Homogeneous Transformations*

(Com S 477/577 Notes)

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Aug 30, 2022

1 Projective Transformations

A *projective transformation* of the projective plane is a mapping $L : \mathbb{P}^2 \to \mathbb{P}^2$ defined as

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} l_{11}u + l_{12}v + l_{13}w \\ l_{21}u + l_{22}v + l_{23}w \\ l_{31}u + l_{32}v + l_{33}w \end{pmatrix}, \tag{1}$$

where the 3×3 matrix formed by the entries $l_{ij} \in \mathbb{R}$ is invertible. This matrix is called a *homogeneous transformation matrix*. When $l_{31} = l_{32} = 0$ and $l_{33} \neq 0$, the mapping L is an *affine transformation* introduced in the previous lecture. Affine transformations correspond to transformations of the Cartesian plane.

Note that homogeneous coordinates (ru, rv, rw) under the mapping (1) has the image

$$\begin{pmatrix} l_{11}ru + l_{12}rv + l_{13}rw \\ l_{21}ru + l_{22}rv + l_{23}rw \\ l_{31}ru + l_{32}rv + l_{33}rw \end{pmatrix}.$$

A division by r gives the image $L(u, v, w)$ of (u, v, w). Thus, $L(ru, rv, rw)$ and $L(u, v, w)$ are equivalent and correspond to the same point in homogeneous coordinates. The definition of a transformation does not depend on the choice of homogeneous coordinates for a given point.

A projective transformation preserves type, that is, it maps points to points and lines to lines. It also preserves incidence, that is, a point on a line has its image point on the image of the line.

1.1 Translation and Scaling

We first describe the homogeneous transformation matrices for translations and scalings, in the plane and the space. Let us start with translation:

$$\mathrm{Trans}(h, k) = \begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}.$$

*Appendices are optional for reading unless specifically required.

Then

$$\begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+h \\ y+k \\ 1 \end{pmatrix},$$

which verifies that the point $\binom{x}{y}$ is translated to $\binom{x+h}{y+k}$.

A translation by a, b, c in the x-, y-, and z-directions, respectively, has the transformation matrix:

$$\mathrm{Trans}(a,b,c) = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The point $\boldsymbol{p} = (x, y, z, 1)^\top$ is translated to the point

$$\begin{aligned} \boldsymbol{p}' &= \mathrm{Trans}(a,b,c)\,\boldsymbol{p} \\ &= \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} x+a \\ y+b \\ z+c \\ 1 \end{pmatrix}. \end{aligned}$$

Accordingly, the point $(x, y, z)^\top$ in the Cartesian space is translated to $(x + a, y + b, z + c)^\top$.

The homogeneous scaling matrix is

$$\mathrm{Scale}(s_x, s_y) = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} s_x x \\ s_y y \\ 1 \end{pmatrix}.$$

Example 1. The unit square with vertices $\binom{1}{1}$, $\binom{2}{1}$, $\binom{2}{2}$, and $\binom{1}{2}$ is scaled about the origin by factors of 4 and 2 in the x- and y- directions, respectively. We have

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 8 & 4 \\ 2 & 2 & 4 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

So the image is a square with vertices $\binom{4}{2}$, $\binom{8}{2}$, $\binom{8}{4}$, and $\binom{4}{4}$.

A scaling about the origin by factors s_x/s_w, s_y/s_w, and s_z/s_w in the x-, y-, and z-directions, respectively, has the transformation matrix (often, s_w is chosen to be 1):

$$\mathrm{Scale}(s_x, s_y, s_z, s_w) = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & s_w \end{pmatrix}.$$

Similar to the cases of translation and scaling, the transformation matrix for a planar rotation about the origin through an angle θ is

$$\mathrm{Rot}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1.2 Planar Rotation About a Point

