

Notes on MGO

Arthur Adriaens

March 21, 2024

1 GO

GO assumes λ is the shortest lengthscale, suppose a stationary wavefield ψ is governed by a linear wave equation

$$\hat{\mathcal{D}}(\mathbf{x}, -i\partial_{\mathbf{x}})\psi(\mathbf{x}) \quad (1)$$

With D the dispersion kernel, and suppose further that ψ can be partitioned into a rapidly varying phase θ and a slowly varying envelope ϕ :

$$\psi(\mathbf{x}) = \phi(\mathbf{x})e^{i\theta(\mathbf{x})} \quad (2)$$

Then it can be shown that θ and ϕ asymptotically satisfy the following two relations:

1. The local dispersion relation:

$$\mathcal{D}[\mathbf{x}, \partial_{\mathbf{x}}\theta(\mathbf{x})] = 0 \quad (3)$$

2. The envelope transport equation

$$2\mathbf{v}(\mathbf{x})^T \partial_{\mathbf{x}}\phi(\mathbf{x}) + [\nabla \cdot \mathbf{v}(\mathbf{x})]\phi(\mathbf{x}) = 0 \quad (4)$$

Where $\mathcal{D}(\mathbf{x}, \mathbf{k})$ is the Weyl symbol of $\hat{\mathcal{D}}(\mathbf{x}, -i\partial_x)$ (appendix A) and

$$\mathbf{v}(\mathbf{x}) \doteq \partial_{\mathbf{k}}\mathcal{D}(\mathbf{x}, \mathbf{k})|_{\mathbf{k}=\partial_{\mathbf{x}}\theta(\mathbf{x})} \quad (5)$$

is proportional to the local group velocity. If we neglect dissipation, thus assuming \hat{D} to be Hermitian and consequently, both \mathcal{D} and \mathbf{v} real. Then, the envelope transport equation can be cast as a conservation relation:

$$\nabla \cdot [|\phi(\mathbf{x})|^2 \mathbf{v}(\mathbf{x})] = 0 \quad (6)$$

I.e conservation of wave action flux.

The local dispersion relation defines a $(2N-1)$ dimensional volume in the $2N$ -D phase space with coordinates (\mathbf{x}, \mathbf{k}) . For coherent wavefields that have a single wavevector $\mathbf{k}(\mathbf{x})$ (or a finite superposition of such wavevectors), one can identify

$$\mathbf{k}(\mathbf{x}) = \partial_{\mathbf{x}}\theta(\mathbf{x}) \quad (7)$$

Such that \mathbf{k} is restricted to an N -D surface contained within the $(2N-1)$ -D volume (which N -D surface is dictated by initial conditions). This N -D surface is called the *ray manifold* which is a Lagrangian manifold.

2 Metaplectic Geometrical Optics

Rather than describing waves as propagating in some configuration space with coordinates \mathbf{x} according to pseudo-differential equation of the form (1), it is more natural to describe waves as state vectors $|\psi\rangle$ in a Hilbert space being acted upon by operators. Then, partial differential equations that govern wavefields can be understood as projections of the invariant wave equations

$$\hat{D}(\hat{\mathbf{x}}, \hat{\mathbf{k}}) |\psi\rangle = |0\rangle \quad (8)$$

on a particular basis.

A Wigner-Weyl transform

The Wigner-Weyl transform (WWT, denoted as \mathbb{W}) maps a given operator $\hat{A}(\hat{\mathbf{z}})$ to a corresponding phase-space function $\mathcal{A}(\mathbf{z})$, called the *Weyl symbol* of \hat{A} , as

$$\mathcal{A}(\mathbf{z}) = \mathbb{W}[\hat{A}(\hat{\mathbf{z}})] \quad (9)$$

$$\doteq \int d\zeta \frac{i\zeta^\top J_{2N} \mathbf{z}}{(2\pi)^N} \text{tr}[e^{-i\zeta^\top J_{2N} \hat{\mathbf{z}}} \hat{A}] \quad (10)$$

With the integral taken over the phase space. and

$$J_{2N} = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix} \quad (11)$$

The inverse WWT maps a phase-space function to an operator:

$$\hat{A}(\mathbf{z}) = \mathbb{W}^{-1}[\mathcal{A}(\mathbf{z})] \quad (12)$$

$$\doteq \int \frac{d\mathbf{z}' d\zeta}{(2\pi)^{2N}} \mathcal{A}(\mathbf{z}') e^{-i\zeta^\top J_{2N} \mathbf{z}' + i\zeta^\top J_{2N} \hat{\mathbf{z}}} \quad (13)$$