Assignment 3 - Numerical Linear Algebra

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Abstract

coming soon

Contents

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1. Introduction

2. Norm Distribution (a)

2.1. The Chi-Square Distribution

Here we construct a theoretical basis for our analysis of the histograms shown in Section 2.2

When we generate a matrix $A \in \mathbb{R}^{m \times n}$, with $A_{ij} \sim N(0,1)$ independent, each column c_i is a gaussian vector in $\mathbb{R}^m$. if

$$x = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{pmatrix} \in \mathbb{R}^m \tag{1}$$

Is a column, then:

$$\begin{gathered} V = \|x\|_2 = \sqrt{\sum_{i=1}^{m} X_i^2} \\ V^2 = \sum_{i=1}^{m} X_i^2 \end{gathered} \tag{2}$$

Is of our interest. The expected value and variance are:

$$\begin{gathered} \mathbb{E}[V^2] = \mathbb{E}\left[\sum_{i=1}^{m} X_i^2\right] = \sum_{i=1}^{m} \mathbb{E}[X_i^2] = m \\ \mathrm{Var}(V^2) = \mathrm{Var}\left(\sum_{i=1}^{m} X_i^2\right) = \sum_{i=1}^{m} \mathrm{Var}(X_i^2) = 2m \end{gathered} \tag{3}$$

But we know that if $X_i \sim N(0,1)$ are independent:

$$\sum_{i=1}^{m} X_i^2 \sim \chi_m^2 \tag{4}$$

where χ_m is the chi-squared distribution with m degreees of freedom, better discussed in Section 2.1.

Taking the square root on eq. (4), we have:

$$V = \|x\|_2 = \sqrt{\sum_{i=1}^{m} X_i^2} \sim \sqrt{\chi_m^2} \sim \chi_m \tag{5}$$

The 2-norm of a vector x is distributed as a chi distribution with m degrees of freedom, in order to understand the distribution for many values of m, we can calculate the expected value and variance of this distribution as a function of m. The PDF of the chi distribution (with m degrees of freedom) is:

$$f_V(\varphi) = \frac{1}{2^{\frac{m}{2}-1} \cdot \Gamma\left(\frac{m}{2}\right)} \varphi^{m-1} e^{-\frac{\varphi^2}{2}} \tag{6}$$

So from this, the expected value is:

$$\mathbb{E}(V) = \sqrt{2} \cdot \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \tag{7}$$

And from this, the variance:

$$\mathrm{Var}(V) = m - \left(\sqrt{2} \cdot \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}\right)^2 \tag{8}$$

The Stirling Approximation provides a good approximation for the expected value and variance:

$$\mathbb{E}(V) \approx \sqrt{m} \cdot \left(1 - \frac{1}{4m} + O\left(\frac{1}{m^2}\right)\right) \tag{9}$$

$$\mathrm{Var}(V) \approx \frac{1}{2} + O\left(\frac{1}{m}\right) \tag{10}$$

2.2. Histograms

The first cell of this notebook has as expected output, with input being matrices with fixed $n = 1000$ and $m \in \{10, 20, 100, 200, 1000, 2000\}$, the following plots:

