Surfaces - Lecture (number of the lecture)

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1. Basics

Definition 1.1: (Parametrized Surface) A *surface* is a function $\varphi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$. φ is called the parametrization of the surface, and can be seen as:

$$\varphi(u, v) = (x(u, v), y(u, v), z(u, v)) \quad (1)$$

And if the following exist and are continuous, then φ is of class C^m:

$$\frac{\partial^m \varphi}{\partial x^m} \text{ and } \frac{\partial^m \varphi}{\partial y^m} \quad (2)$$

Definition 1.2: (Revolution Surface) S

2. Area of a Surface

Definition 2.1: (Area of a Surface) Given $\varphi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ a C^1 class surface with parametrization φ, to find the area, divide D into infinitely man small rectangles, such that the associated portions of the surface are close enough to a paralellogram. And let $(u, u + \delta u) \times (v, v + \delta v)$ be one of these rectangles in the plane uv, so the portion of the surface associated is a paralellogram with sides approximated by the vectors $a = \varphi(u + \delta u, v) - \varphi(u, v)$ and $b = \varphi(v, \delta + v) - \varphi(u, v)$. So the area of the paralellogram is $\|a \times b\|$:

$$\left\| \frac{\varphi(u, u + \delta u) - \varphi(u, v)}{\delta u} \times \frac{\varphi(v, v + \delta v) - \varphi(u, v)}{\delta v} \right\| \delta u \delta v \quad (3)$$

On the limit $\delta u \to 0$ and $\delta v \to 0$, these are the partial derivatives, so area of $S = \varphi(D)$ is:

$$A(S) = \iint_D \| \frac{\partial \varphi}{\partial x} \times \frac{\partial \varphi}{\partial y} \| \quad (4)$$

Here are some examples:

Example 2.1.:

Example 2.2.:

Example 2.3.:

3. Scalar Surface Integrals

Definition 3.1: Given $\varphi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ a parametrized $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$ surface, with $S = \varphi(D)$ and $f : S \subset \mathbb{R}^3 \to \mathbb{R}$ a continuous real function in S, we are interested on the integral of f along $S = \varphi(D)$, which is:

$$\iint_S f dS = \iint_D f(\varphi(u,v)) \left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\| dudv \tag{5}$$

Notice that when $f = 1$, the surface integral becomes the area of the surface. Here are some examples:

Example 3.1.:

Example 3.2.:

4. Tangent Plane to a Surface

Given S a C^1 class surface with parametrization φ, and $p_0 \in S$, we are interested on the *tangent plane* to S in p_0. We could find it by finding 2 independent vectors to S leaving p_0.

Let $p_0 = \varphi(u_0, v_0)$, we know that th partial derivatives point to the direction that the surface grows (when u or v are fixed), so $v_1 = \frac{\partial \varphi}{\partial u}(u_0, v_0)$ and $v_2 = \frac{\partial \varphi}{\partial v}(u_0, v_0)$ are 2 independent vectors and tangent to the surface in p_0, therefore the normal vector generated by v_i generates the plane, so:

Definition 4.1: The plane tangent to a C^1 class surface S, parametrized by φ is:

$$\| v_1 \times v_2 \| \tag{6}$$

Where $p_0 = \varphi(u_0, v_0)$, $v_1 = \frac{\partial \varphi}{\partial u}(u_0, v_0)$ and $v_2 = \frac{\partial \varphi}{\partial v}(u_0, v_0)$

5. Vector Surface Integrals

Given S a parametrized surface, we want to calculate $\int_S F$ over the vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$, we can interpretate this as:

Imagine F is the speed of a fluid, e.g a river. If we toss a *net* to this river, it will form a surface over the river (the field), so $\int_S F$ quantifies **how much of the fluid is flowing through the net**, or how much of the vector field is flowing through the surface.