Subatomic Physics II

Problem set 3

Problem 3.1: SU(2) algebra

The mathematical algebra of Lie group SU(2) is utilised several times in quantum mechanics and particle physics, describing spin (angular momentum), srtong isospin of hadrons, and weak isospin of the electroweak interaction. Often one has to decompose a state $|J, M\rangle$ into contributing direct-product states $|j_1, m_1\rangle\,|j_2, m_2\rangle$, with coefficients $C^{(J,j_1,j_2)}_{(M,m_1,m_2)}$:

$$|J, M\rangle = \sum C^{(J,j_1,j_2)}_{(M,m_1,m_2)}\,|j_1, m_1\rangle\,|j_2, m_2\rangle$$

The coefficients C are called (for SU(2)) Clebsch-Gordan coefficients.

- Determine the ratio σ_1/σ_2 of the cross sections for the processes 1: $\pi^0 d \to pn$ and 2: $\pi^+ d \to pp$. Neglect mass-difference effects. (1pt)

Solution

The strong interaction is invariant under rotations in the isospin space, and conserves the isospin from the initial to the final state. As a consequence, we can break down scattering or decay processes mediated by the strong force in terms of reactions with defined isospin. In other words, we expect reactions with same isospin I, but different third component I_3 to have the same amplitude.

Using this fact, we can analyze and compare all the processes of type $\pi d \to NN$, where π represents any element of the pion isospin triplet

$$\left|\pi^-\right\rangle = |I = 1, I_3 = -1\rangle\,, \qquad \left|\pi^0\right\rangle = |I = 1, I_3 = 0\rangle\,, \qquad \left|\pi^+\right\rangle = |I = 1, I_3 = +1\rangle\,,$$

and N either element of the nucleon doublet (i.e. neutron or proton)

$$|n\rangle = \left|I = \frac{1}{2}, I_3 = -\frac{1}{2}\right\rangle, \qquad |p\rangle = \left|I = \frac{1}{2}, I_3 = \frac{1}{2}\right\rangle.$$

Let's start by decomposing all initial and final states into their isospin components, introducing the Clebsch-Gordan coefficients:

$$\begin{aligned}
\left|\pi^0 d\right\rangle &= |1, 0\rangle \oplus |0, 0\rangle = |1, 0\rangle \\
\left|\pi^+ d\right\rangle &= |1, 1\rangle \oplus |0, 0\rangle = |1, 1\rangle \\
|pn\rangle &= \left|\frac{1}{2}, \frac{1}{2}\right\rangle \oplus \left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}\,|1, 0\rangle + \frac{1}{\sqrt{2}}\,|0, 0\rangle \\
|pp\rangle &= \left|\frac{1}{2}, \frac{1}{2}\right\rangle \oplus \left|\frac{1}{2}, \frac{1}{2}\right\rangle = |1, 1\rangle\,.
\end{aligned}$$

If $\mathcal{O}$ is a generic operator representing the strong interaction between π, d, and N, the scattering amplitudes can be written as

$$
\begin{aligned}
\mathcal{M}(\pi^0 d \to pn) &= \left\langle pn \middle| \mathcal{O} \middle| \pi^0 d \right\rangle = \left(\frac{1}{\sqrt{2}} \left\langle 1,0 \right| + \frac{1}{\sqrt{2}} \left\langle 0,0 \right| \right) \mathcal{O} \left| 1,0 \right\rangle \\
&= \frac{1}{\sqrt{2}} \left\langle 1,0 | \mathcal{O} | 1,0 \right\rangle + \frac{1}{\sqrt{2}} \left\langle 0,0 | \mathcal{O} | 1,0 \right\rangle \\
&= \frac{1}{\sqrt{2}} \left\langle 1,0 | \mathcal{O} | 1,0 \right\rangle = \frac{1}{\sqrt{2}} \mathcal{M}_1
\end{aligned}
$$

$$
\mathcal{M}(\pi^+ d \to pp) = \left\langle pp \middle| \mathcal{O} \middle| \pi^+ d \right\rangle = \left\langle 1,1 | \mathcal{O} | 1,1 \right\rangle = \mathcal{M}_1
$$

As anticipated, we used the invariance of $\mathcal{O}$ under isospin transformations: $\langle 1,0|\mathcal{O}|1,0\rangle = \langle 1,1|\mathcal{O}|1,1\rangle = \mathcal{M}_1$. Only the $I = 1$ channel contributes, since the initial states in both processes are pure $I = 1$ states.

