Option pricing under Model and Parameter Uncertainty using Predictive Densities

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Abstract

The theoretical price of a financial option is given by the expectation of its discounted expiry time payoff. The computation of this expectation depends on the density of the value of the underlying instrument at expiry time. This density depends on both the parametric model assumed for the behaviour of the underlying, and the values of parameters within the model, such as volatility. However neither the model, nor the parameter values are known. Common practice when pricing options is to assume a specific model, such as geometric Brownian Motion, and to use point estimates of the model parameters, thereby precisely defining a density function.

We explicitly acknowledge the uncertainty of model and parameters by constructing the predictive density of the underlying as an average of model predictive densities, weighted by each model's posterior probability. A model's predictive density is constructed by integrating its transition density function by the posterior distribution of its parameters. This is an extension to Bayesian model averaging. Sampling importance-resampling and Monte Carlo algorithms implement the computation. The advantage of this method is that rather than falsely assuming the model and parameter values are known, inherent ignorance is acknowedged and dealt with in a mathematically logical manner, which utilises all information from past and current observations to generate and update option prices. Moreover point estimates for parameters are unnecessary. We use this method to price a European Call option on a share index.

1 Introduction

A European call option is a financial instrument that gives the holder the right, but not the obligation, to buy a specified security, the *underlying*, at a certain price, the *strike*, on a certain date, the *expiry* date, from the writer of the call. This definition implies a *payoff* function (a function of the underlying's expiry price that gives the option value at expiry) of $Max(S_T - K, 0)$ where K is the strike price. A call option is the simplest type of option, and more complex options are defined by more complex payoff functions. The mainstream method of pricing options by no-arbitrage was pioneered by Black, Scholes and Merton in the 1970s [3] and was later refined by Harrison and Kreps [10], and Harrison and Pliska [11]. This method of pricing options assumes a continuous time diffusion model for the stochastic behaviour of the underlying and uses no-arbitrage arguments to derive the option price as the expectation of the payoff function under a risk-neutral probability measure. The existence and uniqueness of such a risk-neutral probability measure is dependent on the structure of the model (see Oksendal [15] for details). The diffusion models used for the underlying are expressed as Ito stochastic differential equations (SDEs) driven by standard Brownian Motions (BMs). These form the most general class of mathematical models for stochastic processes that do not jump, and hence for *complete* market models, i.e., models for which every option can be replicated by a dynamic portfolio of traded instruments. Market models that are not complete do not give unique option prices: generally no-arbitrage arguments may give upper and lower bounds for an option's price and a specific price may only be inferred given an agent's utility function, which is problematic. Thus we only consider complete no-arbitrage market models.

As an example of the standard approach we assume the underlying, S, follows a geometric Brownian Motion model. Thus the SDE for S is

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{1}$$

where W is a standard Brownian Motion under the risk neutral probability measure. Then it can be shown that, under certain market assumptions (see Hull [13]), the price of an option, u, with payoff $f(S_T)$ is

$$u(S_t, t) = e^{-r(T-t)} E_t[f(S_T)] = e^{-r(T-t)} \int_0^\infty f(S_T) p(S_T, T|S_t, t) dS_T \tag{2}$$

where $p(S_T, T|S_t, t)$ is the risk neutral transition probability density function of S. $E_t[f]$ is the expectation of f under the risk-neutral probability measure conditioned on information up to time t. For a call on an underlying modelled by a geometric Brownian Motion, equation (2) can be simplified to the standard Black-Scholes pricing formula.

This approach can be criticised on two related counts:

1. It assumes the model is known with certainty.

2. It assumes the parameters within the model are known with certainty.

With regard to the first assumption of the standard approach, although there has been research on testing models against data, there has been little research on integrating model uncertainty into the actual pricing of options. There have been several approaches to deal with or elaborate on the second assumption, within the framework of specific models. The key parameter in the above model is the volatility of the underlying, σ, as all other variables and parameters are observable. Some of the approaches to deal with parameter uncertainty are:

1. Implied volatility: the option is priced "by the market" and the market price implies a value for σ through the Black-Scholes formula

2. Point estimation of constant volatility using the maximum likelihood estimator or the Bayesian posterior mean

3. Discrete time GARCH models for volatility

4. Discrete and continutous time Stochastic volatility models which specify a stochastic process for volatility to follow

