

Definitions

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t) dt$$

$$f(t) = F^{-1}[F(\omega)] = \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t) \frac{d\omega}{2\pi}$$

Convolution theorem

$$\begin{aligned} f * g &= \int_{-\infty}^{\infty} g(t') f(t - t') dt' = \int_{-\infty}^{\infty} g(t') \left[\int_{-\infty}^{\infty} F(\omega) \exp(i\omega(t - t')) \frac{d\omega}{2\pi} \right] dt' = \int_{-\infty}^{\infty} F(\omega) \left[\int_{-\infty}^{\infty} g(t') \exp(i\omega(t - t')) dt' \right] \frac{d\omega}{2\pi} = \\ &= \int_{-\infty}^{\infty} F(\omega) G(\omega) \exp(i\omega t) \frac{d\omega}{2\pi} = F^{-1}[F(\omega)G(\omega)] \\ F[f * g] &= F(\omega)G(\omega) \end{aligned}$$

Correlation theorem

$$\begin{aligned} f \star g &= \int_{-\infty}^{\infty} \bar{f}(\tau) g(t + \tau) d\tau = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \bar{F}(\omega) \exp(-i\omega\tau) \frac{d\omega}{2\pi} \right] \left[\int_{-\infty}^{\infty} G(\omega') \exp(i\omega'(t + \tau)) \frac{d\omega'}{2\pi} \right] d\tau = \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{F}(\omega) G(\omega') \exp(i\omega't) \left[\int_{-\infty}^{\infty} \exp(i\tau(\omega' - \omega)) \frac{d\tau}{2\pi} \right] \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{F}(\omega) G(\omega') \exp(i\omega't) \delta(\omega - \omega') \frac{d\omega}{2\pi} d\omega' = \\ &= \int_{-\infty}^{\infty} \bar{F}(\omega) G(\omega) \exp(i\omega t) \frac{d\omega}{2\pi} = F^{-1}[\bar{F}(\omega)G(\omega)] \\ F[f \star g] &= \bar{F}(\omega)G(\omega) \end{aligned}$$

Autocorrelation function and Wiener-Khintchine theorem

$$C(t) = f \star f = \int_{-\infty}^{\infty} \bar{f}(\tau) f(t + \tau) d\tau$$

$$F[C(t)] = |F(\omega)|^2 \quad \longleftrightarrow \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{f}(\tau) f(t + \tau) \exp(-i\omega t) dt d\tau = \left| \int_{-\infty}^{\infty} f(t) \exp(i\omega t) dt \right|^2$$

Discrete Fourier transform – toy example

$$\begin{aligned}
 &\text{continuous} & X(\omega) &= \int_{-\infty}^{\infty} x(t) \exp(-2\pi i \omega t) \\
 &\text{discrete} & X_k &= \sum_{n=0}^{N-1} x_n \exp\left(-\frac{2\pi i k n}{N}\right) \\
 && \omega &\cong \frac{k}{N}, \quad n \cong t
 \end{aligned}$$

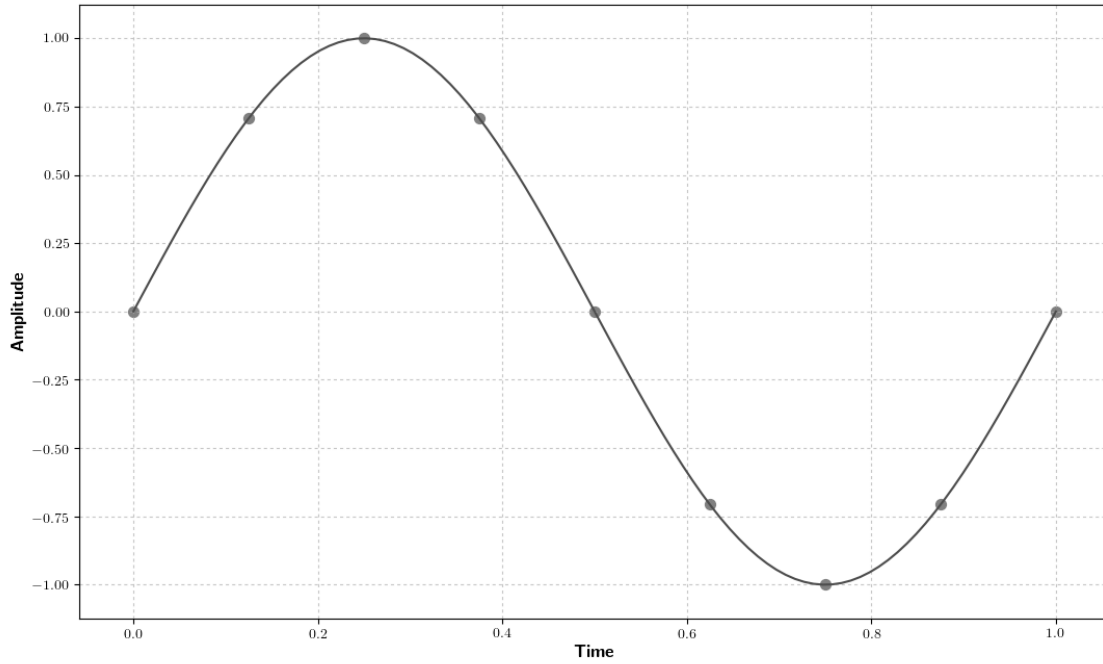


Рис. 1: Sine wave $f(x) = \sin(2\pi x)$: $\omega = 1\text{Hz}$, amplitude = 1. Sampling frequency: 8 Hz

Amplitude in time domain	Amplitude in frequency domain
0.000	0.000
0.707	-4i
1.000	0.000
0.707	0.000
0.000	0.000
-0.707	0.000
-1.000	0.000
-0.707	4i

Frequency resolution = $\frac{\text{Sampling frequency}}{\text{Number of samples}} = \frac{8\text{Hz}}{8} = 1\text{Hz}$.