If we neglect the small mass differences, the phase space for the two processes are identical. Therefore the ratio of the two cross sections is only determined by the matrix elements $\mathcal{M}$, keeping in mind that $\sigma \propto |\mathcal{M}|^2$:

$$
\frac{\sigma(\pi^0 d \to pn)}{\sigma(\pi^+ d \to pp)} = \frac{\left|\mathcal{M}(\pi^0 d \to pn)\right|^2}{\left|\mathcal{M}(\pi^+ d \to pp)\right|^2} = \frac{\left|\frac{1}{\sqrt{2}} \mathcal{M}_1\right|^2}{\left|\mathcal{M}_1\right|^2} = \frac{1}{2}.
$$

- What are the relative probabilities of the strong decays of the $\Delta^+(1232)$ resonance into $n\pi^+$ and $p\pi^0$? Neglect mass-difference effects. (1pt)

Solution

The isospin state of the Δ^+ baryon is $\left|\frac{3}{2}, +\frac{1}{2}\right\rangle$. Since we are studying a decay of type $\Delta \to \pi N$, we have to decompose $|\Delta^+\rangle$ into a combination of $I = 1$ and $I = \frac{1}{2}$ states:

$$
\left|\Delta^+\right\rangle = \left|\frac{3}{2}, +\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}} \left|1,1\right\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}} \left|1,0\right\rangle \left|\frac{1}{2}, +\frac{1}{2}\right\rangle.
$$

We can now compute the amplitude of each decay as:

$$
\begin{aligned}
\left\langle n\pi^+ \middle| \Delta^+ \right\rangle &= \left\langle 1,1 \right| \left\langle \frac{1}{2}, -\frac{1}{2} \right| \left(\sqrt{\frac{1}{3}} \left|1,1\right\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}} \left|1,0\right\rangle \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \right) = \sqrt{\frac{1}{3}} \\
\left\langle p\pi^0 \middle| \Delta^+ \right\rangle &= \left\langle 1,0 \right| \left\langle \frac{1}{2}, \frac{1}{2} \right| \left(\sqrt{\frac{1}{3}} \left|1,1\right\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}} \left|1,0\right\rangle \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \right) = \sqrt{\frac{2}{3}}.
\end{aligned}
$$

Since the phase space for the two decays is the same—neglecting small mass differences—the ratio of their probabilities is:

$$
\frac{\Gamma(\Delta^+ \to n\pi^+)}{\Gamma(\Delta^+ \to p\pi^0)} = \frac{\left|\left\langle n\pi^+ \middle| \Delta^+ \right\rangle\right|^2}{\left|\left\langle p\pi^0 \middle| \Delta^+ \right\rangle\right|^2} = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}.
$$

- The $\Delta^+(1920)$ resonance decays into $p\rho^0$. What is the probability that the proton spin is oriented in the same direction as the Δ-spin? (1pt)

Solution

$\Delta(1920)$ baryons have spin $J = \frac{3}{2}$ and four possible values of spin projection $M = \pm\frac{3}{2}, \pm\frac{1}{2}$). Protons and ρ^0 mesons have spin $\frac{1}{2}$ and 1, respectively. Since it is a decay mediated by the strong interaction, it has to conserve parity as well:

$$P_{\Delta(1920)} = P_p \cdot P_\rho \cdot (-1)^L,$$

where L is the orbital angular momentum of the p–ρ^0 system. Since $P_{\Delta(1920)} = P_p = 1$ and $P_\rho = -1$, we see that L must be odd (1, 3, etc.). In principle, this means that we need to combine three angular momenta in the final state. This is beyond the goal of this exercise—unless you are particularly brave! For simplicity, let's just consider the (unphysical) case where $L = 0$ and the problem reduces to coupling two angular momenta, namely the spins of p and ρ^0.

