EBS Business School

Mathematical Economics
Tutorial assignments

Most problems are taken directly from *Essential Mathematics for Economic Analysis* (EMEA), Fifth Edition, by Knut Sydsæter, Peter Hammond, Arne Strom and Andres Carvajal

When attending the tutorials, please come prepared. In addition, we strongly recommend to work through as many exercises as possible both in the textbook and the MyMathLab online platform. Practice makes perfect ...

Tutorial 1

1. Review Exercises Chapter 6, Problem 7, p. 218
   Differentiate the following functions:

   (a) $f(x) = x\left(x^2 + 1\right)$

   (b) $g(w) = w^{-5}$

   (c) $h(y) = y(y - 1)(y + 1)$

   (d) $G(t) = \frac{2t+1}{t^2+3}$

   (e) $\varphi(\xi) = \frac{2\xi}{\xi^2+2}$

   (f) $F(s) = \frac{s}{s^2+s-2}$

2. Review Exercises Chapter 6, Problem 10, p. 219
   Compute the following:

   (a) $\frac{\mathrm{d}Z}{\mathrm{d}t}$ when $Z = \left(u^2 - 1\right)^3$ and $u = t^3$

   (b) $\frac{\mathrm{d}K}{\mathrm{d}t}$ when $K = \sqrt{L}$ and $L = 1 + \frac{1}{t}$

3. Review Exercises Chapter 6, Problem 15, p. 219
   Find the intervals where the following functions are increasing:

   (a) $y = (\ln x)^2 - 4$

   (b) $y = \ln\left(e^x + e^{-x}\right)$

   (c) $y = x - \frac{3}{2}\ln\left(x^2 + 2\right)$

4. The cost of producing x units of a commodity is given by $C(x) = x^2 + x + 100$.

   (a) Compute $C(0)$, $C(100)$, and $C(101) - C(100)$.

   (b) Compute the incremental cost $C(x + 1) - C(x)$ and explain in words its meaning.

   (c) Compute the marginal cost $C'(x)$ and $C'(100)$. What is the difference between $C'(x)$ and $C(x + 1) - C(x)$?

5. Let $g(x) = 3x^3 - \frac{1}{5}x^5$.

   (a) Find $g'(x)$ and $g''(x)$.

   (b) Check where g is increasing and where it is concave.

   (c) Prove that $g(-x) = -g(x)$. What does this mean geometrically?

   (d) Sketch the graph of g.

Additional problems

1. Section 4.10, Problem 6, p. 136 (to brush up on the rules for logarithms)
2. Review Exercises Chapter 4, Problem 22, p. 138 (exponential functions)

Tutorial 2

1. Section 6.11, Problem 9, p. 218
2. Section 7.1, Problem 8, p. 227
3. Consider the following macroeconomic model

$$(i)\, Y = C + I \quad (ii)\, C = f(Y - T) \quad (iii)\, T = \alpha + \beta Y$$

   where Y is national income, C is consumption, I is investment, T denotes taxes, and α and β are constants. Assume that $f' \in (0, 1)$ and $\beta \in [0, 1]$.

   (a) From equations $(i) - (iii)$ derive the equation $Y = f((1 - \beta)Y - \alpha) + I$.

   (b) Differentiate the equation obtained in (b) implicitly w.r.t. I and find an expression for $\frac{dY}{dI}$.

   (c) Examine the sign of $\frac{dY}{dI}$.

4. Section 7.4, Problem 5, p. 238
5. Section 7.7, Problem 4., p. 250
6. Find the following limits:

   (a) $\lim\limits_{x \to \infty} \frac{x-3}{x^2+1}$

   (b) $\lim\limits_{x \to 0} \frac{e^x - 1 - x - \frac{1}{2}x^2}{3x^3}$

   (c) $\lim\limits_{x \to 2} \frac{2x+3}{x+2}$

   (d) $\lim\limits_{x \to 0} \frac{e^{-3x} - e^{-2x} + x}{x^2}$

Tutorial 3

1. The function f is defined for all x by $f(x) = 2 - \frac{4x}{x^2+3}$

   (a) Find $f'(x)$ and $f''(x)$.

   (b) Find the critical (stationary) points and classify them. Find possible inflection points.

   (c) Determine $\lim_{x\to\pm\infty} f(x)$.

   (d) Sketch the graph of f.

2. Section 8.2, Problem 10, p. 290

3. Consider a student with fixed current wealth w, who expects to earn fixed income y next year but nothing this year. Let $r \geq 0$ be the fixed annual rate of interest at which the student can either borrow or lend, and let δ denote $\frac{1}{1+r}$. Let x denote planned expenditure next year.

   (a) Give an economic interpretation of the expression $w + \delta y$.

   (b) Explain why the student's plan requires that expenditure this year must equal $w + \delta(y - x)$.

