Notes on MGO

Arthur Adriaens

March 25, 2024

1 GO

GO assumes λ is the shortest lenghtscale, suppose a stationary wavefield ψ is governed by a linear wave equation

$$\hat{\mathcal{D}}(\mathbf{x}, -i\partial_{\mathbf{x}})\psi(\mathbf{x}) \tag{1}$$

With D the dispersion kernel, and suppose further that ψ can be partitioned into a rapidly varying phase θ and a slowly varying envelope ϕ:

$$\psi(\mathbf{x}) = \phi(\mathbf{x})e^{i\theta(\mathbf{x})} \tag{2}$$

Then it can be shown that θ and ϕ asymptotically satisfy the following two relations:

1. The local dispersion relation:

$$\mathcal{D}[\mathbf{x}, \partial_{\mathbf{x}}\theta(\mathbf{x})] = 0 \tag{3}$$

2. The envelope transport equation

$$2\mathbf{v}(\mathbf{x})^{\intercal}\partial_{\mathbf{x}}\phi(\mathbf{x}) + [\nabla \cdot \mathbf{v}(\mathbf{x})]\phi(\mathbf{x}) = 0 \tag{4}$$

Where $\mathcal{D}(\mathbf{x}, \mathbf{k})$ is the Weyl symbol of $\hat{\mathcal{D}}(\mathbf{x}, -i\partial_x)$ (appendix A) and

$$\mathbf{v}(\mathbf{x}) \doteq \partial_{\mathbf{k}}\mathcal{D}(\mathbf{x}, \mathbf{k})|_{\mathbf{k}=\partial_{\mathbf{x}}\theta(\mathbf{x})} \tag{5}$$

is proportional to the local group velocity. If we neglect dissipation, thus assuming $\hat{D}$ to be Hermitian and consequently, both $\mathcal{D}$ and $\mathbf{v}$ real. Then, the envelope transport equation can be cast as a conservation relation:

$$\nabla \cdot [|\phi(\mathbf{x})|^2\mathbf{v}(\mathbf{x})] = 0 \tag{6}$$

I.e conservation of wave action flux.

The local dispersion relation defines a (2N-1) dimensional volume in the 2N-D phase space with coordinates $(\mathbf{x}, \mathbf{k})$. For coherent wavefields that have a single wavevector $\mathbf{k}(\mathbf{x})$ (or a finite superposition of such wavevectors), one can identify

$$\mathbf{k}(\mathbf{x}) = \partial_{\mathbf{x}}\theta(\mathbf{x}) \tag{7}$$

Such that $\mathbf{k}$ is restricted to an N-D surface contained within the (2N-1)-D volume (which N-D surface is dictated by initial conditions). This N-D surface is called the *ray manifold* which is a Lagrangian manifold.

2 Metaplectic Geometrical Optics

Rather than describing waves as propagating in some configuration space with coordinates $\mathbf{x}$ according to pseudo-differential equation of the form (1), it is more natural to describe waves as state vectors $|\psi\rangle$ in a Hilbert space being acted upon by operators. Then, partial differential equations that govern wavefields can be understood as projections of the invariant wave equations

$$\hat{D}(\hat{\mathbf{x}}, \hat{\mathbf{k}})\, |\psi\rangle = |0\rangle \tag{8}$$

on a particular basis.

3 Ray Tracing

The GO equations can be solved by finding phase space trajectories $\mathbf{z}(\tau) = (\mathbf{x}(\tau), \mathbf{k}(\tau))^\intercal$ satisfying the local dispersion relation. Such trajectories are called *rays*. Here $\tau = (\tau_1, \tau_\perp)^\intercal$ where τ_1 is the longitudinal time parameter and $\tau_\perp = (x_2^{(0)}, x_3^{(0)})^\intercal$ are the perpendicular initial coordinates of the ray. Given an initial condition $\mathbf{z}(0, \tau_\perp) = (\mathbf{x}_0, \mathbf{k}_0)^\intercal$, a ray can be found from Hamilton's ray equations:

$$\partial_{\tau_1} \mathbf{x}(\tau_1) \quad = -\partial_{\mathbf{k}} \mathcal{D}(\mathbf{x}, \mathbf{k}) \tag{9}$$

$$\partial_{\tau_1} \mathbf{k}(\tau_1) \quad = -\partial_{\mathbf{x}} \mathcal{D}(\mathbf{x}, \mathbf{k}) \tag{10}$$

We can launch a finite family of rays on a discrete $\tau_\perp$-grid

A Wigner-Weyl transform

The Wigner-Weyl transform (WWT, denoted as $\mathbb{W}$) maps a given operator $\hat{A}(\hat{\mathbf{z}})$ to a corresponding phase-space function $\mathcal{A}(\mathbf{z})$, called the *Weyl symbol* of $\hat{A}$, as

$$\mathcal{A}(\mathbf{z}) \qquad = \mathbb{W}[\hat{A}(\hat{\mathbf{z}})] \tag{11}$$

$$\doteq \int d\zeta \tfrac{i\zeta^\intercal \mathrm{J}_{2N}\mathbf{z}}{(2\pi)^N} \mathrm{tr}[e^{-i\zeta^\intercal \mathrm{J}_{2N}\hat{\mathbf{z}}} \hat{A}] \tag{12}$$

With the integral taken over the phase space. and

$$\mathrm{J}_{2N} = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix} \tag{13}$$

The inverse WWT maps a phase-space function to an operator:

$$\hat{A}(\mathbf{z}) \qquad = \mathbb{W}^{-1}[\mathcal{A}(\mathbf{z})] \tag{14}$$

$$\doteq \int \tfrac{d\mathbf{z}' d\zeta}{(2\pi)^{2N}} \mathcal{A}(\mathbf{z}') e^{-i\zeta^\intercal \mathrm{J}_{2N}\mathbf{z}' + i\zeta^\intercal \mathrm{J}_{2N}\hat{\mathbf{z}}} \tag{15}$$