Definitions

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t)\, dt$$

$$f(t) = F^{-1}[F(\omega)] = \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t) \frac{d\omega}{2\pi}$$

Convolution theorem

$$f * g = \int_{-\infty}^{\infty} g(t') f(t-t')\, dt' = \int_{-\infty}^{\infty} g(t') \left[\int_{-\infty}^{\infty} F(\omega) \exp(i\omega(t-t')) \frac{d\omega}{2\pi} \right] dt' = \int_{-\infty}^{\infty} F(\omega) \left[\int_{-\infty}^{\infty} g(t') \exp(i\omega(t-t'))\, dt' \right] \frac{d\omega}{2\pi} =$$

$$= \int_{-\infty}^{\infty} F(\omega)G(\omega) \exp(i\omega t) \frac{d\omega}{2\pi} = F^{-1}[F(\omega)G(\omega)]$$

$$F[f * g] = F(\omega)G(\omega)$$

Correlation theorem

$$f \bigstar g = \int_{-\infty}^{\infty} \bar{f}(\tau) g(t+\tau) d\tau = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \bar{F}(\omega) \exp(-i\omega\tau) \frac{d\omega}{2\pi} \right] \left[\int_{-\infty}^{\infty} G(\omega') \exp(i\omega'(t+\tau)) \frac{d\omega'}{2\pi} \right] d\tau =$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{F}(\omega)G(\omega') \exp(i\omega' t) \left[\int_{-\infty}^{\infty} \exp(i\tau(\omega'-\omega)) \frac{d\tau}{2\pi} \right] \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{F}(\omega)G(\omega') \exp(i\omega' t)\, \delta(\omega-\omega') \frac{d\omega}{2\pi} d\omega' =$$

$$= \int_{-\infty}^{\infty} \bar{F}(\omega)G(\omega) \exp(i\omega t) \frac{d\omega}{2\pi} = F^{-1}\left[\bar{F}(\omega)G(\omega)\right]$$

$$F[f \bigstar g] = \bar{F}(\omega)G(\omega)$$

Autocorrelation function and Wiener-Khintchine theorem

$$C(t) = f \bigstar f = \int_{-\infty}^{\infty} \bar{f}(\tau) f(t+\tau) d\tau$$

$$F[C(t)] = |F(\omega)|^2 \quad \longleftrightarrow \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{f}(\tau) f(t+\tau) \exp(-i\omega t)\, dt\, d\tau = \left| \int_{-\infty}^{\infty} f(t) \exp(i\omega t)\, dt \right|^2$$

Discrete Fourier transform – toy example

$$\text{continuous} \qquad X(\omega) = \int\limits_{-\infty}^{\infty} x(t) \exp\left(-2\pi i \omega t\right)$$

$$\text{discrete} \qquad X_k = \sum_{n=0}^{N-1} x_n \exp\left(-\frac{2\pi i k n}{N}\right)$$

$$\omega \cong \frac{k}{N}, \qquad n \cong t$$

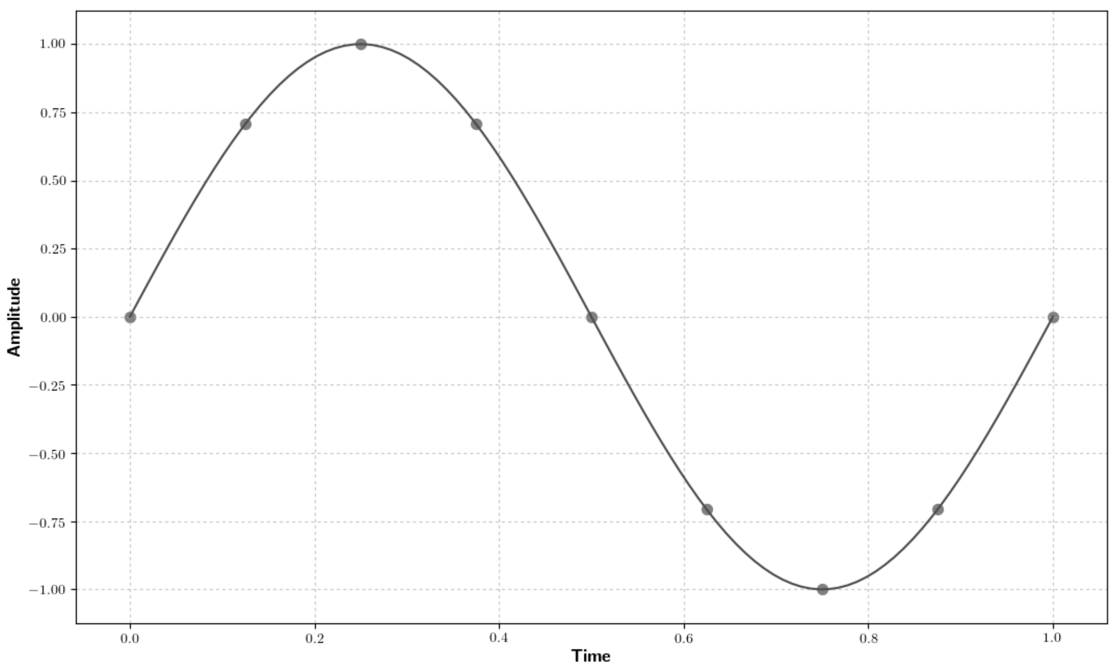

Рис. 1: Sine wave $f(x) = \sin(2\pi x)$: $\omega = 1Hz$, amplitude $= 1$. Sampling frequency: 8 Hz

Amplitude in time domain	Amplitude in frequency domain
0.000	0.000
0.707	-4i
1.000	0.000
0.707	0.000
0.000	0.000
-0.707	0.000
-1.000	0.000
-0.707	4i

$$\text{Frequency resolution} = \frac{\text{Sampling frequency}}{\text{Number of samples}} = \frac{8\text{Hz}}{8} = 1\text{Hz}.$$

Getting rid of all frequencies above Nyquist limit, doubling amplitudes and normalizing amplitudes by number of samples.