   (c) For the utility function $u(x) = \sqrt{w + \delta(y - x)} + \sqrt{x}$, find the derivatives $u'(x)$ and $u''(x)$.

   (d) Find the value x^* of x at which the function $u(x)$ attains a global maximum.

4. Section 8.7, Problem 2, p. 315

Tutorial 4

1. Section 11.2, Problem 5, p. 417

2. Find the domains of the following functions:

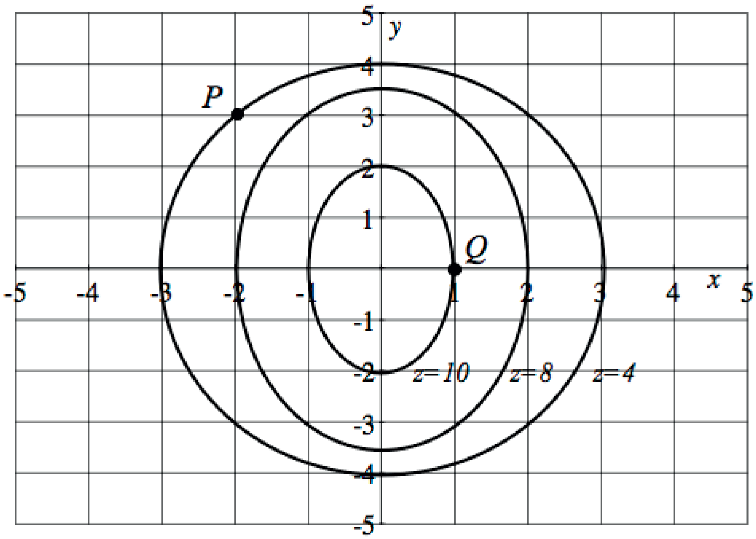
   (a) $f(x, y) = \sqrt{y - x^2} - \sqrt{\sqrt{x} - y}$. Sketch the domain in the xy-plane.

   (b) $f(x, y) = e^{x-2y} - \ln(2x^2 y)$.

3. Section 11.3, Problem 8, p. 424

4. Section 11.7, Problem 1, p. 436

5. Consider a continuous function $z = f(x, y)$. The following figure depicts three level curves of this function for $z = 10$, $z = 8$, and $z = 4$, respectively. In addition, two points are shown, $P(-2, 3)$ and $Q(1, 0)$.



   (a) What are the signs of the first partial derivatives of z w.r.t. x and y at the point Q?

   (b) What are the signs of the first partial derivatives of z w.r.t. x and y at the point P?

   (c) Give an approximation of the value of $f'_x(Q)$.

   (d) Can you say anything about $f''_{xx}(Q)$? Explain, why or why not.

Additional problems

1. Partial derivatives: Review Exercises 8, p. 440
2. Domains: Section 11.1, Problem 6, p. 411
3. Domains: Review Exercises 7, p. 440

Tutorial 5

1. Applying the Chain rule.

   (a) If $z = F(x, y) = x^2 + e^y$, $x = t^3$, $y = 2t$, find $\frac{dz}{dt}$.

   (b) If $Y = F(K, L) = KL^2$, $K = f(t)$, $L = g(t, s)$, find an expression for $\frac{dY}{dt}$ and $\frac{dY}{ds}$.

   (c) If $g(r) = F\left(r, 1 - r, \frac{1}{1-r}\right)$, find an expression for $g'(r)$.

2. Find the elasticities ε

   (a) of z w.r.t. x (ε^z_x) and y (ε^z_y) of $z = 10(x + 2)^2(y + 3)^3$.

   (b) of y w.r.t. x (ε^y_x) when $x^5y^3 = x + 2y$.

3. Suppose that the equation

$$\ln x + 2(\ln x)^2 = \frac{1}{2}\ln K + \frac{1}{3}\ln L$$

   defines x as a differentiable function of K and L.

   (a) Find expressions for $\frac{\partial x}{\partial K}$, $\frac{\partial x}{\partial L}$ and $\frac{\partial^2 x}{\partial K \partial L}$.

   (b) Show that $\varepsilon^x_K + \varepsilon^x_L = \frac{5}{6}\left(\frac{1}{1+4\ln x}\right)$.

4. Section 12.7, Problem 1, p. 473

Additional problems

1. Chain rule: Section 12.2, Problem 1, p. 451
2. Implicit differentiation: Section 12.3., Problems 1 and 5, p. 456
3. Degree of homogeneity: Section 12.6., Problems 3 and 7, p. 427 and Section 12.7, Problem 6, p. 468

Tutorial 6

1. Section 13.1, Problem 1, p. 499
2. Section 13.2, Problem 8, p. 504
3. Section 13.3, Problem 2, p. 508
4. Section 13.5, Problem 6, p. 521
5. The function g is defined by $g(x, y) = 3 + x^3 - x^2 - y^2$. Its domain D is given by $x^2 + y^2 \leq 1$ and $x \geq 0$.
   (a) Sketch the domain D in the xy-plane.
   (b) Find the critical points of the function g, and classify them.
   (c) Find the (global) extreme points of g in D by way of the extreme value theorem.