As said, the Δ baryon can exhibit four distinct states of spin third component M, *a priori* all equally probable. Let's start from the case $M = +\frac{3}{2}$ $\left(\text{i.e. } \left|J = \frac{3}{2}, M = +\frac{3}{2}\right\rangle\right)$ and decompose it into all the possible spin states of p and ρ^0:

$$\left|\frac{3}{2}, +\frac{3}{2}\right\rangle = \left|1, +1;\ \frac{1}{2}, +\frac{1}{2}\right\rangle.$$

In this case $\left(M = +\frac{3}{2}\right)$, the proton spin is always oriented in the same direction as the Δ along the quantization axis: $Prob\left(+\frac{3}{2}, +\frac{1}{2}\right) = 1$. Let's now consider the case $M = +\frac{1}{2}$:

$$\left|\frac{3}{2}, +\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|1, +1;\ \frac{1}{2}, -\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|1, 0;\ \frac{1}{2}, +\frac{1}{2}\right\rangle.$$

The final state where p has the same spin orientation as Δ is $\left|1, 0;\ \frac{1}{2}, +\frac{1}{2}\right\rangle$. The probability for this decay is given by

$$Prob\left(+\frac{1}{2}, +\frac{1}{2}\right) = \left|\left\langle 1, 0;\ \frac{1}{2}, +\frac{1}{2}\middle|\frac{3}{2}, +\frac{1}{2}\right\rangle\right|^2 = \frac{2}{3}.$$

It can be easily seen that the two remaining cases bring to the same results as the previous two: $Prob\left(-\frac{1}{2}, -\frac{1}{2}\right) = Prob\left(+\frac{1}{2}, +\frac{1}{2}\right) = \frac{2}{3}$ and $Prob\left(-\frac{3}{2}, -\frac{1}{2}\right) = Prob\left(+\frac{3}{2}, +\frac{1}{2}\right) = 1$.

Assuming that the four spin states of $\Delta(1920)^+$ are all equally probable (0.25 each), we have that the probability for the p to have the spin oriented as the Δ is

$$Prob_{\text{tot}} = 2 \times \frac{1}{4} \times \left(1 + \frac{2}{3}\right) = \frac{5}{6} \simeq 83.33\%.$$

- Consider the following pion-nucleon scattering reactions:

$$\begin{aligned} \pi^+ p &\rightarrow \pi^+ p \quad (1)\\ \pi^- p &\rightarrow \pi^- p \quad (2)\\ \pi^- p &\rightarrow \pi^0 n \quad (3)\end{aligned}$$

If at a certain energy the reaction proceeds through an intermediate $I = 3/2$ state, what are the relative cross sections? And if it proceeds through an intermediate $I = 1/2$ resonance? (3pt)

Solution

We can use again the invariance of the strong interaction for isospin transformations, and analyze the reactions above in terms of isospin states. First of all, let's combine the interacting particles into isospin states:

$$\begin{aligned}
\left|\pi^+ p\right\rangle &= |1,+1\rangle \oplus \left|\frac{1}{2},+\frac{1}{2}\right\rangle = \left|\frac{3}{2},+\frac{3}{2}\right\rangle \\
\left|\pi^- p\right\rangle &= |1,-1\rangle \oplus \left|\frac{1}{2},+\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \\
\left|\pi^0 n\right\rangle &= |1,0\rangle \oplus \left|\frac{1}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle
\end{aligned}$$

Now we can derive the scattering amplitudes for the three processes.

a. $\pi^+ p \to \pi^+ p$

$$\mathcal{M}_a = \mathcal{M}(\pi^+ p \to \pi^+ p) = \left\langle \pi^+ p \middle| \mathcal{O} \middle| \pi^+ p \right\rangle = \left\langle \frac{3}{2},+\frac{3}{2} \middle| \mathcal{O} \middle| \frac{3}{2},+\frac{3}{2} \right\rangle = \mathcal{M}_{3/2}$$

b. $\pi^- p \to \pi^- p$

$$\begin{aligned}
\mathcal{M}_b = \mathcal{M}(\pi^- p \to \pi^- p) &= \left\langle \pi^- p \middle| \mathcal{O} \middle| \pi^- p \right\rangle \\
&= \left(\sqrt{\frac{1}{3}}\left\langle \frac{3}{2},-\frac{1}{2}\right| - \sqrt{\frac{2}{3}}\left\langle \frac{1}{2},-\frac{1}{2}\right|\right) \mathcal{O} \left(\sqrt{\frac{1}{3}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right) \\
&= \frac{1}{3}\left\langle \frac{3}{2},-\frac{1}{2} \middle| \mathcal{O} \middle| \frac{3}{2},-\frac{1}{2} \right\rangle + \frac{2}{3}\left\langle \frac{1}{2},-\frac{1}{2} \middle| \mathcal{O} \middle| \frac{1}{2},-\frac{1}{2} \right\rangle \\
&= \frac{1}{3}\mathcal{M}_{3/2} + \frac{2}{3}\mathcal{M}_{1/2}
\end{aligned}$$