5. Uncertain volatility models which only specify a range of values which, it is believed, bound volatility

Implied volatility is only applicable to options with liquid markets, and is thus not directly an option pricing method. Point estimates of volatility, either the maximum likelihood estimate or the posterior expectation, merely give an approximation of the predictive density of S:

$$p(S_T, T|S_t, t) \approx p(S_T, T|S_t, t, \hat{\sigma})$$

and the uncertainty surrounding this estimate does not enter into the actual option price calculation. GARCH and stochastic volatility models introduce more unknown and unobservable parameters, and again the practice is to use point estimates for these. However interesting work has been done by Jacquier et al. [14] using Bayesian inference implemented by Markov chain Monte Carlo to systematically estimate the parameters in a discrete time stochastic volatility model using their posterior distribution.

Finally the uncertain volatility approach proposed by Avellaneda et al. [2] has the advantage that little is assumed beyond the range bounding volatility: nothing is assumed about the distribution of volatility within the range, and certainly no point estimate is used.

Thus the uncertainty of parameter values is integrated into the option price calculation. Unfortunately this results in models that are incomplete, and thus to ranges for option prices rather than unique prices. Additionally the specification of the volatility range introduces two unknown parameters σ_{min} and σ_{max}. There exists uncertainty over the values of these parameters which is not dealt with by such an approach.

Our approach is to explicitly acknowledge the uncertainty over models within a set of candidate models, $\mathcal{M} = \{M_k\}_{k=1\ldots n}$, and parameter values within a parameter space, Θ_k, for each model M_k. We do this within a Bayesian framework, so that models have probabilities of being true which are dynamically updated given the evidence of incoming data, and parameters within each model have a probability distribution that is also updated given the data. The updating in each case is performed by the Bayes theorem. This approach is thus an extension of Bayesian model averaging as described by Wasserman [21], Hoetig et al. [12] and Draper [7]. We compute the option price as

$$\begin{aligned}
u(S_t, t|\mathcal{M}) &= e^{-r(T-t)} E_t[f(S_T)|\mathcal{M}] \\
&= e^{-r(T-t)} \int_0^\infty f(S_T) p(S_T, T|S_t, t, \mathcal{M}, D_t) dS_T \\
&= e^{-r(T-t)} \int_0^\infty f(S_T) \sum_{k=1}^{n} p(S_T, T|S_t, t, M_k) p(M_k|\mathcal{M}, D_t) dS_T \\
&= e^{-r(T-t)} \int_0^\infty f(S_T) \sum_{k=1}^{n} \int_{\Theta_k} p(S_T, T|S_t, t, \theta_k) p(\theta_k|D_t) d\theta_k p(M_k|\mathcal{M}, D_t) dS_T
\end{aligned}$$

The rest of the paper discusses the reasoning and computations involved in this approach. In section 2 we discuss dealing with parameter uncertainty by forming the model predictive density $p(S_T, T|S_t, t, M_k)$. This involves sampling from the dynamically updated parameter distributions having density $p(\theta_k)$ using the sampling importance-resampling (SIR) algorithm and then using these samples to sample from the model predictive density $p(S_T, T|S_t, t, M_k)$ using Monte Carlo simulation. Model uncertainty is dealt with in section 3, which describes the computation of model probabilities, their updating given incoming data, and the computation of the expectation (3) using Monte Carlo integration, giving the option price. This approach is applied to pricing a call option on an equity index in section 4 which includes numerical results. Comments and conclusions are given in section 5.

2 Parameter Uncertainty

This section deals with the problem of parameter uncertainty given a specific model. Our approach to this problem is to utilise the standard Bayesian method of integrating over the parameter space. From an option modelling perspective this approach is related to Avellaneda et al.'s uncertain volatility [2] and Jacquier et al.'s [14] approaches. We revisit the option pricing equation (2). In this equation it is implicit that any parameters of the model, θ, are known. The equation may be written

$$u(S_t, t|\theta) = e^{-r(T-t)} E_t[f(S_T)|\theta] = e^{-r(T-t)} \int_0^\infty f(S_T) p(S_T, T|S_t, t, \theta) dS_T \tag{3}$$

where the conditioning on knowledge of θ is explicit. If we accept that we do not have this knowledge, i.e., we do not know the value of $\theta \in \Theta$ with certainty and want to price with this ignorance, we can write the option price as

$$u(S_t, t) = e^{-r(T-t)} E_t[f(S_T)] = e^{-r(T-t)} \int_0^\infty f(S_T) \int_\Theta p(S_T, T|S_t, t, \theta) p(\theta|D_t) d\theta dS_T \tag{4}$$