A rotation through an angle θ about a point $\binom{a}{b}$ is obtained by performing a translation which maps $\binom{a}{b}$ to the origin, followed by a rotation through an angle θ about the origin, and followed by a translation which maps the origin to $\binom{a}{b}$. The rotation matrix is

$$\begin{aligned} \mathrm{Rot}_{(a,b)}(\theta) &= \mathrm{Trans}(a,b)\,\mathrm{Rot}(\theta)\,\mathrm{Trans}(-a,-b) \\ &= \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta & -a\cos\theta + b\sin\theta + a \\ \sin\theta & \cos\theta & -a\sin\theta - b\cos\theta + b \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{2}$$

Example 2. A square has vertices $\boldsymbol{p}_1 = \binom{1}{1}$, $\boldsymbol{p}_2 = \binom{2}{1}$, $\boldsymbol{p}_3 = \binom{2}{2}$, and $\boldsymbol{p}_4 = \binom{1}{2}$. Determine the new vertices of the square after a rotation about $\boldsymbol{p}_2$ through an angle of $\pi/4$. The transformation matrix is

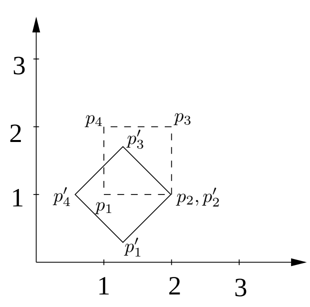


Figure 1: Rotation about p_2.

$$\mathrm{Rot}_{(2,1)}(\pi/4) = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 2 - \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{3\sqrt{2}}{2} + 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Apply the transformation above to the homogeneous coordinates of the vertices:

$$\begin{aligned} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 2 - \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{3\sqrt{2}}{2} + 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} &= \begin{pmatrix} 2 - \frac{\sqrt{2}}{2} & 2 & 2 - \frac{\sqrt{2}}{2} & 2 - \sqrt{2} \\ 1 - \frac{\sqrt{2}}{2} & 1 & 1 + \frac{\sqrt{2}}{2} & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \\ &\approx \begin{pmatrix} 1.2929 & 2 & 1.2929 & 0.5858 \\ 0.2929 & 1 & 1.7071 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \end{aligned}$$

Thus, the new vertices are $\boldsymbol{p}'_1 \approx \binom{1.2929}{0.2929}$, $\boldsymbol{p}'_2 = \binom{2}{1}$, $\boldsymbol{p}'_3 \approx \binom{1.2929}{1.7071}$, and $\boldsymbol{p}'_4 \approx \binom{0.5858}{1.0}$, as illustrated in Figure 1.

1.3 Application — Instancing

A geometric object is created by defining its components. For example, the front of a house in Figure 2 consists of rectangles, which form the walls, windows, and door of the house. The rectangles are scaled from a square, which is an example of a *picture element*. For convenience, picture elements are defined in their own local coordinate systems, and are constructed from *graphical primitives* which are the basic building blocks. Picture elements are defined once but may be used many times in the construction of objects.

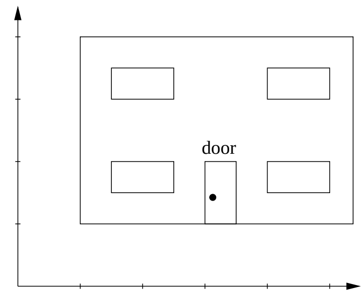


Figure 2: Front of a house obtained from instances of square and point.

For example, a square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ can be obtained using the graphical primitive `Line` for the line segment joining the points $(0,0)$ and $(1,0)$ through rotations and translations. A transformed copy of a graphical primitive or picture element is referred to as an instance. The aforementioned square, denoted `Square`, is defined by four instances of the line.

The completed 'real' object is defined in world coordinates by applying a transformation to each picture element. The house in the figure is defined by six instances of the picture element `Square`, and one instance of the primitive `Point` (for the door handle). In particular, the front door is obtained from `Square` by applying a scaling of 0.5 unit in the x-direction, followed by a translation of 3 units in the x-direction and 1 unit in the y-direction. In homogeneous coordinates, the transformation matrix is given by

$$\mathrm{Trans}(3,1) \circ \mathrm{Scale}(0.5,1) = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The vertices of the door are obtained by applying the above transformation matrix to the vertices of the `Square` primitive, giving

$$\begin{pmatrix} 0.5 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3.5 & 3.5 & 3 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

So the vertices of the door in world coordinates are $\binom{3}{1}$, $\binom{3.5}{1}$, $\binom{3.5}{2}$, and $\binom{3}{2}$.

