Fin404 Derivatives

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Master in Financial Engineering
Spring 2024

Oil futures and storage options

The goal of this project is to study the market for oil futures contract and to analyze the valuation of various contracts related to oil storage.

Part 1. Documentation

Provide a basic presentation (maximum 5 pages) of the market for (crude) oil futures taking as reference the contracts traded on the Chicago Mercantile Exchange (CME). This part of your report is meant as an introductory briefing. As such it should not be technical, but it can include basic equations, numbers, and graphics as necessary. You are free to structure and organize your presentation as you see fit. However, your report should at the minimum address the following points:

- Description of oil futures contracts.
- Illustration of the cash flows associated to a long position in one futures contract and to rollover of long positions in successively maturing contracts.
- Why do oil futures exist? What do they allow investors to achieve?
- Evolution of the *trading volume* and *open interest* on oil futures across maturities. Is the market subject to systematic seasonal variations?
- What is a *convenience yield*? How does the convenience yield influence the spot and futures price of oil?

- What does it mean for the futures market to be in *Backwardation* or in *Contango*? When do these situations occur and what do (may) they signal? Which situation are we in today?
- What other types of oil derivatives can be traded on the CME?

Part 2. Analysis

Consider a financial market that operates in continuous time. Assume that market are complete and that the risk free rate r_t evolves according to

$$dr_t = \lambda_r (\bar{r} - r_t)\, dt + \sigma_r^\top dB_t^{\mathbb{Q}}$$

where $(\lambda_r, \bar{r}, \sigma_r) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3$ are constants and $B^{\mathbb{Q}}$ is a 3d Brownian motion under the the equivalent martingale measure $\mathbb{Q}$.

Spot price modelling

Let $\mathcal{O}_t$ denote the spot price of 1,000 barrels of crude oil at time $t \geq 0$. Following Gibson and Schwartz (1990) we assume that

$$\frac{d\mathcal{O}_t}{\mathcal{O}_t} = (r_t - \delta_t)\, dt + \sigma_{\mathcal{O}}^\top dB_t^{\mathbb{Q}}$$

subject to

$$d\delta_t = \lambda_\delta \left(\bar{\delta} - \delta_t\right) dt + \sigma_\delta^\top dB_t^{\mathbb{Q}}$$

for some constants $(\lambda_\delta, \bar{\delta}, \sigma_\delta, \sigma_{\mathcal{O}}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$.

1. Show that the model can be rewritten as

$$\begin{aligned} d\mathcal{O}_t/\mathcal{O}_t &= \|\sigma_{\mathcal{O}}\| dW_t^{\mathcal{O}} + (r_t - \delta_t)\, dt \\ dr_t &= \|\sigma_r\| dW_t^r + \lambda_r (\bar{r} - r_t)\, dt \\ d\delta_t &= \|\sigma_\delta\| dW_t^\delta + \lambda_\delta \left(\bar{\delta} - \delta_t\right) dt \end{aligned}$$

where $W_t^{\mathcal{O}}$, W_t^r, and W_t^{δ} are scalar $\mathbb{Q}-$Brownian motions such that

$$d\langle W^r, W^{\mathcal{O}}\rangle_t = \rho_{r\mathcal{O}} dt$$
$$d\langle W^r, W^{\delta}\rangle_t = \rho_{r\delta} dt$$
$$d\langle W^{\delta}, W^{\mathcal{O}}\rangle_t = \rho_{\delta\mathcal{O}} dt$$

for some constant correlation coefficients $\rho_{r\mathcal{O}}$, $\rho_{r\delta}$, and $\rho_{\delta\mathcal{O}}$ to be determined. Make sure to verify that the constants you find lie in $[-1, 1]$.

2. Show that the sport price of oil satisfies

$$\mathcal{O}_t = E_t^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r_u du}\delta_s \mathcal{O}_s ds + e^{-\int_t^T r_u du}\mathcal{O}_T\right], \qquad 0 \leq t \leq T < \infty.$$

How do you interpret this relation? Use this relation to explain why the Gibson and Schwartz model is referred to as a stochastic convenience yield model.

