Exercise: Water hammer

Determine time series of water pressure (head) H and flow velocity v for three cases of water hammer in a horizontal pipe (see Fig. 1):

1. instantaneous closure in point C, friction neglected
2. instantaneous closure in point C, friction coefficient $\lambda = 0{,}032$
3. gradual closure in 8 seconds in point C, friction coefficient $\lambda = 0{,}032$.

Water pressure and velocity must be calculated at three locations of the pipe: the inlet (A), middle of the pipe (B), and at the valve (C). Computation can be done in MS Excel or Matlab, C++, etc. In all three cases, the simulation time is 14 seconds.

Pipe length	L	=	2040 m
Initial discharge	Q	=	30 m^3/s
Pipe diameter	D	=	2,67 m
Reservoir level	H_0	=	725 m
Pipe wall thickness	e	=	0,03 m
Pipe elasticity module	E	=	$1{,}962 \times 10^{11}$ N/m^2
Water compressiibility	K	=	$1{,}962 \times 10^{9}$ N/m^2
Pipe inclination	θ	=	0

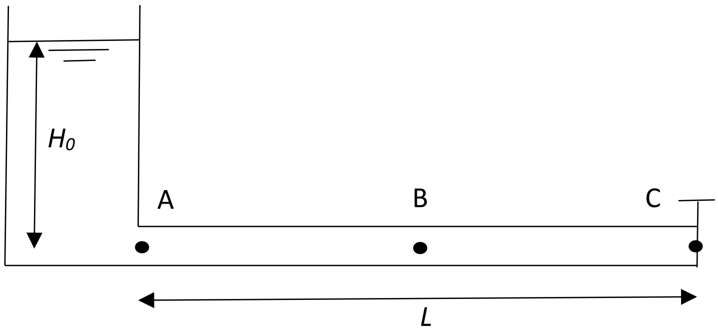


Fig. 1: Reservoir and horizontal pipe with a valve at the outlet

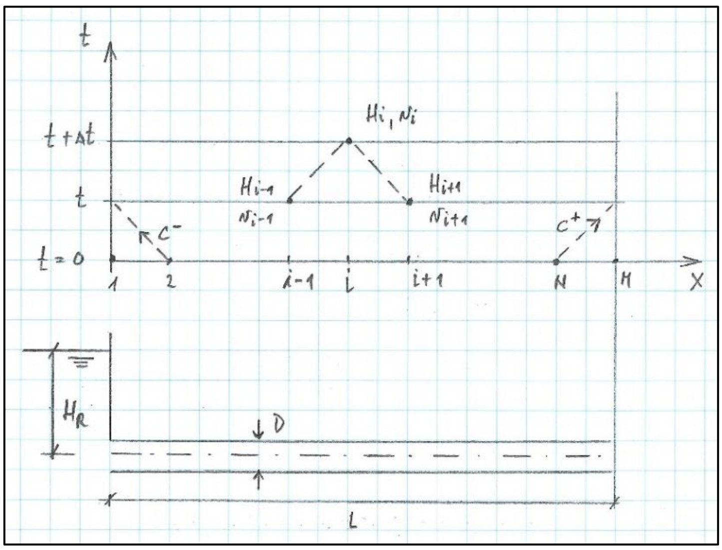


Fig. 2: Basic computational scheme

1) Basic Equations

Two basic equations are the continuity equation and the momentum equation, which are analogous to the corresponding equations of the free surface flow:

$$\frac{a^2}{g}\cdot\frac{\partial v}{\partial x}+v\frac{\partial H}{\partial x}+\frac{\partial H}{\partial t}+v\cdot\sin\theta=0$$

continuity equation for flow under pressure (*a* is the velocity of propagation)

$$\frac{\partial v}{\partial x}+\frac{B}{S}\left(v\frac{\partial h}{\partial x}+\frac{\partial h}{\partial t}\right)=-\frac{v}{S}\left(\frac{\partial S}{\partial x}\right)_{h=konst}$$

continuity equation for free surface flow

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+g\frac{\partial H}{\partial x}+\frac{\lambda v|v|}{2D}=0$$

momentum equation for flow under pressure

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+g\frac{\partial h}{\partial x}=g(I_0-I)$$

momentum equation for free surface flow

Due to the above analogy, the flow in the pipe can be considered as consisting of elementary waves propagating in both directions, as is the case in the free surface flow. This allows us to employ the method of characteristics. Equations of characteristics can be obtain from both basic equations and from the following two relations:

