Assignment 3 - Numerical Linear Algebra

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Abstract

coming soon

Contents

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1. Introduction

2. Norm Distribution (a)

2.1. The Chi-Square Distribution

Here we construct a theoretical basis for our analysis of the histograms shown in Section 2.2

When we generate a matrix $A \in \mathbb{R}^{m \times n}$, with $A_{ij} \sim N(0,1)$ independent, each column c_i is a gaussian vector in $\mathbb{R}^m$. if

$$x = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{pmatrix} \in \mathbb{R}^m \tag{1}$$

Is a column, then:

$$\begin{aligned} V = \|x\|_2 = \sqrt{\sum_{i=1}^{m} X_i^2} \\ V^2 = \sum_{i=1}^{m} X_i^2 \end{aligned} \tag{2}$$

Is of our interest. The expected value and variance are:

$$\begin{aligned} \mathbb{E}[V^2] = \mathbb{E}\left[\sum_{i=1}^{m} X_i^2\right] = \sum_{i=1}^{m} \mathbb{E}[X_i^2] = m \\ \operatorname{Var}(V^2) = \operatorname{Var}\left(\sum_{i=1}^{m} X_i^2\right) = \sum_{i=1}^{m} \operatorname{Var}(X_i^2) = 2m \end{aligned} \tag{3}$$

But we know that if $X_i \sim N(0,1)$ are independent:

$$\sum_{i=1}^{m} X_i^2 \sim \chi_m^2 \tag{4}$$

where χ_m is the chi-squared distribution with m degreees of freedom, better discussed in Section 2.1.

Taking the square root on eq. (4), we have:

$$V = \|x\|_2 = \sqrt{\sum_{i=1}^{m} X_i^2} \sim \sqrt{\chi_m^2} \sim \chi_m \tag{5}$$

The 2-norm of a vector x is distributed as a chi distribution with m degrees of freedom, in order to understand the distribution for many values of m, we can calculate the expected value and variance of this distribution as a function of m. The PDF of the chi distribution (with m degrees of freedom) is:

$$f_V(\varphi) = \frac{1}{2^{\frac{m}{2}-1} \cdot \Gamma\left(\frac{m}{2}\right)} \varphi^{m-1} e^{-\frac{\varphi^2}{2}} \tag{6}$$

So from this, the expected value is:

$$\mathbb{E}(V) = \sqrt{2} \cdot \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \tag{7}$$

And from this, the variance:

$$\mathrm{Var}(V) = m - \left(\sqrt{2} \cdot \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}\right)^2 \tag{8}$$

The Stirling Approximation provides a good approximation for the expected value and variance:

$$\mathbb{E}(V) \approx \sqrt{m} \cdot \left(1 - \frac{1}{4m} + O\left(\frac{1}{m^2}\right)\right) \tag{9}$$

$$\mathrm{Var}(V) \approx \frac{1}{2} + O\left(\frac{1}{m}\right) \tag{10}$$

2.2. Histograms

The first cell of this notebook has as expected output, with input being matrices with fixed $n = 1000$ and $m \in \{10, 20, 100, 200, 1000, 2000\}$, the following plots:

