ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Derivatives (FIN-404)

Oil futures and storage options

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1 Documentation

In the past centuries, commodity sellers have been looking for ways to secure and stabilize their revenue over time. It started with rice traders in Japan, then evolved to the first standardized future contracts for grain in the US and brought us to nowadays large digitalized futures exchanges for commodities, currencies and more.

One of the largest exchange is the Chicago Mercantile Exchange (CME). Many oil derivatives can be traded there, such as crude oil futures, crude oil options, refined products options, crack spread futures, micro oil futures and more. Our focus will be on crude oil futures.

Oil futures are contracts where a buyer agrees to purchase a fixed quantity of crude oil barrels at a set future date for a predetermined price. These contracts are traded on exchanges like the CME. They can be settled by cash payment or by physical delivery of the oil (rare for speculators). These contracts are derivatives as they are based on the price of the underlying asset, which is oil here.

Oil futures allow the seller to remove uncertainty in future oil prices, which is useful for oil companies to secure their revenue. It also brings features for the buyer, whether it is to secure future oil stocks at a predetermined price to remove risk for a consumer like airlines and refineries, or for pure market speculation on future oil prices.

To have a better understanding, we can visualize the cash flows associated to a long position in one futures contract. We would enter this position if we believe that the price of oil at the expiry date will be higher than the current price of the contract. If our belief becomes reality, we would earn a profit on the difference, but if the opposite happens, we make a loss on the difference.

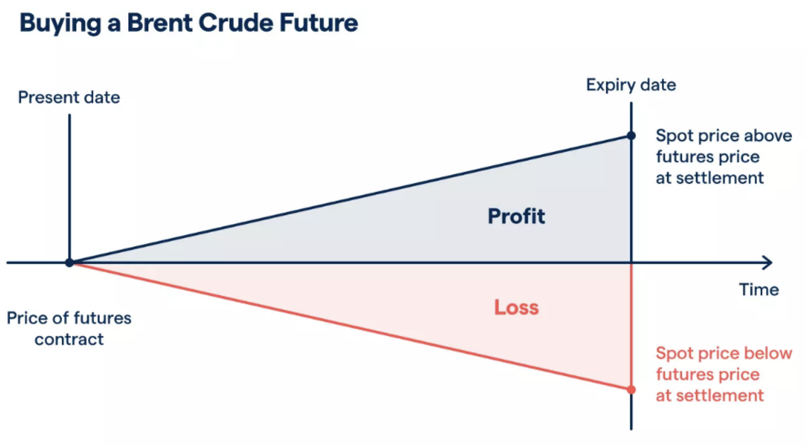

Figure 1. Cash flow associated to a long position in a futures contract over one period. [1]

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On the CME, oil futures mainly expire 3 business day before the 25th calendar day of the month prior to the contract month. If the 25th calendar day is not a business day, trading terminates 4 business days before the 25th calendar day of the month prior to the contract month [2]. When a contract comes to expiration, traders might want to keep exposure for a longer period of time. To do so, they perform rollover, which consists of closing their current futures position before maturity and opening a new one with a later expiration date.

Here is an example with multiple 1 month contract rollovers. We assume that we sell each position right before it expires at the spot price, as in theory the futures price must converge to the spot price at maturity to have no arbitrage opportunities. We also assume that the risk free rate is 0 for cash flow computations to make things easier, see Table 1. Note that $f_a(b)$ refers to the price at time a of a futures contract with expiration date b.

We can see how this allows traders to keep exposure to oil futures, by rolling over their positions successively, without taking positions over very long period of times, which could be risky or undesirable for some.

We will now have a look at the market more generally. To have an overview, we look at trading volumes and open interest in oil futures.

- **Trading volume** refers to the total amount of contracts traded in one day.
- **Open interest** refers to the total number of outstanding contracts that have not been settled by delivery at the end of the trading day. It shows the ongoing commitment in the market.

We can have a look at both of these measures for the past 30 trading days on the CME. (Figure 2)

Date	Action	Futures Price	Spot Price	Cash Flow	Net Position
t_0	Enter long position 1	$f_{t_0}(t_1) = 50$	-	0	Long (1 month)
t_1	Sell position 1	-	$S(t_1) = 52$	$52 - 50 = 2$	-
t_1	Enter new long position 2	$f_{t_1}(t_2) = 51$	-	0	Long (1 month)
t_2	Sell position 2	-	$S(t_2) = 49$	$49 - 51 = -2$	-
t_2	Enter new long position 3	$f_{t_2}(t_3) = 48$	-	0	Long (1 month)
t_3	Sell position 3	-	$S(t_3) = 51$	$51 - 48 = 3$	-

Table 1. Cash flows associated with rolling over 3 long futures positions.

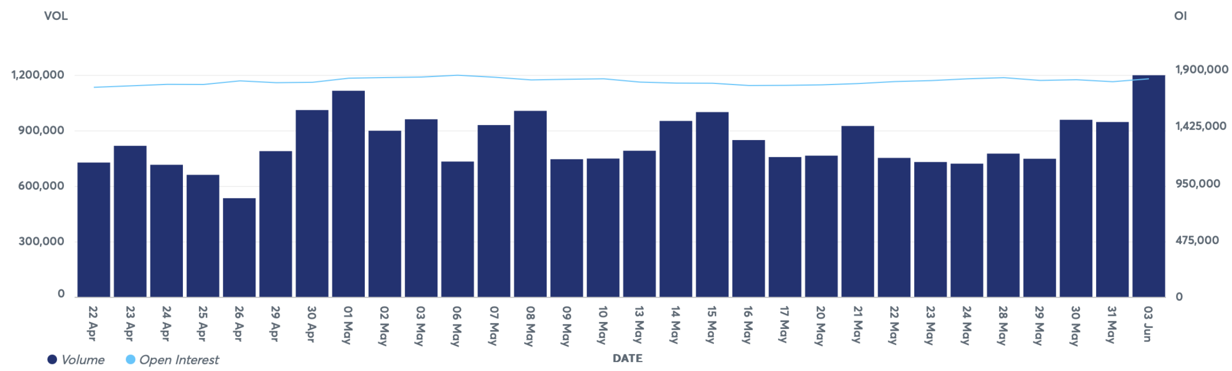


Figure 2. Daily trading volumes and open interest of crude oil futures on the CME, between 22.04.24 and 03.06.24. [3]

As we can see, there are many variations over a 30 days period. The trading volume and open interest in oil futures also vary significantly across different contract maturities, which can be seen on CME group's website. Trading volume tends to be highest in the near-term contracts due to their proximity to the current market conditions and higher liquidity. As the maturity date extends further into the future, the volume generally decreases. Open interest also exhibits a similar trend, though it can also be influenced by the specific strategies of market participants.

Seasonal variations do affect the oil futures market. For example, demand for heating oil futures typically rises during the winter months due to increased heating needs, while gasoline futures may see higher interest in the summer when driving and travel activity peak. These seasonal trends can cause systematic fluctuations in both trading volume and open interest, reflecting the changing needs and anticipations of market participants based on the time of year. Also, as it can be seen in Figure 2, trading volume increases at the end of each month. Contracts expire a few days before the end of each month and many traders perform rollover, which leads to the increase in trading volume for these periods.

To price correctly oil futures, a participant in the market should intuitively compute the expected price of oil at the maturity date of the contract he wants to enter. As the trade happens in the future, many factors that can influence the price of oil must be taken into consideration. Here is a non-exhaustive list: risk-free rate, storage costs of oil, inventory size of oil, geopolitical context, demand and supply. There should be a correction for these factors in the computation.

Geopolitical events have a large impact on the oil futures market, often causing significant price volatility and trading volume fluctuations. Events such as conflicts in oil-producing regions, sanctions, and political instability can disrupt supply chains and create uncertainty about future oil availability. For instance, tensions in the Middle East, which is a major oil-producing region, frequently lead to spikes in oil futures prices as traders react to potential supply disruptions. The war in Ukraine also had a profound impact as Russia is one of the largest oil exporters and it was affected by sanctions. These geopolitical risks force market participants to often reassess their positions due to the randomness and unpredictability of the events.

To account for some of these factors, we use the convenience yield, which refers to the added benefit or premium of holding a physical commodity like oil, rather than a futures contract. This yield becomes particularly important during periods of high demand or supply disruptions, where having the commodity immediately available is valuable. For example, during a cold winter, the value of having physical oil for heating increases, therefore raising the convenience yield.

This yield impacts the spot and futures prices of oil by affecting their differential. When the convenience yield is high, the spot price rises due to the immediate utility of the commodity, while futures prices may not increase as much, or could even fall relative to the spot price. This narrows the gap between the spot and futures prices, reflecting the higher immediate value of the physical commodity.

More generally, variations in the factors listed above induces shifts between different phases in the oil futures market. The two main states are "backwardation" and "contango". They describe the relationship between spot prices and future prices of a commodity.

- **Backwardation** happens when the futures price is lower than the spot price, often due to immediate high demand or short-term supply shortages. It suggests that prices are expected to decrease over time. This can occur with commodities like oil during supply disruptions or increased short-term demand.
- **Contango** is when the futures price is higher than the spot price, reflecting storage costs or expectations of higher future demand. This typically suggests that the market expects prices to rise over time, which is common in normal market conditions.

As of June 2024, many commodities, including oil, are in backwardation. This indicates strong current demand or supply constraints, with an expectation of easing in the near future.

Like any large and impactful market, the oil futures market is heavily regulated to ensure transparency, fairness and stability by organizations like the Commodity Futures Trading Commission (CFTC) in the United States. These regulatory bodies implement and enforce rules to prevent market manipulation, fraud and excessive speculation to maintain market integrity. Additionally, exchanges like the CME Group have their own regulations and monitoring systems to oversee trading activities and ensure compliance with the rules. This regulatory framework provides market participants confidence, ensuring that the market operates smoothly and that prices reflect true supply and demand dynamics. Moreover, environmental policies and sustainability initiatives are increasingly influencing the market with regulations aimed at reducing carbon emissions and promoting renewable energy sources. It will be interesting to observe how the market will evolve in the coming decades and see if there will be innovation in the products traded.