$H = H(x,t)$, therefore: $$dH=\frac{\partial H}{\partial x}dx+\frac{\partial H}{\partial t}dt$$ and

$v = v(x,t)$, therefore: $$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial t}dt$$

2) Derivation of equations of characteristics

We write the system of basic equations in the matrix form:

$$\begin{Bmatrix} g & 0 & v & 1 \\ v & 1 & \frac{a^2}{g} & 0 \\ dx & dt & 0 & 0 \\ 0 & 0 & dx & dt \end{Bmatrix}\cdot\begin{Bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial t} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial t} \end{Bmatrix}=\begin{Bmatrix} -\frac{\lambda v|v|}{2D} \\ -v\cdot\sin\theta \\ dH \\ dv \end{Bmatrix}$$

coefficients unknown variables

The unknown variables can be calculated employing the method of determinants:

$$\frac{\partial H}{\partial x}=\frac{\Delta_1}{\Delta}\qquad\frac{\partial H}{\partial t}=\frac{\Delta_2}{\Delta}\qquad\frac{\partial v}{\partial x}=\frac{\Delta_3}{\Delta}\qquad\frac{\partial v}{\partial t}=\frac{\Delta_4}{\Delta}$$

where: Δ … determinant of the matrix of the coefficients

Δ_i (i = 1,2,3,4) … determinants of a modified matrix of coefficients, in which one column is replaced with the column from the matrix on the right side of the system

We are looking for the propagation of the elementary waves, therefore the derivatives must be undefined (remember the case of free surface flow: where two waves meet the slope of the water surface can't be determined). This means that all determinants must be equal to zero (because 0/0 is undefined expression).

Condition Δ = 0 means:

$$\Delta = dx \cdot \begin{vmatrix} 0 & v & 1 \\ 1 & \frac{a^2}{g} & 0 \\ 0 & dx & dt \end{vmatrix} - dt \cdot \begin{vmatrix} g & v & 1 \\ v & \frac{a^2}{g} & 0 \\ 0 & dx & dt \end{vmatrix} = 0$$

After the multiplication, we divide with dt^2 and we get the quadratic equation for (dx/dt). Results of this quadratic equation are:

$$\left(\frac{dx}{dt}\right)_1 = v + a$$

C_1^+ … first equation of positive characteristic

$$\left(\frac{dx}{dt}\right)_2 = v - a$$

C_1^- … first equation of negative characteristic

These characteristics determine the travel of elementary waves. Using these equations we can determine the location *x* and time *t* of meeting of elementary waves.

The velocity of the propagation *a* is significantly higher than the velocity of the flow *v* in the pipe. As will be shown, the values are: $a \approx 1020$ m/s, *v* is usually 1 to 5 m/s. Therefore, the *v* can be neglected and we can write:

C_1^+: $$\left(\frac{dx}{dt}\right)_1 = +a$$

C_1^-: $$\left(\frac{dx}{dt}\right)_2 = -a$$

This means that the slope of the characteristics is constant. In the *x*-*t* mesh we usually choose constant intervals Δx, thus the time interval $\Delta t = \Delta x/a$ is constant. This means the mesh in the *x*-*t* diagram is fixed and we only have to determine *H* and *v* for these points. We do this by stating the second condition:

$$\Delta_1 = \begin{vmatrix} -\frac{\lambda v|v|}{2D} & 0 & v & 1 \\ -v\sin\theta & 1 & \frac{a^2}{g} & 0 \\ dH & dt & 0 & 0 \\ dv & 0 & dx & dt \end{vmatrix} = 0$$

We get:

$$\Delta_1 = dH \begin{vmatrix} 0 & v & 1 \\ 1 & \frac{a^2}{g} & 0 \\ 0 & dx & dt \end{vmatrix} - dt \cdot \begin{vmatrix} -\frac{\lambda v|v|}{2D} & v & 1 \\ -v\sin\theta & \frac{a^2}{g} & 0 \\ dv & dx & dt \end{vmatrix} = 0$$