c. $\pi^- p \to \pi^0 n$

$$\begin{aligned}
\mathcal{M}_c = \mathcal{M}(\pi^- p \to \pi^0 n) &= \left\langle \pi^0 n \middle| \mathcal{O} \middle| \pi^- p \right\rangle \\
&= \left(\sqrt{\frac{2}{3}}\left\langle \frac{3}{2},-\frac{1}{2}\right| + \sqrt{\frac{1}{3}}\left\langle \frac{1}{2},-\frac{1}{2}\right|\right) \mathcal{O} \left(\sqrt{\frac{1}{3}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right) \\
&= \frac{\sqrt{2}}{3}\left\langle \frac{3}{2},-\frac{1}{2} \middle| \mathcal{O} \middle| \frac{3}{2},-\frac{1}{2} \right\rangle - \frac{\sqrt{2}}{3}\left\langle \frac{1}{2},-\frac{1}{2} \middle| \mathcal{O} \middle| \frac{1}{2},-\frac{1}{2} \right\rangle \\
&= \frac{\sqrt{2}}{3}\left(\mathcal{M}_{3/2} - \mathcal{M}_{1/2}\right).
\end{aligned}$$

In general, the ratios of cross sections for the three processes above are:

$$\sigma_a : \sigma_b : \sigma_c = \left|\mathcal{M}_{3/2}\right|^2 : \left|\frac{1}{3}\mathcal{M}_{3/2} + \frac{2}{3}\mathcal{M}_{1/2}\right|^2 : \left|\frac{\sqrt{2}}{3}\left(\mathcal{M}_{3/2} - \mathcal{M}_{1/2}\right)\right|^2.$$

If the reaction proceeds through an intermediate $I = 3/2$ state, we will have:

$$\sigma_a : \sigma_b : \sigma_c = \left|\mathcal{M}_{3/2}\right|^2 : \left|\frac{1}{3}\mathcal{M}_{3/2}\right|^2 : \left|\frac{\sqrt{2}}{3}\mathcal{M}_{3/2}\right|^2 = 1 : \frac{1}{9} : \frac{2}{9} = 9 : 1 : 2.$$

If, instead, the reaction proceeds through an intermediate $I = 1/2$ state, we will have:

$$\sigma_a : \sigma_b : \sigma_c \;=\; 0 : \left|\frac{2}{3}\mathcal{M}_{1/2}\right|^2 : \left|\frac{\sqrt{2}}{3}\mathcal{M}_{1/2}\right|^2 \;=\; 0 : \frac{4}{9} : \frac{2}{9} \;=\; 0 : 4 : 2.$$

- Baryons are 3q bound states, so in the rest frame of the baryon, corresponding to the centre-of-mass frame of the 3q system, there are two orbital angular momenta associated with the relative motion of the three quarks. The first is conveniently taken to be the orbital angular momentum $\vec{L_{12}}$ of a chosen pair of quarks in their mutual cms frame. The second is then the orbital angular momentum $\vec{L_3}$ of the third quark about the cms of the pair in the overall cms frame. The total orbital angular momentum is given by $\vec{L} = \vec{L_{12}} + \vec{L_3}$, while the spin is the sum of the spins of the three quarks $\vec{S} = \vec{S_1} + \vec{S_2} + \vec{S_3}$. What are the possible baryon states (total angular momentum J, spin S and parity P) for the ground state band ($L = 0$), and the next lowest-lying band ($L = 1$)? (4pt)

Solution

The spin of the baryon can be 3/2 or 1/2.

- The first case of $L = 0$ corresponds to $\vec{L_{12}} = 0$ and $\vec{L_3} = 0$, which leads to the possible values of $J = 3/2,\ 1/2$. The parity of the system is given by $(-1)^{L_{12}+L_3} = +1$, and therefore, it corresponds to $3/2^+$ (S = 3/2) and $1/2^+$ (S = 1/2) states.
- For $L = 1$, the possible states are: one state with $5/2^-$ (S = 3/2), two states with $3/2^-$ (S = 3/2, 1/2), and two states with $1/2^-$ (S = 3/2, 1/2).