2 Rigid Body Transformation in Two Dimensions

A rigid body consists of particles whose relative distances and orientations do not change as they move. A *rigid body transformation* is any change which preserves the relative distance and orientation[1] of all the points inside the body. Every rigid body transformation is composed of a rotation about some reference point $\boldsymbol{p}_0 = (x_0, y_0)^\top$ and a translation of the reference point. It is therefore an affine transformation and subsequently a projective transformation.

[1]Hence, reflection is excluded.

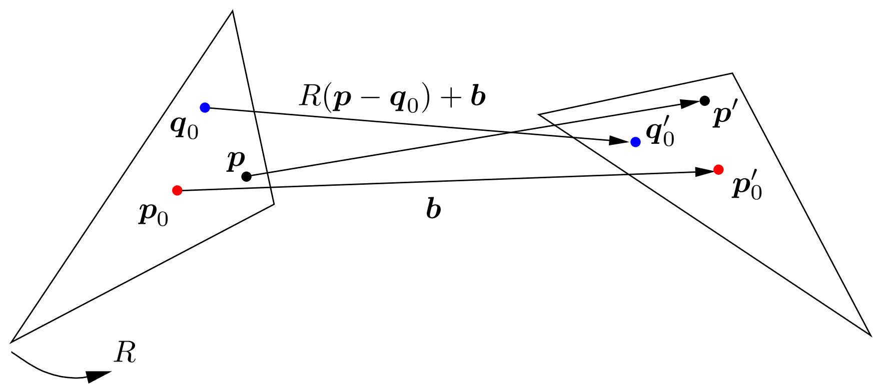


Figure 3: Rotation independent of the reference point in a rigid body transformation.

In this section we will temporarily revert back to the use of Cartesian coordinates. Let

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

be the rotation matrix, and $\boldsymbol{b} = (b_x, b_y)^\top$ be the translation vector. Then a point $\boldsymbol{p}$ is transformed into

$$\boldsymbol{p}' = R(\boldsymbol{p} - \boldsymbol{p}_0) + \boldsymbol{p}_0 + \boldsymbol{b}.$$

It turns out that the rotation part of a rigid body transformation does not depend on the reference point chosen. Suppose that a different point $\boldsymbol{q}_0$ has been chosen as the reference point. Then we rewrite

$$\begin{aligned} \boldsymbol{p}' &= R(\boldsymbol{p} - \boldsymbol{q}_0 + \boldsymbol{q}_0 - \boldsymbol{p}_0) + \boldsymbol{p}_0 + \boldsymbol{b} \\ &= R(\boldsymbol{p} - \boldsymbol{q}_0) + \boldsymbol{q}_0 + \boldsymbol{c}, \end{aligned}$$

where $\boldsymbol{c} = R(\boldsymbol{q}_0 - \boldsymbol{p}_0) + \boldsymbol{p}_0 - \boldsymbol{q}_0 + \boldsymbol{b}$ is the new translation vector.

The same transformation from $\boldsymbol{p}$ to $\boldsymbol{p}'$ can also be regarded as a translation followed by a rotation, for we have

$$\begin{aligned} \boldsymbol{p}' &= R(\boldsymbol{p} - \boldsymbol{p}_0 + R^{-1}\boldsymbol{b}) \\ &= R(\boldsymbol{p} + \boldsymbol{e} - \boldsymbol{r}) \end{aligned}$$

where $\boldsymbol{e} = \boldsymbol{r} - \boldsymbol{p}_0 + R^{-1}\boldsymbol{b}$ is the translation, and the rotation is about the new reference point $\boldsymbol{r}$.

Since rotation does not depend on the choice of reference point and it can be made before translation, we choose the origin as the reference point and represent a rigid body transformation as $T : \boldsymbol{p} \mapsto R\boldsymbol{p} + \boldsymbol{b}$.