Additional problems

1. Critical points: Review Exercises Section 13, Problem 4, p. 529
2. Economic application: Review Exercises Section 13, Problem 3, p. 529
3. Extreme value theorem: Section 13.5, Problem 5, p. 521
4. A firm has the production function $Q = F(K, L) = K^{\frac{1}{2}}L^{\frac{1}{4}}$ for $K > 0$ and $L > 0$, and chooses its inputs K of capital and L of labor in order to maximize its profit $\pi = PQ - rK - wL$.
   (a) Show that F is a homogeneous function, and verify Euler's Theorem.
   (b) Write down the profits as a function of K and L and find the first-order conditions for the maximization problem.
   (c) Show that π is concave.
   (d) Solve the first-order conditions for the firm's input demands as functions $K^*(P, r, w)$ and $L^*(P, r, w)$.

Tutorial 7

1. Review Exercises Chapter 14, Problem 1, p. 578
2. Section 14.7, Problem 1, p. 562
3. Section 14.4, Problem 3, p. 549
4. Section 14.5, Problem 4, p. 552

Additional problems

1. A simple model of labour supply: A worker can divide her time between leisure and work. The day has 24 hours, so let us denote leisure time as x hours and work as $(24 - x)$ hours. The wage the worker receives per hour is given by w and constitutes the only source of income. Apart from leisure the worker also consumes y at the price p. This implies that the time constraint can be formulated as: $py = w(24 - x)$. The utility is represented by the function $U(x, y) = x^{\frac{1}{3}} y^{\frac{2}{3}}$.

   (a) Write down the Lagrangian of the maximization problem.

   (b) Solve the problem.

2. Work through Example 14.1.3 on page 536.

Tutorial 8

1. Section 14.8, Problem 1, p. 568
2. Section 14.8, Problem 4, p. 569
3. Section 14.10, Problem 1, p. 578

Additional problems:

1. Section 14.6, Problem 1, p. 557
2. Consider the following problem:

$$\max \ \frac{1}{2}x - y \qquad \text{subject to } x + \mathrm{e}^{-x} \leq y, \quad y \geq 0$$

   Sketch the admissible set in the xy-plane. Then solve the problem.

Tutorial 9

1. Section 9.1, Problem 10, page 325
2. Section 9.3, Problem 9, p. 336
3. Section 9.4, Problem 4, p. 343
4. Find the integrals:

   (a) $\int_0^1 \frac{4x^3}{\sqrt{4-x^2}}\mathrm{d}x$.

   (b) $\int \frac{\mathrm{e}^{4x}}{\mathrm{e}^{2x}+1}\mathrm{d}x$.

   (c) $\int_0^1 \frac{\ln x}{\sqrt{x}}\mathrm{d}x$.

   (d) $\int_1^\infty \frac{1}{\sqrt{x}}\mathrm{d}x$.

Additional problem

Evaluate $\int_0^2 2x^2(2-x)^2\mathrm{d}x$. Give your answer a rough check by sketching the graph of $f(x) = 2x^2(2-x)^2$ over $[0, 2]$.

Tutorial 10

1. Section 15.5, Problem 1, p. 601
2. Review Exercises Chapter 15, Problem 2, p. 620. Only (a),(b),(c),(e),(h).
3. Section 16.1, Problem 1, p. 626
4. Consider the following matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

   (a) Find the Cofactor matrix $\mathbf{C}$.
   (b) Compute the determinant of $\mathbf{A}$.
   (c) Find the transpose $\mathbf{C}' = \text{adj}(\mathbf{A})$ of the cofactor matrix.
   (d) Compute the inverse $\mathbf{X} = \mathbf{A}^{-1}$ and verify $\mathbf{X} \cdot \mathbf{A} = \mathbf{I}$

Additional problems

1. Multiplication of matrices: Section 15.3, Problem 1, p. 592
2. Determinants: Section 16.1, Problem 5, p. 626
3. Inverse: Section 16.6, Problem 1, p. 648

Solutions

Tutorial 1

1. See EMEA

2. “

3. “

4. $C(x) = x^2 + x + 100$.

   (a) $C(0) = 0^2 + 0 + 100 = 100$
   $C(100) = 100^2 + 100 + 100 = 10,200$
   $C(101) - C(100) = 101^2 + 101 + 100 - 10,200 = 201$.

   (b)
   $$\begin{aligned} C(x+1) - C(x) &= (x+1)^2 + (x+1) + 100 - \left(x^2 + x + 100\right) \\ &= x^2 + 2x + 1 + 1 - x^2 \\ &= 2x + 2 \end{aligned}$$
   This is the incremental cost of increasing production from x to $x + 1$.

