help

Vectorial Calculus A1 Recap

10/04/2025

Spherical and Cylindrical coordinates

Everything described in this section has an easier version in $\mathbb{R}^2$

Cylindrical coordinates

The transformation from the ordinary space $\mathbb{R}^3$ to $C(\mathbb{R}^3)$ is nearly analogous to polar coordinates in $\mathbb{R}^2$ described below:

$$T(x, y, z) = (r\cos\theta, r\sin\theta, z)$$

, and if we were integrating some function $f : \mathbb{R}^3 \to \mathbb{R}$ under the region $A \subset \mathbb{R}^3$ in the space (x, y, z), the equivalent integral in the new system $C(\mathbb{R}^3)$ is:

$$\iiint_{C(A)} f(r\cos\theta, r\sin\theta, z) r dz dr d\theta$$

Spherical coordinates

From the reasoning used in cylindrical coordinates, we have:

$$\iiint_A f(x, y, z) dx dy dz = \iiint_{S(A)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\theta)\rho^2\sin\varphi d\rho d\theta d\varphi$$

Jacobian Determinant For Multiple Integrals

More general change of variables require some technicalities:

If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a bijective function that morphes a region in the space xyz into the space uvw throught the equations that follow:

$$\begin{gathered} x = g(u, v, w) \\ y = h(u,, w) \\ z = k(u, v, w) \end{gathered}$$

Then the **jacobian determinant of T** is:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

And the equivalent integral onto the new system is:

$$\iiint_A f(x, y, z) dx dy dz = \iiint_{T(A)} f(x(u, v, w), y(u, v, w), z(u, v, w))\frac{\partial(x, y, z)}{\partial(u, v, w)} du dv dw$$

Physics

Mass Center and Centroid

Given a plane region $S^2 \subset \mathbb{R}^2$, its **centroid** is the point (x_c, y_c), where:

$$x_c = \frac{1}{\text{area}(S)} \iint xdxdy$$
$$y_c = \frac{1}{\text{area}(S)} \iint_S ydxdy$$

If the region/body/whatever has **constant** density $\mu(x, y), \forall x, y \in \mathbb{R}$, then its centroid is the same as the mass center, which is the point $(\hat{x_c}, \hat{y_c})$ where:

$$\hat{x_c} = \frac{\iint_S \mu(x, y) x dxdy}{\iint_S \mu(x, y) dxdy}$$
$$\hat{y_c} = \frac{\iint_S \mu(x, y) y dxdy}{\iint_S \mu(x, y) dxdy}$$

Notice that the **mass** of S is given by:

$$\iint_S \mu(x, y) dxdy$$

Moment of Inertia

Let $X \subset \mathbb{R}^3$ be a body, rotating over a given axis, let $\mu(x)$ be the mass density in $x, \forall x \in X$, if $r(X)$ is the distance to axis it is rotating over, and $v(x)$ is the speed, $\forall x \in X$, then $|v(x)| = \omega r(x)$, where ω is the angular speed

It follows that the **kinetic energy** of the body is $\frac{mv^2}{2}$:

$$\frac{1}{2} \cdot \int_X \mu(x)\ |v(x)|^2\ dX = \frac{1}{2} \omega^2 \cdot \int_X \mu(x) r(x)^2 dX$$

We define now the **Moment of Inertia** as:

$$I = \int_X \mu(x) r(x)^2 dX.$$

$L = \omega I$ is called the **angular momentum**, it is conserved if there are no external rotational forces.

Vector Calculus

Curves

A Curve is a continuous function $\gamma : [a, b] \to \mathbb{R}^n$, it is of class C^1 if γ' exists and is continuous in [a,b], if $\gamma(a) = \gamma(b)$, the curve is **closed**.

A curve is said to be C^1 *by parts* if there is a partition of $[a, b]$ in a finite number of subintervals such that the curve is C^1 in each of tese subintervals.

