Problem Set 4: Root-Finding and Optimization

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Programming and Numerical Methods for Economics—ECNM10115
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Exercise 1. Testing algorithms with the Rosenbrock function (30 points). Let's use again the Rosenbrock function to test the different algorithms seen in class. For each algoritm report: the value $\mathbf{x}^*$ that minimizes the function and how much time it took to compute the solution. If the computation takes too much time (let's say 5 minutes) stop the computation and report that the algorithm took too much time. [1] The Rosenbrock function is

$$f(\mathbf{x}) = \sum_{i}^{N-1} \left((1 - x_i)^2 + (x_{i+1} - x_i^2)^2 \right)$$

Where N is the number of variables.

a. define the Rosenbrock function for a general number of variables N. [2]

b. **Testing the Brute-force algorithm**. Using a range of $(-2, 2)$ for all x_i, find the minimum of the Rosenbrock function using the brute-force method for the three following cases: $N = 3$, $N = 4$, and $N = 5$. How does computational time increases as the number of variables N increases?

   For the rest of the exercises use an initial value of zero for all the variables. That is $\mathbf{x}_0 = [x_{01}, x_{02}, .., x_{0N}] = [0, 0, ..., 0]$.

c. **Testing local optimization algorithms.** Find the minimum of the Rosenbrock function with $N = 30$ for the following three cases: using the BFGS method, using the Nelder-Mead method, and using the Powell method. Which algorithm performs better?

d. **Let's test the algorithms further**. Minimize the Rosenbrock function for 100 variables, N=100. Use the BFGS method, the Nelder-Mead method and the Powell method. If N=100 takes too much time in your computer, you can use another N as N=50.

e. **Algorithms comparison.** For this exercise we see that there is one algorithm that tends to to do better than the rest (especially when N is large). Which algorithm is and why?

[1] In Spyder you can stop the computation by clicking on the squared button in the console (or just close the console). If some of the fans in your laptop do not work and your laptop gets too hot, also stop the computation.

[2] You need to create the function, if you directly import it from somewhere else you will get a 0 for the whole exercise.

Exercise 2. Solving Cournot markets using root-finding routines (30 points).
Duopoly à la Cournot. Consider a market that is controlled by two firms that compete with each other on quantities. For this duopoly, the inverse of the demand function is given by

$$P(q) = q^{-\alpha}$$

and both firms face quadratic costs

$$C_1 = \frac{1}{2}c_1 q_1^2$$
$$C_2 = \frac{1}{2}c_2 q_2^2$$

. Thus, firm's profits are

$$\pi_1(q_1, q_2) = P(q_1 + q_2)q_1 - C_1(q_1)$$
$$\pi_2(q_1, q_2) = P(q_1 + q_2)q_2 - C_2(q_2)$$

a. Given that each firm maximizes its profits taking as given the other firm's output, find the first order conditions of each firm.

b. Find the Cournot equilibrium (q_1^*, q_2^*, p^*). Solve the system equations given by the previous first order conditions and report the equilibrium quantities of q_1^* and q_2^* and equilbrium price p^*. Provide an interpretation of the results.

Oligopolies à la Cournot. Assume the market is controlled by N firms that compete with each other. Firms face the same inverse demand function as before, the same quadratic costs, and each firm maximizes its profits taking as given the other firms' output.

c. Create a function that for N>1 computes the set of N first order conditions that characterize the Cournot equilibrium quantities.

d. Compute the Cournot equilibrium (q_i^*, p^*) under 3 firms with cost parameters $\mathbf{c} = [0.6, 0.8, 0.5]$ and under 6 firms with $\mathbf{c} = [0.6, 0.8, 0.5, 0.5, 0.4, 0.2]$.

e. Compute the Cournot equilibrium (q_i^*, p^*) for $N = 10$ and $N = 15$. Firms are identical with cost parameter $c_i = 0.6$ for all $i \in N$.

f. Given your previous results, plot the equilibrium prices, p^*, of Cournot oligopolies with number of firms $N = 2, 3, 6, 10, 15$. Provide an interpretation of the results.