This is equivalent to equation (2) when the probability density of S_T is given by the predictive density function of S,

$$p(S_T, T|S_t, t) = \int p(S_T, T|S_t, t, \theta) p(\theta|D_t) d\theta$$

rather than the transition probability density function of S given a known value for parameters. The density $p(\theta|D_t)$ is the posterior density of θ given D_t the observed data values of S. Thus given a specific model we utilise available data on the underlying to update the distribution of the parameters, and use the whole posterior distribution of the parameters to value an option, rather than use a point estimate of the parameter value. In essence, for the predictive distribution of the underlying, we average over the whole parameter space, weighted by the posterior distribution of the parameters. Thus, similarly to Jacquier et al.'s method we employ Bayesian inference to obtain the posterior distribution of parameters, and similarly to the uncertain volatility approach, we avoid making point estimates of parameters, and thus integrate parameter uncertainty directly into the option price calculation.

The distribution of θ is updated dynamically on the arrival of new observations of S. Assume we have a density for θ given a data set D_s of observed values of S for times up to s. Denote this density by $p(\theta|D_s)$. Given $D_{s,t}$, a set of observed values for S from times between s and t, $s < t$, our data set expands to $D_t = D_s \cup D_{s,t}$. Ito diffusions are Markov processes. This implies that, conditional on θ and S_s, $D_{s,t}$ is independent from $D_s - S_s$. The updated density of θ is thus given by Bayes theorem.

$$p(\theta|D_t) = \frac{p(D_{s,t}|\theta, S_s) p(\theta|D_s)}{\int_\Theta p(D_{s,t}|\theta, S_s) p(\theta|D_s) d\theta} \tag{5}$$

There are, however, difficulties with the calculation of (5). First when the model is given as a SDE the likelihood, $p(D|\theta)$ is not always available in closed form. Second, even when the likelihood function is available in closed form, an explicit solution to the integral in equation (5) is usually not available analytically. These problems are dealt with in the following manner.

The observable behaviour of S is modelled by the SDE

$$dS_t = a(S_t, t)dt + b(S_t, t)dZ_t \tag{6}$$

for given functions a and b, where Z_t is a standard Brownian Motion under the "real world" probability measure. Note that when we test parameters and models against observed data we use the SDE (6), which describes the process under the "real world" measure, but when we price options we must use the SDE which describes the process under the equivalent risk-neutral measure, which is derived from the former using Girsanov's theorem [15]. If the SDE can be solved to give S_t as a function, g, of Z_t that has an inverse $h = g^{-1}$, then a closed form for the likelihood is given by the closed form transition probability density function:

$$\begin{aligned}
S_t &= g(t, Z_t, S_0, \theta) \\
Z_t &= h(t, S_t, S_0, \theta) \\
p(S_t, t|S_0, 0, \theta) &= \frac{1}{\sqrt{2\pi t}} exp\{\frac{-h(t, S_t, S_0, \theta)^2}{2t}\}|\frac{\partial h}{\partial S_t}|
\end{aligned}$$

If, however, the SDE cannot be solved, then the likelihood is often not available in closed form. Pedersen [16] uses a simulation algorithm to approximate the transition probability density function based on an Euler discretisation of the SDE. We, however, use the fact that the transition probability density function is the solution to Kolmogorov's forward equation (KFE) which can be solved using the finite difference method. This enables the evaluation of

$$p(D|\theta) = \prod_i p(S_{t_{i+1}}, t_{i+1}|S_{t_i}, t_i, \theta) \tag{7}$$

where each $p(S_{t_{i+1}}, t_{i+1}|S_{t_i}, t_i, \theta)$ is the solution to

$$\frac{\partial p}{\partial t} = \frac{\partial^2(b^2 p)}{\partial S^2} - \frac{\partial(ap)}{\partial S} \tag{8}$$

with dirac-delta intitial condition

$$p(S, t_i|S_{t_i}, t_i) = \delta(S - S_{t_i}).$$

With the ability to evaluate $p(D|\theta)$ comes the ability to sample from $p(\theta|D)$. We intend to iteratively repeat the computational procedure to update the distribution of θ given new observations of S, so that the posterior distribution of θ for the current iteration will become the prior distribution of θ for the next iteration. Therefore we choose the SIR algorithm (see Smith and Gelfand [20] and Pitt and Shepherd [17]), a simulation procedure that requires only the ability to evaluate the likelihood function, and to sample from the prior density of θ, rather than the Metropolis-Hastings algorithm which requires the ability to evaluate the prior density (see Gilks et al. [9]).

