Nuclear instrumentation: Particle Accelerators

Arthur Adriaens

Ghent University

1. Exercise 1: Tumor irradiation

1.1. Triple focusing distance

The momentum p_0 is given by:

$$p_0 c = \sqrt{{E_k}^2 + 2m_0c^2E_k} = 20.5046 MeV \approx 3.2852 \times 10^{-12} J$$

We have that

$$B_0\rho_0 = \frac{p_0}{q}$$

i.e

$$B_0 = \frac{p_0}{q\rho_0} \tag{1}$$

If we define the particle vectors as

$$\boldsymbol{X}(s) = \begin{bmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \\ \delta_1(s) \\ \delta \end{bmatrix}$$

Then the transfer matrix method describes

$$\boldsymbol{X}(s) = M(s)\boldsymbol{X}(0) \tag{2}$$

We can decouple the transfer matrix into 2 matrices:

$$M_H = \begin{bmatrix} m_{11} & m_{12} & m_{16} \\ m_{21} & m_{22} & m_{26} \\ 0 & 0 & 1 \end{bmatrix} M_V = \begin{bmatrix} m_{33} & m_{43} \\ m_{43} & m_{44} \end{bmatrix}$$

I.e if we want to solve an analogous system as equation 2 it'll be for a split-up version of the particle vectors:

$$\boldsymbol{X}_H(s) = \begin{bmatrix} x(s) \\ x'(s) \\ \delta \end{bmatrix} \boldsymbol{X}_V(s) = \begin{bmatrix} y(s) \\ y'(s) \end{bmatrix}$$

Of course in a magnet the momentum doesn't change so δ is independent of the path. For field-free region (drift space) we have the transfer matrices

$$M_H = \begin{bmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M_V = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$$

For the dipole magnet we have the transfer matrices:

$$M_H = \begin{bmatrix} \cos\alpha & \rho_0 \sin\alpha & \rho_0(1-\cos\alpha) \\ \frac{-\sin\alpha}{\rho_0} & \cos\alpha & \sin\alpha \\ 0 & 0 & 1 \end{bmatrix} M_V = \begin{bmatrix} 1 & \rho_0\alpha \\ 0 & 1 \end{bmatrix}$$

It's given that the entrance to the magnet and exit are rotated, implying the transfer matrices

$$M_H = \begin{bmatrix} 1 & 0 & 0 \\ \frac{\tan\beta}{\rho_0} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M_V = \begin{bmatrix} 1 & 0 \\ -\frac{\tan\beta}{\rho_0} & 1 \end{bmatrix}$$

For the entrance it's a rotation of 45 degrees and for the exit a rotation of 32.4 degrees := β_2, α is also given to be 270 degrees. In full, the equations of motion are found by solving:

$$\begin{bmatrix} x(s) \\ x'(s) \\ \delta \end{bmatrix} = \begin{bmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ \frac{\tan\beta_2}{\rho_0} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -\rho_0 & \rho_0 \\ \frac{1}{\rho_0} & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{\rho_0} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x(0) \\ x'(0) \\ \delta \end{bmatrix}$$

And

$$\begin{bmatrix} y(s) \\ y'(s) \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\frac{\tan\beta_2}{\rho_0} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{3\rho_0\pi}{2} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\frac{1}{\rho_0} & 1 \end{bmatrix} \cdot \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