   (c) $C'(x) = 2x + 1$
   $C'(100) = 2 \cdot 100 + 1 = 201$
   $C'(x) - (C(x+1) - C(x)) = 2x + 1 - (2x + 2) = -1$.
   $C'(x)$ is the marginal costs. It gives the slope of the cost curve at point x and thus approximates the increase of costs from x to $x + 1$. The real increase can be calculated by $C(x+1) - C(x)$.

5. $g(x) = 3x^3 - \frac{1}{5}x^5$

   (a) $g'(x) = 9x^2 - x^4 = x^2\left(9 - x^2\right) = x^2(3 - x)(3 + x)$
   $g''(x) = 18x - 4x^3 = 4x\left(\frac{9}{2} - x^2\right) = 4x\left(\frac{3}{2}\sqrt{2} - x\right)\left(\frac{3}{2}\sqrt{2} + x\right)$

   (b) g is increasing when $g'(x) \geq 0$
   $g'(x) = 0$ for $x_1 = -3$, $x_2 = 0$, $x_3 = +3$.

x	$(\infty, -3)$	$(-3, 0)$	$(0, 3)$	$(3, \infty)$
$g'(x)$	-	+	+	-

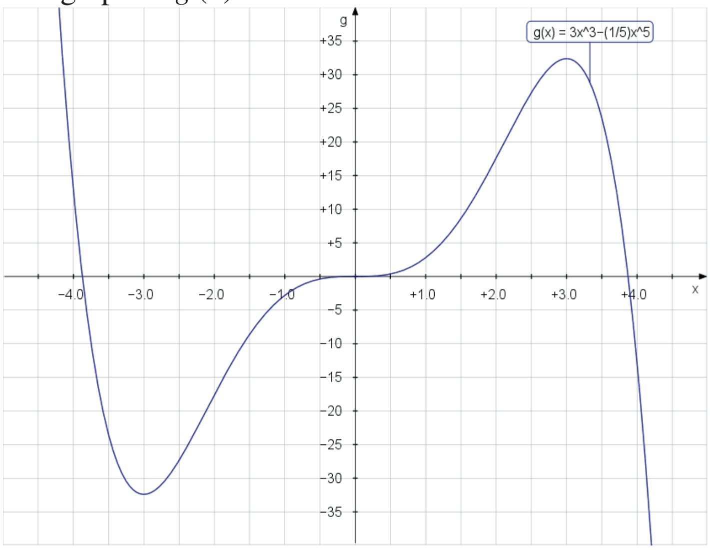
   g is concave when $g''(x) \leq 0$
   $g''(x) = 0$ for $x_1 = -\frac{3}{2}\sqrt{2}$, $x_2 = 0$, $x_3 = +\frac{3}{2}\sqrt{2}$.

x	$(\infty, -\frac{3}{2}\sqrt{2})$	$(-\frac{3}{2}\sqrt{2}, 0)$	$(0, \frac{3}{2}\sqrt{2})$	$(\frac{3}{2}\sqrt{2}, \infty)$
$g''(x)$	+	-	+	-

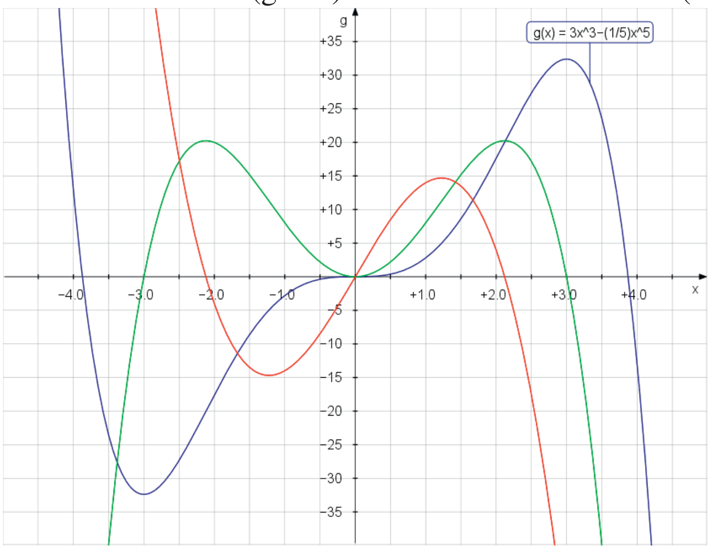
   $x = \pm\frac{3}{2}\sqrt{2}$ are inflection points as is $x = 0$.

   (c) $g(-x) = 3(-x)^3 - \frac{1}{5}(-x)^5 = -\left[3x^3 - \frac{1}{5}x^5\right] = -g(x)$.
   Thus the graph is symmetric about the origin, i.e., if (a, b) belongs to the graph of g, then so does $(-a, -b)$.