2 Analysis

2.1 Spot price modeling

1. Since $B_t^{\mathbb{Q}}$ is a 3-dimensional Brownian motion, $B^{i\mathbb{Q}}$ are independent Brownian motions, $i = 1, 2, 3$. Therefore,

$$\sigma^T B_t^{\mathbb{Q}} = \sum_{i=1}^{3} \sigma^i B_t^{i\mathbb{Q}} \sim \mathcal{N}(0, t||\sigma||_2^2) = ||\sigma||_2 \mathcal{N}(0, t).$$

Hence,

$$W_t := \frac{1}{||\sigma||_2} \sum_{i=1}^{3} \sigma^i B_t^{i\mathbb{Q}} \sim \mathcal{N}(0, t).$$

is a Brownian motion. One can check it with Levy's characterisation of Brownian motion. As $W_0 = 0$, it is a martingale (as a sum of martingale) and

$$[W_t, W_t] = \frac{1}{||\sigma||_2^2} \sum_{i=1}^{3} \sigma^{2,i} t = t.$$

For $j = \{\mathcal{O}, r, \delta\}$,

$$W_t^j := \frac{1}{||\sigma_j||_2} \sum_{i=1}^{3} \sigma_j^i B_t^{i\mathbb{Q}} \implies \sigma_j^T dB_t^{\mathbb{Q}} = ||\sigma_j||_2 dW_t^j.$$

The model can be rewritten as desired:

$$\begin{aligned} \frac{d\mathcal{O}_t}{\mathcal{O}_t} &= ||\sigma_{\mathcal{O}}|| dW_t^{\mathcal{O}} + (r_t - \delta_t)dt, \\ dr_t &= ||\sigma_r|| dW_t^r + \lambda_r(\bar{r} - r_t)dt, \\ d\delta_t &= ||\sigma_\delta|| dW_t^\delta + \lambda_\delta(\bar{\delta} - \delta_t)dt. \end{aligned}$$

Given that W_t^j are Itô's processes and the increments of the Brownian motions $B_t^{\mathbb{Q}}$ are independant. We obtain for $j \neq k \in \{\mathcal{O}, r, \delta\}$:

$$d\langle W^j, W^k \rangle = \underbrace{\frac{1}{||\sigma_j|| \cdot ||\sigma_k||} \langle \sigma^j, \sigma^k \rangle}_{:=\rho_{jk}} dt.$$

Cauchy-Schwarz yields,

$$|\rho_{jk}| = \frac{1}{||\sigma_j|| \cdot ||\sigma_k||} |\langle \sigma^j, \sigma^k \rangle| \leq 1.$$

2. We define

$$S_{ot} := \exp(\int_0^t r_u du) \quad \text{and} \quad \hat{\mathcal{O}}_t := \frac{\mathcal{O}_t}{S_{ot}}.$$

Then with Itô's formula we obtain,

$$d(S_{ot}) = S_{ot} r_t dt \quad \text{and} \quad d(\frac{1}{S_{ot}}) = -\frac{r_t dt}{S_{ot}}.$$

Itô's formula for a product of Itô's process yields,

$$d(\hat{\mathcal{O}}_t) = d(\mathcal{O}_t \cdot \frac{1}{S_{ot}}) = \hat{\mathcal{O}}_t[-r_t dt + (r_t - \delta_t)dt + \sigma_{\mathcal{O}}^T dB_t^{\mathbb{Q}}] = \hat{\mathcal{O}}_t[-\delta_t dt + \sigma_{\mathcal{O}}^T dB_t^{\mathbb{Q}}]$$

Therefore,

$$\hat{\mathcal{O}}_T = \hat{\mathcal{O}}_t + \int_t^T -\delta_s \hat{\mathcal{O}}_s ds + \int_t^T \delta_{\mathcal{O}}^T dB_t^{\mathbb{Q}} \iff \hat{\mathcal{O}}_t = \hat{\mathcal{O}}_T + \int_t^T \delta_s \hat{\mathcal{O}}_s ds - \int_t^T \delta_{\mathcal{O}}^T dB_t^{\mathbb{Q}}.$$

Taking expectation and multiplying by S_{ot} yields,

$$E_t^{\mathbb{Q}}[\mathcal{O}_t] = E_t^{\mathbb{Q}}[\frac{S_{ot}}{S_{oT}}\mathcal{O}_T + \int_t^T \frac{S_{ot}}{S_{os}}\delta_s\mathcal{O}_s ds] = E_t^{\mathbb{Q}}[\int_t^T \exp(-\int_t^s r_u du)\delta_s\mathcal{O}_s ds + \exp(-\int_t^T r_u du)\mathcal{O}_T].$$

The result follows as $\mathcal{O}_t$ is an adapted process.

The relation implies that the spot price of oil at current date t is the sum of the expected discounted value of the spot price at date T and the expected discounted value of the dividend paid over the interval [t,T], for any time $T > t$ smaller than the terminal date. Gibson and Schwartz Model is known as the stochastic convenience yield model as it models the convenience yield (implied return on holding the inventory) as a stochastic process rather than a constant, which is exactly the first term in our sum. It allows fore more accurate pricing by accounting for the mean-reverting tendency of convenience yield of commodities like crude-oil.

3. Assuming $r_t \equiv r$, let us show that the derivative of $\mathcal{O}_t$ is ruled by the following,

$$d\mathcal{O}_t = \mathcal{O}_t[\phi\log(l_t/\mathcal{O}_t)dt + \sigma_o^T dB_t^{\mathbb{Q}}] \quad \text{subject to} \quad dl_t = l_t[\mu_l dt + \sigma_l^T dB_t^{\mathbb{Q}}].$$

for some constants $(\phi, \mu_l, \sigma_l, \sigma_{\mathcal{O}})$ to be defined.

Equating the two expression for the derivative of $\mathcal{O}_t$ yields,

$$r - \delta_t = \phi log(l_t/\mathcal{O}_t) \iff l_t = \exp(\frac{r}{\phi}) \cdot (\mathcal{O}_t \cdot \exp(-\frac{\delta_t}{\phi})).$$

Therefore l_t has to be under the latter form and satisfy its own derivative constraint. Aditionally,

$$d(\exp(-\frac{\delta_t}{\phi})) = \frac{1}{\phi}\exp(-\frac{\delta_t}{\phi})[\underbrace{(-\lambda_\delta(\bar{\delta} - \delta_t) + \frac{1}{2\phi}||\sigma_\delta||_2^2)}_{G_t} dt - \sigma_\delta^T dB_t^{\mathbb{Q}}] = \frac{1}{\phi}\exp(-\frac{\delta_t}{\phi})[G_t dt - \sigma_\delta^T dB_t^{\mathbb{Q}}].$$

Then,

$$dl_t = \exp(\frac{r}{\phi})[\frac{\mathcal{O}_t}{\phi}\exp(-\delta_t/\phi)[G_t dt - \sigma_\delta^T dB_t^{\mathbb{Q}}] + \exp(\frac{-\delta_t}{\phi})\mathcal{O}_t[(r-\delta_t)dt + \sigma_{\mathcal{O}}^T dB_t^{\mathbb{Q}}] - \frac{1}{\phi}\mathcal{O}_t\sigma_\delta^T\sigma_{\mathcal{O}}dt].$$

$$= l_t[\frac{1}{\phi}[G_t dt - \sigma_\delta^T dB_t^{\mathbb{Q}}] + (r-\delta_t)dt + \sigma_{\mathcal{O}}^T dB_t^{\mathbb{Q}} - \sigma_\delta^T\sigma_{\mathcal{O}}dt].$$

$$= l_t[(\frac{G_t}{\phi} + r - \delta_t - \frac{1}{\phi}\sigma_\delta^T\sigma_{\mathcal{O}})dt + (\sigma_{\mathcal{O}} - \frac{\sigma_\delta}{\phi})^T dB_t^{\mathbb{Q}}].$$

$$= l_t[\underbrace{(\delta_t(\frac{\lambda_\delta}{\phi} - 1) - \frac{\lambda_\delta\bar{\delta}}{\phi} + \frac{||\sigma_\delta||_2^2}{2\phi^2} - \frac{1}{\phi}\sigma_\delta^T\sigma_{\mathcal{O}})}_{\mu_l} dt + \underbrace{(\sigma_{\mathcal{O}} - \frac{\sigma_\delta}{\phi})^T}_{\sigma_l^T} dB_t^{\mathbb{Q}}] - \exp(\frac{r}{\phi})\mathcal{O}_t\sigma_\delta^T\sigma_{\mathcal{O}}dt.$$

Using the fact that μ_l has to be a constant, we obtain:

$$\begin{cases} \phi := \lambda_\delta, \\ \sigma_l := \sigma_{\mathcal{O}} - \frac{\sigma_\delta}{\lambda_\delta}, \\ \mu_l := r - \bar{\delta} + \frac{||\sigma_\delta||_2^2}{2\lambda_\delta^2} - \frac{1}{\lambda_\delta}\sigma_\delta^T\sigma_{\mathcal{O}}, \\ \sigma_o = \sigma_{\mathcal{O}}. \end{cases}$$

Now let us show its equivalent to Schwartz and Smith (2000) model, i.e.

$$log(\mathcal{O}_t) = x_t + l_t \implies \mathcal{O}_t = \exp(x_t + l_t).$$

Taking the derivative yields,

$$d(\mathcal{O}_t) = \mathcal{O}_t[dx_t + dl_t + \frac{1}{2}\sigma_x^T\sigma_l dt] = [\underbrace{(\mu_x + \lambda_l(\bar{l} - l_t) + \frac{1}{2}\sigma_x^T\sigma_l)}_{(r-\delta_t)} dt + \underbrace{(\sigma_x + \sigma_l)^T}_{\sigma_{\mathcal{O}}^T} dB_t^{\mathbb{Q}}]$$

Hence,

$$\begin{cases} \sigma_{\mathcal{O}} := \sigma_x + \sigma_l, \\ r := \mu_x + \lambda_l \bar{l} + \frac{1}{2}\sigma_x^T \sigma_l, \\ \delta_t = \lambda_l l_t. \end{cases}$$