If the rigid body transformation T is in the plane, it is called a *planar displacement*. Suppose that T contains a rotation through some angle θ with $\theta \neq 0$. There exists a point $\boldsymbol{s}$ that does not move under the transformation. It is obtained from solving the equation $\boldsymbol{s} = R\boldsymbol{s} + \boldsymbol{b}$:

$$\boldsymbol{s} = (I_2 - R)^{-1}\boldsymbol{b},$$

where

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is the 2×2 identity matrix, and

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is the rotation matrix. We can easily verify that the matrix $I_2 - R$ has determinant $2 - 2\cos\theta$, which is non-zero since $\cos\theta \neq 1$. The fixed point $\boldsymbol{s}$ is called the *pole* of the transformation T. It does not exist if T is a pure translation.

3 Inverse Transformations

The *identity transformation* I is the transformation that leaves all points of the plane unchanged. More precisely, I is the transformation for which $I \circ L = L \circ I = L$, for any planar transformation L. The transformation matrix of the identity transformation in homogeneous coordinates is the 3×3 identity matrix I_3.

The *inverse* of a transformation L, denoted L^{-1}, maps images of L back to the original points. More precisely, the inverse L^{-1} satisfies that $L^{-1} \circ L = L \circ L^{-1} = I$.

Lemma 1 *Let T be the matrix of the homogeneous transformation L. If the inverse transformation L^{-1} exists, then T^{-1} exists and is the transformation matrix of L^{-1}. Conversely, if T^{-1} exists, then the transformation represented by T^{-1} is the inverse transformation of L.*

Proof Suppose L has an inverse L^{-1} with transformation matrix R. The concatenations $L \circ L^{-1}$ and $L^{-1} \circ L$ must be identity transformations. Accordingly, the transformation matrices TR and RT are equal to I_3. Thus R is the inverse of matrix T, that is, $R = T^{-1}$.

Conversely, suppose the matrix T has an inverse T^{-1}, which defines a transformation R. Since $T^{-1}T = TT^{-1} = I_3$, it follows that $R \circ L$ and $L \circ R$ are the identity transformation. By definition, R is the inverse transformation. $\square$

We easily obtain the inverses of translation, rotation, and scaling:

$$\begin{aligned} \mathrm{Trans}(h,k)^{-1} &= \mathrm{Trans}(-h,-k), \\ \mathrm{Rot}(\theta)^{-1} &= \mathrm{Rot}(-\theta), \\ \mathrm{Scale}(s_1,s_2)^{-1} &= \mathrm{Scale}(1/s_1, 1/s_2). \end{aligned}$$

A transformation $L : \mathbb{P}^2 \to \mathbb{P}^2$ with an inverse L^{-1} is called a *non-singular transformation.* Lemma 1 implies that a transformation is non-singular if and only if its transformation matrix is non-singular. Non-singular matrices A and B satisfies $(AB)^{-1} = B^{-1}A^{-1}$.

In homogeneous coordinates, the concatenation of transformations T_1 and T_2, denoted $T_2 \circ T_1$, can be carried out with matrix multiplications alone. For example, a rotation $\mathrm{Rot}(\theta)$ about the origin followed by a translation $\mathrm{Trans}(h,k)$ followed by a scaling $\mathrm{Scale}(s_x,s_y)$ has the homogeneous transformation matrix

$$\mathrm{Scale}(s_x,s_y) \circ \mathrm{Trans}(h,k) \circ \mathrm{Rot}(\theta) = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} s_x \cos\theta & -s_x \sin\theta & s_x h \\ s_y \sin\theta & s_y \cos\theta & s_y k \\ 0 & 0 & 1 \end{pmatrix}$$