(d) The graph of $g(x)$ looks as follows



The first derivative (green) and the second derivative (red):



Tutorial 2

1. See EMEA
2. "
3. (a) Substitute (iii) into (ii) and (ii) into (i)

   (b) $\frac{\mathrm{d}Y}{\mathrm{d}I} = (1-\beta)\, f'((1-\beta)Y - \alpha)\frac{\mathrm{d}Y}{\mathrm{d}I} + 1$. Rearranging yields

   $$\frac{\mathrm{d}Y}{\mathrm{d}I} = \frac{1}{1-(1-\beta)\, f'((1-\beta)Y-\alpha)}$$

   .

   (c) As $f' \in (0,1)$ and $\beta \in [0,1]$ it follows that $\frac{\mathrm{d}Y}{\mathrm{d}I} \geq 1$.
4. See EMEA
5. "
6. It is necessary to use L'Hôpital's rule in (*b*) and (*d*). The limits are

   (a) 0

   (b) $\lim_{x\to 0}\frac{e^x-1-x-\frac{1}{2}x^2}{3x^3} = \frac{0}{0} = \lim_{x\to 0}\frac{e^x-1-x}{9x^2} = \frac{0}{0} = \lim_{x\to 0}\frac{e^x-1}{18x} = \frac{0}{0} = \lim_{x\to 0}\frac{e^x}{18} = \frac{1}{18}$

   (c) $\lim_{x\to 2}\frac{2x+3}{x+2} = \frac{7}{4}$

   (d) $\lim_{x\to 0}\frac{e^{-3x}-e^{-2x}+x}{x^2} = \frac{0}{0} = \lim_{x\to 0}\frac{-3e^{-3x}+2e^{-2x}+1}{2x} = \frac{0}{0} = \lim_{x\to 0}\frac{9e^{-3x}-4e^{-2x}}{2} = \frac{5}{2}$

Tutorial 3

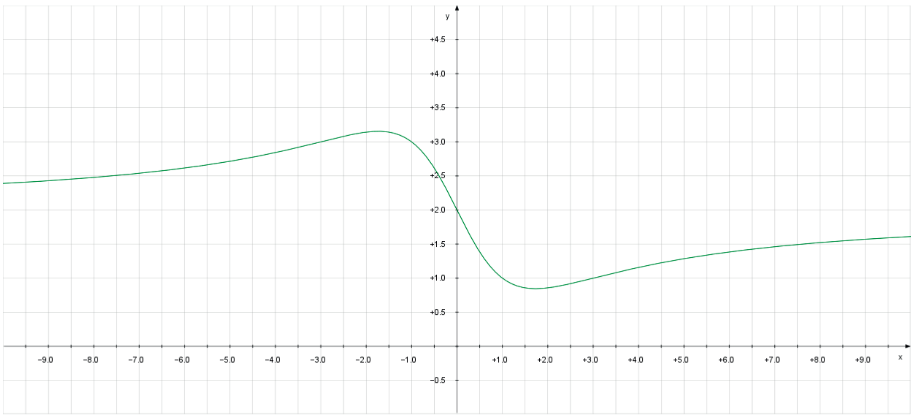
1. (a) $f'(x) = \frac{4(x-\sqrt{3})(x+\sqrt{3})}{(x^2+3)^2}$

   $f''(x) = \frac{8x(3-x)(3+x)}{(x^2+3)^3}$

   (b) $x = -\sqrt{3}$ is a maximum. $x = +\sqrt{3}$ is a minimum.
   $x = -3$, $x = 0$ and $x = +3$ are all inflection points as $f''(x)$ changes signs around these points.

   (c) $\lim_{x\to\pm\infty} f(x) = 2$.

   (d)



2. See EMEA

3. (a) Current wealth + present discounted value of future income.

   (b) Denote expenditure this year as e. Then $x = y + (1+r)(w-e)$ because $w-e$ is savings which earns interest at rate r. Rearranging gives $e = w + \delta(y-x)$.

   (c) $u'(x) = -\frac{1}{2}\delta\left[w+\delta(y-x)\right]^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}$

   $u''(x) = -\frac{1}{4}\delta^2\left[w+\delta(y-x)\right]^{-\frac{3}{2}} - \frac{1}{4}x^{-\frac{3}{2}}$

   (d) $u'(x) = 0 \Leftrightarrow x^* = \delta^{-2}\left[w+\delta(y-x)\right] \Leftrightarrow x^* = \frac{w+\delta y}{\delta(1+\delta)}$.
   This is a maximum if

$$u''(x) \le 0 \Leftrightarrow \underbrace{\underbrace{-\frac{1}{4}\delta^2}_{-}\left[\underbrace{w+\delta(y-x)}_{\textit{should be }+}\right]^{-\frac{3}{2}}}_{-}\underbrace{-\frac{1}{4}\underbrace{x^{-\frac{3}{2}}}_{+}}_{-} \le 0.$$