As, from right to left, we have the initial coordinates, then the 45 degree transfer matrix, then the dipole, then the 32.4 degree exit and then the free streaming for a distance L. Working this out, we get that the transfer matrices are:

$$\boldsymbol{M}_H = \begin{bmatrix} -\left(\frac{L}{\rho_0}(\tan\beta_2 - 1) + 1\right) & -\left(\rho_0 + L\tan\beta_2\right) & \rho_0 + L\tan\beta_2 - L \\ \frac{1-\tan\beta_2}{\rho_0} & -\tan\beta_2 & \tan\beta_2 - 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\boldsymbol{M}_V = \begin{bmatrix} (1-\frac{3\pi}{2})(1-\frac{L\tan\beta_2}{\rho_0}) - \frac{L}{\rho_0} & L\left(1-\frac{3\pi}{2}\tan\beta_2\right) + \frac{3\pi}{2}\rho_0 \\ -\frac{\tan\beta_2}{\rho_0}\left(1-\frac{3\pi}{2}+\frac{1}{\tan\beta_2}\right) & 1-\frac{3\pi}{2}\tan\beta_2 \end{bmatrix}$$

As we wish the beam to be 'triple focussing' we have three conditions:

- parrallel-to-point focussing in both horizontal and vertical directions.

  This implies $m_{11} = 0$

  $$-\left(\frac{L}{\rho_0}(\tan\beta_2 - 1) + 1\right) = 0 \to \frac{L}{\rho_0} = \frac{1}{1-\tan\beta_2} \approx 2.737$$

  and $m_{33} = 0$:

  $$(1-\frac{3\pi}{2})(1-\frac{L\tan\beta_2}{\rho_0}) - \frac{L}{\rho_0} = 0 \to \frac{L}{\rho_0} \approx 2.738$$

  i.e $\frac{L}{\rho_0} \approx 2.74$

- non-dispersive $\rightarrow m_{16} = 0$:

$$\rho_0 + L\tan\beta_2 - L = 0 \rightarrow \frac{L}{\rho_0} = \frac{1}{1-\tan\beta_2} \approx 2.737$$

1.2. Number of coil windings

Supposing that the magnet has a gap between the pole shoes of 3cm, current of 40 A and L=50 cm. The number of coil windings is easy to estimate from

$$NI \approx \frac{B}{\mu_0} g$$

With g the gap length and $\mu_0 = 4\pi 10^{-7}$ Tm/A, i.e

$$N = \frac{0.03}{160\pi 10^{-7}} B \qquad (3)$$

First we calculate ρ_0 from the previously found relation:

$$\frac{L}{\rho_0} \approx 2.74 \rightarrow \rho_0 \approx \frac{L}{2.74} \approx 18.3cm$$

Now looking back at equation 1 and filling in ρ_0 we get that we have a magnetic field of

$$B_0 \approx \frac{3.2852\times 10^{-12}}{2.9979\times 10^{8}\times 0.183\times 1.602176\times 10^{-19}} T \approx 0.37T$$

and filling this into equation 3 gives 223 windings.

2. Exercise 2

2.1. Transfer matrix

The equations of motion are:

$$\begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} = \begin{bmatrix} 1 & 2L \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \cdot \begin{bmatrix} x(0) \\ x'(0) \end{bmatrix}$$

i.e the transfer matrix is given by:

$$\begin{bmatrix} 1 & 2L \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

i.e

$$M = \begin{bmatrix} 1 - \frac{L}{f} - 2\frac{L^2}{f^2} & L + 2\frac{L}{f} + 2L \\ -\frac{L}{f^2} & \frac{L}{f} + 1 \end{bmatrix}$$

2.2. Stable orbit condition

Now if we would make a ring from this, we wish that *infinite* iterations would keep x bound. As explained in appendix A this system is stable if

$$-1 \le \frac{1}{2} trace(M) \le 1$$

i.e

$$-1 \le \frac{1}{2}(2 - 2\frac{L^2}{f^2}) \le 1$$

$$-1 \le (1 - \frac{L^2}{f^2}) \le 1$$

$$-f^2 \le (f^2 - L^2) \le f^2$$

$$-2f^2 \le (1 - L^2) \le 0$$

$$2f^2 \ge (L^2 - 1) \ge 0$$

2.3. Vertical plane

So we have our focussing condition, as it's given that the lenses focus in the horizontal plane but not in the vertical plane, the vertical motion will not be stable.