The SIR algorithm is as follows:

1. Assume samples $\theta_i \sim p(\theta|D_s) \quad i = 1 \ldots n$
2. Given data $D_{s,t}$ compute K=$\frac{1}{n}\sum_i p(D_{s,t}|\theta_i, S_s)$ from the n samples
3. For each θ_i compute the importance weight $w_i = \frac{p(D_{s,t}|\theta_i,S_s)}{nK}$
4. Resample from the θ_i, using the importance weights w_i giving $m < n$ samples from $p(\theta|D_t)$. To resample we:
   (a) Form n intervals in $(0, 1]$ where the ith interval is $(a_i, b_i]$ with $a_i = \sum_{j=1}^{i-1} w_j$ and $b_i = \sum_{j=1}^{i} w_j$
   (b) Generate $(U_k)_{k=1\ldots m}$ a sequence of independent Uniform(0,1] random numbers
   (c) Select as the kth sample θ_i such that $U_k \in (a_i, b_i]$

Not only does this algorithm produce samples from the posterior distribution of θ, but in step 2 it also gives an approximation for the marginal likelihood of the model:

$$p(D_{s,t}|M, S_s) \approx \frac{1}{n}\sum_i p(D_{s,t}|\theta_i, S_s), \;\; \theta_i \sim p(\theta|D_s) \tag{9}$$

which is used in the calculation of the model probabilities (see section 3).

Having samples from the posterior density of θ we now obtain samples from the model predictive density of S. If the closed form solution to the risk neutral SDE is known samples of S are generated without discretisation error using

$$\begin{aligned} S_{Ti} &= g(W_T, S_0, \theta_i) \\ & \quad W_T \sim N(0, T) \end{aligned}$$

Otherwise the risk neutral SDE model is discretised to form the Euler approximation, with first order weak convergence

$$\begin{aligned} S_{t_{j+1}} &= S_{t_{j+1}} + a(S_{t_j}, t_j)\delta t + b(S_{t_j}, t_j)(W_{t_{j+1}} - W_{t_j}) \\ & \quad W_{t_{j+1}} - W_{t_j} \sim N(0, t_{j+1} - t_j). \end{aligned}$$

The simulation algorithm is as follows:

1. Assume samples $\theta_i \sim p(\theta|D_t) \quad i = 1 \dots m$
2. Generate p streams of $N(0,1)$ random numbers
3. For each θ_i use the p streams of standard Normally distributed random numbers to obtain p simulated S values which will be from the model predictive density $p(S_T, T|S_t, t, M)$

Finally these samples are used to solve equation (4) using Monte Carlo integration.

$$\begin{aligned} u(S_t, t) &= e^{-r(T-t)} E_t[f(S_T)] \\ &= e^{-r(T-t)} \int_0^\infty f(S_T) p(S_T, T|S_t, t, M) dS_T \\ &\approx e^{-r(T-t)} \frac{1}{m} \sum_{i=1}^m f(S_i) \end{aligned}$$

with

$$S_i \sim p(S_T, T|S_t, t, M). \tag{10}$$

This then gives the option price conditional on the model, but including the uncertainty over the values of parameters in the model. In some simple but important cases there is a closed form solution of equation (4), the option price conditional on a parameter value. These cases usually arise for simple option types, such as Vanilla calls or puts, with geometric Brownian Motion models for the underlying. In order to utilise the computational advantages of these closed form solutions, whilst maintaining our approach to parameter uncertainty, we apply Fubini's theorem to equation (4). Denoting the closed form solution as $v(S_t, S, \theta) = e^{-r(T-t)} E_t[f(S_T)|\theta]$ we obtain

$$\begin{aligned} u(S_t, t) &= e^{-r(T-t)} \int_\Theta \int_0^\infty f(S_T) p(S_T, T|S_t, t, \theta) dS_T p(\theta|D_t) d\theta \\ &= \int_\Theta e^{-r(T-t)} E_t[f(S_T)|\theta] p(\theta|D_t) d\theta \\ &= \int_\Theta v(S_t, t, \theta) p(\theta|D_t) d\theta \\ &\approx \frac{1}{m} \sum_{i=1}^m v(S_t, t, \theta_i). \end{aligned}$$

with

$$\theta_i \sim p(\theta|D_t). \tag{11}$$

Thus in such cases it suffices to generate posterior samples of θ from the data, then use these samples in a Monte Carlo integration of the closed form parameter-conditional option formula. The next section shows how uncertainty over model can be dealt with and integrated with our approach to parameter uncertainty.

