_{K(n)}, a)$, $\therefore n\to\infty$, $u_i(\bar a,\bar a')\geq u_i(\bar a, a)\Rightarrow \bar a'\in BR_i(\bar a)$; contradiction!!

Thm 2.2.3. (Nash Theorem) If Prop 2.2.2. holds, NE exists.

Def. 2-player-0-sum game: $G=\{(A_1,A_2),(u_1,u_2)\}$, $\forall(a_1,a_2)$,
$u_1(a_1,a_2)+u_2(a_1,a_2)=0$

Def. $G=(A_1,A_2,u_1)$-2P0S game.
$\underline{\Lambda}(a_1)=\inf_{a_2\in A_2} u_1(a_1,a_2)$. $\underline{\lambda}:=\sup_{a_1\in A_1}\underline{\Lambda}(a_1)$ = lower value
$\bar\Lambda(a_2):=\sup_{a_1\in A_1} u_1(a_1,a_2)$, $\bar\lambda:=\inf_{a_2\in A_2}\bar\Lambda(a_2)$ = upper value.

Def: 2P0S game is strictly determind or have value if $V:=\underline{\lambda}=\bar\lambda$.

Def. 2P0S game is with value V, $a_1\in A$ is optimal for P_1 if $V=\underline{\Lambda}(a_1)$ (Similiar for $a_2\in A_2$)

Prop 2.3.1. If G a 2P0S, (a_1^*,a_2^*) be NE.
$\Rightarrow$ ① G is strictly determind ② a_1^*, a_2^* is optimal for P_1,P_2 ③ $V=u_1(a_1^*,a_2^*)$

Pf. $\underline{\lambda}=\sup\underline{\Lambda}(a_1)\geq\underline{\Lambda}(a_1^*)=\inf_{a_2} u_1(a_1^*,a_2)\geq u_1(a_1^*,a_2^*)\geq\sup_{a_1} u_1(a_1,a_2^*)$
$=\bar\Lambda(a_2^*)\geq\inf_{a_2\in A_2}\bar\Lambda(a_2)=\bar\lambda$. $\Rightarrow\bar\lambda\geq\underline{\lambda}$, all equation holds.

Prop 2.3.2: G is 2P0S, strictly determindal. a_1,a_2 optimal. (a_1,a_2) is NE.
P.f. $V=\begin{matrix}\bar\Lambda(a_1)\leq u_1(a_1,\tilde a_2)\\ \underline{V}(a_2)\geq u_1(\tilde a_1,a_2)\end{matrix}$. Take $(\tilde a_1,\tilde a_2)(a_1,a_2)$, We have $V=u_1(a_1,a_2)$

r.k. (a_1,a_2), (a_1',a_2') NE of 2P0S, so will (a_1,a_2'), (a_1',a_2) be.

r.k. If $u_1+u_2(a_1,a_2)=k$, call constant sum game. All properties 2P0S can a

Def: G is finite game if $|A_i|<\infty$, $\forall i\in N$.

Def: G is finite game. $E(G):=(S,u)$ is its mixed extension.
$S_i:=\Delta A_i$, $S:=\prod S_i$, $s\in S$, $a\in A$. $s(a):=s_1(a_1)\cdot\ldots s_n(a_n)$
$u_i(s):=\sum_{a\in A} u_i(a)s(a)$

Thm 2.4.1 G is a finite game $\Rightarrow E(G)$ has at least one NE. $\Rightarrow$ simple.

Def. ① support of s_i: $\varphi(s):=\{a_i\in A_i, s_i(a_i)>0\}$. ② If $\varphi(s_i)=A_i$, it is completely mixed ③ Pure Best Replies $PBR(s_i)$:= best pure strategy

Prop 2.4.2. G is finite game:
① $s_i\in BR(s_i)\Leftrightarrow\varphi(s_i)\subset PBR_i(s_{-i})$ (NE of $E(G)$)
② s is NE $\Leftrightarrow\varphi(s)\subset PBR(s)$ ③ s is NE $\Leftrightarrow u_i(s)\geq u_i(s_{-i},\bar a_i)$

Def: Bimatrix game is $G=(\{L,M\},u)$'s mix extension: $(\{S_\ell,S_m\},u)$
① $S_\ell:=\Delta L$, $S_m:=\Delta M$. x for P_1, y for P_2 (probability)
② $u_1(x,y)=xAy^t$, $u_2(x,y)=xBy^t$, A,B is the matrix.
③ $B_1:=\{(x,y)\in S_\ell\times S_m \mid x\in BR_1(y)\}$, similiar for B_2. NE $=B_1\cap B_2$

Def: Matrix game is 2P0S Game, represented like Bimatrix Game.

Prop 2.6.1: A is the matrix of P_1, $(x,y)\in(S_\ell,S_m)$.
$\Rightarrow\underline{\Lambda}(x)=\min_{j\in M} x\cdot a_j$, $\bar\Lambda(y)=\max_{i\in L} a_i\cdot y^t$

P.f. $\underline{\Lambda}(x)=\inf_{y\in S_m} xAy^t\leq\min_{j\in M} xAe_j^t=\min_{j\in M} xa_j$.
$xAy^t\geq\sum_{k\in M}(\min_{j\in M} xa_j)y_k=(\min_{j\in M} xa_j)\sum y_k \overset{=1}{}$. equation holds

Thm 2.6.2: (min-max) Every matrix game is strictly determined

P.f. ① Since $\underline{\lambda}\leq\bar\lambda$, we only need $\underline{\lambda}\geq\bar\lambda$.
② Suppose $\underline{\lambda}=c<\bar\lambda=\inf_{y\in Y}\bar\Lambda(y)=\inf_{y\in Y}(\max a_i\cdot y^t)$
Then, by the fact A is linear, $\exists x_i$ s.t. $c<\inf\sum x_i a_i y^t$
$\leq\inf_{y\in Y}\sum_{i=1}^n a_i(x_i y^t)\leq\sup_{x\in X}\inf_{y\in Y} xAy^t=\underline{\lambda}\Rightarrow$ contradiction!!

Deute: Matrix game A, optimal strategies set: $O_1(A)$, $O_2(A)$

For 2x2 matrix, $V=\max_{x\in[0,1]}\underline{\Lambda}(x)=\max_{x\in[0,1]}\min_{j\in M}(x,1-x)a_j$
$=\max_{x\in[0,1]}\min_{j\in M}(a_{2j}+(a_{1j}-a_{2j})x)$.