Or the other way around using the contraint $d(\delta_t) = \lambda_l d(l_t)$ with Itô's Lemma:

$$\begin{cases} l_t := \frac{1}{\sqrt{\lambda_\delta}}\delta_t \\ \lambda_l := \lambda_\delta, \\ \bar{l} = \frac{1}{\lambda_\delta}\bar{\delta}, \\ \sigma_l = \frac{1}{\lambda_\delta}\sigma_\delta, \\ \sigma_x := \sigma_{\mathcal{O}} - \frac{1}{\lambda_\delta}\sigma_\delta, \\ \mu_x := r - \lambda_\delta \bar{\delta} - \frac{1}{2\lambda_\delta}\sigma_{\mathcal{O}}^T \sigma_\delta - \frac{1}{2\lambda_\delta^2}||\sigma_\delta||_2^2. \end{cases}$$

2.2 Bond pricing

4. We know that payoff of zero-coupon bond is $B_T(T) = 1$. Its arbitrage price is

$$B_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{S_{0t}B_t(T)}{S_{0T}}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{S_{0t}}{S_{0T}}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,ds}\right]$$

where $\mathbb{Q}$ the unique EMM. Since r_t solves an autonomous SDE under $\mathbb{Q}$, it is a $\mathbb{Q}$-markov process, which in turn implies the bond price is

$$B_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,ds}\right] = \mathcal{B}(t, r_t)$$

for some function $\mathcal{B}$ such that $e^{-\int_0^t r_s\,ds}\mathcal{B}(t, r_t)$ is a $\mathbb{Q}$-martingale and $\mathcal{B}(T, r_t) = 1$. Applying Itô's lemma we have:

$$d\left(e^{-\int_0^t r_s\,ds}\mathcal{B}(t, r_t)\right) = e^{-\int_0^t r_s\,ds}\frac{\partial \mathcal{B}}{\partial r_t}\sigma^T dB_t^{\mathbb{Q}} + e^{-\int_0^t r_s\,ds}\left(\frac{\partial \mathcal{B}}{\partial t} + \lambda(\bar{r} - r_t)\frac{\partial \mathcal{B}}{\partial r_t} + \frac{1}{2}\|\sigma\|^2\frac{\partial^2 \mathcal{B}}{\partial r_t^2} - r_t\mathcal{B}\right)dt.$$

As the process is a martingale, meaning it drift is equal to zero. This gives the following PDE:

$$r_t\mathcal{B} = \frac{\partial \mathcal{B}}{\partial t} + \lambda(\bar{r} - r_t)\frac{\partial \mathcal{B}}{\partial r} + \frac{1}{2}\|\sigma\|^2\frac{\partial^2 \mathcal{B}}{\partial r^2}$$

subject to boundary condition $\mathcal{B}(T, r) = 1$.

5. Assuming

$$B_t(T) = \mathcal{B}(t, r_t) = e^{b_0(T-t)+b_r(T-t)r_t}$$

and inserting into the PDE, we find:

$$r_t e^{b_0(T-t)+b_r(T-t)r_t} = (-b_0'(T-t) - b_r'(T-t)r_t)e^{b_0(T-t)+b_r(T-t)r_t} + \lambda(\bar{r} - r_t)b_r(T-t)e^{b_0(T-t)+b_r(T-t)r_t}$$
$$+\frac{1}{2}\|\sigma\|^2 b_r^2(T-t)e^{b_0(T-t)+b_r(T-t)r_t},$$

with terminal condition $e^{b_0(0)+b_r(0)r_t} = 1$. This gives the following equations for b_0 and b_r:

$$\begin{cases} 1 = -b_r' - \lambda b_r, \\ 0 = -b_0' + \lambda\bar{r}b_r + 1/2\|\sigma\|^2 b_r^2. \end{cases}$$

with the terminal conditions $b_0(0) = 0$ and $b_r(0) = 0$.

6. Solving the PDE for b_r we find:

$$b_r(T-t) = 1/\lambda((e^{-\lambda(T-t))} - 1)).$$

Now, for b_0 we use Mathematica to solve the following integral:

$$b_0(T-t) = \int(\lambda\bar{r}b_r(T-t) + 1/2\|\sigma\|^2 b_r^2(T-t))d(T-t).$$

This gives us:

$$b_0(T-t) = \int \bar{r}\left(e^{-\lambda(T-t)} - 1\right) + \frac{1}{2}\|\sigma\|^2 \frac{1}{\lambda^2}\left(e^{-\lambda(T-t)} - 1\right)^2 d(T-t)$$
$$= \bar{r}\left(\frac{\exp(-\lambda(T-t)) - 1}{\lambda} + (T-t)\right) - \frac{\|\sigma\|^2}{4\lambda^3}\exp(-2\lambda(T-t)) + \frac{\|\sigma\|^2}{\lambda^3}\exp(-\lambda(T-t)) + \frac{\|\sigma\|^2}{2\lambda^2}(T-t) + C$$

Using, $b_0(0) = 0$, we find: $C = \frac{3\|\sigma\|^2}{4\lambda^3}$.

2.3 Futures pricing

7. The relationship between the spot price $\mathcal{O}_T$ and the future price $f_t := f_t(T)$ at date $t \leq T$ for delivery at date T is given by,

$$f_t(T) = E_t^{\mathbb{Q}}[\mathcal{O}_T]$$

As $(r_t, \delta_t, \mathcal{O}_t)$ is the solution of a one dimensional SDE under $\mathbb{Q}$ it is a Markov Process. Therefore,

$$f_t(T) \equiv f(t, r_t, \delta_t, \mathcal{O}_t) \quad \text{with} \quad f(T, r_T, \delta_T, \mathcal{O}_T) = \mathcal{O}_T,$$

and f is a solution to the following PDE:

$$\text{Drift}^{\mathbb{Q}}(d(f(t, r_t, \delta_t, \mathcal{O}_t))) = 0$$

Where we used that the future price is a martingale under the risk neutral measure, i.e. its drift is zero.

$$\begin{aligned} d(f(t, r_t, \delta_t, \mathcal{O}_t)) = &\frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \delta}d\delta + \frac{\partial f}{\partial \mathcal{O}}d\mathcal{O} \\ &+ \frac{1}{2}\left[\frac{\partial^2 f}{\partial r\, \partial r}||\sigma_r||_2^2 + \frac{\partial^2 f}{\partial \mathcal{O}\, \partial \mathcal{O}}\mathcal{O}_t^2||\sigma_{\mathcal{O}}||_2^2 + \frac{\partial^2 f}{\partial \delta\, \partial \delta}||\sigma_\delta||_2^2 \right. \\ &\left. + \frac{\partial^2 f}{\partial r\, \partial \mathcal{O}}\rho_{r\mathcal{O}}\mathcal{O}_t + \frac{\partial^2 f}{\partial r\, \partial \delta}\rho_{r\delta} + \frac{\partial^2 f}{\partial \mathcal{O}\, \partial \delta}\rho_{\mathcal{O}\delta}\mathcal{O}_t\right] dt. \end{aligned}$$

Explicitly the PDE is,

$$\begin{aligned} &\frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial r}\lambda_r(\bar{r} - r_t) + \frac{\partial f}{\partial \delta}\lambda_\delta(\bar{\delta} - \delta_t) + \frac{\partial f}{\partial \mathcal{O}}(r_t - \delta_t)\mathcal{O}_t \\ &+ \frac{1}{2}\left[\frac{\partial^2 f}{\partial r\, \partial r}||\sigma_r||_2^2 + \frac{\partial^2 f}{\partial \mathcal{O}\, \partial \mathcal{O}}\mathcal{O}_t^2||\sigma_{\mathcal{O}}||_2^2 + \frac{\partial^2 f}{\partial \delta\, \partial \delta}||\sigma_\delta||_2^2 \right. \\ &\left. + \frac{\partial^2 f}{\partial r\, \partial \mathcal{O}}\rho_{r\mathcal{O}}\mathcal{O}_t + \frac{\partial^2 f}{\partial r\, \partial \delta}\rho_{r\delta} + \frac{\partial^2 f}{\partial \mathcal{O}\, \partial \delta}\rho_{\mathcal{O}\delta}\mathcal{O}_t\right] = 0. \end{aligned}$$

8. To solve the PDE we will use a separation of variables argument, more precisely we assume the following,

$$f_t = \exp(\phi_0(T-t) + \phi_r(T-t)r_t + \phi_\delta(T-t)\delta_t)\mathcal{O}_t.$$

Let $\tau := (T-t)$, then substituting f_t in the PDE yields,

$$\begin{aligned} &f_t\left[-\phi_0'(\tau) - \phi_r'(\tau)r_t - \phi_\delta'(\tau)\delta_t + \phi_r(\tau)(\bar{r} - r_t) + \phi_\delta(\tau)(\bar{\delta} - \delta) + (r_t - \delta_t)\right] \\ &\quad + \underbrace{\frac{1}{2}\left[\phi_r^2(\tau)||\sigma_r||_2^2 + \phi_\delta^2(\tau)||\sigma_\delta||_2^2 + \phi_r(\tau)\rho_{r\mathcal{O}} + \phi_r(\tau)\phi_\delta(\tau)\rho_{r\delta} + \rho_{\mathcal{O}\delta}\phi_\delta(\tau)\right]}_{C} = 0. \end{aligned}$$

After rewriting we obtain,

$$f_t[\underbrace{r_t(-\phi_r'(\tau) - \phi_r(\tau)\lambda_r + 1)}_{1} + \underbrace{\delta_t(-\phi_\delta'(\tau) - \phi_\delta(\tau)\lambda_\delta - 1)}_{2} + \underbrace{(-\phi_0'(\tau) + \phi_r(\tau)\lambda_r\bar{r} + \phi_\delta(\tau)\lambda_\delta\bar{\delta} + C)}_{3}] = 0.$$

Since this equation has to hold for all $r_t, \delta_t, \mathcal{O}_t$, we are tasked to solve a system of ODE for $\phi_0, \phi_r, \phi_\delta$:

$$\begin{cases} \phi_r'(\tau) = 1 - \lambda_r \phi_r(\tau) \\ \phi_\delta'(\tau) = -1 - \lambda_\delta \phi_\delta(\tau) \\ \phi_0'(\tau) = \phi_r(\tau)\lambda_r \bar{r} + \phi_\delta(\tau)\lambda_\delta \bar{\delta} + \underbrace{\frac{1}{2}\left[\phi_r^2(\tau)||\sigma_r||_2^2 + \phi_\delta^2(\tau)||\sigma_\delta||_2^2 + \phi_r(\tau)\rho_{r\mathcal{O}} + \phi_r(\tau)\phi_\delta(\tau)\rho_{r\delta} + \rho_{\mathcal{O}\delta}\phi_\delta(\tau)\right]}_{C} \end{cases}.$$

As $f_T = \mathcal{O}_T$, we obtain $\phi_0(0) = \phi_r(0) = \phi_\delta(0) = 0$.