EXAMPLE 3. Determine the transformation matrix of the inverse of the concatenation of transformations $\mathrm{Trans}(-2,5) \circ \mathrm{Rot}(-\pi/3)$. The transformation matrix L^{-1} of the inverse is the inverse of the corresponding matrix product:

$$\begin{aligned} L^{-1} &= \Big(\mathrm{Trans}(-2,5)\,\mathrm{Rot}(-\pi/3)\Big)^{-1} \\ &= \mathrm{Rot}(-\pi/3)^{-1}\,\mathrm{Trans}(-2,5)^{-1} \\ &= \mathrm{Rot}(\pi/3)\,\mathrm{Trans}(2,-5) \\ &= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 1+\frac{5\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & -\frac{5}{2}+\sqrt{3} \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

4 Reflection in an Arbitrary Line

How to determine the transformation matrix for reflection in an arbitrary line $ax + by + c = 0$? If $c = 0$ and either $a = 0$ or $b = 0$, then it is the reflection in either x- or y-axis that we considered before. The matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

In general, the reflection is obtained by transforming the line to one of the axes, reflecting in that axis, and then taking the inverse of the first transformation. Suppose $b \neq 0$. More specifically, the reflection is accomplished in the following five steps:

1. The line intersects the y-axis in the point $(0, -c/b)$.
2. Make a translation that maps $(0, -c/b)$ to the origin.
3. The slope of the line is $\tan\theta = -a/b$, where the angle θ made by the line with the x-axis remains the same after the translation. Rotate the line about the origin through an angle $-\theta$. This maps the line to the x-axis.
4. Apply a reflection in the x-axis.
5. Rotate about the origin by $-\theta$ and then translate by $(0, -c/b)$.

The concatenation of the above transformation is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -c/b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c/b \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta & \frac{2c}{b}\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta & -\frac{2c}{b}\cos^2\theta \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\tan\theta = -a/b$, it follows that $\cos^2\theta = 1/(1+\tan^2\theta) = b^2/(a^2+b^2)$ and $\sin^2\theta = 1-\cos^2\theta = a^2/(a^2+b^2)$. So $\sin\theta\cos\theta = \tan\theta\cos^2\theta = -ab/(a^2+b^2)$. Substituting these expressions into (4) yields

$$\begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} & -\frac{2ac}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} & -\frac{2bc}{a^2+b^2} \\ 0 & 0 & 1 \end{pmatrix}.$$

In homogeneous coordinates, multiplication by a factor does not change the point. So the above matrix can be scaled by a factor a^2+b^2 to remove all the denominators in the entries, yielding the reflection matrix

$$\mathrm{Ref}_{(a,b,c)} = \begin{pmatrix} b^2-a^2 & -2ab & -2ac \\ -2ab & a^2-b^2 & -2bc \\ 0 & 0 & a^2+b^2 \end{pmatrix}.$$

Note that the above matrix agrees with (3) in the cases of reflection in x- and y-axes. Thus we have removed the assumption $b \neq 0$ made for deriving the reflection matrix.

5 Rotations About the Coordinate Axes

A *transformation* of the projective space is a mapping $M : \mathbb{P}^3 \to \mathbb{P}^3$ given by

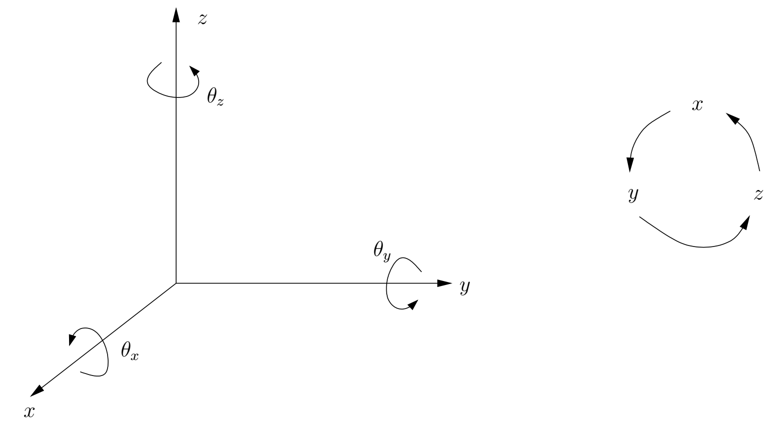
$$\begin{pmatrix} s \\ u \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} s \\ u \\ v \\ w \end{pmatrix},$$

The 4×4 matrix (m_{ij}) is called the *homogeneous transformation matrix* of M. If the matrix is non-singular, then M is called a *non-singular* transformation. If $m_{41} = m_{42} = m_{43} = 0$ and $m_{44} \neq 0$, then M is said to be an *affine transformation*. Affine transformations correspond to translations, scalings, rotations, reflections, etc. of the three-dimensional space.

