Notes on MGO

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1 GO

GO assumes λ is the shortest lenghtscale, suppose a stationary wavefield ψ is governed by a linear wave equation

$$\hat{\mathcal{D}}(\mathbf{x}, -i\partial_{\mathbf{x}})\psi(\mathbf{x}) \tag{1}$$

With D the dispersion kernel, and suppose further that ψ can be partitioned into a rapidly varying phase θ and a slowly varying envelope ϕ:

$$\psi(\mathbf{x}) = \phi(\mathbf{x})e^{i\theta(\mathbf{x})} \tag{2}$$

Then it can be shown that θ and ϕ asymptotically satisfy the following two relations:

1. The local dispersion relation:

$$\mathcal{D}[\mathbf{x}, \partial_{\mathbf{x}}\theta(\mathbf{x})] = 0 \tag{3}$$

2. The envelope transport equation

$$2\mathbf{v}(\mathbf{x})^\intercal \partial_{\mathbf{x}}\phi(\mathbf{x}) + [\nabla \cdot \mathbf{v}(\mathbf{x})]\phi(\mathbf{x}) = 0 \tag{4}$$

Where $\mathcal{D}(\mathbf{x}, \mathbf{k})$ is the Weyl symbol of $\hat{\mathcal{D}}(\mathbf{x}, -i\partial_x)$ (appendix A) and

$$\mathbf{v}(\mathbf{x}) \doteq \partial_{\mathbf{k}}\mathcal{D}(\mathbf{x}, \mathbf{k})|_{\mathbf{k}=\partial_{\mathbf{x}}\theta(\mathbf{x})} \tag{5}$$

is proportional to the local group velocity. If we neglect dissipation, thus assuming $\hat{D}$ to be Hermitian and consequently, both $\mathcal{D}$ and $\mathbf{v}$ real. Then, the envelope transport equation can be cast as a conservation relation:

$$\nabla \cdot [|\phi(\mathbf{x})|^2\mathbf{v}(\mathbf{x})] = 0 \tag{6}$$

I.e conservation of wave action flux.

The local dispersion relation defines a (2N-1) dimensional volume in the 2N-D phase space with coordinates $(\mathbf{x}, \mathbf{k})$. For coherent wavefields that have a single wavevector $\mathbf{k}(\mathbf{x})$ (or a finite superposition of such wavevectors), one can identify

$$\mathbf{k}(\mathbf{x}) = \partial_{\mathbf{x}}\theta(\mathbf{x}) \tag{7}$$

Such that $\mathbf{k}$ is restricted to an N-D surface contained within the (2N-1)-D volume (which N-D surface is dictated by initial conditions). This N-D surface is called the *ray manifold* which is a Lagrangian manifold. In particular this means that all vectors $\{\mathbf{T}_j\}$ tangent to it satisfy

$$\mathbf{T}_j^\intercal \mathrm{J}_{2N} \mathbf{T}_{j'} = 0 \tag{8}$$

Where we have introduced the 2Nx2N matrix

$$\mathrm{J}_{2N} = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix} \tag{9}$$

The ray manifold is a central object in GO and MGO, as such it will be useful to have an explicit construction of it. This is provided by the ray (Hamilton's) equations:

$$\begin{aligned} \partial_\xi \mathbf{x} &= \partial_{\mathbf{k}} \mathcal{D}(\mathbf{x}, \mathbf{k}) && (10) \\ \partial_\xi \mathbf{k} &= -\partial_{\mathbf{x}} \mathcal{D}(\mathbf{x}, \mathbf{k}) && (11) \\ & && (12) \end{aligned}$$

The family of solution trajectories $[\mathbf{x}(\xi), \mathbf{k}(\xi)]$ for a corresponding family of initial conditions $[\mathbf{x}(0), \mathbf{k}(0)]$ then trace out the ray manifold. Since the ray manifold is N-D, let's introduce a set of N-D coordinates τ such that it can be paramterized as $[\mathbf{x}(\tau), \mathbf{k}(\tau)]$. We shall choose $\tau_1 = \xi$ as a "longitudinal" coordinate along each ray and the remaining $\tau_\perp \doteq (\tau_2, ..., \tau_N)$ as "transverse" coordinates that describe the different initial conditions of each ray.