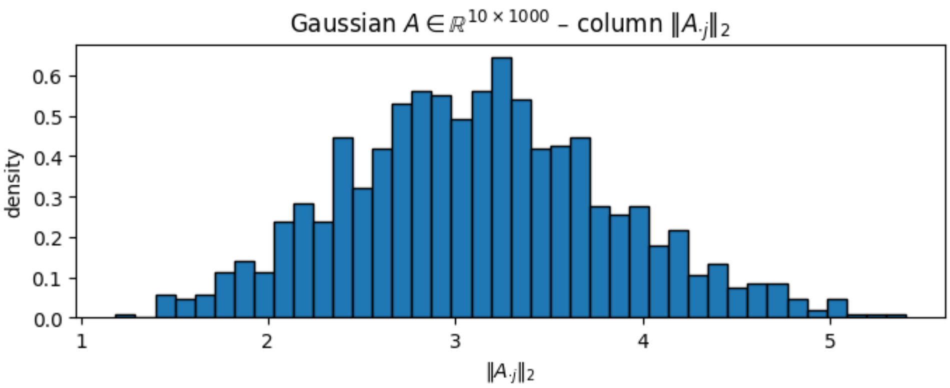


Figure 1: 10×1000 gaussian matrix

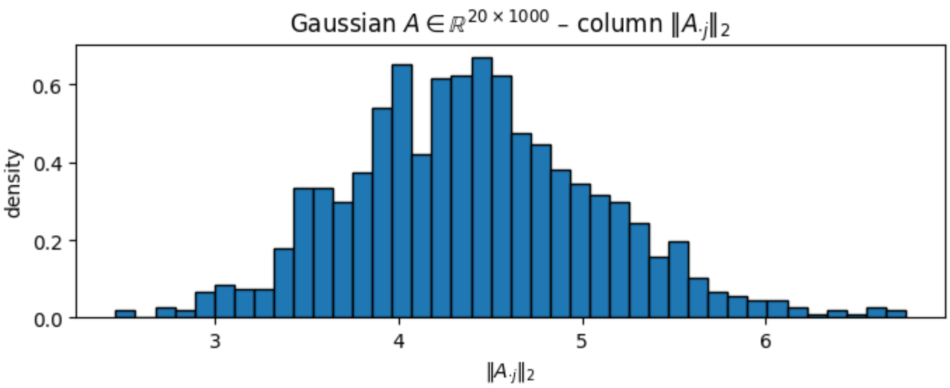


Figure 2: 20×1000 gaussian matrix

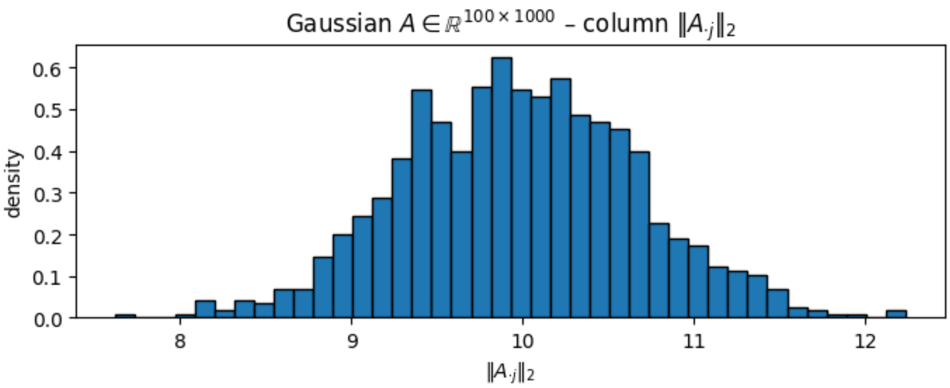


Figure 3: 100×1000 gaussian matrix

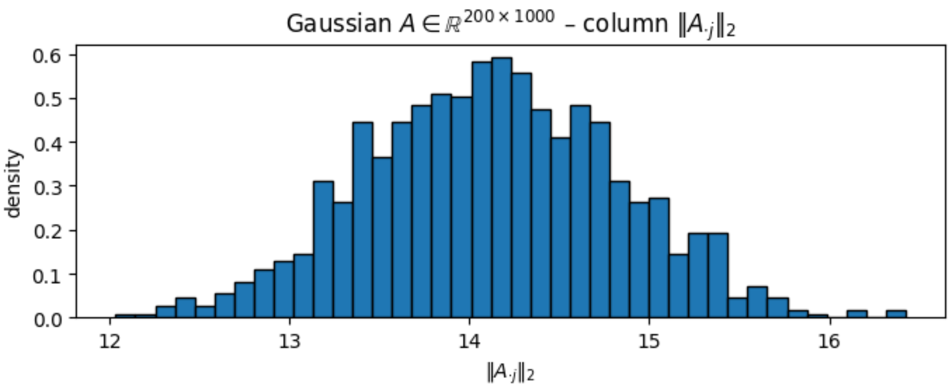


Figure 4: 200×1000 gaussian matrix

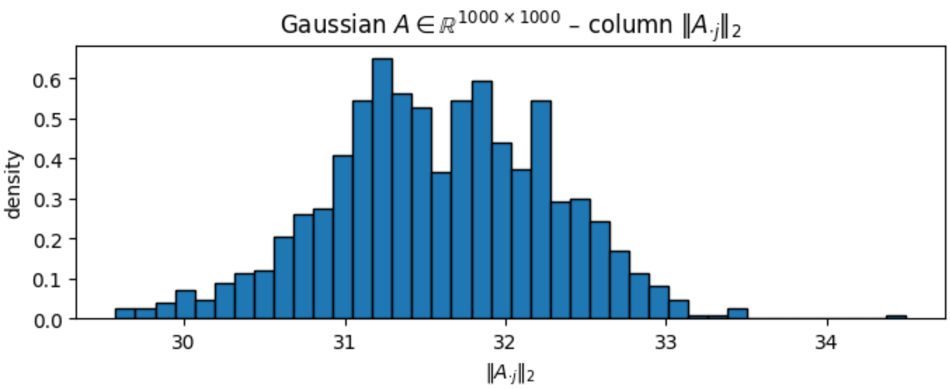


Figure 5: 1000×1000 gaussian matrix

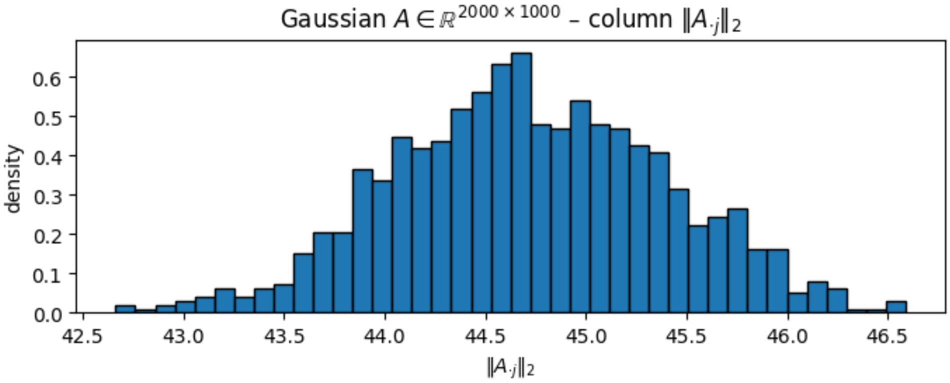


Figure 6: 2000×1000 gaussian matrix

m	**approximate μ_m (theory)**	**$[\mu \pm 3\sigma]$ (theory)**	**observed spike**	**visual range**
10	3.08	$1.0 - 5.18$	≈ 3.1	$1.2 - 5.1$
20	4.42	$2.3 - 6.52$	$\approx 4.3 - 4.4$	$2.9 - 6.4$
100	9.98	$7.9 - 12.1$	$\approx 9.9 - 10.0$	$7.8 - 12.0$
200	14.12	$12.0 - 16.2$	≈ 14.1	$12.2 - 16.2$
1000	31.61	$29.5 - 33.7$	$\approx 31.7 - 32.0$	$29.9 - 33.3$
2000	44.72	$42.6 - 46.8$	$\approx 44.5 - 45.0$	$42.8 - 46.6$

This table illustrates the expected value μ_m and the range $[\mu - 3\sigma, \mu + 3\sigma]$ for Figure 1 to Figure 6.