9. Therefore,

$$\begin{cases} \phi_r(\tau) = \frac{1-\exp(-\lambda_r \tau)}{\lambda_r}, \\ \phi_\delta(\tau) = \frac{-1+\exp(-\lambda_\delta \tau)}{\lambda_\delta} \\ \phi_0(\tau) = \int_0^\tau \left[\phi_r(s)\lambda_r \bar{r} + \phi_\delta(s)\lambda_\delta \bar{\delta} + \frac{1}{2}\left[\phi_r^2(s)||\sigma_r||_2^2 + \phi_\delta^2(s)||\sigma_\delta||_2^2 + \phi_r(s)\rho_{r\mathcal{O}} + \phi_r(s)\phi_\delta(s)\rho_{r\delta} + \rho_{\mathcal{O}\delta}\phi_\delta(s)\right]\right] ds. \end{cases}.$$

To describe the evolution of the future price under $\mathbb{Q}$, we only need to consider the stochastic part (as it is a martingale under $\mathbb{Q}$). We obtain,

$$\begin{aligned} d(f_t) &= \frac{\partial f}{\partial r}\sigma_r^T dB_t^{\mathbb{Q}} + \frac{\partial f}{\partial \delta}\sigma_\delta^T dB_t^{\mathbb{Q}} + \frac{\partial f}{\partial \mathcal{O}}\sigma_{\mathcal{O}}^T \mathcal{O}_t dB_t^{\mathbb{Q}} \\ &= [\frac{\partial f}{\partial r}\sigma_r + \frac{\partial f}{\partial \delta}\sigma_\delta + \frac{\partial f}{\partial \mathcal{O}}\sigma_{\mathcal{O}}\mathcal{O}_t]^T dB_t^{\mathbb{Q}} \\ &= f_t[\phi_r(\tau)\sigma_r + \phi_\delta(\tau)\sigma_\delta + \sigma_{\mathcal{O}}]^T dB_t^{\mathbb{Q}}. \end{aligned}$$

10. Transitions between periods of backwardation and contango can happen for multiple reasons. Here are a few ones.

- Storage costs: falling storage costs can shift the market from contango to backwardation.
- Changes in supply and demand: shortages in supply due to unexpected events lead to higher spot prices, potentially causing backwardation.
- Changes in interest rates: the cost of carry is influenced by the interest rate, both increasing or decreasing depending on central bank decisions. This can cause transitions between contango and backwardation.
- Inventory levels: large inventory level changes can impact the current state, with low inventory levels leading potentially to higher spot prices and backwardation.

We can see that assuming constant interest rate and convenience yield would be unrealistic here as many factors influence then through time before the futures contract comes to maturity. Furthermore, not modeling correctly shifts between contango and backwardation would fail to capture the true dynamics of the futures market, potentially leading to inaccurate pricing and creating arbitrage opportunities for others as the same futures could be trading both above and under spot price.

Therefore, to correctly model futures contracts, we require stochastic interest rates and convenience yields.

11. Let us assume $\lambda_r = ||\sigma_r|| = 0$. Then $dr_t = 0$, which implies $r_t \equiv r$, $r \in \mathbb{R}$. In particular,

$$\phi_r(\tau) = \lim_{\lambda_r \to 0} \frac{1-\exp(-\lambda_r \tau)}{\lambda_r} \overset{B.H}{=} \tau.$$

The implicit convenience yield is defined as $f_t^i(T) := \exp((r - Y_t(T))\tau)\mathcal{O}_t$. Its relationship with the instantaneous convenience yield is defined as,

$$f_t^i(T) = f_t \iff (r - Y_t(T))\tau = \phi_0(\tau) + \tau r + \phi_\delta(\tau)\delta_t \iff Y_t(T) = \frac{-1}{\tau}(\phi_0(\tau) + \phi_\delta(\tau)\delta_t)$$

Therefore,

$$Y_t(T) = \delta_t \iff \phi_0(\tau) + \delta_t(\tau + \phi_\delta(\tau)) = 0.$$

Assuming $Y_t(T) = \delta_t$, then $\phi_0(\tau) = -\delta_t(\tau + \phi_\delta(\tau))$. We get,

$$f_t = \exp(\phi_0(T-t) + \phi_r(T-t)r_t + \phi_\delta(T-t)\delta_t)\mathcal{O}_t.$$

$$= \exp(\tau(r-\delta_t)).$$

2.4 Storage options

12. From question 2, we know that

$$\mathcal{O}_{T_0} = \mathbb{E}^{\mathbb{Q}}_{T_0}\left(\int_{T_0}^{T_1} \exp\left\{-\int_{T_0}^{s} r_u \mathrm{d}u\right\}\delta_s\mathcal{O}_s + \exp\left\{-\int_{T_0}^{T_1} r_u\mathrm{d}u\right\}O_{T_1}\right)$$

Therefore,

$$\begin{aligned}\mathcal{P}_{T_0} &= \mathbb{E}^{\mathbb{Q}}_{T_0}\left(\exp\left\{-\int_{T_0}^{T_1} r_u\mathrm{d}u\right\}O_{T_1}\right) - O_{T_0} - \mathbb{E}^{\mathbb{Q}}_{T_0}\left(\int_{T_0}^{T_1}\exp\left\{-\int_{T_0}^{s} r_u\mathrm{d}u\right\}\xi_s\mathrm{d}s\right)\\ &= -\mathbb{E}^{\mathbb{Q}}_{T_0}\left(\int_{T_0}^{T_1}\exp\left\{-\int_{T_0}^{s} r_u\mathrm{d}u\right\}(\xi_s+\delta_s\mathcal{O}_s)\mathrm{d}s\right).\end{aligned}$$

This then gives

$$\mathcal{P}^+_{T_0} = \max\{0,\mathcal{P}_{T_0}\} = \max\left\{0, -\mathbb{E}^{\mathbb{Q}}_{T_0}\left(\int_{T_0}^{T_1}\exp\left\{-\int_{T_0}^{s} r_u\mathrm{d}u\right\}(\xi_s+\delta_s\mathcal{O}_s)\d{s}\right)\right\}$$

Given the evolution of the spot price $\mathcal{O}_t$, $\exp\left\{-\int_{T_0}^{s} r_u\mathrm{d}u\right\}\delta_s\mathcal{O}_s$ is the expected infinitesimal change of the discounted sport price, and $\exp\left\{-\int_{T_0}^{s} r_u\mathrm{d}u\right\}\xi_s$ is the discounted infinitesimal storage cost. This means that $\int_{T_0}^{T_1}\exp\left\{-\int_{T_0}^{s} r_u\mathrm{d}u\right\}(\xi_s+\delta_s\mathcal{O}_s)\d{s}$ is the cost of storing oil from T_0 to T_1 with the specified flow cost.
So the relation we derived means that the payoff at time T_0 is the expected cost of storing oil from T_0 to T_1 with the specified flow cost.

13. Denote $Z_t = \mathcal{O}_t\exp\left\{-\int_0^t r_u - \delta_u\mathrm{d}u\right\}$.
Then we have the desired property that

$$\exp\left\{-\int_{T_0}^{T_1} r_u\mathrm{d}u\right\}\mathcal{O}_{T_1} = \exp\left\{-\int_{T_0}^{T_1}\delta_u\mathrm{d}u\right\}\mathcal{O}_{T_0}\frac{Z_{T_1}}{Z_{T_0}}$$

with (Z_t) positive and a $\mathbb{Q}$-martingale.
Indeed, $\mathcal{O}_t = \mathcal{O}_0 e^{\|\sigma_O\|W_t^{\mathcal{O}}+\int_0^t r_u-\delta_u\mathrm{d}u-\frac{\|\sigma_O\|^2}{2}}$ which in turn implies that $Z_t = \mathcal{O}_0 e^{\|\sigma_O\|W_t^{\mathcal{O}}-\frac{\|\sigma_O\|^2}{2}t} = \mathcal{O}_0 e^{\int_0^t\sigma_O^T dB_s^{\mathcal{Q}}-\frac{1}{2}\int_0^t\|\sigma_O\|^2\mathrm{d}s}$ which is the exponential martingale associated to the BM $-W_t^{\mathcal{O}}$ for $Y_s=\sigma_{\mathcal{O}}$. Since Y_s is bounded, Novikov's condition is verified, and the process is a martingale.

Then, we define for any measurable set A the measure $\bar{\mathbb{Q}}(A) = \mathbb{E}^{\mathbb{Q}}\left(\mathbb{1}_A\frac{Z_{T_1}}{Z_{T_0}}\right)$, which we know from the lectures is a probability measure.