3 Model Uncertainty

We deal with model uncertainty in an analogous manner. In equation (2) it is implicit that the model for the behaviour of S is given. To make this explicit we rewrite equation (2) as

$$u(S_t, t|M) = e^{-r(T-t)} E_t[f(S_T)|M] = e^{-r(T-t)} \int_0^\infty f(S_T) p(S_T, T|S_t, t, M) dS_T \tag{12}$$

so that the option price obtained is conditioned on the knowledge, or choice of, model, M. If, however, we have a set of candidate models, $\mathcal{M} = \{M_i\}_{i=1\ldots n}$, and are not certain which model to use for the behaviour of the underlying, we can write the option price as

$$\begin{aligned} u(S_t, t|\mathcal{M}) &= e^{-r(T-t)} E_t[f(S_T)|\mathcal{M}] \\ &= e^{-r(T-t)} \int_0^\infty f(S_T) \sum_{i=1}^n p(S_T, T|S_t, t, M_i) p(M_i|D_t) dS_T \\ &= e^{-r(T-t)} \sum_{i=1}^n \int_0^\infty f(S_T) p(S_T, T|S_t, t, M_i) dS_T p(M_i|D_t) \\ &= e^{-r(T-t)} \sum_{i=1}^n E_t[f(S_T)|M_i] p(M_i|D_t) \end{aligned} \tag{13}$$

where $p(M_i|D_t)$ is the posterior probability of M_i being true. In effect we use the predictive density of S given the set of models $\mathcal{M}$, which is the average of the predictive densities of each individual model $M_i \in \mathcal{M}$ weighted by that model's probability. From equation (13) it is clear that for option pricing purposes this is equivalent to taking the average of the option prices given by individual models, weighted by each model's probability. The probabilities of the models are updated by new observations of S using Bayes theorem

$$p(M_i|D_t) = \frac{p(D_{s,t}|M_i, S_s) p(M_i|D_s)}{\sum_k^n p(D_{s,t}|M_k, S_s) p(M_k|D_s)} \tag{14}$$

This technique is known as Bayesian model averaging (BMA). The advantage of using BMA is that the model uncertainty is explicit and directly affects the calculation of expectations involving S, in this case the calculation of the option price.

Since an approximation of the marginal likelihood $p(D_{s,t}|M_i, S_s)$ for each model is calculated in the SIR algorithm (see section 2), the calculation of the model probabilities is straightforward. All that we need for initiating the iterative process of calculating model probabilities given the arrival of data on S, are initial prior model probabilities. These can be based on the modeller's subjective knowledge. For example each model can be assigned equal initial probabilities.

4 Option pricing results

We illustrate our approach by pricing a one year call option on an equity index, the FTSE 100, with strike price $K = 5500$, interest rate $r = 0.075$, and $S_0 = 5669.1$ (the closing FTSE level on 9 December 1998). The call payoff function is $Max(S_T - K, 0)$. The data are 50 weekly closing levels of the FTSE 100 equity index from 30 December 1997 to 9 December 1998. This was taken from the Economist [1]. For simplicity we ignore the dividend yield on the index and use just two candidate models for the behaviour of the index. M_1 is the standard geometric Brownian Motion model, referred to in the introduction. This is defined by

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dZ_t \\ &= r S_t dt + \sigma S_t dW_t \end{aligned}$$

where $W_t = Z_t + \lambda t$ and $\lambda = \frac{\mu - r}{\sigma}$ is the market price of risk. M_2 is the constant elasticity of variance (CEV) model proposed by Cox and Ross [6]. This is defined by

$$\begin{aligned} dS_t &= \mu S_t dt + v S_t^{1-\alpha} dZ_t \\ &= r S_t dt + v S_t^{1-\alpha} dW_t. \end{aligned}$$

In this model the elasticity of instantaneous return variance with respect to price is a constant -2α (in economics the elasticity of y with respect to x is defined as dy/dx x/y) so that with $\alpha \in (0, 1]$ price and volatility $v S_t^{-\alpha}$ are inversely related. This models the theoretical argument that increases in a company's share price decreases its debt-equity ratio, and thus should reduce the variance of its share price return.