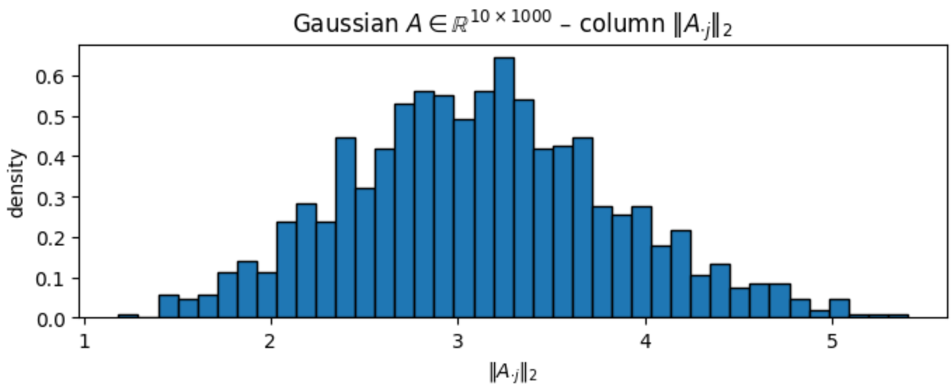


Figure 1: 10×1000 gaussian matrix

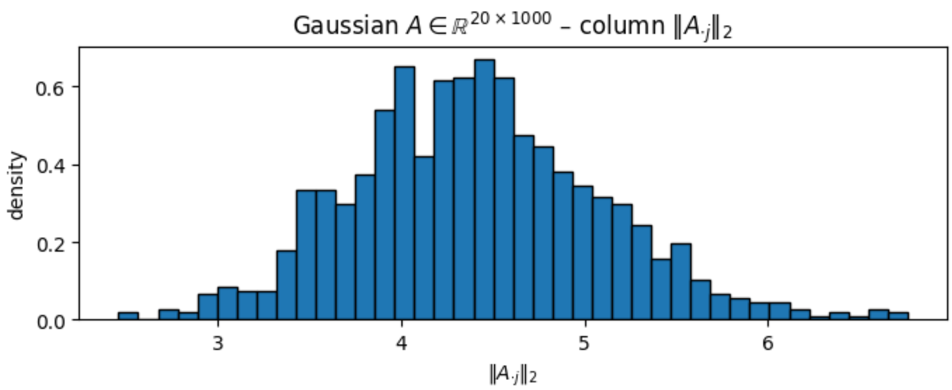


Figure 2: 20×1000 gaussian matrix

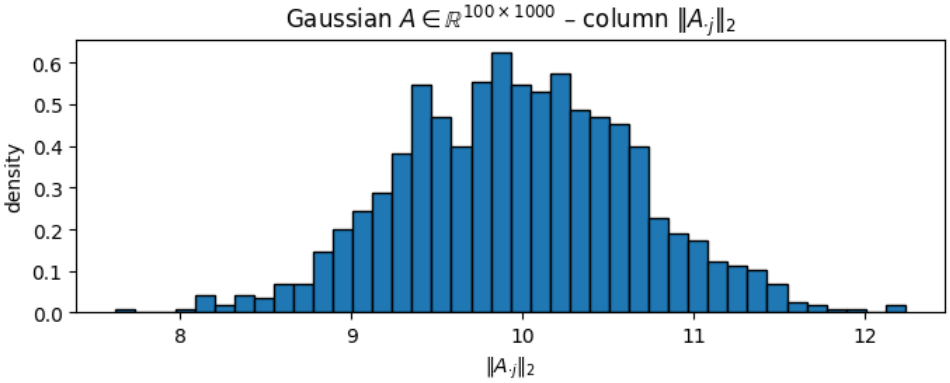


Figure 3: 100×1000 gaussian matrix

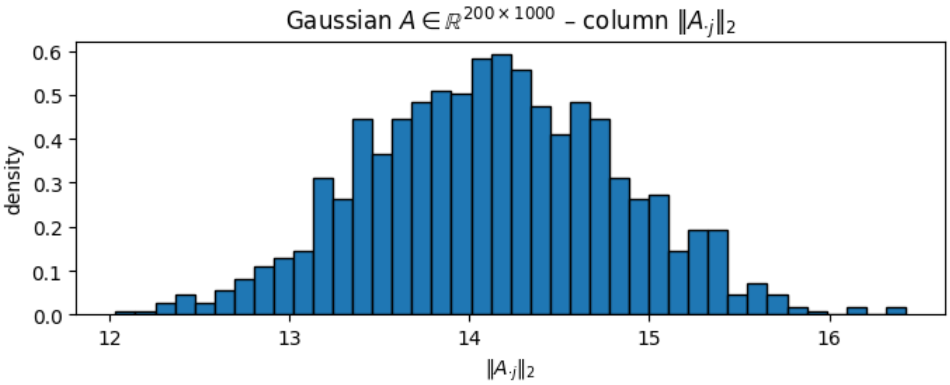


Figure 4: 200×1000 gaussian matrix

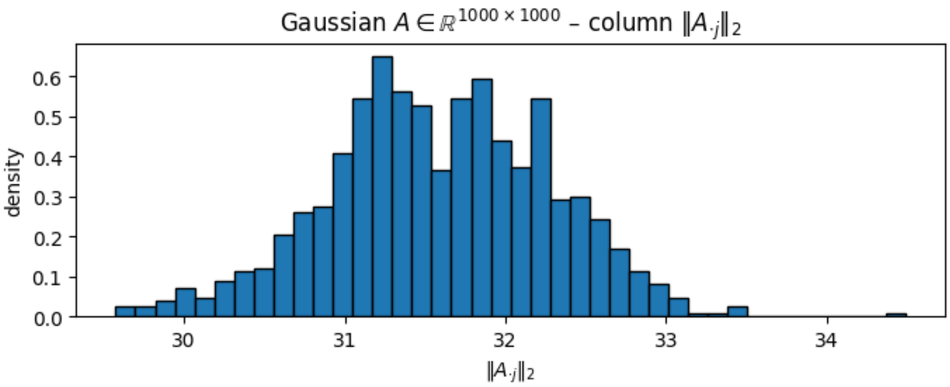


Figure 5: 1000×1000 gaussian matrix

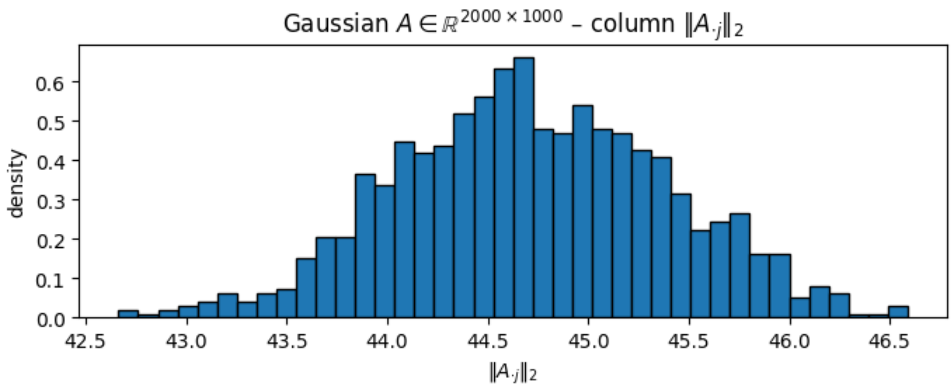


Figure 6: 2000×1000 gaussian matrix

m	**approximate μ_m (theory)**	**$[\mu \pm 3\sigma]$ (theory)**	**observed spike**	**visual range**
10	3.08	$1.0 - 5.18$	≈ 3.1	$1.2 - 5.1$
20	4.42	$2.3 - 6.52$	$\approx 4.3 - 4.4$	$2.9 - 6.4$
100	9.98	$7.9 - 12.1$	$\approx 9.9 - 10.0$	$7.8 - 12.0$
200	14.12	$12.0 - 16.2$	≈ 14.1	$12.2 - 16.2$
1000	31.61	$29.5 - 33.7$	$\approx 31.7 - 32.0$	$29.9 - 33.3$
2000	44.72	$42.6 - 46.8$	$\approx 44.5 - 45.0$	$42.8 - 46.6$

This table illustrates the expected value μ_m and the range $[\mu - 3\sigma, \mu + 3\sigma]$ for Figure 1 to Figure 6.

So apparently as m grows, the size of the gaussian vectors rapidly converge to $\sqrt{m}$, with small errors.

3. Inner Products (b)

Here we construct a theoretical basis for our analysis of the inner products shown in Section 3.1.