So apparently as m grows, the size of the gaussian vectors rapidly converge to $\sqrt{m}$, with small errors.

3. Inner Products (b)

Here we construct a theoretical basis for our analysis of the inner products shown in Section 3.1.

When we generate a matrix $A \in \mathbb{R}^{m \times n}$, with $A_{ij} \sim N(0,1)$ independent, each column c_i is a gaussian vector in $\mathbb{R}^m$. If The inner product of two gaussian vectors $x = (X_1, ..., X_n), y = (Y_1, ..., Y_n)$ is:

$$Z = \langle x, y \rangle = \sum_{i=1}^{m} X_i Y_i \tag{11}$$

With $X, Y \sim N(0,1)$. Since X_i, Y_j are independent, we have:

$$\begin{aligned} \mathbb{E}[Z] &= \sum_{i=1}^{m} \mathbb{E}[X_i Y_i] = \sum_{i=1}^{m} \mathbb{E}[X_i]\mathbb{E}[Y_i] = 0 \\ \mathrm{Var}(Z) &= \sum_{i=1}^{m} \mathrm{Var}(X_i Y_i) = \sum_{i=1}^{m} \mathbb{E}[X_i^2]\mathbb{E}[Y_i^2] = \sum_{i=1}^{m} 1 = m \end{aligned} \tag{12}$$

