Final Assignment - Probability Theory

Arthur Rabello Oliveira[1]

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Abstract

coming soon

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[1]Escola de Matemática Aplicada, Fundação Getúlio Vargas (FGV/EMAp), email: arthur.oliveira.1@fgv.edu.br

1. Introduction

2. User Parameters

3. Problem 1

3.1. Problem 1 (a)

3.2. Problem 1 (b)

We have 2 kinds of customers, so the waiting time S is a combination of distributions:

$$\mathbb{E}(S) = f\mathbb{E}(S_{\text{fast}}) + (1-f)\mathbb{E}(S_{\text{slow}}) \tag{1}$$

From Section 2 we have a $f \approx 0.35$ of fast customers and a PDF of:

$$g(\varphi) = 0.25 + c(\varphi - 4), 2 < \varphi < 6, c = 0.02 \tag{2}$$

The expected value is, therefore:

$$\begin{aligned} \mathbb{E}(S_{\text{fast}}) &= \int_{-\infty}^{\infty} \varphi g(\varphi) d\varphi = \int_{2}^{6} \varphi(0.25 + 0.02(\varphi - 4)) d\varphi \\ &= \int_{2}^{6} 0.25\varphi d\varphi + \int_{2}^{6} 0.02\varphi^2 d\varphi - \int_{2}^{6} 0.08\varphi d\varphi \\ &= 0.25 \int_{2}^{6} \varphi d\varphi + 0.02 \int_{2}^{6} \varphi^2 d\varphi - 0.08 \int_{2}^{6} \varphi d\varphi \\ &= 0.25 \left[\frac{\varphi^2}{2}\right]_2^6 + 0.02 \left[\frac{\varphi^3}{3}\right]_2^6 - 0.08 \left[\frac{\varphi^2}{2}\right]_2^6 \\ &= 0.25(36-4) + 0.02(216-8) - 0.08(36-4) \\ &= \frac{32}{4} + 0.02 \cdot \frac{208}{3} - 0.08 \cdot \frac{32}{4} \approx 4.10 \end{aligned} \tag{3}$$

The remaining $S_{\text{slow}} \sim \text{Exp}(m = 5.5)$ customers have an expected value of $\mathbb{E}(S_{\text{slow}}) = 5.5$, so the total expected waiting time is:

$$\mathbb{E}(S) = 0.35 \cdot 4.10 + 0.65 \cdot 5.5 \approx 5.01 \text{ minutes} \tag{4}$$

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On Section 3.1 our simulation shows an average waiting time of 5.060 min, close enough to the analytical expected value.

Now for the standard deviation, we calculate the variance of S as follows:

$$
\begin{aligned}
E(S_{\text{fast}}) &= 4.10 \\
E\left(S_{\text{fast}}^2\right) &= \int_{-\infty}^{\infty} \varphi^2 g(\varphi) d\varphi = \int_2^6 \varphi^2 (0.25 + 0.02(\varphi - 4)) d\varphi \\
&= 0.25 \int_2^6 \varphi^2 d\varphi + 0.02 \int_2^6 \varphi^3 d\varphi - 0.08 \int_2^6 \varphi^2 d\varphi \\
&= 0.25 \left[\frac{\varphi^3}{3}\right]_2^6 + 0.02 \left[\frac{\varphi^4}{4}\right]_2^6 - 0.08 \left[\frac{\varphi^3}{3}\right]_2^6 \approx 18.18
\end{aligned} \tag{5}
$$

And now for the slow customers $S_{\text{slow}} \sim \text{Exp}(5.5)$:

$$
\begin{aligned}
\mathbb{E}(S_{\text{slow}}) &= 5.5 \\
\mathbb{E}(S_{\text{slow}}^2) &= 2m^2 = 2 \cdot 5.5^2 = 60.5
\end{aligned} \tag{6}
$$

Therefore the total variance is:

$$
\begin{aligned}
\mathbb{E}(S) &= 5.01 \\
\mathbb{E}(S^2) &= f\mathbb{E}\left(S_{\text{fast}}^2\right) + (1 - f)\mathbb{E}\left(S_{\text{slow}}^2\right) \\
&= 0.35 \cdot 18.18 + 0.65 \cdot 60.5 \approx 45.69 \\
\text{Var}(S) &= \mathbb{E}(S^2) - \mathbb{E}(S)^2 = 45.69 - 5.01^2 \approx 20.56
\end{aligned} \tag{7}
$$

And finally the standard deviation is:

$$
\sigma_S = \sqrt{\text{Var}(S)} \approx 4.53 \text{ minutes} \tag{8}
$$

On Section 3 our simulation shows a standard deviation of 4.508 minutes, which is close enough to the analytical value.

4. Problem 2

4.1. Problem 2 (a)

4.2. Problem 2 (b)

4.3. Problem 2 (c)

4.4. Problem 2 (d)