   $w + \delta(y-x) > 0 \Leftrightarrow x < y + (1+r)w$ which is the largest value of x as $e > 0$ (compare with (b))

4. See EMEA

Tutorial 4

1. See EMEA.
2. (a) $D = \left\{(x, y) : \ 0 \leq x \leq 1; \ \sqrt{x} \geq y \geq x^2\right\}$
   (b) $D = \{(x, y) : \ y > 0, \ x \neq 0\}$
3. See EMEA.
4. “
5. (a) $f'_x(Q) < 0$, $f'_y(Q) = 0$
   (b) $f'_x(P) > 0$, $f'_y(P) < 0$
   (c) $f'_x(Q) \approx -2$
   (d) $f''_{xx}(Q) \leq 0$

Tutorial 5

1. (a) $\frac{dz}{dt} = 2x \cdot 3t^2 + 2e^y = 6t^5 + 2e^{2t}$

   (b) $\frac{dY}{dt} = L^2 f'(t) + 2KLg'_t(t, s)$
   $\frac{dY}{ds} = 2KLg'_s(t, s)$.

   (c) $g'(r) = F'_1(\cdot) - F'_2(\cdot) + \frac{F'_3(\cdot)}{(1-r)^2}$.

2. (a) $\varepsilon^z_x = \frac{\partial z}{\partial x} \cdot \frac{x}{z} = \frac{2x}{x+2}$,
   $\varepsilon^z_y = \frac{\partial z}{\partial y} \cdot \frac{y}{z} = \frac{3y}{y+3}$.

   (b) $\varepsilon^y_x = \frac{\partial y}{\partial x} \cdot \frac{x}{y} = \frac{x-5x^5y^3}{3x^5y^3-2y}$, then inserting $x^5y^3 = x + 2y$
   $\varepsilon^y_x = \frac{-4x-10y}{3x+4y}$.

3. (a) $\frac{\partial x}{\partial K} = \frac{x}{2K(1+4\ln x)}$
   $\frac{\partial x}{\partial L} = \frac{x}{3L(1+4\ln x)}$
   $\frac{\partial^2 x}{\partial K \partial L} = \frac{\partial^2 x}{\partial L \partial K} = \frac{x(4\ln x-3)}{6KL(1+4\ln x)^3}$.

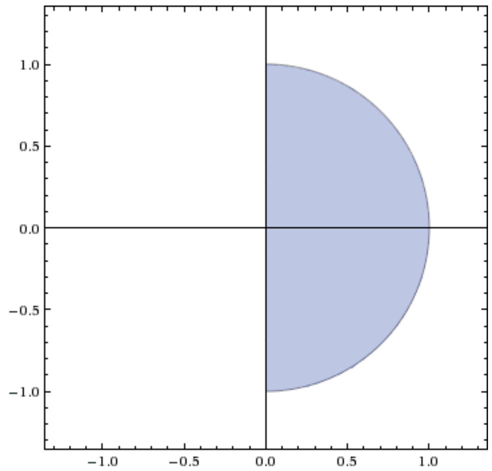
   (b) $\varepsilon^x_K + \varepsilon^x_L = \frac{\partial x}{\partial K}\frac{K}{x} + \frac{\partial x}{\partial L}\frac{L}{x} = \frac{x}{2K(1+4\ln x)}\frac{K}{x} + \frac{x}{3L(1+4\ln x)}\frac{L}{x} = \frac{5}{6}\left(\frac{1}{1+4\ln x}\right)$.

4. See EMEA.

Tutorial 6

1. See EMEA.
2. “
3. “
4. “
5.

   (a)



   (b) $g'_x = 3x^2 - 2x$
   $g'_y = -2y$, which gives two stationary points, $(0, 0)$ and $(\frac{2}{3}, 0)$.
   $g''_{xx} = 6x - 2$
   $g''_{yy} = -2$
   $g''_{xy} = 0$

(x, y)	g''_{xx}	g''_{yy}	g''_{xy}	D	Type
$(0, 0)$	-2	-2	0	4	local maximum
$(\frac{2}{3}, 0)$	2	-2	0	-4	saddle point

   $D = (g''_{xx})\left(g''_{yy}\right) - \left(g''_{xy}\right)^2$

   (c) Examine the boundaries:
   On the circle we have $x^2+y^2 = 1$, so $h(x) = 2+x^3$, $[0, 1]$, which has a maximum at $x = 1$ and a minimum at $x = 0$. This gives solution candidates $(1, 0)$ and $(0, \pm 1)$.
   On the axis, we have $x = 0$, so $f(y) = 3 - y^2$, $[-1, 1]$, which has a maximum at $y = 0$ and minima at $y = \pm 1$. So we get candidates $(0, 0)$ and $(0, \pm 1)$.
   Evaluate the candidates: $g(0, 0) = 3$ and $g(1, 0) = 3$ are maxima and $g(0, 1) = 2$ and $g(0, -1) = 2$ are minima.