If $W = X_i Y_i$, we have:

$$M_W(\varphi) = \mathbb{E}[e^{\varphi W}] = \frac{1}{\sqrt{1 - \varphi^2}}, |\varphi| < 1 \tag{13}$$

Over all $W_i = X_i Y_i$:

$$M_Z(\varphi) = \mathbb{E}[e^{\varphi Z}] = (M_W(\varphi))^m = \left(\frac{1}{\sqrt{1-\varphi^2}}\right)^m = (1-\varphi^2)^{-\frac{m}{2}}, |\varphi| < 1 \tag{14}$$

And magically:

$$M_{\frac{Z}{\sqrt{m}}}(\varphi) = \left(1 - \frac{\varphi^2}{m}\right)^{-\frac{m}{2}} \Rightarrow \lim_{m\to\infty} M_{\frac{Z}{\sqrt{m}}}(\varphi) = e^{\frac{\varphi^2}{2}} \tag{15}$$

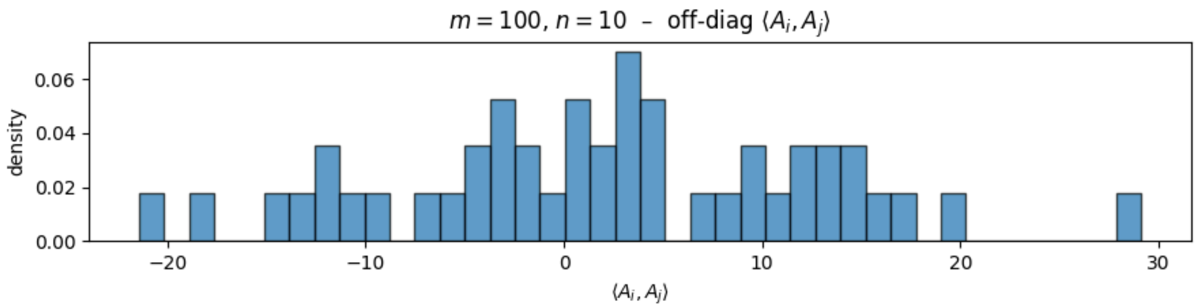
Precisely the moment generating function of a standard normal distribution, so as $m \to \infty$:

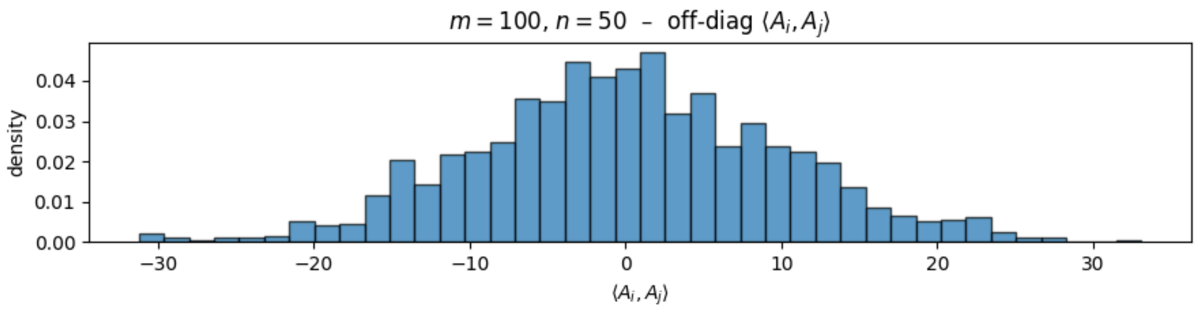
$$\frac{Z}{\sqrt{m}} \sim N(0,1) \tag{16}$$