3. Verify that *if the risk free rate r_t is constant* ($\lambda_r = \|\sigma_r\| = 0$) then the model is equivalent to both the Gabillon (1991) model in which

$$\frac{d\mathcal{O}_t}{\mathcal{O}_t} = \phi \log\left(\ell_t/\mathcal{O}_t\right) dt + \sigma_o^{\top} dB_t^{\mathbb{Q}}$$

subject to

$$\frac{d\ell_t}{\ell_t} = \mu_\ell dt + \sigma_\ell^{\top} dB_t^{\mathbb{Q}}$$

for some constants $(\phi, \mu_\ell, \sigma_\ell, \sigma_{\mathcal{O}}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$; and to the Schwartz and Smith (2000) model in which

$$\log \mathcal{O}_t = x_t + \ell_t$$

where x_t and ℓ_t capture short and long term sources of price variations that evolve according to

$$dx_t = \mu_x dt + \sigma_x^{\top} dB_t^{\mathbb{Q}}$$
$$d\ell_t = \lambda_\ell \left(\bar{\ell} - \ell_t\right) dt + \sigma_\ell^{\top} dB_t^{\mathbb{Q}}$$

Bond pricing

4. Show that the price of a zero-coupon bond with maturity date T is

$$B_t(T) = \mathcal{B}(t, r_t)$$

for some function $\mathcal{B}$ that solves a PDE to be determined. Make sure to also provide the terminal boundary condition.

5. Use a separation of variables argument to show that the zero-coupon bond price is explicitly given by

$$B_t(T) = \mathcal{B}(t, r_t) = e^{b_0(T-t)+b_r(T-t)r_t}$$

for some functions b_0 and b_r that solve a system of ODEs to be determined. Make sure to also specify the boundary condition at the point $T - t = 0$.

6. Solve for the functions b_0 and b_r in closed form.

Futures pricing

7. What is the relation between the *spot* price $\mathcal{O}_T$ and the *futures* price $f_t = f_t(T)$ at date $t \leq T$ for delivery at date T? Show that this relation implies that the futures prices is given by

$$f_t = f(t, r_t, \delta_t, S_t)$$

for some function f that solves a PDE to be determined. Make sure to also specify the terminal boundary condition.

8. Use a separation of variables argument to show that the futures price for delivery at date T is explicitly given by

$$f_t = f(t, r_t, \delta_t, \mathcal{O}_t) = e^{\phi_0(T-t)+\phi_r(T-t)r_t+\phi_\delta(T-t)\delta_t}\mathcal{O}_t$$

for some functions ϕ_0, ϕ_r, and ϕ_δ that solve a system of ODEs to be determined. Make sure to also specify an appropriate boundary condition for these two functions at the point $T - t = 0$.

9. Solve for the functions (ϕ_r, ϕ_δ) in closed form and provide solution to the differential equation for the function ϕ_0 in the form of an integral. What is the evolution of the futures price under $\mathbb{Q}$?

10. Explain why having either a stochastic convenience yield or a stochastic interest rate (or both) is necessary to account for the observed transitions between periods of backwardation and contango.

11. When the risk free rate is constant ($\lambda_r = \|\sigma_r\| = 0$) one can define the implicit convenience yield for maturity T by the formula

$$f_t(T) = e^{(r - Y_t(T))(T-t)}\mathcal{O}_t.$$

What is the relation between the implied and instantaneous yields? Under what condition(s) do we have $Y_t(T) = \delta_t$ and what is the main feature of the instantaneous convenience yield process under $\mathbb{Q}$ in this case?

Storage options

Oil is a commodity not a financial asset. In particular, oil itself cannot be shorted and physically owning oil incurs a number of costs (transportation, evaporation, storage) that are collectively referred to below as storage costs.