Additional problems

4. (a) $Q = F(tK, tL) = t^{\frac{1}{2}}K^{\frac{1}{2}}t^{\frac{1}{4}}L^{\frac{1}{4}} = t^{\frac{3}{4}}F$. Homogeneous of degree $\frac{3}{4}$.
$\frac{1}{2}K^{-\frac{1}{2}}L^{\frac{1}{4}}K + \frac{1}{4}K^{\frac{1}{2}}L^{-\frac{3}{4}}L = \frac{3}{4}F$.

(b) $\pi = PK^{\frac{1}{2}}L^{\frac{1}{4}} - rK - wL$ for $K > 0$ and $L > 0$.
$\pi'_K(K, L) = \frac{1}{2}PK^{-\frac{1}{2}}L^{\frac{1}{4}} - r = 0$; $\pi'_L(K, L) = \frac{1}{4}PK^{\frac{1}{2}}L^{-\frac{3}{4}} - w = 0$

(c) $\pi''_{KK}(K, L) = -\frac{1}{4}PK^{-\frac{3}{2}}L^{\frac{1}{4}} < 0$
$\pi''_{LL}(K, L) = -\frac{3}{16}PK^{\frac{1}{2}}L^{-\frac{7}{4}} < 0$
$\pi''_{KL}(K, L) = \frac{1}{8}PK^{-\frac{1}{2}}L^{-\frac{3}{4}}$
$\pi''_{KK}\pi''_{LL} - \left(\pi''_{KL}\right)^2 = \frac{P^2}{32KL^{\frac{3}{2}}} > 0$. π is strictly concave.

(d) $K^*(P, r, w) = \frac{1}{32r^3w}P^4$
$L^*(P, r, w) = \frac{1}{64r^2w^2}P^4$.

Tutorial 7

1. See EMEA

2. “

3. “

4. “

Additional problems:

1. (a) $\mathcal{L} = x^{\frac{1}{3}} y^{\frac{2}{3}} - \lambda\,(py - w(24 - x))$

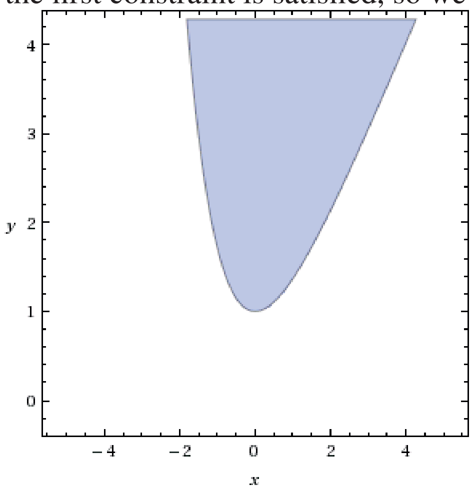
   (b) $x^* = 8$, $y^* = 16\frac{w}{p}$, $\lambda^* = \frac{1}{3} 2^{\frac{2}{3}} p^{-\frac{2}{3}} w^{-\frac{1}{3}}$

2. See EMEA.

Tutorial 8

1. See EMEA
2. “
3. “

Additional problems:
2.
Skecth the admissible set to see that the constraint $y \geq 0$ is automatically satisfied if the first constraint is satisfied, so we can exclude it:



The solution to the problem is given by $x^* = \ln 2$, $y^* = \ln 2 + \frac{1}{2}$, $\lambda^* = 1$

Tutorial 9

1. See EMEA.

2. See EMEA.

3. “

4.

(a) Set $u = \sqrt{4-x^2}$. Then $u^2 = 4 - x^2$and $2u\,\mathrm{d}u = -2x\,\mathrm{d}x$. Then

$$\int_0^1 \frac{4x^3}{\sqrt{4-x^2}}\mathrm{d}x = \left|_0^1 -\frac{4}{3}\sqrt{4-x^2}(x^2+8) = \frac{64}{3} - 12\sqrt{3}\right.$$

(b) Set $u = e^{2x} + 1$. Then $\mathrm{d}u = 2e^{2x}\,\mathrm{d}x$. Then

$$\frac{1}{2}\int \frac{e^{2x}}{u}\mathrm{d}u = \frac{1}{2}\int \frac{u-1}{u}\mathrm{d}u = \frac{1}{2}e^{2x} - \frac{1}{2}\ln(e^{2x}+1) + C$$

(c) Let’s look only at the indefinite integral first:
$f = \ln x$, $g' = \frac{1}{\sqrt{x}}$. We then have

$$\int \frac{\ln x}{\sqrt{x}}\mathrm{d}x = 2\sqrt{x}\ln x - 2\int \frac{1}{\sqrt{x}}\mathrm{d}x = 2\sqrt{x}\ln x - 4\sqrt{x} + C$$