Exercise 3. Optimal life-cycle consumption paths: (40 points). Consider the problem of an individual that maximizes his/her lifetime utility by choosing consumption levels across periods. The per period utility funciton takes a CRRA form

$$u(c_t) = \frac{c_t^{1-\theta}}{1-\theta}$$

where we assume a constant relative risk aversion parameter of $\theta = 1.5$. The individual discounts future consumption at discount rate $\beta = 0.96$. The individual can save and borrow in a risk-free asset a_t at an interest rate r. For simplicity, let's assume $r = \frac{1}{\beta} - 1$.

a. **A two-period model**: consider the problem of an individual that lives for two periods ($t = 0, t = 1$) and maximizes his/her lifetime utility, subject that the budget constraint at each period hold. The maximization problem is

$$\max_{c_0, c_1, a_1} \{u(c_0) + \beta u(c_1)\}$$
$$\text{st:}$$
$$c_0 + a_1 = a_0 + y_0$$
$$c_1 = (1+r)a_1 + y_1$$

Where a_0 is an initial assets endowment, y_0, y_1 are the income flows at period 0 and period 1 respectively. Using the following parameter values $a_0 = 0, y_0 = 10, y_1 = 5$, find the optimal consumption c_0^*, c_1^* that solves the previous problem. plot the life-cycle of income and optimal consumption. That is a line plot where the x-axis are the time periods: $t = 0, 1$, line-1 is y_0, y_1, and line-2 is c_0^*, c_1^*.

b. **A life-cycle model with 4 periods**. Solve the following maximization problem

$$\max_{\mathbf{c},\mathbf{a}}\{u(c_0) + \beta u(c_1) + \beta^2 u(c_3) + \beta^3 u(c_2)\}$$

$$\begin{aligned}
&\text{st:} \\
c_0 + a_1 &= a_0 + y_0 \\
c_1 + a_2 &= (1+r)a_1 + y_1 \\
c_2 + a_3 &= (1+r)a_2 + y_2 \\
c_3 &= (1+r)a_3 + y_3
\end{aligned}$$

and plot the life-cycle of income and optimal consumption. Use the following parameter values: $a_0 = 0$, and $\mathbf{y} = [y_0, y_1, y_2, y_3] = [5, 10, 15, 0]$

c. **Comparative statics.** Solve the problem in b) for r=0.01 (case 1), for r=0.04 (case 2) and for r=0.08 (case 3). Plot the optimal consumption path along the 4 periods for each of the three cases. How does the optimal life-cycle consumption changes as r changes?

d. **Optimal consumption paths under income risk**. Now let's solve the original 2-period model with uncertainty. Suppose that period 1 income, y_1, is unknown: with probability $p^l = 0.5$ it takes a low realization $y_1^l = 2.5$, with probability $p^h = 0.5$ it takes a high realization $y_1^h = 7.5$. The rest of parameters and assumptions are exactly as in exercise a). solve the optimal consumption and saving decisions that maximize the expected lifetime utility. That is c_0, c_1, s_1 that solves the following problem

$$\begin{aligned}
&\max_{c_0,c_1,a_1}\{u(c_0) + \beta\mathbb{E}\left[u(c_1)\right]\} = \\
= &\max_{c_0,c_1,a_1}\{u(c_0) + \beta\left[p^l u(c_1(y_1^l)) + p^h u(c_1(y_1^h))\right]\} \\
&\text{st:} \\
&c_0 + a_1 = a_0 + y_0 \\
&c_1 = (1+r)a_1 + y_1
\end{aligned}$$

Plot the income and optimal consumption paths when y_t takes a low realization and when y_t takes a high realization.

e. **Precautionary Savings.** Compare the savings in a world with certainty (case a), with respect to the world with uncertainty (d). Why do savings change as there is income risk? Provide an economic or "real life" intuition.