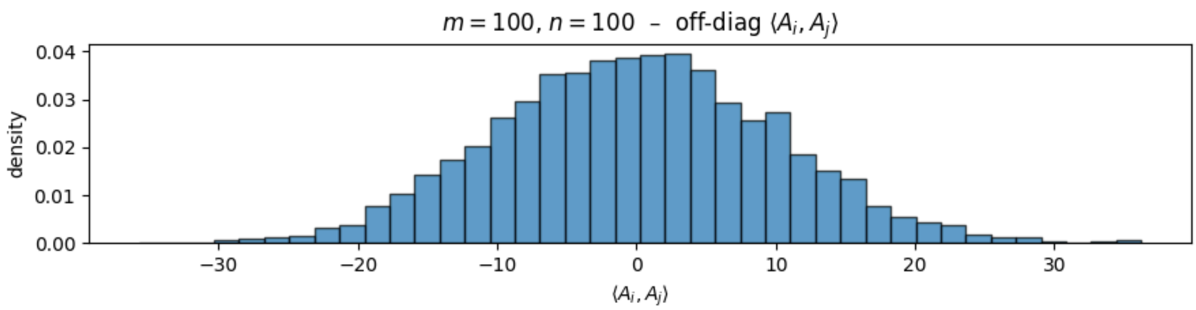
With a fixed $m = 100$, when $n \to \infty$ we can see the distribution approaching $N(0,1)$, as shown in Section 3.1

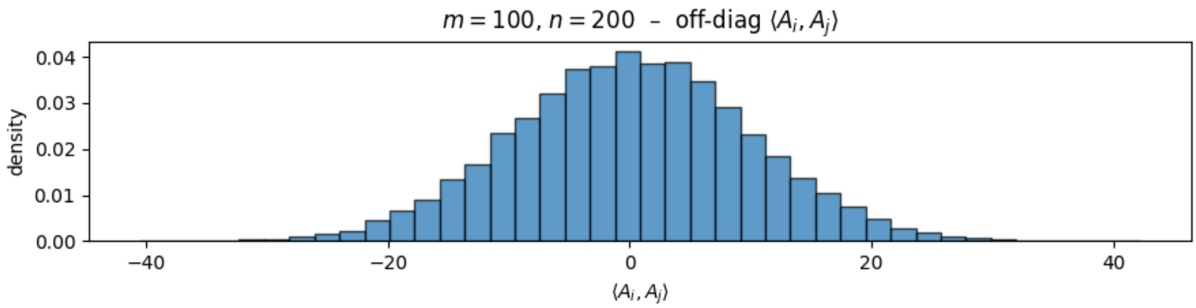
3.1. Histograms

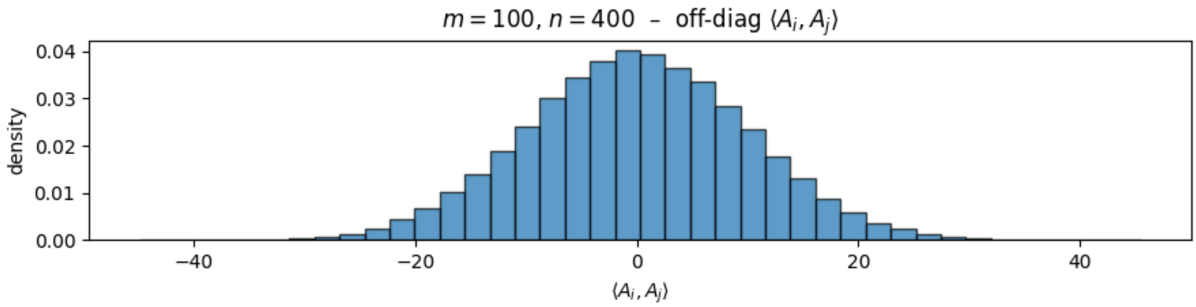
The following plots are an expected output for the second cell of this notebook, with input $m = 100, n \in \{10, 20, 30, 40, 50, 60, ..., 1000\}$:

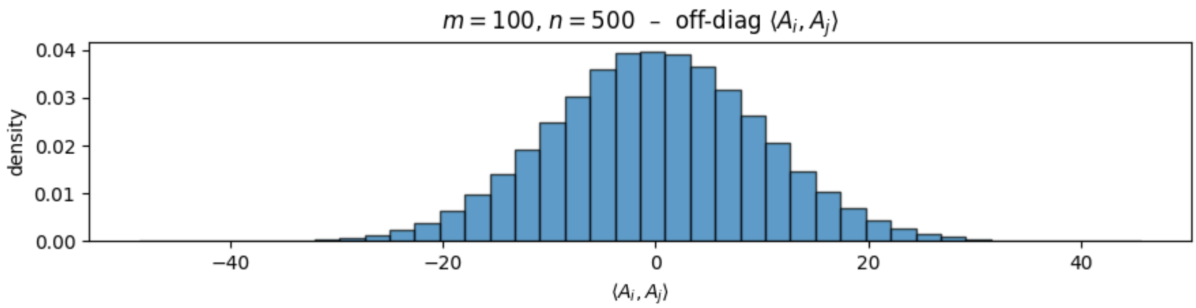


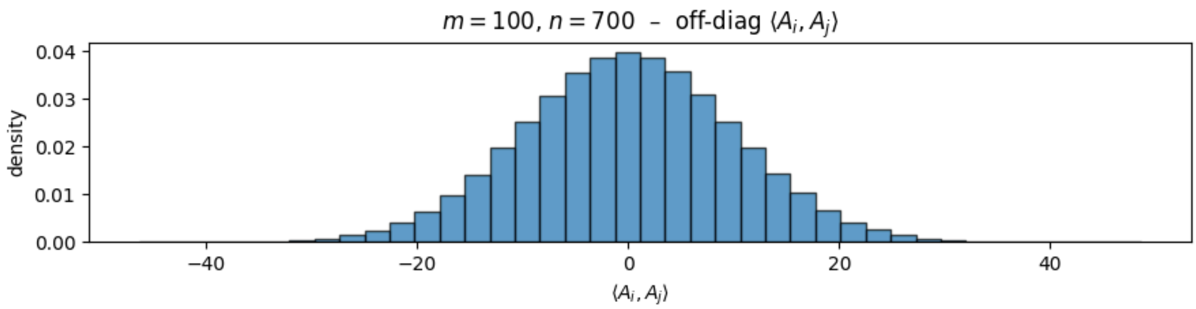












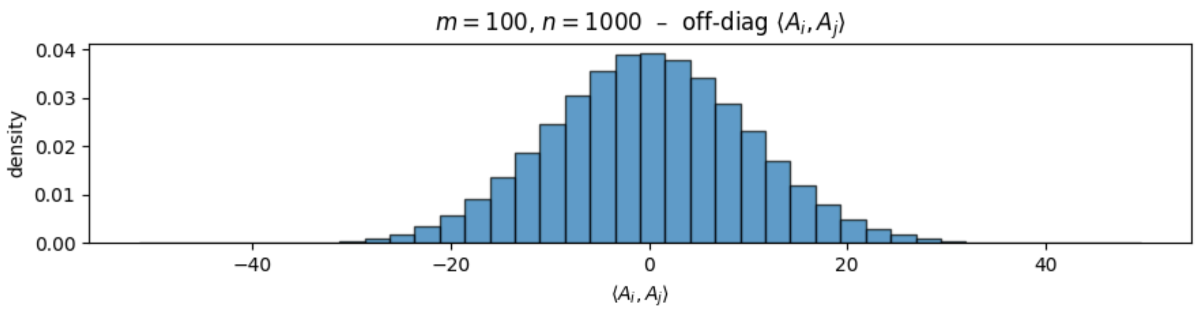


Figure 7 $\rightarrow$ Figure 14 shows that the distribution indeed approaches $N(0, 1)$

4. The Maximum Distribution (c)

In this section, we analyze the distribution of the maximum non-orthogonality between columns of a Gaussian matrix. This non-orthogonality is quantified by the maximum absolute value of the cosine similarity between any two distinct column vectors. Specifically, for a matrix $A \in \mathbb{R}^{m \times n}$, we study the distribution of the random variable:

$$M = \max_{i \neq j} \frac{|\langle A_i, A_j \rangle|}{\|A_i\| \|A_j\|} \tag{17}$$

Our experiment generates K independent realizations of this value, $M_1, M_2, ..., M_K$, by creating K different Gaussian matrices of size $m = 100, n = 300$. The histograms later shown in Section 4.2 display the empirical probability density function of this collection of maxima.

4.1. Theoretical Framework and the Gumbel Distribution

Let $C_{ij} = \frac{\langle A_i, A_j \rangle}{\|A_i\| \|A_j\|}$. For a given matrix A, we are examining the maximum of $N = \frac{n(n-1)}{2}$ random variables, $\left\{|C_{ij}|\right\}_{1 \leq i < j \leq n}$. For $m = 100$ and $n = 300$, this is the maximum of $N = 44850$ values.

