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THE ACCURACY OF NUMEROV'S METHOD FOR EIGENVALUES

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Abstract.

A simpler proof is given of a slightly stronger version of a convergence result of Chawla and Katti.

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Let $\lambda_1 < \lambda_2 < \ldots$ be the eigenvalues and $y_1, y_2, \ldots$ corresponding eigenvectors of the regular Sturm-Liouville problem

$$-y'' + py = \lambda q y, \tag{1a}$$

$$y(a) = y(b) = 0. \tag{1b}$$

Numerov's method, with uniform mesh length $h := (b-a)/(n+1)$, approximates the λ_k by eigenvalues Λ_k of the matrix equation

$$-A\boldsymbol{u} + BP\boldsymbol{u} = \Lambda BQ\boldsymbol{u}, \tag{2}$$

where $A := (a_{ij})$ is symmetric tridiagonal with $a_{ii} := -2/h^2$ $(i = 1, \ldots, n)$, $a_{i,i+1} := 1/h^2$ $(i = 1, \ldots, n-1)$, $B := 12I + h^2 A$, $P := \mathrm{diag}(p(x_1), \ldots, p(x_n))/12$, $Q := \mathrm{diag}(q(x_1), \ldots, q(x_n))/12$, I is the identity matrix and $x_i := a + ih$ $(i = 1, \ldots, n)$.

It is proved in [3], Theorem 2 that, when $p = 0$, the approximation for fixed k is $O(h^4)$ as $h \to 0$. The proof in [3] is mainly devoted to establishing a uniform bound for $\|B^{-1}\|$ as $n \to \infty$. In this note, a simpler proof is given of a sharper (and simpler) bound for $\|B^{-1}\|_2$, and this is used to prove the following slightly stronger result.

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THEOREM 1. *For all real-valued $p, q \in C^4[a,b]$ with $q > 0$, there exists a number c (independent of k and n) sucht that, for all k and all sufficiently large n, there is an eigenvalue Λ of (2) for which*

$$|\lambda_k - \Lambda| \leqslant ck^6h^4.$$

PROOF. Direct substitution shows that

$$Bs_k = \mu_k s_k, \qquad k = 1,\ldots,n, \tag{3}$$

where $s_k := (\sin(k\varrho),\ldots,\sin(kn\varrho))^T$, $\mu_k := 10+2\cos(k\varrho)$ and $\varrho := \pi/(n+1)$. Hence B is positive definite and, since B is symmetric, $\|B^{-1}\|_2$ is the largest eigenvalue of B^{-1}. Hence by (3),

$$\|B^{-1}\|_2 < 1/8. \tag{4}$$

This is the best possible uniform bound. By (1) and Taylor's theorem, the ith component of $(-A+BP-\lambda_k BQ)y_k$ is $[5y^{(6)}(\xi_{i+})-2y^{(6)}(\xi_{i-})]h^4/720$, where $|\xi_{i\pm}-x_i| < h$ $(i = 1,\ldots,n)$ and $y_k := (y_k(x_1),\ldots,y_k(x_n))^T$. Hence, since $y \in C^6[a,b]$,

$$\|(-A+BP-\lambda_k BQ)y_k\|_2^2/\|y_k\|_2^2 \to (h^4/240)^2 \int_a^b (y^{(6)})^2(x)dx \Big/ \int_a^b y^2(x)dx \text{ as } n \to \infty.$$

Hence $\|(-A+BP-\lambda_k BQ)y_k\|_2 = O(k^6h^4\|y_k\|_2)$ since, by (1), $\|y_k^{(6)}\|_2/\|y_k\|_2 = O(\lambda_k^3) = O(k^6)$. The result now follows from (4) and Theorem 5.3.3 of [5] (see also [4]).

REMARK 1. Theorem 1 generalizes equation (7) of [1] which was deduced from the bound of [3] for $\|B^{-1}\|$. In [1] it is shown that this $O(k^6h^4)$ error is reduced to $O(k^3h^4)$ when a correction suggested in [6] is used.

REMARK 2. Equation (3) is a special case of the result that, if $U := (u_{ij})$ and $V := (v_{ij})$ are any $n \times n$ symmetric tridiagonal Toeplitz matrices with V nonsingular then

$$Us_k = \nu_k V s_k, \qquad k = 1,\ldots,n, \tag{5}$$

where $\nu_k := [u_{11}+2u_{12}\cos(k\varrho)]/[v_{11}+2v_{12}\cos(k\varrho)]$ (with $u_{12} = v_{12} = 0$ if $n = 1$). An application of another special case of (5) is given in [2].

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