Subatomic Physics II: Problem set 4

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4.1 Q^2, x and y in terms of E_e, E_p, E_0 and θ_e

We have the following four-vectors:

$$\begin{aligned} P_e &\approx (30GeV, 30GeV, 0, 0) && (1)\\ P_p &\approx (900GeV, -900GeV, 0, 0) && (2)\\ P'_e &\approx (E_0, E_0\sin(\theta_c), E_0\cos(\theta_c)) && (3) \end{aligned}$$

Now per definition Q^2 is given by [2]:

$$Q^2 := -(P_e - P'_e)^2 \approx 2(EE' - |\boldsymbol{p}||\boldsymbol{p'}|\cos(\theta_c)) = 2(E_eE_0 - E_eE_0\cos(\theta_c)) = 4E_eE_0\sin^2\left(\frac{\theta_c}{2}\right) \quad (4)$$

And the Bjorken scaling variable x is given by:

$$x := \frac{Q^2}{2Pq} = \frac{4E_eE_0\sin^2\left(\frac{\theta_c}{2}\right)}{2P_p(P_e - P'_e)} \approx \frac{4E_eE_0\sin^2\left(\frac{\theta_c}{2}\right)}{4E_eE_p - 2E_0E_p(1+\sin(\theta_c))} \quad (5)$$

Where the approximation $|\boldsymbol{p_p}| \approx E_p$ was used. Now finally the inelasticity y is given by:

$$y := \frac{Pq}{Pp} \approx \frac{2E_eE_p - E_0E_p(1+\sin(\theta_c))}{2E_eE_p} = 1 - \frac{E_0E_p(1+\sin(\theta_c))}{2E_eE_p} \quad (6)$$

4.2 Variables Q^2, x, y and W^2 in terms of the Mandelstam variables s,t and u

The Mandelstam variables are [3]:

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 && (7)\\ t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 && (8)\\ u &= (p_1 - p_4)^2 = (p_2 - p_3)^2 && (9) \end{aligned}$$

naming the initial electron and proton number 1 and 2 respectively and the final electron and X numbers 3 and 4 we can proceed.
Q^2 is by far the easiest one as $Q^2 = -(P_e - P'_e)^2 = -(P_1 - P_3)^2 = -t$.
Now to write x in terms of the Mandelstam variables we'll take a look at Pq:

$$Pq = P_2(P_1 - P_3) = P_2P_1 - P_2P_3 \quad (10)$$

Now we have that

$$s + u = P_1^2 + 2P_2^2 + P_3^2 + 2(P_1P_2 - P_2P_3) \Rightarrow \frac{s+u}{2} = m_e^2 + m_p^2 + P_2P_1 - P_2P_3 \approx m_p^2 + P_2P_1 - P_2P_3 \quad (11)$$

And thus:

$$x = -\frac{t}{s + u - 2m_p^2} \quad (12)$$

Now to get the inelasticity y we'll first look at Pp, i.e:

$$s = P_1^2 + P_2^2 + 2P_1P_2 \approx m_p^2 + 2P_1P_2 \rightarrow Pp = P_2P_1 \approx \frac{s - m_p^2}{2} \quad (13)$$

From which we have:

$$y = \frac{s + u - 2m_p^2}{s - m_p^2} \quad (14)$$

Now lastly W^2 is the squared invariant mass and it is calculated from the four momenta of the exchanged photon q and of the incoming proton P:

$$\begin{aligned} W^2 = (P+q)^2 = (P_p + (P_e - P_e'))^2 &= P_1^2 + P_2^2 + P_3^2 + 2(P_1P_2 - P_2P_3 - P_1P_3) & (15)\\ &\approx m_p^2 + 2(P_1P_2 - P_2P_3 - P_1P_3) & (16)\\ &= s + u + t - 2m_p^2 & (17)\end{aligned}$$

as $s + u + t = 3m_p^2 + 2(P_1P_2 - P_2P_3 - P_1P_3)$.
So summing up:

- $Q^2 = -t$
- $x = -\frac{t}{s+u-2m_p^2}$
- $y = \frac{s+u-2m_p^2}{s-m_p^2}$
- $W^2 = s + u + t - 2m_p^2$

4.3 Energy reached by an electron beam hitting a solid proton target to have the same centre-of-mass energy as HERA

For this we'll first calculate the center of mass energy in this situation:

$$s = (P_p + P_e)^2 \approx m_p^2 + 2 * (30 * 900\text{GeV}^2 + 900 * 30\text{GeV}^2) \Rightarrow \sqrt{s} \approx 328.64\text{GeV} \quad (18)$$

Where I have used 938 MeV as the proton mass [1].
Now if the proton would be standing still then:

$$s = m_p^2 + 2m_pE_e = 108000.88\text{GeV}^2 \Rightarrow E_e = \frac{108000.88\text{GeV}^2 - m_p^2}{2m_p} = 57569.3\text{GeV} \quad (19)$$

So the electron would need an energy of 58 TeV to match the center of mass energy.