We are interested in the maximum of $\left\{|C_{ij}|\right\}$. As established in previous sections:

- From part (a) (Section 2), for large m, $\|A_i\|$ concentrates around $\sqrt{m}$.
- From part (b) (Section 3), $Z_{ij} = \langle A_i, A_j \rangle$ is approximately $N(0, m)$.

Let's first characterize the distribution of a single variable C_{ij}.

$$C_{ij} = \frac{Z_{ij}}{\|A_i\| \|A_j\|} \approx \frac{N(0, m)}{\sqrt{m} \cdot \sqrt{m}} = \frac{N(0, m)}{m} \tag{18}$$

If a random variable $X \sim N(0, \sigma^2)$, then $\frac{X}{c} \sim N\left(0, \frac{\sigma^2}{c^2}\right)$. Thus:

$$C_{ij} \approx N\left(0, \frac{m}{m^2}\right) = N\left(0, \frac{1}{m}\right) \tag{19}$$

So, the individual correlation values are approximately drawn from a normal distribution with mean 0 and a small variance of $\frac{1}{m}$.

Our analysis, however, concerns the variable $M = \max_{i \neq j} |C_{ij}|$. The parent distribution is therefore not $N\left(0, \frac{1}{m}\right)$, but rather its absolute value, $\left|N\left(0, \frac{1}{m}\right)\right|$. This is known as a **folded normal distribution**.

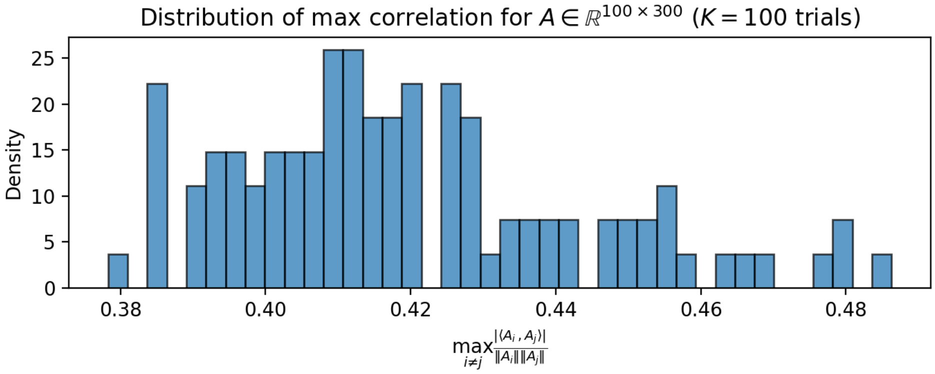
The tail of the folded normal distribution behaves identically to the tail of the underlying normal distribution. According to **Extreme Value Theory**, the limiting distribution for the maximum of many i.i.d. variables from a parent distribution with an exponential tail (like the normal distribution) is the **Gumbel distribution**.

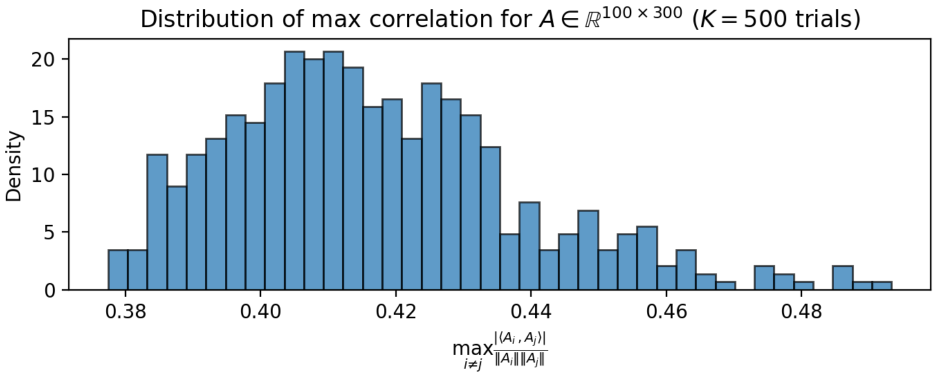
The probability density function (PDF) for the Gumbel distribution is given by:

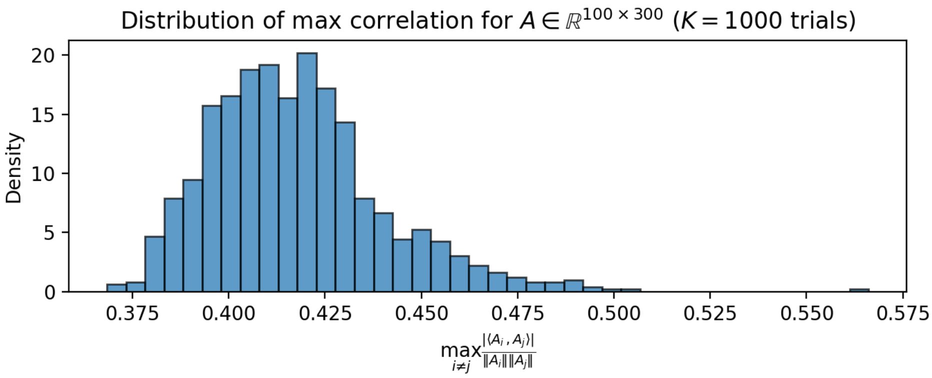
$$\begin{gathered} f(x; \mu, \beta) = \frac{1}{\beta} e^{-(z + e^{-z})} \\ z = \frac{x - \mu}{\beta} \end{gathered} \tag{20}$$

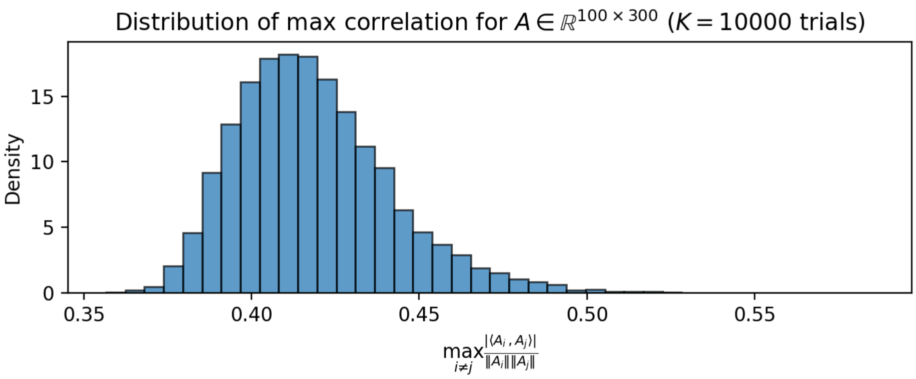
where μ is the mode of the distribution (location parameter) and β is the scale parameter (proportional to the standard deviation).

4.2. Analysis of the Histograms









The histograms generated, especially for large K (e.g., $K = 1000$ and $K = 10000$ as shown in Figure 17 and Figure 18, respectively), exhibit the distinct features of a Gumbel distribution:

- A single peak (unimodal).
- Asymmetry with a more extended tail on the right side.

As we can observe, growing K *(number of trials)* leads to a smoother plot and a clearer shape of the distribution, which aligns with the theoretical expectations of the Gumbel distribution.

The observed mode of the distribution is around 0.43, which is consistent with theoretical predictions. The location parameter μ can be approximated by:

$$\mu \approx \sqrt{\frac{2\ln(N)}{m}} = \sqrt{\frac{2\ln\left(\frac{n(n-1)}{2}\right)}{m}} \tag{21}$$

This formula arises from the well-known approximation for the expected maximum of N standard normal variables ($\sqrt{2\ln N}$), applied to our standardized variables $\left\{|\sqrt{m}C_{ij}|\right\}$.

For $m = 100$ and $n = 300$, we have $N = 44850$:

$$\mu \approx \sqrt{\frac{2\ln(44850)}{100}} \approx \sqrt{\frac{2 \cdot 10.71}{100}} = \sqrt{0.214} \approx 0.462 \tag{22}$$

This theoretical approximation gives a value in the general vicinity of the observed peak (around 0.42). The discrepancy arises because the variables $\left\{C_{ij}\right\}$ are not perfectly independent (for instance, $C_{1,2}$ and $C_{1,3}$ both depend on column A_1) and their distribution is only approximately normal. Nonetheless, this formula correctly shows that the peak of the distribution is determined by the dimensions m and n.

In conclusion, the observed distribution is a **Gumbel distribution**. This arises because we are plotting the maximum of a very large number of approximately independent, normally-distributed random variables (the cosine similarities).

5. Complexity

6. Another Maximum Distribution

7. Conclusion

Bibliography