Then, by the generalized Bayes' rule,

$$\mathbb{E}^{\mathbb{Q}}_{T_0}\left(\exp\left\{-\int_{T_0}^{T_1} r_u\mathrm{d}u\right\}\mathcal{O}_{T_1}\right) = \mathbb{E}^{\mathbb{Q}}_{T_0}\left(\exp\left\{-\int_{T_0}^{T_1}\delta_u\mathrm{d}u\right\}\mathcal{O}_{T_0}\frac{Z_{T_1}}{Z_{T_0}}\right) = \mathcal{O}_{T_0}\mathbb{E}^{\bar{\mathbb{Q}}}_{T_0}\left(\exp\left\{-\int_{T_0}^{T_1}\delta_u\mathrm{d}u\right\}\right)$$

Moreover, by Girsanov's theorem, we know that the process $B_t^{\bar{\mathbb{Q}}} := -B_t^{\mathbb{Q}} + t\sigma_{\mathcal{O}}$ is a $\bar{\mathbb{Q}}$ standard brownian motion. Therefore, δ_t now has the dynamic:

$$\mathrm{d}\delta_t = \lambda_\delta(\bar\delta-\delta_t)\mathrm{d}t + \sigma_\delta^T\mathrm{d}B_t^{\mathbb{Q}} = \lambda_\delta(\bar\delta + \frac{\sigma_{\mathcal{O}}^T\sigma_\delta}{\lambda_\delta} - \delta_t)\mathrm{d}t - \sigma_\delta^T\mathrm{d}B_t^{\bar{\mathbb{Q}}} = \lambda_\delta(\bar\delta + \frac{\sigma_{\mathcal{O}}^T\sigma_\delta}{\lambda_\delta} - \delta_t)\mathrm{d}t + (-\|\sigma_\delta\|\mathrm{d}W_t^{\bar{\mathbb{Q}}}),$$

where $W_t^{\bar{\mathbb{Q}}}$ is a one-dimensional BM that we construct like in question 1. Notice we have a similar dynamic, simply changed constants.

14. We want to compute the conditional expectation $\mathbb{E}^{\bar{\mathbb{Q}}}_{T_0}(\exp\left\{-\int_{T_0}^{T_1}\delta_u \mathrm{d}u\right\})$ where δ_t is a stochastic process that satisfies

$$\mathrm{d}\delta_t = \lambda_\delta(\bar{\delta} + \frac{\sigma_{\mathcal{O}}^T\sigma_\delta}{\lambda_\delta} - \delta_t)\mathrm{d}t + (-\|\sigma_\delta\|\mathrm{d}W_t^{\bar{\mathbb{Q}}}).$$

This is exactly the same setup as the bond pricing questions but for different constants.

This means we can repeat the reasoning of questions 5. and 6. to get that

$$\mathbb{E}^{\bar{\mathbb{Q}}}_{T_0}(\exp\left\{-\int_{T_0}^{T_1}\delta_u \mathrm{d}u\right\}) = \exp\left\{\psi_0(\Delta) + \psi_\delta(\Delta)\delta_{T_0}\right\}$$

for two functions

$$\begin{aligned}
\psi_0(T-t) &= (\bar{\delta} + \frac{\sigma_{\mathcal{O}}^T\sigma_\delta}{\lambda_\delta})\left(\frac{\exp(-\lambda_\delta(T-t))-1}{\lambda_\delta} + (T-t)\right) - \frac{\|\sigma_\delta\|^2}{4\lambda_\delta^3}\exp(-2\lambda_\delta(T-t)) \\
&\quad + \frac{\|\sigma_\delta\|^2}{\lambda_\delta^3}\exp(-\lambda_\delta(T-t)) + \frac{\|\sigma_\delta\|^2}{2\lambda_\delta^2}(T-t) + \frac{3\|\sigma_\delta\|^2}{4\lambda_\delta^3} \\
\psi_\delta(T-t) &= \frac{1}{\lambda_\delta}(e^{-\lambda_\delta(T-t)} - 1)
\end{aligned}$$

Notice that ψ_δ is negative when $T-t>0$ and decreasing.

15. By the generalized Bayes' rule, $\mathbb{E}^{\mathbb{Q}}_{T_0}\left(\exp\left\{-\int_{T_0}^{s} r_u \mathrm{d}u\right\}\mathcal{O}_s\right) = \mathcal{O}_{T_0}\mathbb{E}^{\bar{\mathbb{Q}}}_{T_0}\left(\exp\left\{-\int_{T_0}^{s}\delta_u \mathrm{d}u\right\}\right)$, so

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}_{T_0}\left(\int_{T_0}^{T_1}\exp\left\{-\int_{T_0}^{s} r_u \mathrm{d}u\right\}\xi_s \mathrm{d}s\right) &= \alpha\int_{T_0}^{T_1}\mathbb{E}^{\mathbb{Q}}_{T_0}\left(\exp\left\{-\int_{T_0}^{s} r_u \mathrm{d}u\right\}\mathcal{O}_s\right)\mathrm{d}s \\
&= \alpha\int_{T_0}^{T_1}\mathcal{O}_{T_0}\mathbb{E}^{\bar{\mathbb{Q}}}_{T_0}\left(\exp\left\{-\int_{T_0}^{s}\delta_u \mathrm{d}u\right\}\right)\mathrm{d}s \\
&= \alpha\int_{T_0}^{T_1}\mathcal{O}_{T_0}\exp\left\{\psi_0(s-T_0) + \psi_\delta(s-T_0)\delta_{T_0}\right\}\mathrm{d}s \\
&= \alpha\mathcal{O}_{T_0}\int_0^{\Delta}\exp\left\{\psi_0(\tau) + \psi_\delta(\tau)\delta_{T_0}\right\}\mathrm{d}\tau
\end{aligned}$$

From this, it follows

$$\begin{aligned}
\mathcal{P}_{T_0} &= \mathbb{E}^{\mathbb{Q}}_{T_0}\left(\exp\left\{-\int_{T_0}^{T_1} r_u \mathrm{d}u\right\}O_{T_1}\right) - O_{T_0} - \mathbb{E}^{\mathbb{Q}}_{T_0}\left(\int_{T_0}^{T_1}\exp\left\{-\int_{T_0}^{s} r_u \mathrm{d}u\right\}\xi_s \mathrm{d}s\right) \\
&= \left[\exp\left\{\psi_0(\Delta) + \psi_\delta(\Delta)\delta_{T_0}\right\} - 1 - \alpha\int_0^{\Delta}\exp\left\{\psi_0(\tau) + \psi_\delta(\tau)\delta_{T_0}\right\}\mathrm{d}\tau\right]\mathcal{O}_{T_0} \\
&= H(\delta_{T_0})\mathcal{O}_{T_0}
\end{aligned}$$

where we denote $H(\delta_{T_0}) = \exp\left\{\psi_0(\Delta) + \psi_\delta(\Delta)\delta_t\right\} - 1 - \alpha\int_0^{\Delta}\exp\left\{\psi_0(\tau) + \psi_\delta(\tau)\delta_{T_0}\right\}\mathrm{d}\tau$
Then, by the generalized Bayes' rule,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}_t\left(\exp\{-\int_t^{T_0} r_u \mathrm{d}u\}\mathcal{P}^+_{T_0}\right) &= \mathbb{E}^{\mathbb{Q}}_t\left(\frac{Z_t}{Z_{T_0}}\frac{Z_{T_0}}{Z_t}\exp\left\{-\int_t^{T_0} r_u \mathrm{d}u\right\}\mathcal{P}^+_{T_0}\right) \\
&= \mathbb{E}^{\bar{\mathbb{Q}}}_t\left(\frac{\mathcal{O}_t}{\mathcal{O}_{T_0}}\exp\left\{-\int_t^{T_0}\delta_u \mathrm{d}u\right\}\mathcal{P}^+_{T_0}\right) \\
&= \mathcal{O}_t\mathbb{E}^{\bar{\mathbb{Q}}}_t\left(\exp\left\{-\int_t^{T_0}\delta_u \mathrm{d}u\right\}H^+(\delta_{T_0})\right)
\end{aligned}$$

In this specification, the storage cost is proportional to the spot price, so a priori the price of the option depends on the risk free rate. However to compute the price we evaluate the expected discounted storage cost, whose evolution does not depend on the risk free rate. This is why the final price does not depend on the risk free rate.
16. Since ψ_δ is decreasing,

$$\begin{aligned} H'(\delta) &= \psi_\delta(\Delta)\exp\{\psi_0(\Delta)+\psi_\delta(\Delta)\delta\} - \alpha\int_0^\Delta \psi_\delta(\tau)\exp\{\psi_0(\tau)+\psi_\delta(\tau)\delta\} \\ &\leq \psi_\delta(\Delta)\exp\{\psi_0(\Delta)+\psi_\delta(\Delta)\delta\} - \alpha\int_0^\Delta \psi_\delta(\Delta)\exp\{\psi_0(\tau)+\psi_\delta(\tau)\delta\} \\ &\leq \psi_\delta(\Delta)(H(\delta)+1) < \psi_\delta(\Delta)H(\delta) \end{aligned}$$

since $\psi_\delta(\Delta) < 0$.

By the Grönwall lemma, we deduce that for any $\delta > \delta_0$,

$$H(\delta) \leq H(\delta_0)\exp\{\psi_\delta(\Delta)(\delta-\delta_0)\} < H(\delta_0)$$

which means H is a decreasing function. Therefore, the set where H is strictly positive is necessarily of the form $(-\infty, \delta^*)$.

Using Lemma 1, and we know the random variable $\left(\delta_{T_0}, \int_t^{T_0}\delta_s \mathrm{d}s\right)$ conditional on δ_t is normal.
For $a = \lambda_\delta, b = \overline{\delta} + \frac{\sigma_{\mathcal{O}}^T\sigma_\delta}{\lambda_\delta}, \sigma = -\|\sigma_\delta\|, \tau = T_0 - t$:

$$m(\delta_t) = \begin{pmatrix} e^{-a\tau}\delta_t + (1-e^{-a\tau})b \\ \frac{b\tau + (1-e^{-a\tau})(\delta_t - b)}{a} \end{pmatrix}$$

and covariance matrix

$$V = \begin{pmatrix} \frac{\sigma^2(1-e^{-2a\tau})}{2a} & \frac{\sigma^2(1-e^{-a\tau})^2}{2a^2} \\ \frac{\sigma^2(1-e^{-a\tau})^2}{2a^2} & \frac{\sigma^2 e^{-2a\tau}(e^{2a\tau}(2a\tau-3)+4e^{a\tau}-1)}{2a^3} \end{pmatrix}$$

Using question 15., we get the formula for the price:

