Problem Set 5, Analytic Geometry (Solutions)

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Problem 1 (Introductory)

Calculate the determinant of the matrix A below using Sarrus's rule:

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 5 \\ 1 & 4 & 3 \end{pmatrix}$$

Solution:

The expanded matrix is:

$$A' = \left(\begin{array}{ccc|cc} 2 & 1 & 3 & 2 & 1 \\ 3 & 2 & 5 & 3 & 2 \\ 1 & 4 & 3 & 1 & 4 \end{array}\right)$$

And the determinant is:

$$\begin{aligned} \det(A) &= 2 \cdot 2 \cdot 3 + 1 \cdot 5 \cdot 1 + 3 \cdot 3 \cdot 4 - (3 \cdot 2 \cdot 1 + 2 \cdot 5 \cdot 4 + 1 \cdot 3 \cdot 2) \\ &= 12 + 5 + 36 - (6 + 40 + 6) = 53 - 52 = 1. \end{aligned}$$

Problem 2 (Introductory)

Calculate the determinant and the trace of the following matrices:

a)

$$\begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}$$

b)

$$\begin{pmatrix} 1 & 3 & 4 \\ 5 & 2 & -3 \\ 1 & 4 & 2 \end{pmatrix}$$

c)

$$\begin{pmatrix} -1 & -4 & -6 \\ 0 & -2 & -5 \\ 0 & 0 & -3 \end{pmatrix}$$

Solution:

a)

The determinant is:

$$\det\left(\begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}\right) = 3 \cdot 2 - (-1) \cdot 4 = 6 + 4 = 10.$$

The trace is $3 + 2 = 5$.

b)

The determinant is:

$$\begin{aligned} \det\left(\begin{pmatrix} 1 & 3 & 4 \\ 5 & 2 & -3 \\ 1 & 4 & 2 \end{pmatrix}\right) &= \det\left(\left(\begin{array}{ccc|cc} 1 & 3 & 4 & 1 & 3 \\ 5 & 2 & -3 & 5 & 2 \\ 1 & 4 & 2 & 1 & 4 \end{array}\right)\right) \\ = 1 \cdot 2 \cdot 2 + 3 \cdot -3 \cdot 1 + 4 \cdot 5 \cdot 4 &- (4 \cdot 2 \cdot 1 + 3 \cdot -3 \cdot 5 + 1 \cdot 2 \cdot 3) \\ &= 49. \end{aligned}$$

The trace is $1 + 2 + 2 = 5$.

c)

The determinant is:

$$\begin{aligned} \det\left(\begin{pmatrix} -1 & -4 & -6 \\ 0 & -2 & -5 \\ 0 & 0 & -3 \end{pmatrix}\right) &= \det\left(\left(\begin{array}{ccc|cc} -1 & -4 & -6 & -1 & -4 \\ 0 & -2 & -5 & 0 & -2 \\ 0 & 0 & -3 & 0 & 0 \end{array}\right)\right) \\ = -1 \cdot -2 \cdot -3 + -4 \cdot -5 \cdot 0 + -6 \cdot 0 \cdot 0 &- (0 \cdot -2 \cdot 0 + -4 \cdot -5 \cdot 0 + -6 \cdot -2 \cdot 0) \\ &= -6 + 0 + 0 - (0 + 0 + 0) = -6. \end{aligned}$$

The trace is $-1 - 2 - 3 = -6$.

Problem 3 (Introductory)

If $\det(A) = -3$, find:
- $\det(A^2)$
- $\det(A^3)$
- $\det(A^{-1})$
- $\det(A^T)$

Solution

Since $\det(A \cdot B) = \det(A) \cdot \det(B)$, and $\det(A) = \det(A^T)$ we have:

- $\det(A^2) = \det(A) \cdot \det(A) = (-3) \cdot (-3) = 9$.
- $\det(A^3) = \det(A) \cdot \det(A) \cdot \det(A) = (-3) \cdot (-3) \cdot (-3) = -27$.
- $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{-3}$.
- $\det(A^T) = \det(A) = -3$.

Problem 4

If A and B are $n \times n$ matrices such that $\det(A) = -2$ and $\det(B) = 3$, calculate $\det(A^T \cdot B^{-1})$

Solution

We have:

$$\det(A^T \cdot B^{-1}) = \det(A^T) \cdot \det(B^{-1}) = \det(A) \cdot \frac{1}{\det(B)} = -\frac{2}{3}.$$

Problem 1 (In-Depth)

Find all $\lambda \in \mathbb{R}$ s.t $\det(A - \lambda I) = 0$, in:

a)

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

b)

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & 2 & -2 \end{pmatrix}$$

Solution

a)

The characteristic polynomial is:

$$\det(A - \lambda I) = \det\left(\begin{pmatrix} -\lambda & 1 & 2 \\ 0 & -\lambda & 3 \\ 0 & 0 & -\lambda \end{pmatrix}\right) = (-\lambda) \cdot (-\lambda) \cdot (-\lambda) = -\lambda^3.$$

