Problem Set 3: Random Variables and Simulation Methods

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Exercise 1. The OLS asymptotic behavior. (20 points) Prove that the OLS estimator is distributed asymptotically normal using Monte Carlo simulation.

$$\sqrt{n}\left(\hat{\beta}-\beta\right) \overset{as}{\sim} N\left(0, \sigma^2\left(\mathbb{E}(x_i x_i')\right)^{-1}\right)$$

To prove that, consider the model or data generating process seen in class

$$y_i = 2 + 0.5x_{2,i} + e_i$$
$$e_i \sim N(0,25)$$

a. As a preliminary step, create a sample with size N=100, a fictitious $x_2 \sim N(0,1)$, and estimate the model by OLS.

b. Now, following the steps in lecture 4, slide 12, run a Monte Carlo simulation to show the asymptotic properties of $\hat{\beta}_{OLS}$ and **replicate figure 1 and figure 2 from slide 13**. Set the number of simulations equal to T=10000. For each repetition of the experiment (step 3) use a sample size of $N = 100$. What is the type of distribution shown? what is the Monte-Carlo approximation of the $\mathbb{E}(\hat{\beta}_{OLS})$ and the approximation of $Var(\hat{\beta}_{OLS})$? [1]

c. Redo exercise a) with a sample size $N = 1000$. How do the $\hat{\beta}_{OLS}$ asymptotic distributions change when the sample size increases?

Exercise 2. Solving the earnings distribution in an Aiyagari economy (50 points). In the Aiyagari model—one of the fundamental models in quantitative macroeconomics—"there is a very large number of households" whose log of labor earnings (y_t) follows the next AR(1) process

$$y_t = \rho y_{t-1} + \varepsilon_t \quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2) \tag{1}$$

For this exercise let's use the following parameterization of the AR(1) process:
$\rho = 0.95$, $\sigma_\varepsilon = 0.25$ and an initial value $y_0 = 0$.

a. Simulate and plot the AR(1) process given by equation (1) for T=50 periods.

b. Create a function that simulates N AR(1) processes for T periods.

c. Simulate and plot 5 AR(1) processes given by equation (1) for t=50 periods. What are the stationary values of $\mathbb{E}(y_t)$ and $Var(y_t)$? What is the sample mean of y_t and its sample variance?

d. **The stationary earnings distribution.** Simulate for a large T (like T=1000) the AR(1) process of 10000 individuals. Use the result of the last period to plot the stationary distribution. Comment whether the stationary distributions of log-earnings y_t and earnings e^{y_t} resemble to distributions seen in in previous problem sets or lectures.

e. From the stationary distribution you previously computed, compute the mean and variance of the log of earnings (y_t) and compute the Gini coefficient of earnings (e^{y_t}). Do the sample analogs approximate well the true values of mean and variange of the log of earnings? Does this income process generate a high or low income inequality?

[1] Note that I use vector notation for the OLS coefficients. That is $\hat{\beta}_{OLS} = [\hat{\beta}_0 \quad \hat{\beta}_1]'$

f. For computational reasons, Aiyagari discretizes the AR(1) process into a Markov process of 7 states. Using the Rouwenhorst method, discretize the AR(1) process of this exercise into a 3 states Markov process. What is the resulting transition matrix P? What is the resulting stationary distribution ψ^*?

Exercise 3. Simulating distributions and computing expectations (20 points).

a. Simulate a binomial distribution with $n = 4$ and $p = 0.5$. Plot the resulting distribution.

b. Compute the expected value of a function $g(x) = x^2$ where x follows a Poisson distribution with $\lambda = 2$. Use Monte Carlo integration.
Is your result equal to $g(\mathbb{E}(x))$? where $\mathbb{E}(x) = \lambda = 2$ if $x \sim Poisson(\lambda)$? Why?

Exercise 4. Simulating and computing expectations in an economy (10 points) Consider an economy where individual's income (y) follows a log-normal distribution. That is $log(y) \sim N(\mu, \sigma^2)$ where $\mu = 7.5$, $\sigma = 0.8$.

a. Simulate y for $N = 100000$ and compute the average, the variance, and the Gini of y.

b. Now consider that individuals follow a consumption rule that takes the following functional form:

$$c_i = (y_i)^{0.8} + 0.5y_i + 500$$

Compute the average, the variance, and the Gini coefficient of c.

c. Plot the distribution of y and c in the same graph.