The transition probability density function for M_1 is available in closed form

$$p(S_T, T|S_t, t) = \frac{1}{\sqrt{2\pi(T-t)}\sigma S_T} exp\{-\frac{(\log \frac{S_T}{S_t} - (\mu - \frac{\sigma^2}{2})(T-t))^2}{2\sigma^2(T-t)}\} \tag{15}$$

which is the density function of a log-normally distributed variable.

The transition probability density function for M_2 is also available in closed form (see Emanuel and MacBeth [8] or Schroder [18])

$$p(S_T, T|S_t, t) = 2\alpha k^{\frac{1}{2\alpha}}(zw^{4\alpha-3})^{\frac{1}{4\alpha}} e^{-z-w} I_{1/2\alpha}(2\sqrt{zw}) \tag{16}$$

where

$$\begin{aligned} k &= \frac{2\mu}{v^2(2\alpha)(e^{\mu 2\alpha(T-t)} - 1)} \\ z &= kS_t^{2\alpha} e^{\mu 2\alpha(T-t)} \\ w &= kS_T^{2\alpha} \end{aligned}$$

where $I_q()$ is the modified Bessel function of the first kind of order q.

We provide priors for $\theta_1 = \sigma$ in M_1 and for $\theta_2 = (v, \alpha)$ in M_2. For simplicity we set μ equal to 0.1 for both models. We used

$$\begin{aligned} \sigma &\sim Uniform(0.1, 0.3) \\ \alpha &\sim Uniform(0, 1) \\ \log v &\sim N(5\alpha, 0.1) \end{aligned}$$

These are simple choices that restrict the parameters to economically feasible values, and enforces a positive correlation between the two parameters in M_2. We used 10,000 samples from the prior distribution of σ and 1000 samples from the prior distribution of (v, α), and obtained 5,000 and 500 samples from the posterior distributions respectively. We set $p(M_1) = p(M_2) = 1/2$ to reflect initial indifference between models. Table 1 shows this information and the resulting model marginal likelihoods, model posterior probabilities, the computed option price for each model and the combined Bayesian model average option price. As can be seen M_1 has slightly higher posterior probability and consequently a larger weight in the Bayesian model average option price. However, M_2 higher option price does contribute significantly to the Bayesian model average option price. In this example, therefore, the initial symmetric uncertainty over models within $\mathcal{M}$ is modified to favour M_1 based on the data and the prior distributions of the parameters.

Figures 1, 2 and 3 show the posterior distributions of the parameters. These were computed using a kernel density estimation as described by Silverman [19]. Interestingly while the posterior for σ shows good agreement with the prior, concentrating around the mid value, the marginal posterior for α shows larger mass around higher values, perhaps suggesting that our prior was a poor choice. An empirical Bayes methodology (see Carlin and Louis [4]) could utilise this empirical information to respecify the prior to one that

Table 1: FTSE100 model and call option results

Model	prior prob.	mar. likelihood	post. prob.	call price	BMA call price
M_1	0.5	2.74692e-138	0.602823	766.628	779.924
M_2	0.5	1.80984e-138	0.397177	800.124	779.924

better fits the data. In any case, as we would use this posterior as the prior for the next iteration, we would expect M_2 to improve its overall performance, as measured by its marginal likelihood, as the parameter distributions receive more information from the data.

5 Conclusions

We have demonstrated how the uncertainty over model and parameters can be integrated into the pricing of options using a Bayesian approach that utilises data to update both the distributions of model parameters and the probabilities of the models. Numerical and simulation algorithms were presented to implement this approach when the models are SDEs. This approach includes as a special case the standard approach which ignores model and parameter uncertainty by selecting one model and selecting fixed values (perhaps point estimates from data) for parameters. The standard approach may be recovered by

- setting the selected model's prior probability to 1, and hence all other models' prior probabilities to 0 and
- using dirac-delta parameter prior density functions, $p(\theta) = \delta(\theta - \theta^*)$ which equates to using the point value θ^*.

A combination of approaches is obtained by following either of the above.