Anticipating future needs for physical oil one may wish to acquire *storage options* that allow to secure the cost of physically owning oil over a fixed period of time in the future. Specifically, such an option fixes a start date T_0, an end date T_1, and a the flow cost $\{\xi_t : t \in [T_0, T_1]\}$ to be paid continuously over the storage period. The size of the contract, i.e. the quantity to be stored, is normalized to 1,000 barrels but costs and prices scale linearly with size.

To value the storage option let us start by determining its payoff at the start date T_0. If the buyer does not exercize the option then the payoff is simply 0. If the buyer exercizes the option he needs to first buy oil at the spot price $\mathcal{O}_{T_0}$. This oil is then stored at a flow cost of ξ_t until T_1 at which point it can be resold on the spot market at a price of $\mathcal{O}_{T_1}$. Since markets are complete the value at date T_0 of the different cash flows generated by these operations is given by

$$\mathcal{P}_{T_0} \equiv E^{\mathbb{Q}}_{T_0}\left[e^{-\int_{T_0}^{T_1} r_u du}\mathcal{O}_{T_1}\right] - \mathcal{O}_{T_0} - E^{\mathbb{Q}}_{T_0}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u du}\xi_s ds\right]$$

and it follows that the buyer exercizes if and only if $\mathcal{P}_{T_0} > 0$. In particular, the value of the option at date T_0 is simply $\mathcal{P}^+_{T_0}$.

12. Show that

$$\mathcal{P}^+_{T_0} = \max\left\{0, -E^{\mathbb{Q}}_{T_0}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u du}\left(\xi_s + \delta_s \mathcal{O}_s\right) ds\right]\right\}$$

How do you interpret this expression in view of the definition of δ_t?

13. Show that

$$e^{-\int_{T_0}^{T_1} r_u du}\mathcal{O}_{T_1} = e^{-\int_{T_0}^{T_1} \delta_u du}\mathcal{O}_{T_0}\frac{Z_{T_1}}{Z_{T_0}}$$

for some strictly positive $\mathbb{Q}-$martingale Z_t to be determined. Use this process to construct a probability measure $\overline{\mathbb{Q}}$ such that

$$E^{\mathbb{Q}}_{T_0}\left[e^{-\int_{T_0}^{T_1} r_u du}\mathcal{O}_{T_1}\right] = \mathcal{O}_{T_0}E^{\overline{\mathbb{Q}}}_{T_0}\left[e^{-\int_{T_0}^{T_1} \delta_u du}\right]$$

and determine the evolution of δ_t under this new probability measure.

14. Use an analogy with the derivation of the zero-coupon bond price in questions **5.** and **6.** to show that

$$E^{\overline{\mathbb{Q}}}_{T_0}\left[e^{-\int_{T_0}^{T_1} \delta_u du}\right] = e^{\psi_0(\Delta)+\psi_\delta(\Delta)\delta_t}$$

where $\Delta \equiv T_1 - T_0$ for some deterministic functions ψ_0 and ψ_δ to computed in closed form.

15. As a first specification assume that the flow cost is given by $\xi_s = \alpha\mathcal{O}_s$ for some constant $\alpha > 0$. Show that in this case

$$E^{\mathbb{Q}}_{T_0}\left[\int_{T_0}^{T_1} e^{-\int_{T_0}^{s} r_u du}\xi_s ds\right] = \alpha\mathcal{O}_{T_0}\int_0^{\Delta} e^{\psi_0(\tau)+\psi_\delta(\tau)\delta_{T_0}} d\tau$$

with the same functions ψ_0 and ψ_δ as in the previous question. Combine this result with those of the three previous questions to show that at date $t \leq T_0$ the value of the storage option is

$$E^{\mathbb{Q}}_t\left[e^{-\int_t^{T_0} r_u du}\mathcal{P}^+_{T_0}\right] = \mathcal{O}_t E^{\overline{\mathbb{Q}}}_t\left[e^{-\int_t^{T_0} \delta_u du}H\left(\delta_{T_0}\right)^+\right]$$

for some function H to be determined. Explain intuitively why the price of the storage option is completely independent of the specification of the risk free rate process under this form of the storage cost.