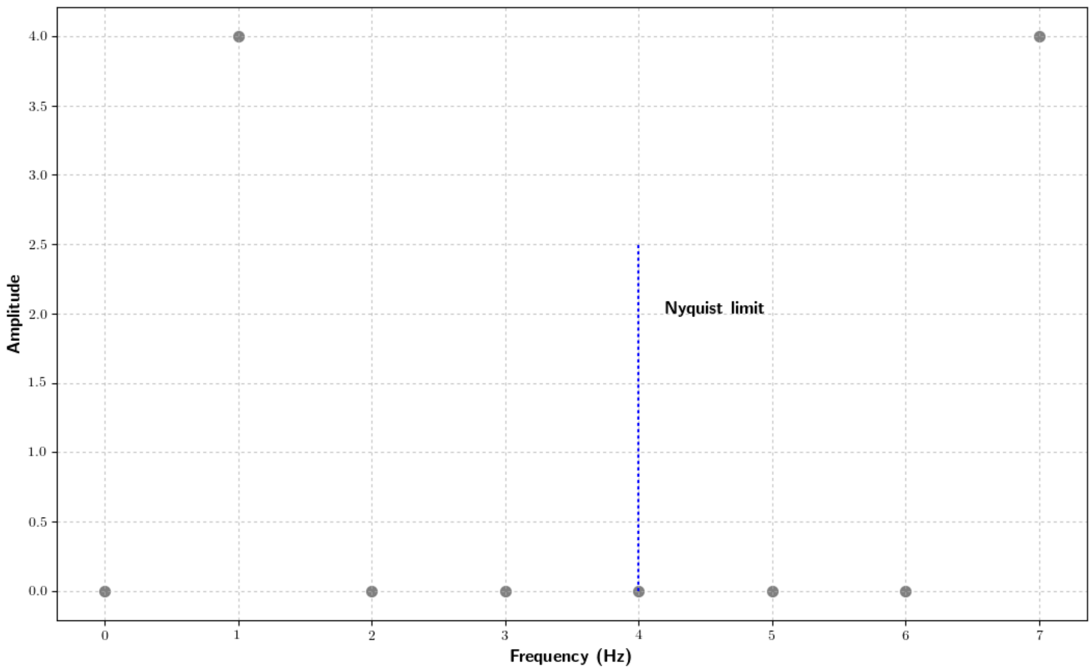

Рис. 2: Power spectrum. $\omega_i - |X_i|$.

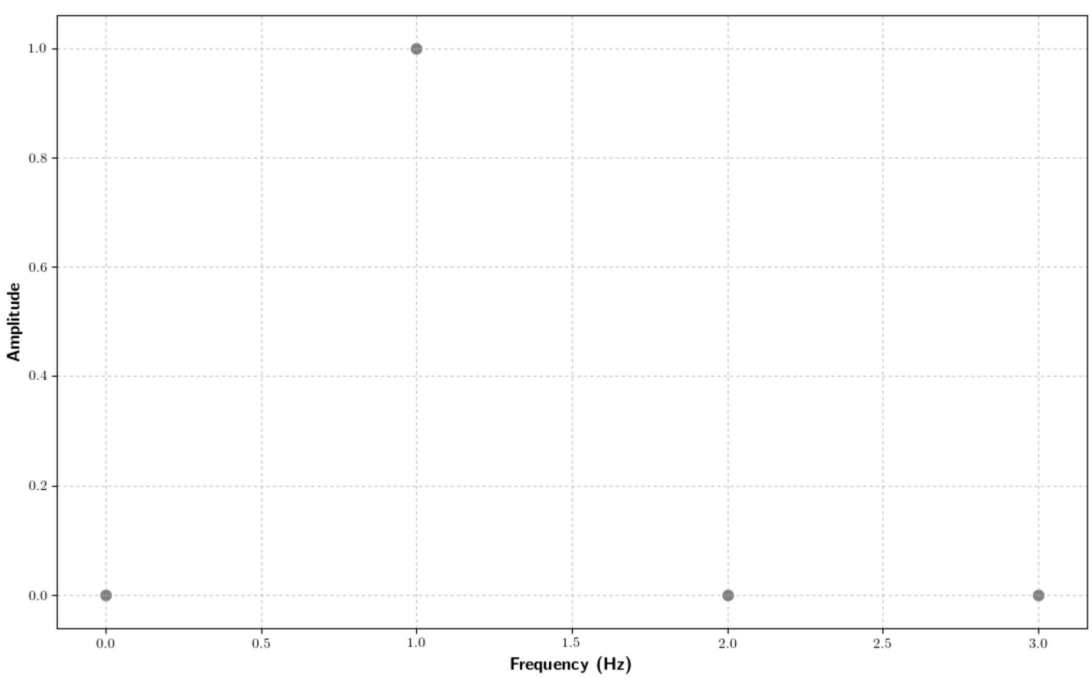

Рис. 3: Normalized power spectrum