We multiply, then divide the result with dt^2 and insert expressions for dx/dt:

$$\left(\frac{dx}{dt}\right)_{1,2} = v \pm a$$

We get:

$$\mp \frac{dH}{dt} a - \frac{a^2}{g} \cdot \frac{dv}{dt} \mp av\sin\theta - \frac{\lambda v|v|a^2}{2Dg} = 0$$

That is:

$$\frac{dH}{dt} + \frac{a}{g} \cdot \frac{dv}{dt} + v\sin\theta + \frac{\lambda v|v|a}{2Dg} = 0$$ C_2^+ … second equation of positive characteristic

$$\frac{dH}{dt} - \frac{a}{g} \cdot \frac{dv}{dt} + v\sin\theta - \frac{\lambda v|v|a}{2Dg} = 0$$ C_2^- … second equation of negative characteristic

Those characteristics determine the head *H* and the velocity *v* at the location of meeting of elementary waves.

2.1) Simplification of equations of characteristics

We have a horizontal pipe without friction. This means the equations C_2^+ in C_2^- include $sin\theta = 0$ and $\lambda = 0$. These lead to simplified forms of equations (after we multiply with *dt* and divide with *dv*):

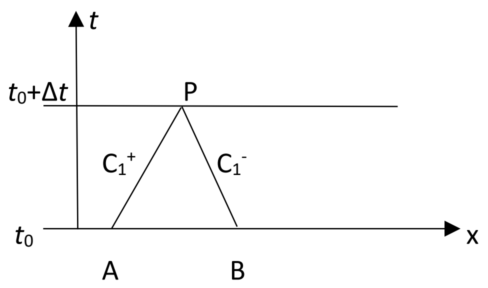
$$\frac{dH}{dv} = -\frac{a}{g}$$ simplified C_2^+

$$\frac{dH}{dv} = +\frac{a}{g}$$ simplified C_2^-

3) Derivation of equations of characteristics in differential form

For the practical use, the derived equations C_2 need to be written in differential form.

We assume we know all the quantities (x, t, v, H) in two points, denoted A and B (they are not the locations mentioned in the introduction to this exercise!). Conditions in these two points can be known as a result from previous calculation or as boundary conditions. We are looking for values of H and v in point P, which is located between A and B (in terms of x), but in a later time, as shown below:



We multiply the equations C_2^+ and C_2^- with dt and integrate:

$$\int_A^P dH + \int_A^P \frac{a}{g} dv + \int_A^P v \sin\theta dt + \int_A^P \frac{\lambda v|v|a}{2Dg} dt = 0$$

$$\int_B^P dH - \int_B^P \frac{a}{g} dv + \int_B^P v \sin\theta dt - \int_B^P \frac{\lambda v|v|a}{2Dg} dt = 0$$

We get:

$$H_P - H_A + \frac{a}{g}(v_P - v_A) + v_A \sin\theta \cdot \Delta t + \frac{\lambda v_A |v_A| a}{2Dg} \Delta t = 0$$ … C_2^+ in differential form

$$H_P - H_B - \frac{a}{g}(v_P - v_B) + v_B \sin\theta \cdot \Delta t - \frac{\lambda v_B |v_B| a}{2Dg} \Delta t = 0$$ … C_2^- in differential form

4) The velocity of water hammer propagation

We use the following equation:

$$a = \sqrt{\frac{K}{\rho\left(1+\frac{KD}{eE}\right)}} = \sqrt{\frac{1{,}962\cdot 10^{9}\,N/m^{2}}{10^{3}kg/m^{3}\left(1+\frac{1{,}962\cdot 10^{9}\,N/m^{2}\cdot 2{,}67m}{0{,}03m\cdot 1{,}962\cdot 10^{11}N/m^{2}}\right)}} = 1018{,}87\text{ m/s} \approx 1020\text{ m/s}$$

5) Maximum pressure in the system

In case of instantaneous closing of the valve, the water hammer will develop to its full measure, which means we have to calculate maximum pressure ΔH_{max}. Obtained value ΔH_{max} needs to be added to the value of the static head to determine the highest head, and subtracted to determine the lowest head.

$$\Delta H_{max} = \frac{av}{g} = \frac{1018{,}87\cdot 5{,}36}{9{,}81} = 556{,}7\text{ m}$$

where: $v = \frac{Q_0}{S} = \frac{Q_0}{\frac{\Pi D^2}{4}} = 5{,}36$ m/s

6) Graphs *x-t*, *v-H* and *H*(*t*)

6.1) Graph *x–t* (lines of characteristics)

1) Draw the horizontal axis *x* and vertical axis *t*.
Scale of axis *x*: 2040 m … 8 cm
Scale of axis *t*: 2 s … 1 cm

2) At time $t = 0$ we are at the location where the disturbance starts, that is at the end of the pipe, in point C.