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}}\left(\exp\{-\int_t^{T_0} r_u \mathrm{d}u\}\mathcal{P}_{T_0}^+\right) &= \mathcal{O}_t\mathbb{E}_t^{\bar{\mathbb{Q}}}\left(\exp\left\{-\int_t^{T_0}\delta_u\mathrm{d}u\right\}H^+(\delta_{T_0})\right) \\ &= \mathcal{O}_t\int_{\mathbb{R}^2}\exp\{-y\}H^+(x)\frac{1}{2\pi\sqrt{\det V}}\exp\left\{-\frac{1}{2}(\begin{pmatrix}x & y\end{pmatrix} - m(\delta_t)^T)V^{-1}(\begin{pmatrix}x \\ y\end{pmatrix} - m(\delta_t)\right\}\mathrm{d}x\mathrm{d}y \\ &= \mathcal{O}_t\int_{\mathbb{R}}\int_{-\infty}^{\delta^*}\exp\{-y\}H(x)\frac{1}{2\pi\sqrt{\det V}}\exp\left\{-\frac{1}{2}(\begin{pmatrix}x & y\end{pmatrix} - m(\delta_t)^T)V^{-1}(\begin{pmatrix}x \\ y\end{pmatrix} - m(\delta_t)\right\}\mathrm{d}x\mathrm{d}y \end{aligned}$$

17. In this case,

$$\begin{aligned} \mathcal{P}_{T_0} &= \mathbb{E}_{T_0}^{\mathbb{Q}}\left(\exp\left\{-\int_{T_0}^{T_1} r_u\mathrm{d}u\right\}O_{T_1}\right) - O_{T_0} - \mathbb{E}_{T_0}^{\mathbb{Q}}\left(\int_{T_0}^{T_1}\exp\left\{-\int_{T_0}^{s} r_u\mathrm{d}u\right\}\xi_s\mathrm{d}s\right) \\ &= \mathbb{E}_{T_0}^{\mathbb{Q}}\left(\exp\left\{-\int_{T_0}^{T_1} r_u\mathrm{d}u\right\}O_{T_1}\right) - O_{T_0} - \mathbb{E}_{T_0}^{\mathbb{Q}}\left(\int_{T_0}^{T_1}\exp\{-r(s-T_0)\}\alpha\mathcal{O}_{T_0}\mathrm{d}s\right) \\ &= \left[\exp\{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_{T_0}\} - 1 - \alpha\int_0^\Delta\exp\{-r\tau\}\mathrm{d}\tau\right]\mathcal{O}_{T_0} \\ &= \left[\exp\{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_{T_0}\} - 1 + \frac{\alpha}{r}(\exp\{-r\Delta\}-1)\right]\mathcal{O}_{T_0} \\ &= \bar{H}(\delta_0)\mathcal{O}_{T_0} \end{aligned}$$

where we denote $\bar{H}(\delta_{T_0}) = \exp\{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_{T_0}\} - 1 + \frac{\alpha}{r}(\exp\{-r\Delta\}-1)$. It is a decreasing function since $\psi_\delta(\Delta) < 0$.

From here, we have a formula similar to question 15, so following the same reasoning, we get

$$\mathbb{E}_t^{\mathbb{Q}}\left(\exp\{-\int_t^{T_0} r_u \mathrm{d}u\}\mathcal{P}_{T_0}^{+}\right) = \mathcal{O}_t\mathbb{E}_t^{\bar{\mathbb{Q}}}\left(\exp\left\{-\int_t^{T_0}\delta_u \mathrm{d}u\right\}\bar{H}^{+}(\delta_{T_0})\right)$$

Same as in question 16., using Lemma 1, we know the random variable $\left(\delta_{T_0}, \int_t^{T_0}\delta_s \mathrm{d}s\right)$ conditional on δ_t is normal.
For $a = \lambda_\delta, b = \overline{\delta} + \frac{\sigma_{\mathcal{O}}^T\sigma_\delta}{\lambda_\delta}, \sigma = -\|\sigma_\delta\|, \tau = T_0 - t$:

$$m(\delta_t) = \begin{pmatrix} e^{-a\tau}\delta_t + (1-e^{-a\tau})b \\ \frac{b\tau + (1-e^{-a\tau})(\delta_t - b)}{a}\end{pmatrix}$$

and covariance matrix

$$V = \begin{pmatrix} \frac{\sigma^2(1-e^{-2a\tau})}{2a} & \frac{\sigma^2(1-e^{-a\tau})^2}{2a^2} \\ \frac{\sigma^2(1-e^{-a\tau})^2}{2a^2} & \frac{\sigma^2 e^{-2a\tau}(e^{2a\tau}(2a\tau-3)+4e^{a\tau}-1)}{2a^3}\end{pmatrix}$$

Notice that

$$\begin{aligned}
&\overline{H}(\delta_{T_0}) \geq 0 \\
\iff &\exp\{\psi_0(\Delta) + \psi_\delta(\Delta)\delta_{T_0}\} - 1 + \frac{\alpha}{r}(\exp\{-r\Delta\} - 1) \geq 0 \\
\iff &\delta_{T_0} \leq \frac{\log(1 + \frac{\alpha}{r} - \frac{\alpha}{r}\exp\{-r\Delta\}) - \psi_0(\Delta)}{\psi_\delta(\Delta)} =: \delta^*
\end{aligned}$$

We get the formula for the price:

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}\left(\exp\{-\int_t^{T_0} r_u \mathrm{d}u\}\mathcal{P}_{T_0}^{+}\right) &= \mathcal{O}_t\mathbb{E}_t^{\bar{\mathbb{Q}}}\left(\exp\left\{-\int_t^{T_0}\delta_u \mathrm{d}u\right\}\overline{H}^{+}(\delta_{T_0})\right) \\
&= \mathcal{O}_t\mathbb{E}_t^{\bar{\mathbb{Q}}}\left(\exp\left\{-\int_t^{T_0}\delta_u \mathrm{d}u\right\}\overline{H}(\delta_{T_0})\mathbf{1}_{\delta_{T_0}\leq\delta^*}\right) \\
&= \mathcal{O}_t\mathbb{E}\left(\exp\{-Y\}\overline{H}(X)\mathbf{1}_{X\leq\delta^*}\right)
\end{aligned}$$

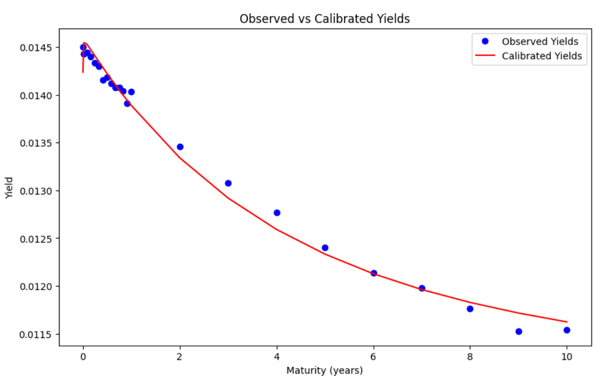
where $(X, Y) \sim \mathcal{N}(m(\delta_t), V)$.
Using mathematica, we then get the following closed form solution:

$$\begin{aligned}
&\mathcal{O}_t\frac{1}{r\,\mathrm{erfc}\left(\frac{m(\delta_t)_1-\delta^*}{\sqrt{2}\sqrt{V_{11}}}\right)}\times \\
&\quad e^{\frac{1}{2}(V_{22}-2(m(\delta_t)_2+\Delta r))}\left(re^{\psi_\delta(\Delta)(m(\delta_t)_1-V_{12})+\Delta r+\frac{1}{2}V_{11}\psi_\delta(\Delta)^2+\psi_0(\Delta)}\left(\mathrm{erf}\left(\frac{\delta^*-m(\delta_t)_1-V_{11}\psi_\delta(\Delta)+V_{12}}{\sqrt{2}\sqrt{V_{11}}}\right)+1\right)\right. \\
&\quad \left.+\left((\alpha+r)e^{\Delta r}-\alpha\right)\left(\mathrm{erfc}\left(\frac{\delta^*-m(\delta_t)_1+V_{12}}{\sqrt{2}\sqrt{V_{11}}}\right)-2\right)\right)
\end{aligned}$$

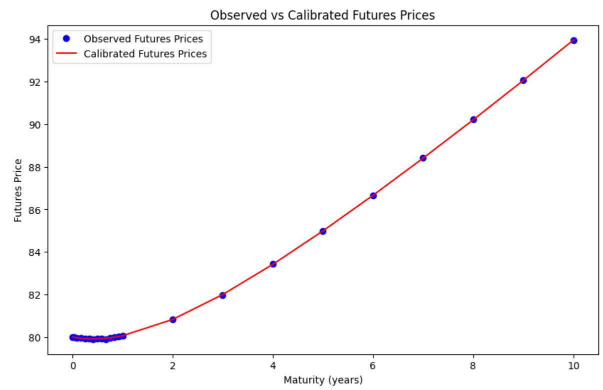
With a stochastic risk free rate it would be much harder to solve. Indeed, in this specification the storage cost is proportional to the sport price at time T_0. This means that when discounting, we still end up with a quantity that depends on the risk free rate, which would prevent us from using the same change of measure trick we used in this question and in question 15.

3 Calibration and implementation

For questions 18 (and 19), we calibrate the parameters by minimizing the Mean Squared Error (MSE) of the observed yields (futures prices) with the calibrated yields (futures prices).
Afterwards, we verify the accuracy of the calibration by computing several metrics : SSE, RMSE, MAE and R^2.
18. 19. Here are the results of our calibrations :



(a) Observed vs Calibrated Yields



(b) Observed vs Calibrated Futures

Figure 3. Comparison of observed and calibrated data

In table 2 you can see the calibrated parameters:

Parameter	**Value**
λ_r	0.42486434416120067
$\bar{r}$	−0.010721343143568865
r_0	0.014612546606659636
λ_δ	0.5054689110381777
$\bar{\delta}$	−0.030614394753573436
δ_0	0.02004131401304637

Table 2. Calibrated Parameters

Finally, we computed the accuracy of calibration using R^2:

- For question 18, we have: $R^2 = 0.98899$.
- For question 19, we have: $R^2 = 0.99998$.

Making us confident about our calibrated parameters. For complete details including other accuracy metrics, see the Jupyter Notebook named calibrations.ipynb.

20. We implement the formulas proven in the questions about storage options. For the storage option price, instead of computing the integral, we used a monte-carlo approach because of numerical stability issues (the covariance matrix is badly scaled for some parameter values). For complete details, see the notebook monte-carlo.ipynb.