Like a rotation in the plane, a rotation in the space takes about a line referred to as its *rotation axis*. Any rotation can be decomposed into three primary rotations about the x-, and y-, and z-axes:

$$\begin{aligned} \mathrm{Rot}_x(\theta_x) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ \mathrm{Rot}_y(\theta_y) &= \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ \mathrm{Rot}_z(\theta_z) &= \begin{pmatrix} \cos\theta_z & -\sin\theta_z & 0 & 0 \\ \sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Figure 4(a) shows the direction in which the primary rotations take when the rotation angle is positive. Figure 4(b) is a mnemonic that helps to remember the directions. For instance, the



(a) (b)

Figure 4: Rotations about the coordinate axes.

positive sense of a rotation about the y-axis has the effect of moving points on the z-axis toward the x-axis.

Example 4. A rotation through an angle $\pi/6$ about the y-axis followed by a translation by 1, -1, 2 respectively along the x-, y-, and z-axes has the transformation matrix

$$\mathrm{Trans}(1,-1,2)\,\mathrm{Rot}_y(\pi/6) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & -1 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A Reflection in an Arbitrary Plane

Reflection in a plane $ax + by + cz + d = 0$ is obtained by transforming the reflection plane to one of the xy-, xz-, or yz-planes, reflecting in that plane, and finally transforming the plane back to the reflection plane. In this appendix, we are back to the use of homogeneous coordinates so all the matrices are 4×4. More specifically, the transformation is obtained in the following steps.

1. Choose a point $\boldsymbol{p} = (p_1, p_2, p_3)$ on the plane. Translate this point to the origin so that the reflection plane becomes $ax + by + cz = 0$. Denote $\boldsymbol{r} = (a, b, c)$.

2. Then following steps 2–3 of the method of general rotation in Section B there are two angles θ_x and θ_y such that the composition of rotations $\mathrm{Rot}_y(-\theta_y) \circ \mathrm{Rot}_x(\theta_x)$ aligns the vector $\boldsymbol{r}$ with the z-axis, and maps the translated reflection plane to the xy-plane. We have $\theta_x = 0$ if $r_2 = r_3 = 0$; otherwise, $\sin\theta_x$ and $\cos\theta_x$ are defined in (5) and (6), respectively. Meanwhile, $\cos\theta_y$ and $\sin\theta_y$ are given by (7) and (8).

3. Apply the reflection in the xy-plane.

4. Apply the inverse of the transformations in steps 1–2 in reverse order.

The general reflection matrix is thus

$$\mathrm{Trans}(p_1, p_2, p_3)\,\mathrm{Rot}_x(-\theta_x)\,\mathrm{Rot}_y(\theta_y)\,\mathrm{Ref}_{xy}\,\mathrm{Rot}_y(-\theta_y)\,\mathrm{Rot}_x(\theta_x)\,\mathrm{Trans}(-p_1, -p_2, -p_3). \tag{4}$$

We can easily verify the above reflection matrix in the special cases where the plane is parallel to the yz-plane, xy-plane, or xz-plane. In the first case, the matrix (4) reduces to

$$\mathrm{Trans}(p_1, p_2, p_3)\,\mathrm{Ref}_{yz}\,\mathrm{Trans}(-p_1, -p_2, -p_3).$$

Example 6. Let us determine the transformation matrix for a reflection in the plane $2x - y + 2z - 2 = 0$. Pick a point, say, $(1, 0, 0)$, in the plane and translate it to the origin. The translated plane is $2x - y + 2z = 0$ which has a normal $(2, -1, 2)$. Next, we determine that

$$\begin{aligned}
\sin\theta_x &= -\frac{1}{\sqrt{5}}, \\
\cos\theta_x &= \frac{2}{\sqrt{5}}, \\
\sin(-\theta_y) &= -\sin\theta_y = -\frac{2}{3}. \\
\cos(-\theta_y) &= \cos\theta_y = \frac{\sqrt{5}}{3}.
\end{aligned}$$