3) After the valve is closed, the disturbance travels upstream with the velocity given by equation C_1^-: $dx/dt = v - a$, therefore we draw a line from point C with a slope of $-a$ (we neglect *v*):
- we calculated $a = 1020$ m/s
- we know $L = 2040$ m

→ line of the characteristic crosses the *t* axis at $t = 2$ s. There we obtain a point, which we mark as A2 (location A, time 2 s).

4) When the disturbance reaches the upper end of the pipe, it bounces back and travels with practically the same velocity (the difference of $2v$ can be neglected) back to the outlet part of the pipe. From $x = 0$, $t = 2$ s we draw a straight line of C_1^+ with slope $+a$.

5) When the disturbance reaches $x = L$, it turns once again. This point can be denoted C4. The time needed for the travel from outlet to inlet and back $t = 2L/a$ (= 4 s) is called the phase of the system.

We continue this procedure to get points A6, C8, A10, C12 and A14, finally getting to the point $x = 0$, $t = 14$ s (i.e. the time of the simulation).

6.2) Graph *v–H* (lines of characteristics)

1) Draw the horizontal axis *v* and vertical axis *H*. The origin of the coordinate system is at the point $v = 0$, $H = H_0 = 725$ m!
Scale of axis *v*: 1 m/s … 1 cm
Scale of axis *H*: 100 m … 1 cm (values will be in the range from 168 to 1282)

2) At starting point, the velocity is equal to 0, while the head is equal to the calculated maximum, i.e.: H = 725 + 557 = 1282 m.

3) When the valve is closed, the relation between H and v follows the equations C_2 in simplified form. From point v = 0, H = 1282 m we draw a line with the slope C_2^-: $dH/dv = a/g$. Where this line crosses the axis v we get the point A2 (v = -5,36 m/s, H = 725 m).

4) When the disturbance reaches the inlet part of the pipe, it bounces back. From point A2 we draw a line with the slope C_2^+: $dH/dv = - a/g$. Where this line crosses the axis H we get the point C4 (v = 0, H = 168 m).

5) From point C4 we draw a line with the slope C_2^- to the point A6 (v = 5,36 m/s, H = 725 m).

6) We draw a line C_2^+ to the starting point v = 0, H = 1282 m. This is the point C8.

6.3) Graph *H(t)* for the point C (time series of the head at the valve)

1) Draw the horizontal axis t and vertical axis H.
Scale of axis t: 1 s … 1 cm (values from 0 do 14 s)
Scale of axis H: 100 m … 1 cm (values from 0 to 1300 m)

2) During the first four seconds, when the water hammer travels upstream and back to the valve C (i.e. during the phase), the head in point C remains $H_0 + \Delta H_{max}$ = 1282 m.

3) After the mentioned four seconds the positive pressure turns into negative one, resulting in the value $H_0 - \Delta H_{max}$ = 168 m. This can be seen also from the diagram v-H, point C4. This value stays constant for the next four seconds.

4) At t = 8 s the head changes its sign again and stays H = 1282 m until t = 12 s.

5) We continue this procedure until we reach t = 14 s.

6.4) Diagram *H(t)* for the point B

Diagram $H(t)$ for the point B is similar to the one for the point C. It has the same range of values and a similar dynamics, but more gradual transitions of head.

1) At first, the head in point B is equal to H_0, while the increase of pressure starts after ¼ of the phase, i.e. after one second, when the disturbance travels from point C to point B.
2) Head becomes H = 1282 m and stays such for the next ½ of the phase, i.e. two seconds.
3) At t = 3 s the disturbance passes point B again, but this time the head is decreasing and becomes $H = H_0$ again.
4) This lasts for two seconds, when the disturbance returns and causes H = 168 m.
5) This pattern continues until t = 14 s.

7) Calculation of the water hammer

The pipe of length L is separated to N equal intervals with length Δx. Heads and velocities will be calculated in all points at the ends of these intervals in every time interval Δt.