Alternatively integration by substitution: Set $u = \sqrt{x}$. Then $\mathrm{d}u = \frac{1}{2}x^{-0.5}\mathrm{d}x$, or $\mathrm{d}x = 2u\,\mathrm{d}u$. Then, $\int \frac{\ln x}{u} 2u\,\mathrm{d}u = 4\int \ln u\,\mathrm{d}u = 4(u\ln u - u) + C$. (For the last step you use integration by parts.) So,

$$\int \frac{\ln x}{\sqrt{x}}\mathrm{d}x = 2\sqrt{x}\ln x - 4\sqrt{x} + C$$

For the definite integral: Note that the function is not defined at $x = 0$. So we write

$$\lim_{h\to 0}\int_{0+h}^{1} \frac{\ln x}{\sqrt{x}}\mathrm{d}x = \lim_{h\to 0}\left|_{0+h}^{1} 2\sqrt{x}\ln x - 4\sqrt{x}\right. = \lim_{h\to 0}(-4 - 2\sqrt{h}\ln h - 4\sqrt{h}) = -4$$

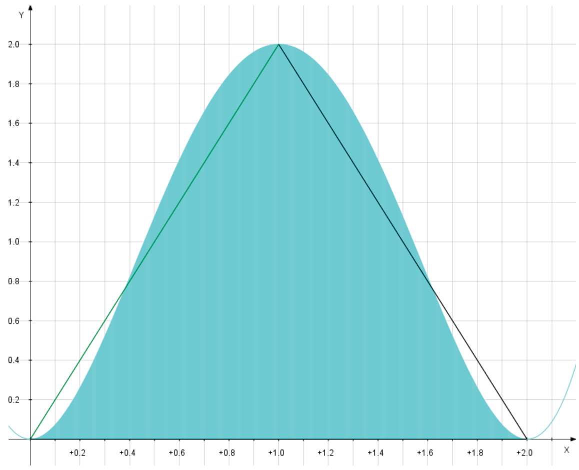
(Note that $\sqrt{h}\ln h = \frac{\ln h}{h^{-0.5}}$ tends to zero by l’Hopital’s rule).

(d) $\int_1^\infty \frac{1}{\sqrt{x}}\mathrm{d}x = \lim_{a\to\infty}\int_1^a \frac{1}{\sqrt{x}}\mathrm{d}x = \lim_{a\to\infty}\left|_1^a 2\sqrt{x}\right. = \lim_{a\to\infty}(2\sqrt{a} - 2) = \infty$.

Additional problem:

$$\int_0^2 2x^2(2-x)^2\mathrm{d}x = \Big|_0^2 \tfrac{8}{3}x^3 - 2x^4 + \tfrac{2}{5}x^5 = 2\tfrac{2}{15}.$$

We see that the triangle between $(0,0)$, $(1,2)$ and $(2,1)$ approximately encloses an area of 2.



Tutorial 10

1. See EMEA.

2. "

3. "

4. (a)

$$\mathbf{C} = \begin{pmatrix} 1 & -3 & 3 \\ 2 & 1 & -1 \\ -2 & 6 & 1 \end{pmatrix}$$

(b) Simply use the cofactor matrix. For example, one can take the first row of **A** and the first row of **C**, multiply each element and take the sum: det(**A**) = 7

(c) The transpose $\boldsymbol{C}'$ is:

$$\mathbf{C}' = \text{adj}(\mathbf{A}) = \begin{pmatrix} 1 & 2 & -2 \\ -3 & 1 & 6 \\ 3 & -1 & 1 \end{pmatrix}$$

(d)

$$\mathbf{X} = \mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \cdot \text{adj}(\mathbf{A})$$

Therefore,

$$\mathbf{X} = \mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{7} & \frac{2}{7} & -\frac{2}{7} \\ -\frac{3}{7} & \frac{1}{7} & \frac{6}{7} \\ \frac{3}{7} & -\frac{1}{7} & \frac{1}{7} \end{pmatrix}$$

Then, one can verify:

$$\mathbf{AX} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{7} & \frac{2}{7} & -\frac{2}{7} \\ -\frac{3}{7} & \frac{1}{7} & \frac{6}{7} \\ \frac{3}{7} & -\frac{1}{7} & \frac{1}{7} \end{pmatrix} = \begin{pmatrix} \frac{1}{7} + \frac{6}{7} & \frac{2}{7} - \frac{2}{7} & -\frac{2}{7} + \frac{2}{7} \\ \frac{3}{7} - \frac{3}{7} & \frac{6}{7} + \frac{1}{7} & -\frac{6}{7} + \frac{6}{7} \\ -\frac{3}{7} + \frac{3}{7} & \frac{1}{7} - \frac{1}{7} & \frac{6}{7} + \frac{1}{7} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$