The reflection matrix (4) becomes

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} \frac{\sqrt{5}}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{2}{3} & 0 & \frac{\sqrt{5}}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} \frac{\sqrt{5}}{3} & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{\sqrt{5}}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \frac{1}{9}\begin{pmatrix} 1 & 4 & -8 & 8 \\ 4 & 7 & 4 & -4 \\ -8 & 4 & 1 & 8 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

We can simply remove the multiplier $\frac{1}{9}$ in front of the matrix above since homogeneous coordinates are used.

B Rotation About an Arbitrary Line

When the rotation axis is an arbitrary line, we obtain the transformation matrix as follows. Firstly, transform the rotation axis to one of the coordinate axes. Secondly, perform a rotation of the required angle θ about the coordinate axis. Finally, transform the coordinate axis back to the original rotation axis. More specifically, let the rotation axis be the line ℓ through the points $\boldsymbol{p} = (p_1, p_2, p_3)$ and $\boldsymbol{q} = (q_1, q_2, q_3)$. Denote $\boldsymbol{r} = \boldsymbol{q} - \boldsymbol{p} = (r_1, r_2, r_3)$. Then we perform the following steps:

1. Translate the point $\boldsymbol{p}$ by $(-p_1, -p_2, -p_3)$ to the origin O and the rotation axis to the line $\overline{O\boldsymbol{r}}$ through O and the point $\boldsymbol{r}$.

2. Rotate the vector $\boldsymbol{r}$ about the x-axis until it lies in the xz-plane. This is shown in Figure 5(a). Suppose that the line $\overline{O\boldsymbol{r}}$ makes an angle θ_x with the xz-plane. If $r_2 = r_3 = 0$, then the line

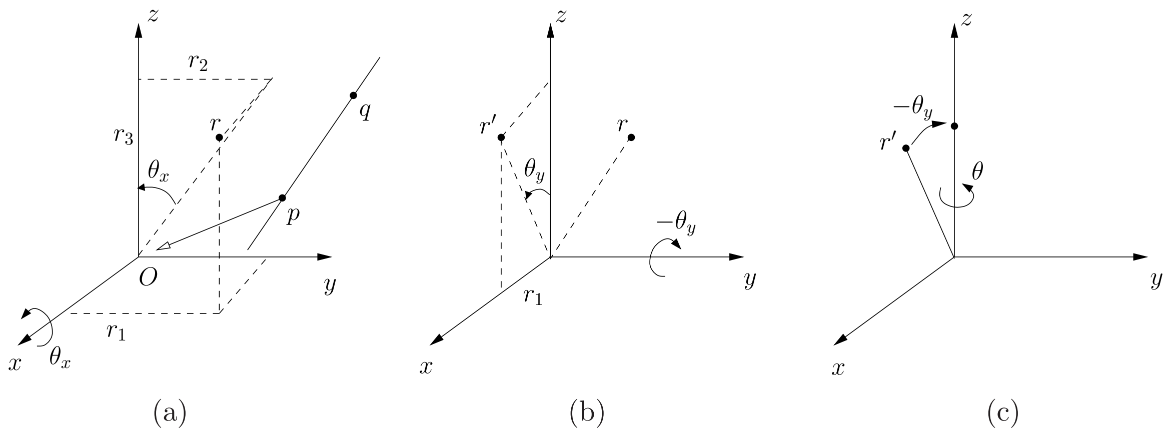


Figure 5: Rotation about an arbitrary axis performed by transforming the axis to the z-axis, applying the rotation, and transforming back to the original axis.

is aligned with the x-axis and $\theta_x = 0$. Otherwise, we have

$$\sin\theta_x = \frac{r_2}{\sqrt{r_2^2 + r_3^2}}, \tag{5}$$

$$\cos\theta_x = \frac{r_3}{\sqrt{r_2^2 + r_3^2}}. \tag{6}$$

The desired rotation $\mathrm{Rot}_x(\theta_x)$ maps $\boldsymbol{r}$ to the point $\boldsymbol{r}' = (r_1, 0, \sqrt{r_2^2 + r_3^2})$ shown in Figure 5(b).