The calculation has to be performed in the following four stages:

a) initial condition, i.e. conditions before the closure of the valve, i.e. *H* and *v* in all points of the pipe at time $t = 0$
b) internal points (in diagram *x-t*) at the next time step
c) left boundary points (i.e. in profile 1, see Fig. 2) at the next time step
d) right boundary points (i.e. in profile M, see Fig. 2), at the next time step.

Steps b), c), and d) need to be repeated until the end of simulation.

7.1) Initial conditions

Before the closure of the valve in time $t = 0$ the velocity in the pipe is constant in all points from 1 to M, i.e. $v(i) = v_0 = Q_0/S$.

The head $H(i)$ for each profile is thus calculated:

$$H(i) = HR - \frac{v_0^2}{2g} - \lambda \frac{L}{D}\frac{v_0^2}{2g}\frac{(i-1)\Delta x}{L} + (i-1)\Delta x sin\theta$$

HR	…	head of the reservoir (see figure)
$\frac{v_0^2}{2g}$	…	local energy loss at the pipe entrance
$\lambda \frac{L}{D}\frac{v_0^2}{2g}\frac{(i-1)\Delta x}{L}$	…	losses along a distance to the observed location
$(i-1)\Delta x sin\Theta$	…	pipe slope (in our case 0, as the pipe is horizontal).

7.2) Internal points (from *i* = 2 to N = M – 1)

If we know the values of *x*, *t*, *H* and *v* in time *T*, we can use the equations of characteristics to calculate the values of variables in the profiles 2 to N = M – 1 in time $T + \Delta T$. Values of head and velocity are determined with the following two expressions:

$$H_{Pi} = 0{,}5\left[H_{i-1} + H_{i+1} + \frac{a}{g}(v_{i-1} - v_{i+1}) - sin\theta\Delta t(v_{i-1} + v_{i+1}) - \frac{a\lambda\Delta t}{2gD}(v_{i-1}|v_{i-1}| - v_{i+1}|v_{i+1}|)\right]$$

$$v_{Pi} = 0{,}5\left[v_{i-1} + v_{i+1} + \frac{g}{a}(H_{i-1} - H_{i+1}) - \frac{g}{a}sin\theta\Delta t(v_{i-1} - v_{i+1}) - \frac{\lambda\Delta t}{2D}(v_{i-1}|v_{i-1}| + v_{i+1}|v_{i+1}|)\right]$$

This equation shows we are using adjacent locations in the previous time step.
Example: Let's calculate point B in the middle of the pipe, i.e. in the profile $i = 11$. At $t = 0{,}1$ s the *H* is calculated as:

$$H_{Pi} = 0{,}5\left[H_{i=10,t=0} + H_{i=12,t=0} + \frac{a}{g}\left(v_{i=10,t=0} - v_{i=12,t=0}\right) - 0 - \frac{a\lambda \cdot 0{,}1}{2gD}\left(v_{i=10,t=0}\left|v_{i=10,t=0}\right| - v_{i=12,t=0}\left|v_{i=12,t=0}\right|\right)\right]$$

In the boundary profiles 1 and M the procedure is different, as explained below.

7.3) Left boundary points (profile 1)

For the left boundary point (x = 1, $T = T + \Delta T$) we have two unknowns: H_1 and v_1.

We need two equations:
1) In the above equation for v_{Pi} we use i = 1 and i = 2 in the previous time step.
2) The pipe starts in the reservoir, where the depth HR is known. Therefore, we can use the equations for the steady flow, assuming that during short time interval ΔT the flow is steady:

$$H_{P1} = HR - \frac{v_0^2}{2g}$$

7.4) Right boundary points (profile M)

We have two unknowns: H_M and v_M.

We need two equations:
1) In the above equation for H_{Pi} we use i = M and i = N.
2) We employ the equation of the flow through the valve to get:

$$\frac{v_M}{v_0} = \tau \sqrt{\frac{H_M}{H_0}}$$

where:

τ … non-dimensional parameter of the opening of the valve (when the valve is open: τ = 1, when the valve is closed: τ = 0)

H_0 … head at the valve (for the steady flow).

In case of linear (gradual) closing of the valve, we use: $\tau = 1 - \frac{T}{T_{close}}$

where T_{close} represents the total time of closing.