21. To study the storage dependence of the option price, we will look at how the price varies on the parameters. As for the exercise threshold, we know that the sign of the payoff is the sign of $H(\delta_{T_0})$, which is a decreasing function which starts positive and becomes negative from $\delta^\star$. Therefore we will plot this critical delta for varying parameters.

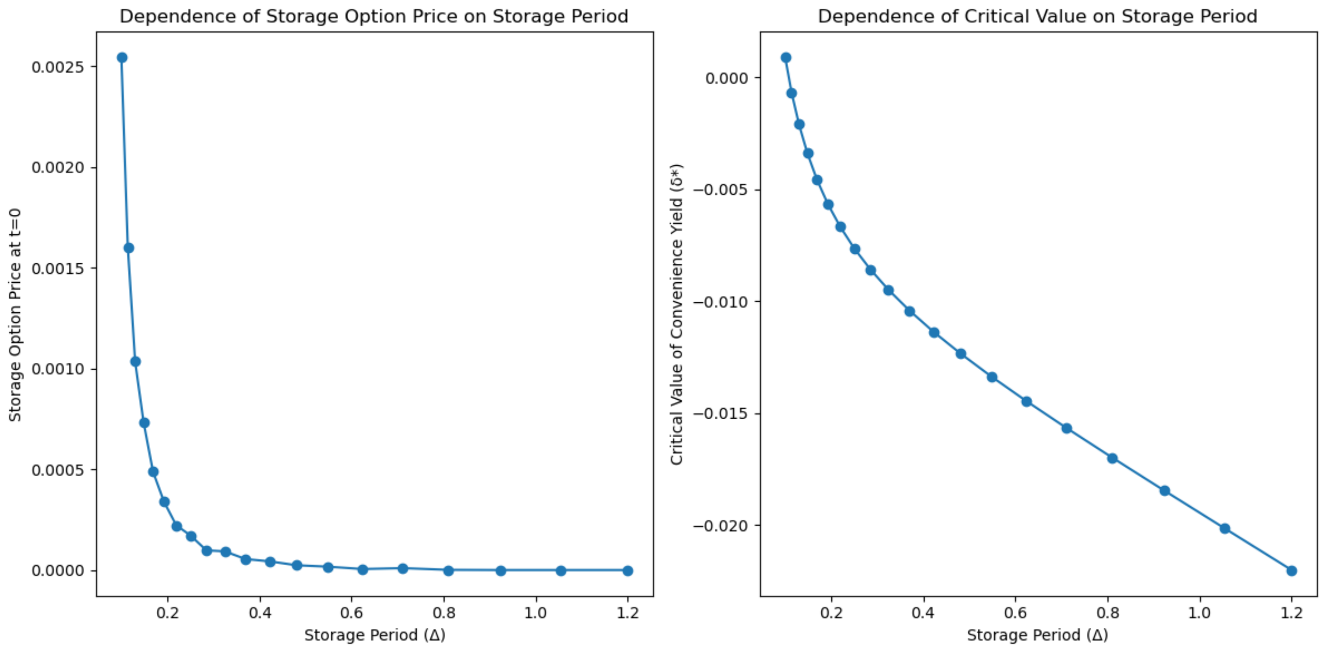


Figure 4. Dependence on the storage period.

Looking at figure 4, we see that both option price and critical delta are decreasing in the storage period. This is logical since a longer storage period means more storage cost. This is also confirmed by the formula, for example the component $\psi_\delta(\Delta)$ is decreasing which leads to a lower option price.

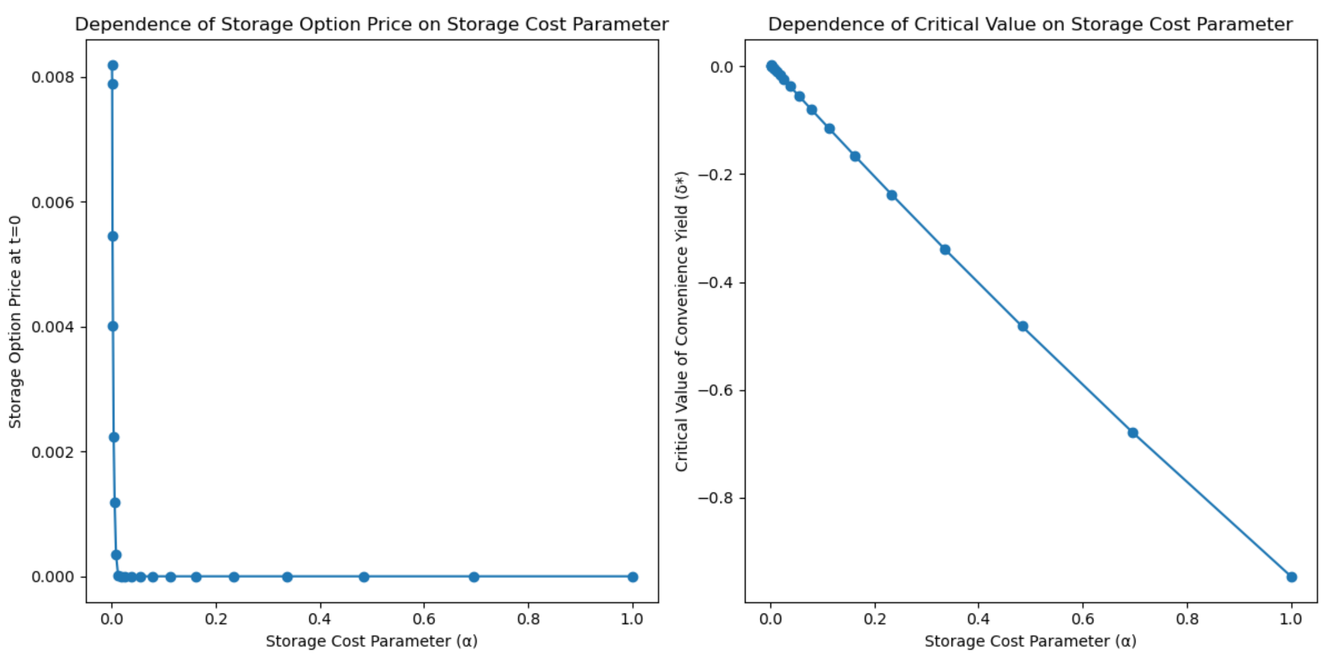


Figure 5. Dependence on alpha.

Looking at figure 5 we see that both option price and critical delta are decreasing in α. This is logical since a higher alpha directly implies higher storage costs. The linear decrease of the critical delta can be confirmed from the formula since the function H is linear in α.

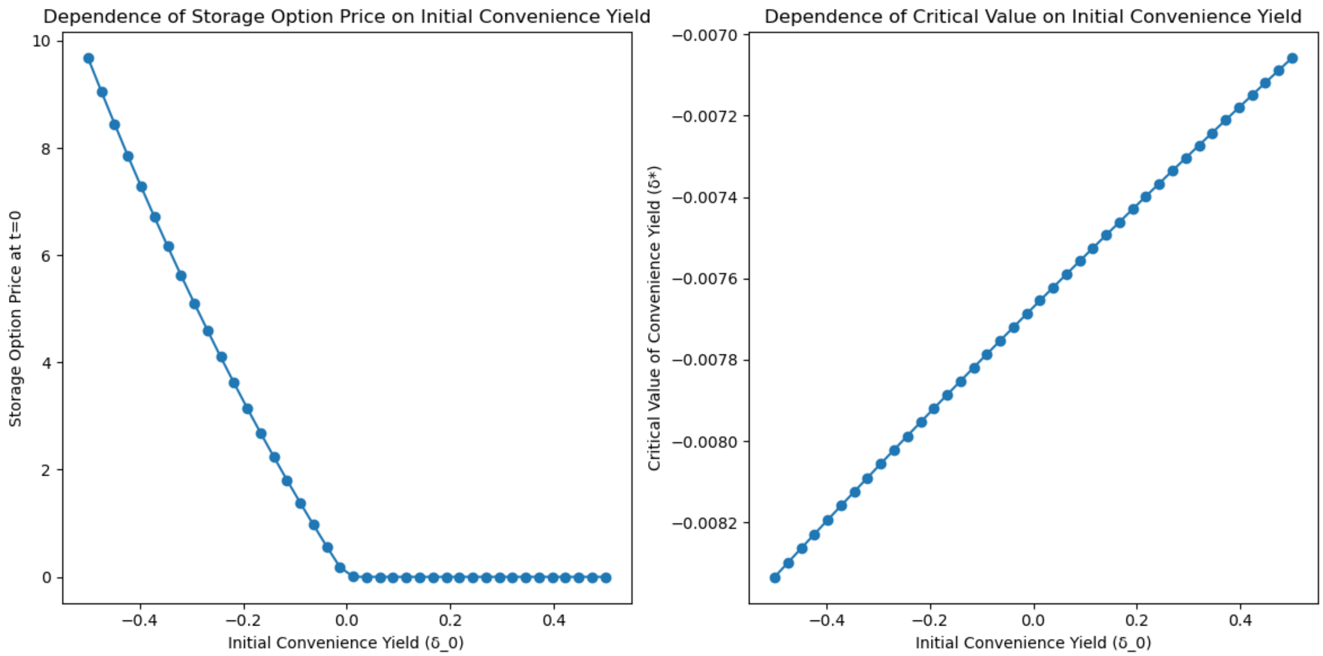


Figure 6. Dependence on the initial convenience yield δ_0

Looking at figure 6 we see that the option price decreases for a higher initial convenience yield. This is logical since a higher initial convenience yield means that at the start date the convenience yield is high too and the option price is therefore lower. We can confirm with the formulas from question 16: the first coordinate of the mean $m(\delta_0)$ is higher for a higher δ_0, so when taking the expectation, more of the mass is past the critical convenience yield making the price lower.
However the critical delta gets higher for a higher initial convenience yield. Indeed, a higher initial yield means at the start date, the yield has a bigger probability of being big. Looking at the formula for the payoff in question 12, this means that the part $\delta_s \mathcal{O}_s$ of the expectation is higher. This overall drives the expected payoff down which means the critical delta gets higher. Note that a higher critical delta drives the option price up but in this case as seen and explained before this is countered by the shifting of the mean $m(\delta_0)$.