Along this family of rays, the envelope transport equation takes the form

$$2j(\tau)\partial_{\tau_1}\phi(\tau) + \phi(\tau)\partial_{\tau_1} j(\tau) \tag{13}$$

Where we have introduced the Jacobian determinant of the ray trajectories

$$j(\tau) \doteq \det \partial_\tau \mathbf{x}(\tau) \tag{14}$$

Equation 13 can be formally solved to yield

$$\phi(\tau) = \phi_0(\tau_\perp)\sqrt{\frac{j_0(\tau_\perp)}{j(\tau)}} \tag{15}$$

This thus states that

$$|\phi(\tau)|^2 |\mathbf{v}| \mathrm{d}A \tag{16}$$

is constant along the ray, i.e action is conserved for an infinitesimal "ray tube". Having determined ϕ from equation 15 and θ from integrating the rays using

Hamilton's equations and then using equation 7, the full field Ψ can be constructed by summing over all rays that arrive at a given $\mathbf{x}$, i.e

$$\Psi(\mathbf{x}) = \sum_{t\in\tau(\mathbf{x})} \phi(\mathbf{t})e^{i\theta(\mathbf{t})} \tag{17}$$

$$\equiv \sum_{t\in\tau(\mathbf{x})} \phi_0(\tau_\perp)\sqrt{\tfrac{j_0(\tau_\perp)}{j(\tau)}}e^{i\int \mathbf{k}^\intercal \mathrm{d}\mathbf{x}} \tag{18}$$

With $\tau(\mathbf{x})$ the multi-valued formal function inverse of $\mathbf{x}(\tau)$, clearly the GO field diverges where the jacobian determinant tends to zero, or equivalently where

$$\det\, \partial_{\mathbf{x}}\mathbf{k} \equiv \det\, \partial_{\mathbf{xx}}\theta \to \infty \tag{19}$$

such locations are called *caustics*.

2 Metaplectic Geometrical Optics

Rather than describing waves as propagating in some configuration space with coordinates $\mathbf{x}$ according to pseudo-differential equation of the form (1), it is more natural to describe waves as state vectors $|\psi\rangle$ in a Hilbert space being acted upon by operators. Then, partial differential equations that govern wavefields can be understood as projections of the invariant wave equations

$$\hat{D}(\hat{\mathbf{x}}, \hat{\mathbf{k}})\,|\psi\rangle = |0\rangle \tag{20}$$

on a particular basis.

3 Ray Tracing

The GO equations can be solved by finding phase space trajectories $\mathbf{z}(\tau) = (\mathbf{x}(\tau), \mathbf{k}(\tau))^\intercal$ satisfying the local dispersion relation. Such trajectories are called *rays*. Here $\tau = (\tau_1, \tau_\perp)^\intercal$ where τ_1 is the longitudinal time parameter and $\tau_\perp = (x_2^{(0)}, x_3^{(0)})^\intercal$ are the perpendicular initial coordinates of the ray. Given an initial condition $\mathbf{z}(0, \tau_\perp) = (\mathbf{x}_0, \mathbf{k}_0)^\intercal$, a ray can be found from Hamilton's ray equations:

$$\partial_{\tau_1}\mathbf{x}(\tau_1) = -\partial_{\mathbf{k}}\mathcal{D}(\mathbf{x}, \mathbf{k}) \tag{21}$$

$$\partial_{\tau_1}\mathbf{k}(\tau_1) = -\partial_{\mathbf{x}}\mathcal{D}(\mathbf{x}, \mathbf{k}) \tag{22}$$

We can launch a finite family of rays on a discrete $\tau_\perp$-grid, thus populating the space

A Wigner-Weyl transform

The Wigner-Weyl transform (WWT, denoted as $\mathbb{W}$) maps a given operator $\hat{A}(\hat{\mathbf{z}})$ to a corresponding phase-space function $\mathcal{A}(\mathbf{z})$, called the *Weyl symbol* of $\hat{A}$, as

$$\mathcal{A}(\mathbf{z}) = \mathbb{W}[\hat{A}(\hat{\mathbf{z}})] \tag{23}$$

$$\doteq \int d\zeta \tfrac{i\zeta^\intercal \mathtt{J}_{2N}\mathbf{z}}{(2\pi)^N}\mathrm{tr}[e^{-i\zeta^\intercal \mathtt{J}_{2N}\hat{\mathbf{z}}}\hat{A}] \tag{24}$$

With the integral taken over the phase space. and

$$\mathsf{J}_{2N} = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix} \tag{25}$$

The inverse WWT maps a phase-space function to an operator:

$$\hat{A}(\mathbf{z}) = \mathbb{W}^{-1}[\mathcal{A}(\mathbf{z})] \tag{26}$$

$$\doteq \int \frac{d\mathbf{z}' d\zeta}{(2\pi)^{2N}} \mathcal{A}(\mathbf{z}') e^{-i\zeta^{\intercal} \mathsf{J}_{2N} \mathbf{z}' + i\zeta^{\intercal} \mathsf{J}_{2N} \hat{\mathbf{z}}} \tag{27}$$