Assignment 3 - Numerical Linear Algebra

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Abstract

We design and test a function `to_hessemberg(A)` that reduces an arbitrary square matrix to (upper) Hessenberg form with Householder reflectors, returns the reflector vectors, the compact Hessenberg matrix H, and the accumulated orthogonal factor Q, verifying numerically that $A = QHQ^*$ and $Q^* Q = I$ for symmetric and nonsymmetric inputs of orders $10 - 10000$. Timings confirm the expected $O(\text{something})$ cost and reveal the $2 \times$ speed-up attainable for symmetric matrices through trivial bandwidth savings. Leveraging this routine, we investigate the spectral structure of orthogonal matrices: we show that all eigenvalues lie on the unit circle, analyse the consequences for the power method and inverse iteration, and obtain a closed-form spectrum for generic 2×2 orthogonals. Random 4×4 orthogonal matrices generated via QR factorisation are then reduced to Hessenberg form; the eigenvalues of their trailing 2×2 blocks are computed analytically and reused as fixed shifts in the QR iteration, where experiments demonstrate markedly faster convergence. Throughout, every algorithm is documented and supported by commented plots that corroborate the theoretical claims.

Contents

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1. Introduction

One could calculate the eigenvalues of a square matrix using the following algorithm:

1. Compute the n-th degree polynomial $\det(A - \lambda I) = 0$,
2. Solve for λ (somehow).

On step 2, the eigenvalue problem would have been reduced to a polynomial root-finding problem, which is awful and extremely ill-conditioned. From the previous assignment we know that in the denominator of the relative condition number $\kappa(x)$ there's a $|x - n|$. So $\kappa(x) \to \infty$ when $x \to 0$. As an example, consider the polynomial

$$\begin{aligned} p(x) = (x-2)^9 = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 \\ -4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512 \end{aligned} \tag{1}$$

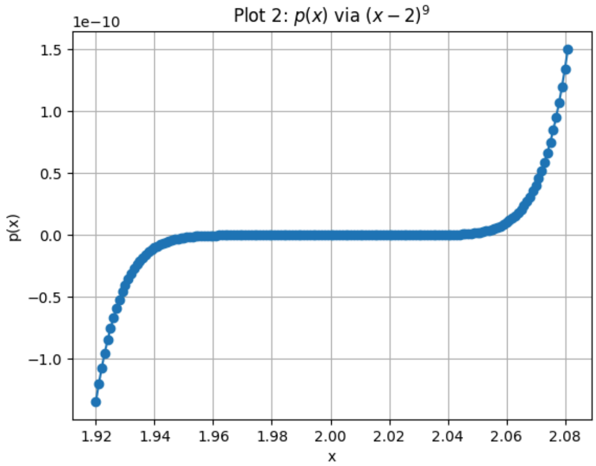

Figure 1: $p(x)$ via the coefficients in eq. (1)

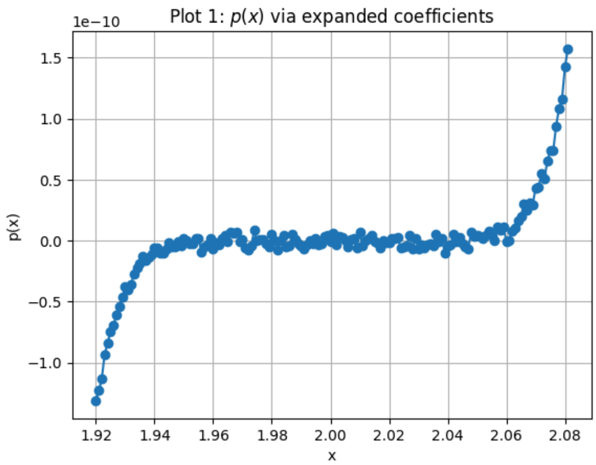

Figure 2: $p(x)$ via $(x-2)^9$

Figure 2 shows a smooth curve, while Figure 1 shows a weird oscillation around $x = 0$ (And pretty much everywhere else if the reader is sufficiently persistent).

This is due to the round-off errors when $x \approx 0$ and the big coefficients of the polynomial. In general, polynomial are very sensitive to perturbations in the coefficients, which is why rootfinding is a bad idea to find eigenvalues.

One could also try the following:

1. Compute A^*A
2.

2. Hessemberg Reduction (Problem 1)

2.1. Calculating the Householder Reflectors (a)

2.2. Checking the Result (b)

2.3. Complexity (c)

2.4. The Symmetric Case (d)

3. Orthogonal Matrices (Problem 2)

3.1. Eigenvalues and Iterative Methods (a)

3.2. The 2×2 Case (b)

3.3. Random Orthogonal Matrices (c)

3.4. Shift With an Eigenvalue (d)

4. Conclusion

Bibliography