Structure functions

4.4 Sum rules

The structure functions of the proton and neutron for electron-nucleon inelastic scattering are respectively given by:

$$\begin{aligned} F_2^{e,p}(x) &= x \cdot \left[\frac{4}{9}(u_v^p + u_s^p + \bar{u}_s^p) + \frac{1}{9}(d_v^p + d_s^p + \bar{d}_s^p) + \frac{1}{9}(s_s^p + \bar{s}_s^p)\right] & (20)\\ F_2^{e,n}(x) &= x \cdot \left[\frac{4}{9}(u_v^n + u_s^n + \bar{u}_s^n) + \frac{1}{9}(d_v^n + d_s^n + \bar{d}_s^n) + \frac{1}{9}(s_s^n + \bar{s}_s^n)\right] & (21)\end{aligned}$$

Where $u_v^{p,n}(x)$ denotes the distribution of valence u-quarks in the proton and neutron respectively and the s in $u_s^{p,n}(x)$ denoting sea-quark. Now because of isospin symmetry

$$\begin{aligned} u_{v,s}^p(x) &= d_{v,s}^n(x) := u_{v,s}(x) & (22)\\ d_{v,s}^p(x) &= u_{v,s}^n(x) := d_{v,s}(x) & (23)\end{aligned}$$

As such we get that the difference of the proton and neutron structure functions is given by:

$$F_2^{e,p} - F_2^{e,p} = x \cdot \left[\frac{1}{3}(u_v(x) - d_v(x)) + \frac{2}{3}(\bar{u}_s(x) - \bar{d}_s(x))\right] \quad (24)$$

The parton distribution function is defined in such a way that $u^p(x)dx$ represents the number of up-quarks within the proton with momentum fraction between x and x+dx, because of we have that

$$\int_0^1 u_v^p = 2 \quad (25)$$

Now evaluating Gottfried's integral:

$$S_G = \int_0^1 \frac{dx}{x}(F_2^{e,p} - F_2^{e,n}) = \int_0^1 \left(\frac{1}{3}(u_v - d_v) + \frac{2}{3}(\bar{u}_s(x) - \bar{d}_s(x))\right) = \frac{1}{3}(2-1) + \frac{2}{3}\int_0^1 (\bar{u}_s(x) - \bar{d}_s(x)) \quad (26)$$

Now if we also take the amount of sea quarks to be equal because of isospin invariance we finally expect:

$$S_G = \int_0^1 \frac{dx}{x}(F_2^{e,p} - F_2^{e,n}) = \frac{1}{3} \quad (27)$$

The structure functions of the proton and neutron for neutrino-nucleon interactions are given by:

$$\begin{aligned} F_2^{\nu,p}(x) &= 2x \cdot [d(x) + \bar{u}(x)] \quad (28) \\ F_2^{\nu,n}(x) &= 2x \cdot \left[u(x) + \bar{d}(x)\right] \quad (29) \end{aligned}$$

Adler's sum rule is thus:

$$\begin{aligned} S_A = \int_0^1 \frac{dx}{x}(F_2^{\nu,n} - F_2^{\nu,p}) &= 2\int_0^1 (u(x) + \bar{d}(x) - d(x) - \bar{u}(x)) \quad (30) \\ &= 2\int_0^1 (u_v(x) + u_s(x) + \bar{d}_s(x) - d_v(x) - d_s(x) - \bar{u}_s(x)) \quad (31) \\ &= 2\int_0^1 (u_v(x) - d_v(x)) = 2 \quad (32) \end{aligned}$$

4.5 Why is the Adler sum rule better satisfied than the Gottfried sum rule?

A possible flaw with the Gottfried Sum Rule is that it's assumed (contrary to Adler's rule in which the sea quarks vanish naturally) that $\bar{d}_s(x) = \bar{u}_s(x)$ and thus that the quark-antiquark sea is symmetric which doesn't have to be the case.

References

[1] Peter J Mohr. 2014 codata recommended values of the fundamental constants of physics and chemistry. *REVIEWS OF MODERN PHYSICS*, Sep 2016.

[2] B. Povh, M. Lavelle, K. Rith, C. Scholz, and F. Zetsche. *Particles and Nuclei: An Introduction to the Physical Concepts*. Springer Berlin Heidelberg, 2015.

[3] Mark Thomson. *Modern Particle Physics*. 2013.