Scalar Line Integrals

Given $f : \mathbb{R}^n \to \mathbb{R}$ a function and $\gamma : [a, b] \to \mathbb{R}^n$ a C^1 class curve in $\mathbb{R}^n$, the **scalar line integral** of f along γ is:

$$\int_\gamma f ds = \int_a^b f(\gamma(t)) \; \|\gamma'(y)\| \; dt$$

If γ is C^1 *by parts*, we integrate on the C^1 partition-subintervals and sum each odf the smaller integrals.

Centroid and Mass Center of a Curve

Mass Cernter

Given $\gamma \subset \mathbb{R}^3$ a curve, and let $f(x, y, z)$ be the mass density per unit of lenght of γ, we know that the γ's mass is given by:

$$M = \int_\gamma f(x, y, z) ds.$$

So the mass center of γ is the point $c_m = (x_c, y_c, z_c)$ s.t:

$$x_c = \frac{\int_\gamma x f(x, y, z) ds}{M}$$
$$y_c = \frac{\int_\gamma y f(x, y, z) ds}{M}$$
$$z_c = \frac{\int_\gamma z f(x, y, z) ds}{M}$$

If f is constant (homogeneous curve), then the mass center is the centroid as well

Centroid

Let $\gamma \subset \mathbb{R}^3$ be a smooth or piecewise smooth curve, parameterized by $\gamma(t) : [a, b] \to \mathbb{R}^3$. If the curve is **homogeneous**, meaning the mass density per unit length is constant, then its **centroid** is the point (x_c, y_c, z_c) given by:

$$x_c = \frac{1}{L} \int_\gamma x ds$$
$$y_c = \frac{1}{L} \int_\gamma y ds$$
$$z_c = \frac{1}{L} \int_\gamma z ds$$

Where L is the total arc length of the curve:

$$L = \int_\gamma ds = \int_a^b \|\gamma'(t)\| \; dt$$

Vectorial Line Integrals

Consider now $F : \mathbb{R}^n \to \mathbb{R}^n$, usually called a *vector field*, and a class C^1 curve $\gamma : [a, b] \to \mathbb{R}^n$ in this field.

The integral of F *along* γ is:

$$\int_\gamma F = \int_a^b F(\gamma(t))\gamma'(t)dt$$

This line integral is linear: $\int_\gamma (aF + bG) = a\int_\gamma F + b\int_\gamma G$

Conservative Vector Fields and Angle Variation

Foreword

We have been trained in the mysterious and dark arts of newtonian and lagrangian mechanics by Master Paulo Verdasca Amorim himself, this is nothing to us.

Conservative Vector Fields

A field $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, Ω an open and connected set, is said to be conservative if $\exists U : \Omega \to \mathbb{R}$ s.t $F = \Delta U, U$ is called F's **potential**

Theorem: Let $F : \Omega \to \mathbb{R}^n$ be a vector field and $f : \Omega \to \mathbb{R}$ be its potential, i.e $F = \Delta f$. Let $A, B \in \Omega$ and $c : [a, b] \to \mathbb{R}^n$ be a C^1 by parts curve s.t c(a) = A and c(b) = B, then:

$$\int_c F = f(B) - f(A)$$

This looks like Calculus' Fundamental Theorem

Angle Variation

We conclude this section on line integrals with a counterexample: a vector field that satisfies $\partial\frac{F_1}{\partial}y = \partial\frac{F_2}{\partial}x$ but is not conservative.

Let $F(x, y) = \frac{1}{x^2+y^2}(-y, x)$, defined over $\mathbb{R}^2 \setminus 0$. Note that:

$$\partial\frac{F_1}{\partial}y = \partial\frac{F_2}{\partial}x = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

So the field meets the symmetry of mixed partials, but F is still not conservative — the domain is not simply connected.

Take the closed curve $c : [0, 2\pi] \to \mathbb{R}^2$ given by:

$$c(t) = (\cos t, \sin t)$$

Then:

$$\int_c F = \int_0^{\{2\pi\}} (-\sin t, \cos t) \cdot (-\sin t, \cos t) dt = \int_0^{\{2\pi\}} 1dt = 2\pi \neq 0$$

Since the line integral over a closed curve is nonzero, F is not conservative.