When we generate a matrix $A \in \mathbb{R}^{m \times n}$, with $A_{ij} \sim N(0,1)$ independent, each column c_i is a gaussian vector in $\mathbb{R}^m$. If The inner product of two gaussian vectors $x = (X_1, ..., X_n), y = (Y_1, ..., Y_n)$ is:

$$Z = \langle x, y \rangle = \sum_{i=1}^{m} X_i Y_i \tag{11}$$

With $X, Y \sim N(0,1)$. Since X_i, Y_j are independent, we have:

$$\begin{gathered} \mathbb{E}[Z] = \sum_{i=1}^{m} \mathbb{E}[X_i Y_i] = \sum_{i=1}^{m} \mathbb{E}[X_i]\mathbb{E}[Y_i] = 0 \\ \operatorname{Var}(Z) = \sum_{i=1}^{m} \operatorname{Var}(X_i Y_i) = \sum_{i=1}^{m} \mathbb{E}[X_i^2]\mathbb{E}[Y_i^2] = \sum_{i=1}^{m} 1 = m \end{gathered} \tag{12}$$

If $W = X_i Y_i$, we have:

$$M_W(\varphi) = \mathbb{E}[e^{\varphi W}] = \frac{1}{\sqrt{1 - \varphi^2}}, |\varphi| < 1 \tag{13}$$

Over all $W_i = X_i Y_i$:

$$M_Z(\varphi) = \mathbb{E}[e^{\varphi Z}] = (M_W(\varphi))^m = \left(\frac{1}{\sqrt{1-\varphi^2}}\right)^m = (1-\varphi^2)^{-\frac{m}{2}}, |\varphi| < 1 \tag{14}$$

And magically:

$$M_{\frac{Z}{\sqrt{m}}}(\varphi) = \left(1 - \frac{\varphi^2}{m}\right)^{-\frac{m}{2}} \Rightarrow \lim_{m\to\infty} M_{\frac{Z}{\sqrt{m}}}(\varphi) = e^{\frac{\varphi^2}{2}} \tag{15}$$

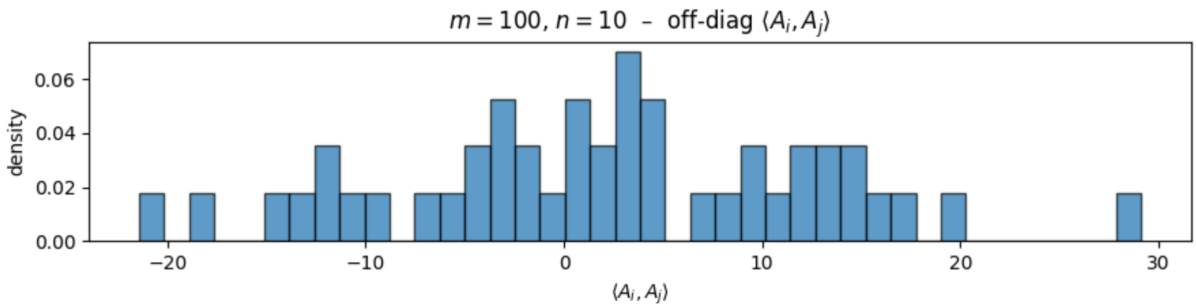
Precisely the moment generating function of a standard normal distribution, so as $m \to \infty$:

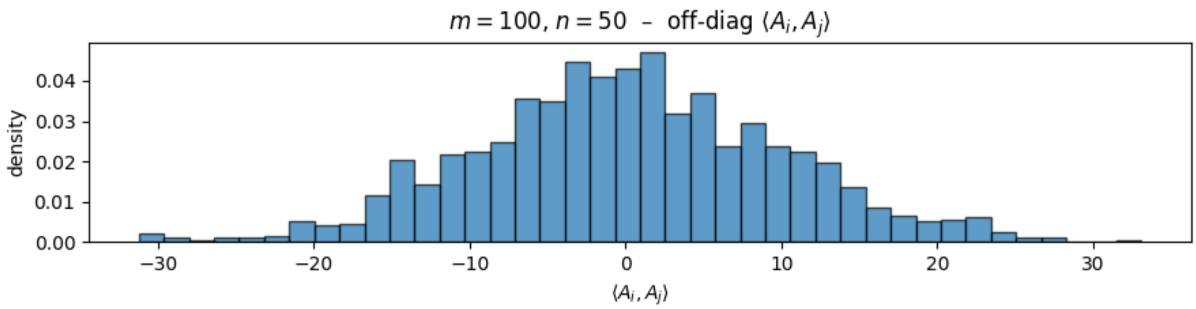
$$\frac{Z}{\sqrt{m}} \sim N(0,1) \tag{16}$$

