1.13 Prove that Fib(n) is the closest integer to $\frac{\varphi^n}{\sqrt{5}}$, where $\varphi = \frac{(1+\sqrt{5})}{2}$.

Hint: Let $\psi = \frac{1-\sqrt{5}}{2}$. Use induction and the definition of the Fibonacci numbers (see 1.2.2) to prove that $Fib(n) = \frac{(\varphi^n - \psi^n)}{\sqrt{5}}$

$$Fib(n) = \begin{cases} 0 & \text{if } n=0, \\ 1 & \text{if } n=1, \\ Fib(n-1) + Fib(n-2) & \text{otherwise.} \end{cases}$$

Induction proof of $Fib(n) = \frac{(\varphi^n - \psi^n)}{\sqrt{5}}$

Base cases are $n=0$ and $n=1$.

$$Fib(0) = \frac{\varphi^0 - \psi^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0 \quad \checkmark$$

$$Fib(1) = \frac{\varphi^1 - \psi^1}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}}$$

$$= \frac{\frac{1+\sqrt{5}-1+\sqrt{5}}{2}}{\sqrt{5}} = \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1 \quad \checkmark$$

We see that the base cases hold.

By the induction hypothesis, we assume the case also holds for $n=k$.

$$Fib(k) = \frac{\varphi^k - \psi^k}{\sqrt{5}}$$

. Then, it should also hold for $n = k+1$.

By the definition of the $Fib(n)$, we see that

$Fib(k+1) = Fib(k) + Fib(k-1)$.

Therefore, we add $\frac{\varphi^{k-1} - \psi^{k-1}}{\sqrt{5}}$ to both sides of the equation

$$Fib(k) + \frac{\varphi^{k-1} - \psi^{k-1}}{\sqrt{5}} = \frac{\varphi^k - \psi^k}{\sqrt{5}} + \frac{\varphi^{k-1} - \psi^{k-1}}{\sqrt{5}}$$

$$Fib(k+1) = \frac{\varphi^k + \varphi^{k-1} - \psi^k - \psi^{k-1}}{\sqrt{5}}$$

$$= \frac{\varphi^{k-1}(\varphi+1) - \psi^{k-1}(\psi+1)}{\sqrt{5}}$$

$$\varphi^2 = \frac{1+\sqrt{5}}{2} \cdot \frac{1+\sqrt{5}}{2}$$
$$= \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4}$$
$$= \frac{3+\sqrt{5}}{2}$$

$$\psi^2 = \frac{1-\sqrt{5}}{2} \cdot \frac{1-\sqrt{5}}{2}$$
$$= \frac{1-2\sqrt{5}+5}{4} = \frac{6-2\sqrt{5}}{4}$$
$$= \frac{3-\sqrt{5}}{2}$$

$$\varphi+1 = \frac{1+\sqrt{5}}{2} + \frac{2}{2}$$
$$= \frac{3+\sqrt{5}}{2}$$

$$\psi+1 = \frac{1-\sqrt{5}}{2} + \frac{2}{2}$$
$$= \frac{3-\sqrt{5}}{2}$$

Therefore, $Fib(k+1) = \frac{\varphi^{k-1} \cdot \varphi^2 - \psi^{k-1} \cdot \psi^2}{\sqrt{5}}$

$$= \frac{\varphi^{k+1} - \psi^{k+1}}{\sqrt{5}}.$$

$\boxed{\text{Thus, we have proved by induction that } Fib(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}}$

Original task: Prove that $Fib(n)$ is the closest integer $\frac{\varphi^n}{\sqrt{5}}$.

Since $Fib(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}} = \frac{\varphi^n}{\sqrt{5}} - \frac{\psi^n}{\sqrt{5}}$, we have that $Fib(n) + \frac{\psi^n}{\sqrt{5}} = \frac{\varphi^n}{\sqrt{5}}$.

If $-0.5 < \frac{\psi^n}{\sqrt{5}} < 0.5$ for all n, then the distance from $Fib(n)$ to $\frac{\varphi^n}{\sqrt{5}}$ must be within the ~~the~~ 0.5 range, thus making $Fib(n)$ the closest integer to $\frac{\varphi^n}{\sqrt{5}}$.

We presume $\left|\frac{\psi^n}{\sqrt{5}}\right| < \frac{1}{2}$. Recall: $\psi^n = \frac{1-\sqrt{5}}{2}$

$|\psi^n| < \frac{\sqrt{5}}{2} \Rightarrow \left|\left(\frac{1-\sqrt{5}}{2}\right)^n\right| < \frac{\sqrt{5}}{2}$

We see that this holds for the cases $n=0$ and $n=1$.

Also, we know from the definition $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

that if $b > |a|$ then $\left|\left(\frac{a}{b}\right)^n\right|$ will shrink as n increases

So $\left|\left(\frac{1-\sqrt{5}}{2}\right)^n\right| < \frac{\sqrt{5}}{2}$ holds if

$|1-\sqrt{5}| < 2$.

Let's inspect:

$$|1-\sqrt{5}| < 2$$

$$-2 < 1-\sqrt{5} < 2$$

$$1-\sqrt{5} < 2$$

$$\sqrt{5} < 1 \quad ✓$$

$$1-\sqrt{5} > -2$$

$$3-\sqrt{5} > 0 \quad ✓$$

Proved!

Thus, we see that $|1-\sqrt{5}| < 2$, meaning $\left(\frac{1-\sqrt{5}}{2}\right)^n$ will shrink as n grows, meaning that $\left|\frac{\psi^n}{\sqrt{5}}\right| < \frac{1}{2}$, which means that since $\frac{\varphi^n}{\sqrt{5}} - \frac{\psi^n}{\sqrt{5}} = Fib(n)$, there can't be an integer closer to $\frac{\varphi^n}{\sqrt{5}}$ than Fib(n.)