3. Exercise 3

3.1. Transfer matrix for one rectangular matrix

The transfer matrix for one rectangular matrix is given by:

$$M_H = \begin{bmatrix} \cos\alpha & \rho_0 \sin\alpha & \rho_0(1-\cos\alpha) \\ \frac{-\sin\alpha}{\rho_0} & \cos\alpha & \sin\alpha \\ 0 & 0 & 1 \end{bmatrix} M_V = \begin{bmatrix} 1 & \rho_0\alpha \\ 0 & 1 \end{bmatrix}$$

3.2. Show that the full system is achromatic

The full system can be calculated by first considering the first magnet and half a distance L traveled:

$$\begin{bmatrix} 1 & L/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ \frac{\tan\alpha}{\rho_0} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha & \rho_0 \sin\alpha & \rho_0(1-\cos\alpha) \\ \frac{-\sin\alpha}{\rho_0} & \cos\alpha & \sin\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & L/2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\frac{\tan\alpha}{\rho_0} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \rho_0\alpha \\ 0 & 1 \end{bmatrix}$$

i.e first the rotation, then an exit at an angle α, then free travel for $L/2$. working these out we get:

$$\begin{bmatrix} \cos\alpha & \rho_0 \sin\alpha + \frac{L}{2\cos\alpha} & \rho_0(1-\cos\alpha) + \frac{L}{2}\tan\alpha \\ 0 & \frac{1}{\cos\alpha} & \tan\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 - \frac{L}{2\rho_0}\tan\alpha & \rho_0\alpha + \frac{L}{2}(1 - \alpha\tan\alpha) \\ -\frac{\tan\alpha}{\rho_0} & 1 - \alpha\tan\alpha \end{bmatrix}$$

We'll now continue for the horizontal part only as that's the only relevant part. Now if the system is mirror antisymmetric (which is the case here), if the first part (above mentioned part) is given by

$$\begin{bmatrix} a & b & m \\ c & d & n \\ 0 & 0 & 1 \end{bmatrix}$$

The full system is given by

$$\begin{bmatrix} ad+bc & 2bd & 2md \\ 2ac & ad+bc & 2mc \\ 0 & 0 & 1 \end{bmatrix}$$

So doing this calculation gives the transfer matrix from 1 to the exit of 2:

$$\begin{bmatrix} 1 & 2\rho_0 \tan\alpha + \dfrac{L}{\cos^2\alpha} & 2\dfrac{\rho_0}{\cos\alpha}(1-\cos\alpha) + L\dfrac{\sin\alpha}{\cos^2\alpha} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we have a symmetric system (1 to 2 is mirror symmetric with 3 to 4) so we'll use:

$$\begin{bmatrix} ad+bc & 2bd & 2nb \\ 2ac & ad+bc & 2na \\ 0 & 0 & 1 \end{bmatrix}$$

i.e the full system is given by:

$$\begin{bmatrix} 1 & 4\rho_0 \tan\alpha + \dfrac{2L}{\cos^2\alpha} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We see that $m_{16} = 0 = m_{26}$, implying that this system is achromatic, irrespective of the distance L.

3.3. Transfer matrix of the system if $L = 0$

We just take our full system and set L to zero:

$$\begin{bmatrix} 1 & 4\rho_0 \tan\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To now obtain the undulator parameter, we wish to know the wavelength λ_u. It can bee seen that this is the total x distance traveled over the 4 magnets, the distance in the x direction traveled in magnet 1 is given by $\rho_0 \tan\alpha$ so the total wavelength is 4 times this or:

$$\lambda_u = 4\rho_0 \tan\alpha \qquad (4)$$

making the undulator parameter:

$$K = \frac{\lambda_u}{2\pi\rho_0} = \frac{2}{\pi}\tan\alpha$$

3.4. wavelength of a FEL

Filling the given values into equation 4, we get a wavelength of 64cm.