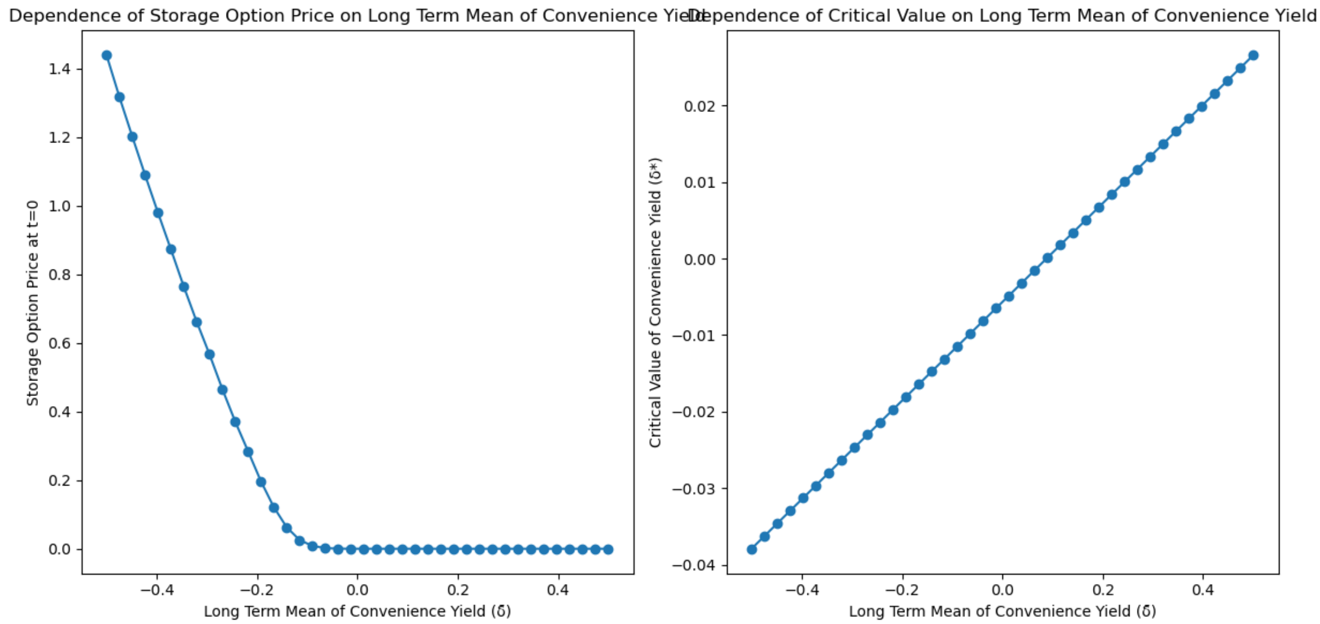


Figure 7. Dependence on the long term convenience yield $\overline{\delta}$

Looking at figure 7, we observe dependences similar to figure 6. Indeed they can be explained in the same way. Option price gets lower for higher $\overline{\delta}$ because as time goes on, the convenience yiel approaches $\overline{\delta}$, which if high means the option is less valuable (look a question 12). We can find this in the formulas of question 15: higher $\overline{\delta}$ shifts the first coordinate of the mean $m(\delta_0)$ higher which means when taking the expectation later on more of the mass is past the critical yield, driving the option price lower.
The critical yield is higher for higher $\overline{\delta}$. Indeed $\overline{\delta}$ is the long term average of δ_s. So if it is high, looking at the formula for the payoff in question 12, this means that the part $\delta_s \mathcal{O}_s$ of the expectation is higher. This overall drives the expected payoff down which means the critical delta gets higher. Note that a higher critical delta drives the option price up but in this case like in figure 6, this is countered by the shifting of the mean $m(\delta_0)$.

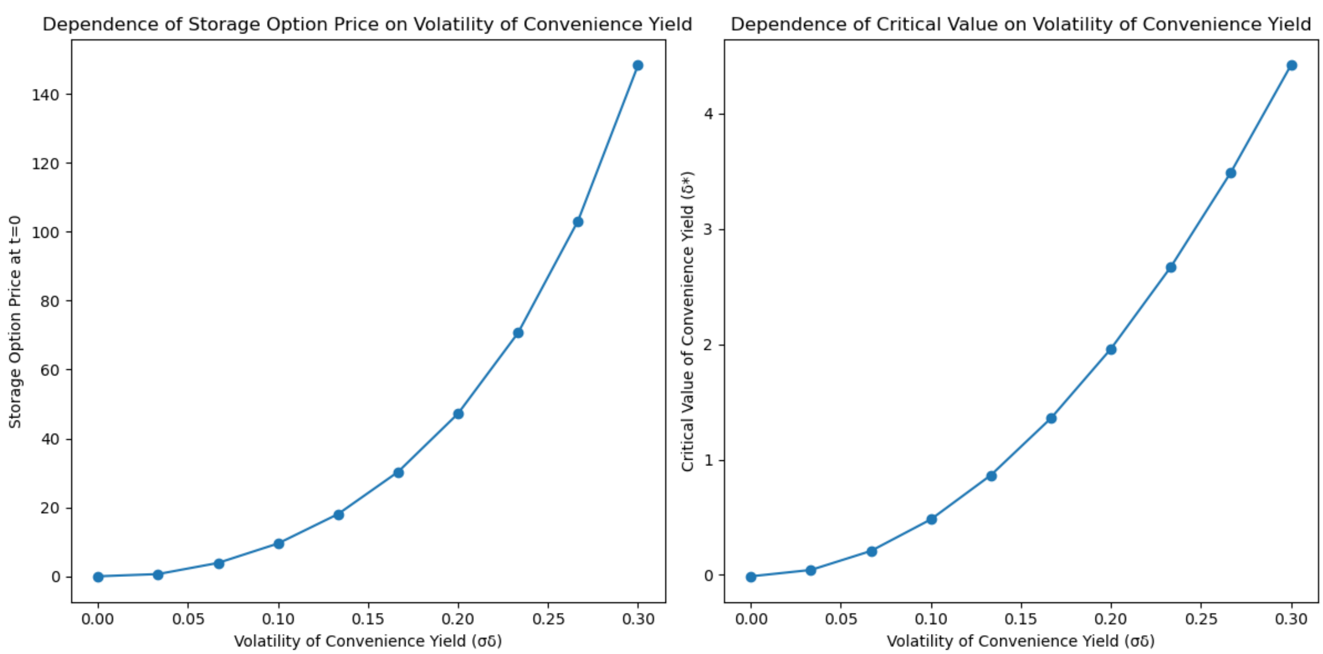


Figure 8. Dependence on the volatility of δ

Looking at figure 8, we see that the volatility of the convenience yield drives both option price and critical yield up. The critical yield is higher because from the formulas of question 14, $\psi_0(\Delta)$ is higher for a higher $\|\sigma_\delta\|$.
A high critical yield drives the option price up, moreover, a higher volatility means there is a higher chance the convenience yield goes under the critical yield which drives the option price up. To confirm this through the formulas, look at the covariance matrix of question 15.

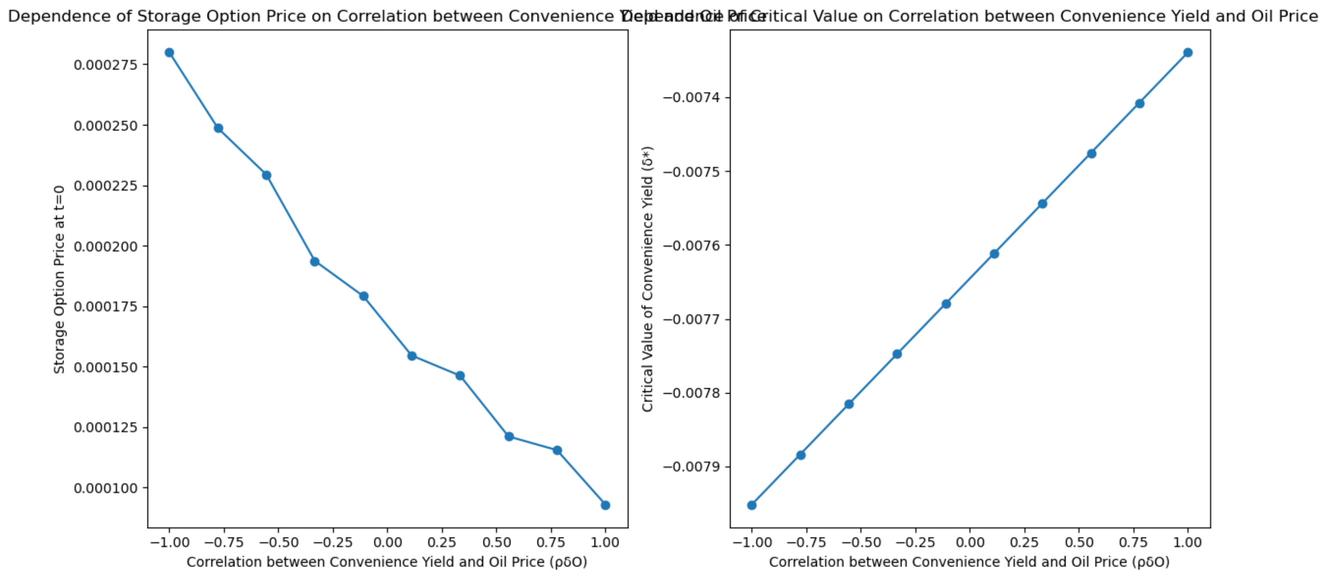


Figure 9. Dependence on the correlation between convenience yield and sport price

Looking at figure 9, we see that the option price decreases and the critical yield increases as the correlation gets higher. The higher critical yield can be seen from the formulas of question 15: it causes $\psi_0(\Delta)$ to be higher which pushes $\delta^\star$ up.
If the critical yield is higher we would expect the option price to increase. However, this is countered by the fact that the first coordinate of the mean $m(\delta_0)$ is pushed up which means when taking the expectation more of the mass is past the critical yield, driving the option price down.

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