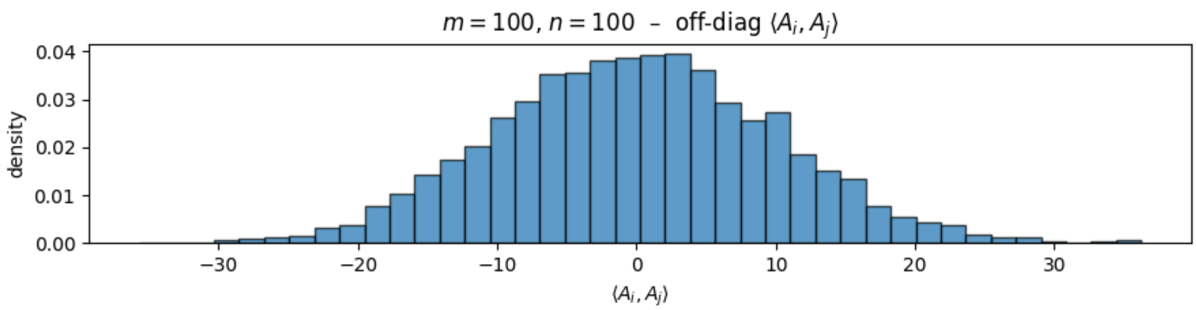
With a fixed $m = 100$, when $n \to \infty$ we can see the distribution approaching $N(0,1)$, as shown in Section 3.1

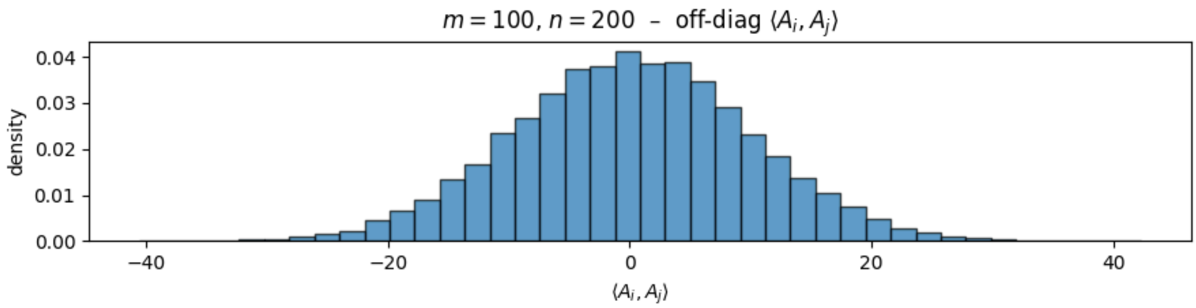
3.1. Histograms

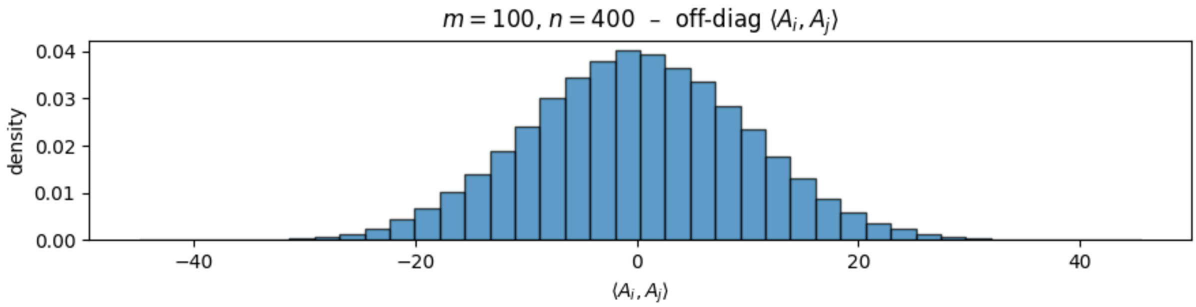
The following plots are an expected output for the second cell of this notebook, with input $m = 100, n \in \{10, 20, 30, 40, 50, 60, ..., 1000\}$:

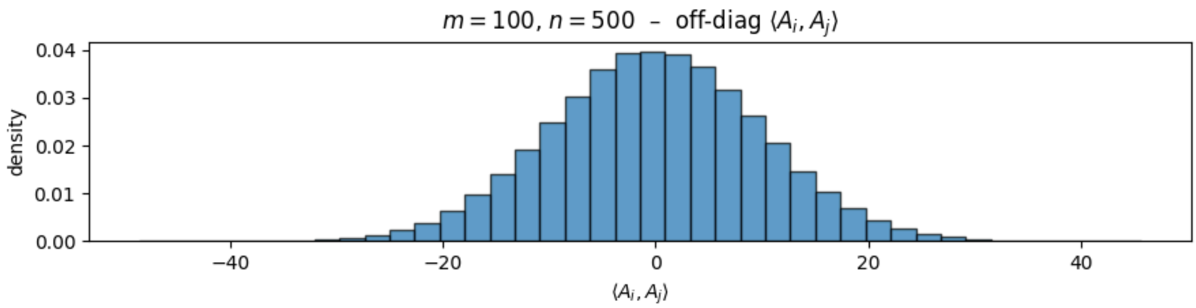


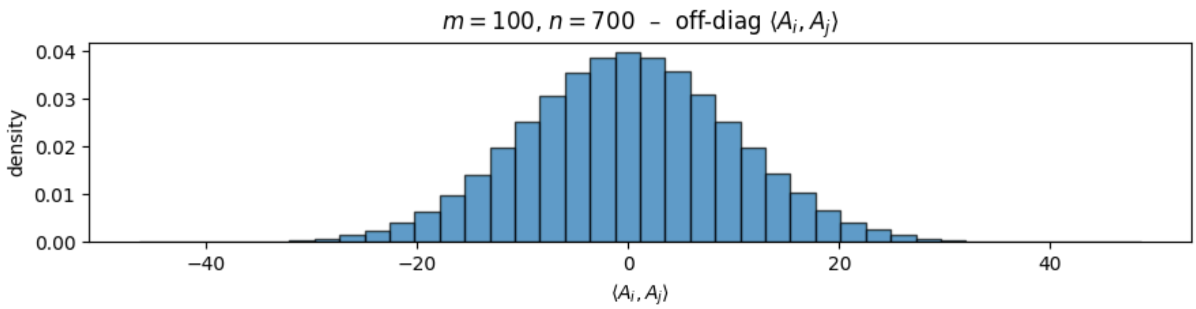












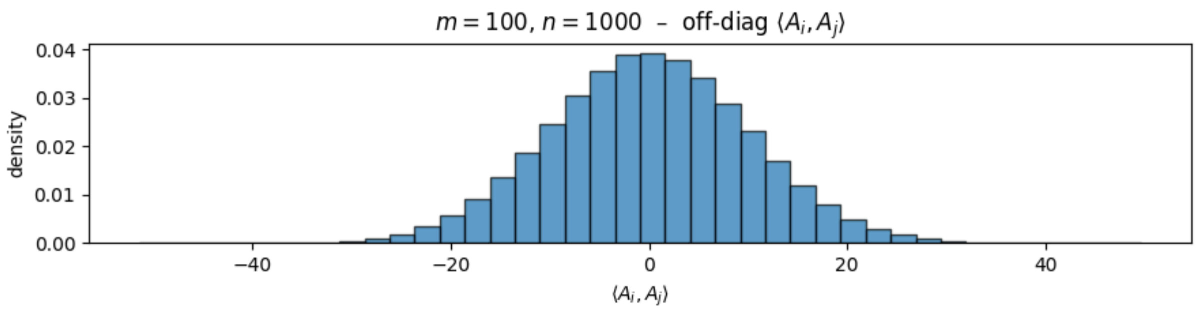


Figure 7 $\to$ Figure 14 shows that the distribution indeed approaches $N(0, 1)$

4. The Maximum Distribution

5. Complexity

6. Another Maximum Distribution

7. Conclusion

Bibliography