There are, however, significant theoretical and algorithmic problems in the implementation of this approach. The theoretical problems lie in the standard disagreements over how to use subjective prior information, thus how to choose the set of candidate models $\mathcal{M}$, how to assign prior probabilites to models in $\mathcal{M}$, and how to choose prior distributions for parameters. These issues have been extensively discussed elsewhere (for example see Carlin and Louis [4] and Cox and Hinkley [5]). We just note that in this case we view a 'model' as being the SDE and the prior distributions of its parameters, and that given enough data, which should not be a problem in finance, and a wide enough set of candidate models, the Bayesian approach we suggest provides important information on models, and quantitatively utilises this information to form the option price. The automatic reduction of a model's contribution to the BMA option price when the data does not support the model is a key advantage.

An algorithmic problem with SIR is that the number of samples decreases in each batch of posterior samples. This could be overcome by forming a density estimate (see Silverman [19]), rather than just obtaining samples from the density, and then produce a new larger batch of samples from the density estimate, rather than using the resampled batch. The computation of $P(D|M)$ is intensive when D is large, θ is multidimensional and especially when $P(D|\theta)$ must be evaluated numerically, but it only needs to be done once for each model per new data set, and should benefit from parallelisation. Thus further research will be directed to systematic methods of choosing priors, candidate models, and also any algorithmic improvements to speed up $P(D|M)$ calculation when $P(D|\theta)$ is not in closed form.

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Figure 1. Posterior density of sigma in model 1

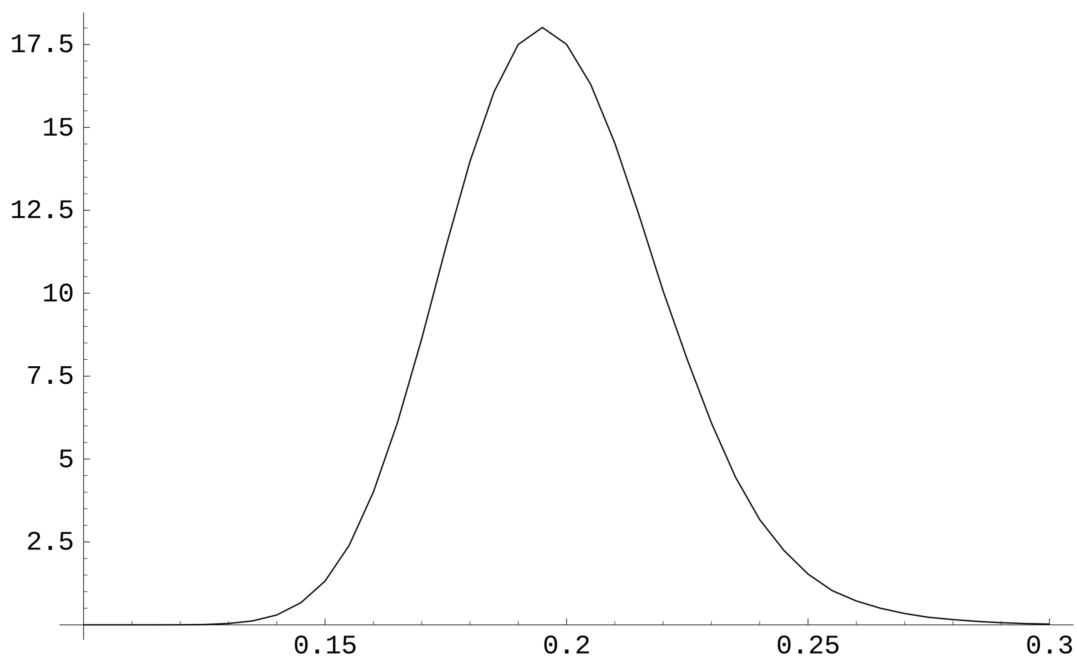
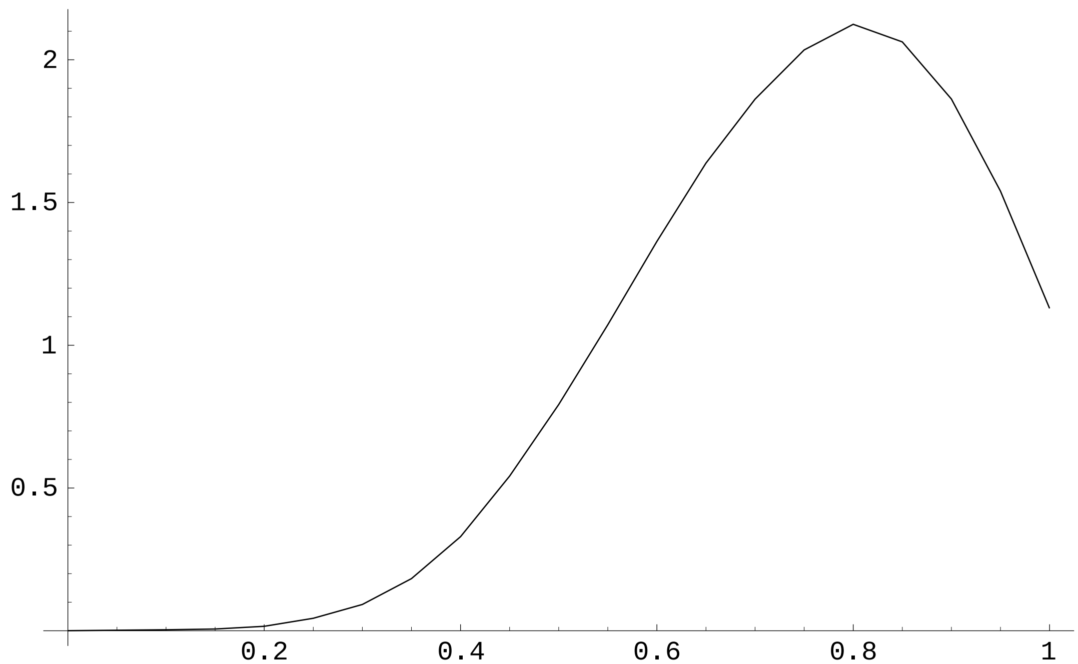