16. Show that

$$H'(\delta) < \psi_\delta(\Delta)H(\delta)$$

and use this inequality to prove that

$$\{\delta \in \mathbb{R} : H(\delta) > 0\} = (-\infty, \delta^*)$$

for some threshold δ^* (no need to try to calculate it at this point—you will do so numerically in the last part of the project). Combine this result with Lemma 1 in the appendix to derive a closed-form solution up to an integral for the storage option at date $t \leq T_0$ as a function of t, δ_t, and $\mathcal{O}_t$. Could you imagine another approach to this calculation, e.g. using a further change of probability measure?

17. As a second specification assume that the risk free rate is constant ($\lambda_r = \|\sigma_r\| = 0$) and that the flow cost is given by $\xi_s = \alpha \mathcal{O}_{T_0}$ for some $\alpha > 0$. Show that in this case the value of the storage option is

$$E_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r_u du}\mathcal{P}_{T_0}^+\right] = \mathcal{O}_t E_t^{\overline{\mathbb{Q}}}\left[e^{-\int_t^{T_0} \delta_u du}\overline{H}\left(\delta_{T_0}\right)^+\right]$$

for some decreasing function $\overline{H}$ to be determined in closed form. Combine this result with Lemma 1 in the appendix to derive a fully explicit solution for the price of the storage option as a function of $t \leq T_0$, δ_t, and $\mathcal{O}_t$. Why would this specification be much harder to solve with a stochastic risk-free rate?

Part 3. Calibration and implementation

18. Use the result of question 6. and the zero-coupon yields

$$Y_0(\tau) \equiv -\frac{1}{\tau}\log B_0(\tau)$$

in *Project-Data.csv* to calibrate the mean reversion intensity λ_r, the long-term mean $\overline{r}$, and the initial value r_0 of the risk free rate taking as given that its volatility is

$\|\sigma_r\| = 0.00023$. Provide complete details of the calibration method you use and discuss the accuracy of your calibration.

19. Use the result of question **9.** together with the estimates of the previous question and the futures prices in *Project-Data.csv* to calibrate the mean reversion intensity λ_δ, the long term mean $\bar{\delta}$, and the initial value δ_0 of the convenience yield taking as given that the quadratic variation matrix of the process

$$X_t = \begin{pmatrix} r_t \\ \delta_t \\ \log \mathcal{O}_t \end{pmatrix}$$

is given by

$$\langle X \rangle_t = \left(\langle X_i, X_j \rangle_t\right)_{i,j=1}^{3} = \begin{pmatrix} 0.000529 & 0.00003312 & 0.44712 \\ 0.00003312 & 0.000144 & 0.08352 \\ 0.44712 & 0.08352 & 576. \end{pmatrix} \cdot 10^{-4} t$$

and that the initial spot price of oil is \$80/barrel. Provide complete details of the calibration method you use and discuss the accuracy of your calibration.

20. Implement the specification of the storage option in question **15.** in a code that calculates the payoff of the storage option at the start date, the critical value of the convenience yield, and the initial price of the storage option as a functions of the start date T_0, the storage period Δ, and the storage cost α in the calibrated model.

21. Study graphically and interpret the storage dependence of the option price and the exercise threshold on the storage period, the storage cost parameter α, the initial convenience yield δ_0, the long term mean of the convenience yield $\bar{\delta}$, the volatility of the convenience yield $\|\sigma_\delta\|$ and the correlation coefficient $\rho_{\delta\mathcal{O}}$ using as benchmark the calibrated model and a storage option with start date $T_0 = 0.5$, storage perdio $\Delta = \frac{1}{4}$, and cost parameter $\alpha = 0.01$.