Getting rid of all frequencies above Nyquist limit, doubling amplitudes and normalizing amplitudes by number of samples.

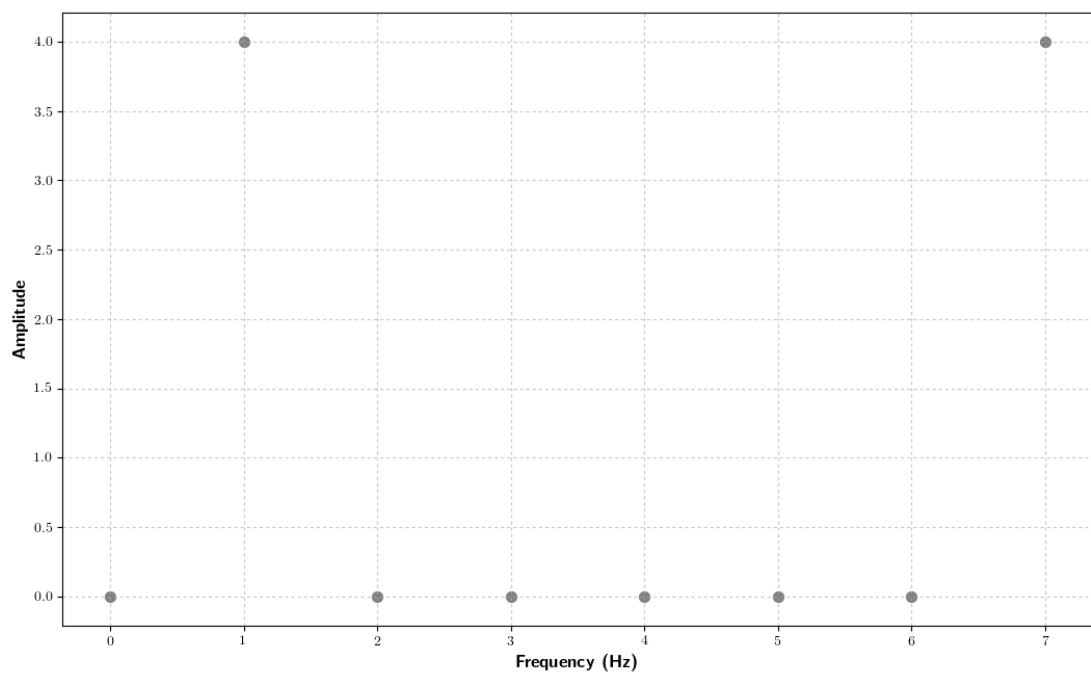


Рис. 2: Power spectrum. $\omega_i - |X_i|$.

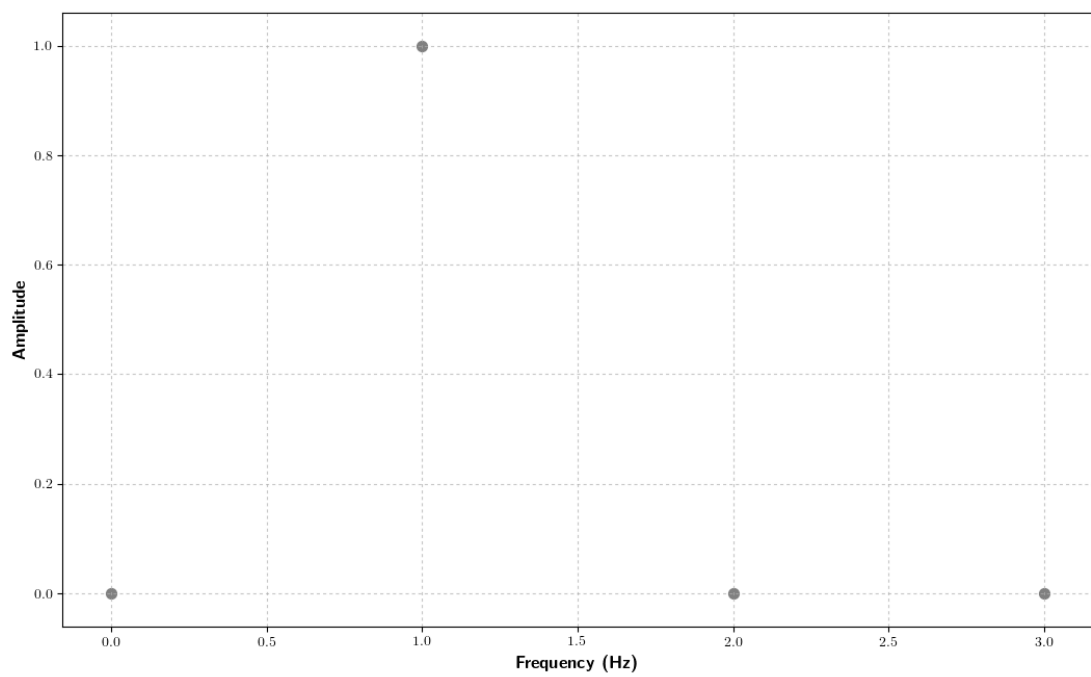


Рис. 3: Normalized power spectrum