$\lambda = 0$ is the root.

b)

The characteristic polynomial is:

$$\begin{aligned} \det(A - \lambda I) = \det\left(\begin{pmatrix} 1-\lambda & 0 & 0 \\ -1 & 3-\lambda & 0 \\ 3 & 2 & -2-\lambda \end{pmatrix}\right) &= (1-\lambda) \cdot (3-\lambda) \cdot (-2-\lambda) + 0 + 0 \\ &= (1-\lambda) \cdot (3-\lambda) \cdot (-2-\lambda). \end{aligned}$$

The roots are $\lambda = 1$, $\lambda = 3$ and $\lambda = -2$.

Problem 2 (In-Depth)

Solve using Cramer's rule:

a)

$$\begin{aligned} x + 3y - z &= 0 \\ 2y + 2z &= 0 \\ x + y + z &= 0 \end{aligned}$$

b)

$$\begin{aligned} x + y - z &= 0 \\ 2x + y + z &= 1 \\ 3x - y + z &= 1 \end{aligned}$$

Solution

a)

Since $b = 0$, all determinants D_i are 0, therefore the only solution is $x = y = z = 0$.

b)

The determinant of the main matrix is (using cofactor expansion):

$$\begin{aligned} \det(A) &= 1 \cdot \det\left(\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\right) - 1 * \det\left(\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}\right) + (-1) \cdot \det\left(\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}\right) \\ &= 1 \cdot (1 + 1) - 1 \cdot (2 - 3) - 1 \cdot (2 - 3) = 2 - (-1) + 5 = 8. \end{aligned}$$

And the determinants D_x, D_y, D_z are:

$$D_x = \det\left(\begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}\right) = 2,$$

$$D_y = \det\left(\begin{pmatrix} 1 & 0 & -1 \\ 2 & 2 & 1 \\ 3 & -1 & 1 \end{pmatrix}\right) = 1,$$

$$D_z = \det\left(\begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 2 \\ 3 & -1 & 3 \end{pmatrix}\right) = 3.$$

Therefore:

$$\begin{aligned} x &= \frac{D_x}{D} = \frac{2}{8} = \frac{1}{4}, \\ y &= \frac{D_y}{D} = \frac{1}{8} = \frac{1}{8}, \\ z &= \frac{D_z}{D} = \frac{3}{8} = \frac{3}{8}. \end{aligned}$$

Problem 3 (In-Depth)

For which values of $k \in \mathbb{R}$ the matrices below are singular (non-invertible)?

a)

$$\begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & 3 & 2 \end{pmatrix}$$

b)

$$\begin{pmatrix} k-3 & -2 \\ -2 & k-2 \end{pmatrix}$$

Solution

a)

$$\det\left(\begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & 3 & 2 \end{pmatrix}\right) = 8k + 8, \text{with } k = -1 \text{ the determinant is } 0.$$

b)

The determinant of the 2×2 matrix is clearly:

$$\det(A) = k^2 - 5k + 2$$

Finding the roots:

$$k^2 - 5k + 2 = 0$$
$$k = \frac{5 \pm \sqrt{17}}{2}.$$

Problem 1 (Advanced)

Answer with true or false, justifying your answer:

a)

If $B = AA^T A^{-1}$, then $\det(A) = \det(B)$.

b)

$\det(A + B) = \det(A) + \det(B)$

Solution

a)

It is True:

$$\det(A) = \det(AA^T A^{-1}) = \det(A) \cdot \det(A^T) \cdot \det(A^{-1}) = \det(A) \cdot \det(A) \cdot \frac{1}{\det(A)} = \det(A).$$

b)

It is false, a counter-example is:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Then:

$$\det(A + B) = \det\left(\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right) = 9 \neq \det(A) + \det(B) = 1 + 4 = 5.$$

Problem 2 (Optional)

$\boxed{\text{Show that } A^T = A^{-1} \Rightarrow \det(A) = \pm 1.}$

Solution

If $A^T = A^{-1}$, then:

$$\begin{aligned} \det(A^T) = \det(A^{-1}) &= \frac{1}{\det(A)} \\ \Leftrightarrow \det(A^T) = \det(A) &= \frac{1}{\det(A)} \\ \Leftrightarrow \det(A)^2 &= 1 \\ \Leftrightarrow \det(A) &= \pm 1. \end{aligned}$$

Problem 3 (Optional)

$\boxed{\text{Show that } \det(\alpha A) = \alpha^n \det(A) \text{, where } \alpha \in \mathbb{R}, A \in \mathbb{R}^{n \times n}}$

Solution

We know that:

$$\det(\alpha A) = \det(\alpha I \cdot A) = \det(\alpha I) \cdot \det(A) = \alpha^n \cdot \det(A).$$