Figure 2. Marginal post. density of alpha in model2



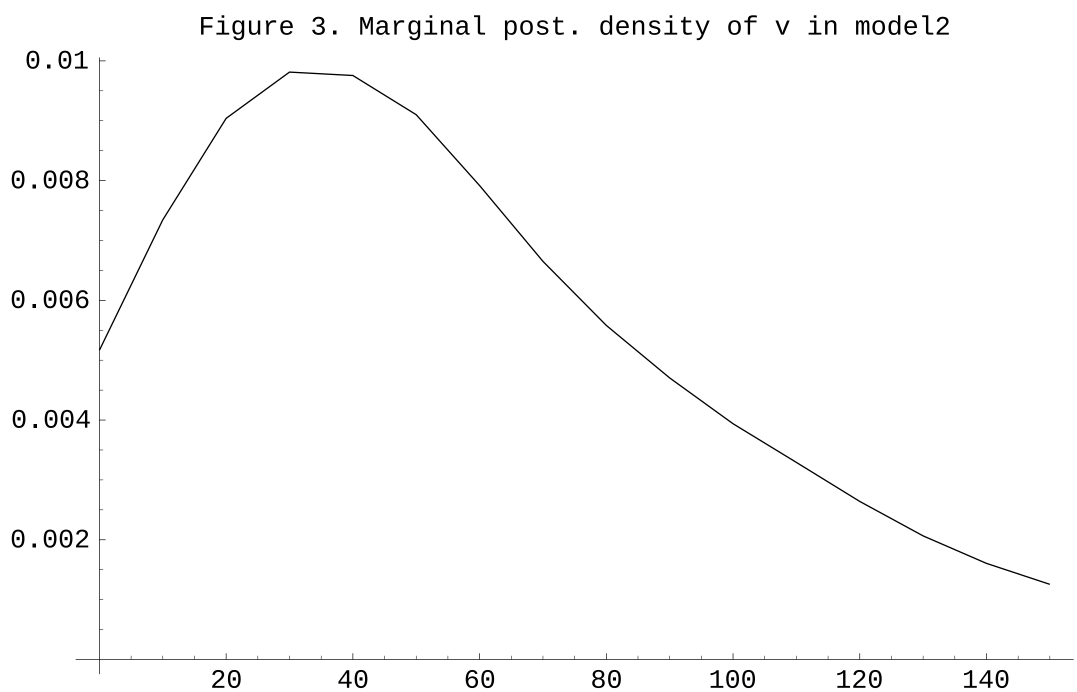


Figure 3. Marginal post. density of v in model2
0.01
0.008
0.006
0.004
0.002
20
40
60
80
100
120
140