3. Rotate the vector $\boldsymbol{r}'$ about the y-axis to align it with the z-axis. This step is shown in Figure 5(b). The required angle is found to be $-\theta_y$ where

$$\sin\theta_y = \frac{r_1}{\sqrt{r_1^2 + r_2^2 + r_3^2}}, \tag{7}$$

$$\cos\theta_y = \sqrt{\frac{r_2^2 + r_3^2}{r_1^2 + r_2^2 + r_3^2}}. \tag{8}$$

4. Apply a rotation through an angle θ about the z-axis, as shown in Figure 5(c).

5. Apply the inverses of the transformations in steps 1–3 in reverse order.

Thus, the general rotation through an angle θ about the line through two points $\boldsymbol{p} = (p_1, p_2, p_3)$ and $\boldsymbol{q} = (q_1, q_2, q_3)$ has the transformation matrix

$$\mathrm{Trans}(p_1, p_2, p_3)\,\mathrm{Rot}_x(-\theta_x)\,\mathrm{Rot}_y(\theta_y)\mathrm{Rot}_z(\theta)\,\mathrm{Rot}_y(-\theta_y)\,\mathrm{Rot}_x(\theta_x)\,\mathrm{Trans}(-p_1, -p_2, -p_3), \tag{9}$$

where $\sin\theta_x$, $\cos\theta_x$, $\sin\theta_y$, and $\cos\theta_y$ are given in (5)–(8) with $(r_1, r_2, r_3) = \boldsymbol{q} - \boldsymbol{p}$.

Example 5. Compute the transformation matrix of the rotation through an angle θ about the line through the points $\boldsymbol{p} = (2, 1, 5)$ and $\boldsymbol{q} = (4, 7, 2)$. We have

$$\boldsymbol{r} = \boldsymbol{q} - \boldsymbol{p} = (2, 6, -3).$$

So $\sqrt{r_2^2 + r_3^2} = 3\sqrt{5}$, and $\sin\theta_x = \frac{2}{\sqrt{5}}$, $\cos\theta_x = -\frac{1}{\sqrt{5}}$, $\sin\theta_y = \frac{2}{7}$, and $\cos\theta_y = \frac{3}{7}\sqrt{5}$. The rotation matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3\sqrt{5}}{7} & 0 & \frac{2}{7} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{2}{7} & 0 & \frac{3\sqrt{5}}{7} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{3\sqrt{5}}{7} & 0 & -\frac{2}{7} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{2}{7} & 0 & \frac{3\sqrt{5}}{7} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ 0 & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \frac{45}{49}\cos\theta + \frac{4}{49} & -\frac{12}{49}\cos\theta + \frac{3}{7}\sin\theta + \frac{12}{49} & \frac{6}{49}\cos\theta + \frac{6}{7}\sin\theta - \frac{6}{49} & -\frac{108}{49}\cos\theta - \frac{33}{7}\sin\theta + \frac{108}{49} \\ -\frac{12}{49}\cos\theta - \frac{3}{7}\sin\theta + \frac{12}{49} & \frac{13}{49}\cos\theta + \frac{36}{49} & \frac{18}{49}\cos\theta - \frac{2}{7}\sin\theta - \frac{18}{49} & -\frac{79}{49}\cos\theta + \frac{16}{7}\sin\theta + \frac{79}{49} \\ \frac{6}{49}\cos\theta - \frac{6}{7}\sin\theta - \frac{6}{49} & \frac{18}{49}\cos\theta + \frac{2}{7}\sin\theta - \frac{18}{49} & \frac{40}{49}\cos\theta + \frac{9}{49} & -\frac{230}{49}\cos\theta + \frac{10}{7}\sin\theta + \frac{230}{49} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

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