Appendix

Lemma 1. *Assume that*

$$dx_t = a\,(b - x_t)\,dt + \sigma dW_t$$

for some constants $(a, b, \sigma) \in \mathbb{R}^3$ *and some Brownian motion* W_t. *Then the distribution of the pair*

$$\left(x_{t+\Delta}, \int_t^{t+\Delta} x_s ds\right) \tag{4}$$

conditional on x_t *is normal with mean*

$$m(x_t) \equiv \begin{pmatrix} e^{-a\tau}x_t + \left(1 - e^{-a\tau}\right) b \\ \frac{b\tau + \left(1-e^{-a\tau}\right)(x_t - b)}{a} \end{pmatrix}$$

and variance-covariance matrix

$$\mathbf{V} \equiv \begin{pmatrix} \frac{\sigma^2\left(1-e^{-2a\tau}\right)}{2a} & \frac{\sigma^2\left(1-e^{-a\tau}\right)^2}{2a^2} \\ \frac{\sigma^2\left(1-e^{-a\tau}\right)^2}{2a^2} & \frac{\sigma^2 e^{-2a\tau}\left(e^{2a\tau}(2a\tau-3)+4e^{a\tau}-1\right)}{2a^3} \end{pmatrix}.$$

Sketch of proof. Consider the function

$$\mathcal{L}(s, x_s) \equiv E_s\left[e^{\alpha x_{t+\Delta} + \beta \int_s^{t+\Delta} x_u du}\right], \qquad s \le t + \Delta,$$

for some $\alpha, \beta \in \mathbb{C}$ and observe that

$$\mathcal{L}(t, x_t) = E_t\left[e^{\alpha x_{t+\Delta} + \beta \int_t^{t+\Delta} x_u du}\right]$$

gives the Fourier transform of joint distribution of the pair in (4) at the point (α, β) conditional on x_t. By definition, the process

$$e^{\beta \int_0^s x_u du}\mathcal{L}(s, x_s) \equiv E_s\left[e^{\alpha x_{t+\Delta} + \beta \int_0^{t+\Delta} x_u du}\right], \qquad s \le t + \Delta.$$

is a martingale. In particular, its drift must be equal to zero and it follows that the function $\mathcal{L}(s,x)$ solves

$$-\beta x\mathcal{L}(s,x) = \mathcal{L}_s(s,x) + a\,(b-x)\,\mathcal{L}_x(s,x) + \frac{1}{2}\sigma^2\mathcal{L}_{xx}(s,x)$$

subject to

$$\mathcal{L}(t+\Delta,x) = e^{\alpha x}.$$

where the $-$ on the left hand side comes from the fact that in the definition of $\mathcal{L}$ we are capitalizing at rate βx_t which is equivalent o discounting at rate $-\beta x_t$. Proceeding as in the futures pricing questions then shows that

$$\mathcal{L}(t,x) = e^{A_0(\Delta)+A_x(\Delta)x}$$

for some functions A_0 and A_x that solve a system of first order differential equations subject to

$$A_0(0) = A_x(0) - \alpha = 0$$

and the result follows by solving these differential equations and verifying that their solution satisfy the Gaussian identity

$$A_0(\Delta) + A_x(\Delta)x = \left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)^\top m(x_t) + \frac{1}{2}\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)^\top \mathbf{V}\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$$

with the mean and variance-covariance matrix of the statement. ■

General guidelines

1. The project is to be done in groups of **at least 4 and no more than 5 students** *without* communications between the groups.

2. The name and sciper number of each member of the group should appear clearly on the final report.

3. The final report should be typeset on no more than 15 A4 pages in 12pt font with 1.5cm margins and standard linespacing.

4. The code should be ready to run in one or more standalone files. If there are multiple files please also provide a readme with clear instructions.

5. The code should be commented with clear explanations for each line/block preferably in the code itself.

6. The code should use any external financial engineering libraries or functions. Please develop your own. It may however include built-in optimization, differential equation solving, and root finding functions.

7. Your code and the final report should be uploaded on the moodle.

8. Only one final report and one code archive per group cab be uploaded. No need to also send your work by email.

9. Make sure to not simply copy existing material and to cite your sources. Your work will automatically be scanned by an antiplagiarism software.